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Existence Results for Nonlinear Coupled Hilfer Fractional Differential Equations with Nonlocal Riemann–Liouville and Hadamard-Type Iterated Integral Boundary Conditions

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Abstract: We introduce and study a new class of nonlinear coupled Hilfer differential equations with nonlocal boundary conditions involving Riemann–Liouville and Hadamard-type iterated fractional integral operators. By applying the Leray–Schauder alternative and Krasnosel'skiĭ's fixed point theorem, two results presenting different criteria for the existence of solutions to the given problem are proven. The third result provides a sufficient criterion for the existence of a unique solution to the problem at hand. Numerical examples are constructed to demonstrate the application of the results obtained. Two graphs show asymmetric solutions when a Hilfer parameter is varied. The work presented in this paper is novel and significantly enriches the literature on the topic.

Keywords: coupled system; Hilfer fractional derivative; Riemann–Liouville and Hadamard iterated fractional integrals; nonlocal boundary conditions; existence; fixed point

MSC: 34A08; 34B10

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1. Introduction

The tools of fractional calculus are found to be of great utility in improving the mathematical modeling of many real-world processes and phenomena occurring in natural and social sciences. Examples include fractional calculus in financial economics [1], fractional dynamics [2], fractional viscoelastic fluid model [3], ecology [4], the fractional Cattaneo–Friedrich Maxwell model [5], bio-engineering [6], diffusion-thermo phenomena in a Darcy medium [7], COVID-19 infection and epithelial cells [8], fractional advection–reaction–diffusion equations with a Rabotnov fractional-exponential kernel [9], the fractional-order model of the Navier–Stokes equation [10], vaccination for COVID-19 with the fear factor [11], etc. For a theoretical background of the topic, for instance, see the monographs [12–17]. Unlike the classical derivative, there exist a variety of fractional derivatives due to Riemann–Liouville, Caputo, Hadamard, Hilfer, and derivative of a function with respect to another function, etc.; for details, see [12,15].

Fractional order boundary value problems (FBVPs) have been extensively investigated in the literature. One can find a detailed account of some recent works on FBVPs involving Caputo, Riemann–Liouville, Hadamard, Hadamard–Caputo, and generalized fractional derivative operators and different kinds of boundary conditions in [18–27] and the references cited therein.

In particular, the Hilfer fractional derivative operator [28] gained much interest as it includes both Riemann–Liouville as well as Caputo fractional derivative operators. The

Hilfer-type fractional differential equations appear in the mathematical modeling of filtration processes [29,30], advection-diffusion phenomena [31], glass forming materials [32], etc. For more details on application of Hilfer fractional differential equations, for instance, see [33–35], while some recent theoretical results on Hilfer fractional neutral evolution equations and functional integro-differential equations can, respectively, be found in [36,37].

Let us now dwell on some recent works dealing with theoretical aspects of Hilfer-fractional differential equations. In [38], the authors discussed the existence of solutions for a Hilfer-fractional differential equation supplemented with nonlocal multi-point integral boundary conditions. Recently, in [39], a boundary value problem involving Riemann–Liouville and Hadamard–Caputo-type sequential fractional derivatives and iterated fractional integral boundary conditions was investigated. More recently, the authors studied a nonlinear Hilfer iterated-integro-differential equation combined with Riemann–Liouville and Hadamard-type iterated fractional integral boundary conditions in [40].

On the other hand, coupled systems of fractional differential equations are also of significant importance as they appear in the mathematical models of several natural phenomena such as chaos synchronization [41], anomalous diffusion [42], ecological effects [4], disease models [43], etc.

Motivated by the work established in [40], we enrich the literature on this class of problems by introducing a new class of coupled systems of nonlinear Hilfer iterated-integro-differential equations equipped with multi-point iterated Riemann–Liouville and Hadamard fractional integral boundary conditions. Precisely, we explore the criteria ensuring the existence and uniqueness of solutions for the following problem:

$$\begin{cases} ({}^{H}D^{\alpha_1,\beta_1}x)(t)+\lambda_1({}^{H}D^{\alpha_1-1,\beta_1}x)(t)=f\Big(t,x(t),R^{(\delta_q,\cdots,\delta_1)}x(t),y(t)\Big),t\in(0,T]\\ ({}^{H}D^{\alpha_2,\beta_2}y)(t)+\lambda_2({}^{H}D^{\alpha_2-1,\beta_2}y)(t)=g\Big(t,x(t),y(t),R^{(\zeta_p,\cdots,\zeta_1)}y(t)\Big),t\in(0,T]\\ x(0)=0,\ \ x(T)=\displaystyle\sum_{i=1}^{m}\varepsilon_i R^{(\mu_\rho,\cdots,\mu_1)}y(\eta_i),\ \ \eta_i\in(0,T),\\ y(0)=0,\ \ y(T)=\displaystyle\sum_{j=1}^{n}\theta_j R^{(\nu_\rho,\cdots,\nu_1)}x(\xi_j),\ \ \xi_i\in(0,T), \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha_l,\beta_l}$ is the Hilfer fractional derivative operator of order α_l with parameters β_l, $l\in\{1,2\}, 1<\alpha_l<2, 0\le\beta_l\le1, \lambda_1,\lambda_2,\varepsilon_i,\theta_j\in\mathbb{R}\backslash\{0\}, i=1,2,\cdots,m, j=1,2,\cdots,n, f,g:$ $[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are nonlinear continuous functions, and $R^{(\phi_\tau,\cdots,\phi_1)}$, $\phi_r\in\{\delta,\zeta,\mu,\nu\}$, $r\in\{q,p,\rho|\ q,p,\rho\in\mathbb{N}\}$, involves the iterated Riemann–Liouville and Hadamard fractional integral operators defined by

$$R^{(\phi_r,\ldots,\phi_1)}x(t)=\begin{cases} {}^{H}I^{\phi_r}\,I^{\phi_{r-1}}\,{}^{H}I^{\phi_{r-2}}\,I^{\phi_{r-3}}\ldots{}^{H}I^{\phi_4}\,I^{\phi_3}\,{}^{H}I^{\phi_2}\,I^{\phi_1}x(t);\ \ r \text{ is even,}\\ I^{\phi_r}\,{}^{H}I^{\phi_{r-1}}\,I^{\phi_{r-2}}\,{}^{H}I^{\phi_{r-3}}\ldots{}^{H}I^{\phi_4}\,I^{\phi_3}\,{}^{H}I^{\phi_2}\,I^{\phi_1}x(t);\ \ r \text{ is odd,}\end{cases}$$

$I^{\phi_{(\cdot)}}$ and ${}^{H}I^{\phi_{(\cdot)}}$, respectively, represent the Riemann–Liouville and Hadamard fractional integral operators of order $\phi_{(\cdot)}>0$.

Here, we emphasize that the problem investigated in this paper is novel in the sense that it consists of coupled multi-term Hilfer fractional differential equations with nonlinearities and nonlocal boundary data depending upon the iterated Riemann–Liouville and Hadamard-type fractional integral operators. Two results (Theorems 3.1 and 3.2) containing different criteria for the existence of solutions for the given problem are presented. In the third result, we provide a sufficient criterion for the the unique solution of the problem at hand. We believe that the work accomplished in this paper is a useful contribution to the existing literature on Hilfer-type fractional boundary value problems in view of the fact that the Hilfer fractional derivative reduces to Riemann–Liouville and Caputo fractional derivatives for $\beta=0$ and $\beta=1$, respectively.

The remaining part of this manuscript is outlined as follows. Section 2 is devoted to some basic concepts of fractional calculus. Section 3 contains the main results, which rely on the standard tools of the fixed point theory. In Section 4, we present examples illustrating the main results.

2. Preliminaries

This section is devoted to the basic concepts of fractional calculus related to our work.

Definition 1 ([12]). *The Riemann–Liouville and Hadamard fractional integrals of order $\alpha > 0$ for an integrable function $f : [0,\infty) \to \mathbb{R}$ are, respectively, defined as*

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ \ 0 < t < \infty,$$

$$({}^{H}I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \ \ 0 < t < \infty,$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2 ([28]). *For $0 \le n-1 < \alpha < n$ ($n \in \mathbb{N}$) and $0 \le \beta \le 1$, the Hilfer fractional derivative of order α with parameter β for a function f on $[0,\infty)$ is defined as*

$$({}^{H}D^{\alpha,\beta} f)(t) = I^{\beta(n-\alpha)} \left(\frac{d}{dt}\right)^{n} I^{(1-\beta)(n-\alpha)} f(t),$$

where $I^{(\cdot)}$ denotes the Riemann–Liouville fractional integral operator of order $(.)$.

We use the following known results in the sequel.

Lemma 1 ([44]). *Let $f \in L^1(0,T)$ and $I^{(n-\gamma)} f \in C^n([0,T],\mathbb{R})$, $\alpha \in (n-1,n]$, $n \in \mathbb{N}$, $\beta \in [0,1]$, $\gamma = \alpha + n\beta - \alpha\beta$. Then,*

$$\left(I^{\alpha} \, {}^{H}D^{\alpha,\beta} f\right)(t) = f(t) - \sum_{j=0}^{n-1} \frac{t^{j-(n-\gamma)}}{\Gamma(j-(n-\gamma)+1)} \lim_{t\to 0^+} \frac{d^k}{dt^k} (I^{n-\gamma} f)(t), j = 0,1,\ldots,n-1.$$

Lemma 2 ([12]). *For positive real numbers α and m, ${}^{H}I^{\alpha} t^m = m^{-\alpha} t^m$.*

Lemma 3 ([39]). *Foe $m > -1$, $\mu_i > 0$, $i = 1,2,\cdots,n$, we have*

$$R^{(\mu_n,\ldots,\mu_1)} t^m = \Gamma(m+1) \frac{\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(m + \sum_{k=1}^{i} \mu_{2k-1}\right)^{-\mu_{2i}}}{\Gamma\left(m + 1 + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}\right)} \times t^{m + \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \mu_{2k-1}}, \tag{2}$$

where $\lceil n \rceil$, $\lfloor n \rfloor$, respectively, represent the ceiling and floor functions of n.

The following lemma, related to the linear variant of the system (1), is of fundamental importance to convert the problem (1) into a fixed point problem.

Lemma 4. *Let $1 < \alpha_1, \alpha_2 < 2$, $0 \le \beta_1, \beta_2 \le 1$, $\gamma_i = \alpha_i + \beta_i(2-\alpha_i)$, $i = 1,2$, $\Delta \ne 0$ and $h_1, h_2 \in C([0,T],\mathbb{R})$. Then, the pair (x,y) is a solution of the coupled system is*

$$\begin{cases} ({}^H D^{\alpha_1,\beta_1} x)(t) + \lambda_1 ({}^H D^{\alpha_1-1,\beta_1} x)(t) = h_1(t),\ t \in [0,T], \\ ({}^H D^{\alpha_2,\beta_2} y)(t) + \lambda_2 ({}^H D^{\alpha_2-1,\beta_2} y)(t) = h_2(t),\ t \in [0,T], \\ x(0) = 0,\ \ x(T) = \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1)} y(\eta_i),\ \ \eta_i \in (0,T), \\ y(0) = 0,\ \ y(T) = \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_1)} x(\xi_j),\ \ \xi_i \in (0,T), \end{cases} \tag{3}$$

if and only if

$$\begin{aligned} x(t) \quad = \quad & \frac{t^{\gamma_1-1}}{\Delta\Gamma(\gamma_1)}\Bigg[C\Bigg(\lambda_1 \int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)} \int_0^T (T-s)^{\alpha_1-1} h_1(s)ds \\ & -\lambda_2 \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} h_2(\eta_i)\Bigg) \\ & +B\Bigg(\lambda_2 \int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)} \int_0^T (T-s)^{\alpha_2-1} h_2(s)ds \\ & -\lambda_1 \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} h_1(\xi_j)\Bigg)\Bigg] \\ & -\lambda_1 \int_0^t x(s)ds + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} h_1(s)ds, \end{aligned} \tag{4}$$

$$\begin{aligned} y(t) \quad = \quad & \frac{t^{\gamma_2-1}}{\Delta\Gamma(\gamma_2)}\Bigg[D\Bigg(\lambda_1 \int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)} \int_0^T (T-s)^{\alpha_1-1} h_1(s)ds \\ & -\lambda_2 \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} h_2(\eta_i)\Bigg) \\ & +A\Bigg(\lambda_2 \int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)} \int_0^T (T-s)^{\alpha_2-1} h_2(s)ds \\ & -\lambda_1 \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} h_1(\xi_j)\Bigg)\Bigg] \\ & -\lambda_2 \int_0^t y(s)ds + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} h_2(s)ds, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Lambda_0 \quad &= \quad \Gamma\Bigg(\gamma_2 + \sum_{l=1}^{\lceil \frac{\rho}{2} \rceil} \mu_{2l-1}\Bigg), \qquad \Lambda_1 = \Gamma\Bigg(\gamma_1 + \sum_{l=1}^{\lceil \frac{\rho}{2} \rceil} \nu_{2l-1}\Bigg), \\ A \quad &= \quad \frac{T^{\gamma_1-1}}{\Gamma(\gamma_1)},\ B = \frac{1}{\Lambda_0} \sum_{i=1}^{m} \varepsilon_i \prod_{k=1}^{\lfloor \frac{\rho}{2} \rfloor} \Bigg(\gamma_2 - 1 + \sum_{l=1}^{k} \mu_{2l-1}\Bigg)^{-\mu_{2k}} \times \eta_i^{\gamma_2-1+\sum_{l=1}^{\lceil \frac{\rho}{2} \rceil} \mu_{2l-1}}, \\ C \quad &= \quad \frac{T^{\gamma_2-1}}{\Gamma(\gamma_2)},\ D = \frac{1}{\Lambda_1} \sum_{j=1}^{n} \theta_j \prod_{k=1}^{\lfloor \frac{\rho}{2} \rfloor} \Bigg(\gamma_1 - 1 + \sum_{l=1}^{k} \nu_{2l-1}\Bigg)^{-\nu_{2k}} \times \xi_j^{\gamma_1-1+\sum_{l=1}^{\lceil \frac{\rho}{2} \rceil} \nu_{2l-1}}, \end{aligned} \tag{6}$$

and

$$\Delta = AC - BD.$$

Proof. Let the pair (x, y) be a solution of the system (3). Operating the Riemann–Liouville fractional integral operators of orders α_1 and α_2 on both sides of the first and second equations of (3), respectively, we obtain

$$x(t)-\frac{c_0t^{\gamma_1-2}}{\Gamma(\gamma_1-1)}-\frac{c_1t^{\gamma_1-1}}{\Gamma(\gamma_1)}+\lambda_1\int_0^t x(s)ds=\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}h_1(s)ds, \tag{7}$$

and

$$y(t)-\frac{d_0t^{\gamma_2-2}}{\Gamma(\gamma_2-1)}-\frac{d_1t^{\gamma_2-1}}{\Gamma(\gamma_2)}+\lambda_2\int_0^t y(s)ds=\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}h_2(s)ds, \tag{8}$$

where
$c_0=(I^{2-\gamma_1}x)(t)\big|_{t=0}$, $c_1=(I^{1-\gamma_1}x)(t)\big|_{t=0}$, $d_0=(I^{2-\gamma_2}y)(t)\big|_{t=0}$ and $d_1=(I^{1-\gamma_2}y)(t)\big|_{t=0}$.

Using the conditions $x(0)=0$ and $y(0)=0$, respectively, in (7) and (8), we have $c_0=0$ and $d_0=0$. Consequently, (7) and (8) become

$$\begin{aligned} x(t) &= \frac{c_1t^{\gamma_1-1}}{\Gamma(\gamma_1)}-\lambda_1\int_0^t x(s)ds+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}h_1(s)ds, && (9)\\ y(t) &= \frac{d_1t^{\gamma_2-1}}{\Gamma(\gamma_2)}-\lambda_2\int_0^t y(s)ds+\frac{1}{\Gamma(\alpha_2)}\int_0^t(t-s)^{\alpha_2-1}h_2(s)ds, && (10)\end{aligned}$$

which, after insertion into $x(T)=\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1)}y(\eta_i)$ and $y(T)=\sum_{j=1}^{n}\theta_jR^{(\nu_\rho,\cdots,\nu_1)}x(\xi_i)$, yields

$$\begin{aligned}&\frac{c_1T^{\gamma_1-1}}{\Gamma(\gamma_1)}-\lambda_1\int_0^T x(s)ds+\frac{1}{\Gamma(\alpha_1)}\int_0^T(T-s)^{\alpha_1-1}h_1(s)ds\\ &=\frac{d_1}{\Gamma(\gamma_2)}\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1)}\eta_i^{\gamma_2-1}-\lambda_2\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+1)}y(\eta_i)+\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+\alpha_2)}h_2(\eta_i),\\ &=\frac{d_1}{\Lambda_0}\sum_{i=1}^{m}\varepsilon_i\prod_{k=1}^{\lfloor\frac{\rho}{2}\rfloor}\left(\gamma_2-1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}}\times\eta_i^{\gamma_2-1+\sum_{l=1}^{\lceil\frac{\rho}{2}\rceil}\mu_{2l-1}}-\lambda_2\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+1)}y(\eta_i)\\ &+\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+\alpha_2)}h_2(\eta_i),\end{aligned}$$

and

$$\begin{aligned}&\frac{d_1T^{\gamma_2-1}}{\Gamma(\gamma_2)}-\lambda_2\int_0^T y(s)ds+\frac{1}{\Gamma(\alpha_2)}\int_0^T(T-s)^{\alpha_2-1}h_2(s)ds\\ &=\frac{c_1}{\Lambda_1}\sum_{j=1}^{n}\theta_j\prod_{k=1}^{\lfloor\frac{\rho}{2}\rfloor}\left(\gamma_1-1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{\gamma_1-1+\sum_{l=1}^{\lceil\frac{\rho}{2}\rceil}\nu_{2l-1}}-\lambda_1\sum_{j=1}^{n}\theta_jR^{(\nu_\rho,\cdots,\nu_1+1)}x(\xi_j)\\ &+\sum_{j=1}^{n}\theta_jR^{(\nu_\rho,\cdots,\nu_1+\alpha_1)}h_1(\xi_j).\end{aligned}$$

In view of the notation (6), the above system can be written as

$$c_1A-d_1B=P,\ \ -c_1D+d_1C=Q, \tag{11}$$

where

$$\begin{aligned}P&=\lambda_1\int_0^T x(s)ds-\frac{1}{\Gamma(\alpha_1)}\int_0^T(T-s)^{\alpha_1-1}h_1(s)ds-\lambda_2\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+1)}y(\eta_i)\\ &+\sum_{i=1}^{m}\varepsilon_iR^{(\mu_\rho,\cdots,\mu_1+\alpha_2)}h_2(\eta_i),\end{aligned}$$

and

$$Q = \lambda_2 \int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)} \int_0^T (T-s)^{\alpha_2-1} h_2(s)ds - \lambda_1 \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_1+1)} x(\xi_j)$$
$$+ \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_1+\alpha_1)} h_1(\xi_j).$$

Solving the system (11) for c_1 and d_1, we have

$$c_1 = \frac{CP + BQ}{AC - BD} \quad \text{and} \quad d_1 = \frac{DP + AQ}{AC - BD}.$$

Substituting the values of c_1 and d_1 in (9) and (10), respectively, we obtain the solution (5) and (6). The converse of the lemma follows by direct computation. The proof is finished. □

3. Existence and Uniqueness Results

Throughout the paper, we denote by $\mathcal{J}$ the Banach space of all functions $x \in \mathcal{C}([0,T],\mathbb{R})$ with the norm $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. Obviously, the product space $\mathcal{J} \times \mathcal{J}$ is a Banach space endowed with the norm: $\|(x_1,x_2)\| = \|x_1\| + \|x_2\|$, $(x_1,x_2) \in \mathcal{J} \times \mathcal{J}$.

In view of Lemma 4, we define an operator $G : \mathcal{J} \times \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ associated with the problem (1) as follows:

$$G(x,y)(t) = (G_1(x,y)(t), G_2(x,y)(t)), \tag{12}$$

where

$$\begin{aligned} G_1(x,y)(t) = {} & \frac{t^{\gamma_1-1}}{\Delta\Gamma(\gamma_1)} \Bigg[C\Bigg(\lambda_1 \int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)} \int_0^T (T-s)^{\alpha_1-1} \widehat{f}(x,y)(s)ds \\ & - \lambda_2 \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} \widehat{g}(x,y)(\eta_i) \Bigg) \\ & + B\Bigg(\lambda_2 \int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)} \int_0^T (T-s)^{\alpha_2-1} \widehat{g}(x,y)(s)ds \\ & - \lambda_1 \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} \widehat{f}(x,y)(\xi_j) \Bigg) \Bigg] \\ & - \lambda_1 \int_0^t x(s)ds + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \widehat{f}(x,y)(s)ds, \ t \in [0,T], \end{aligned} \tag{13}$$

$$\begin{aligned} G_2(x,y)(t) = {} & \frac{t^{\gamma_2-1}}{\Delta\Gamma(\gamma_2)} \Bigg[D\Bigg(\lambda_1 \int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)} \int_0^T (T-s)^{\alpha_1-1} \widehat{f}(x,y)(s)ds \\ & - \lambda_2 \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} \widehat{g}(x,y)(\eta_i) \Bigg) \\ & + A\Bigg(\lambda_2 \int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)} \int_0^T (T-s)^{\alpha_2-1} \widehat{g}(x,y)(s)ds \\ & - \lambda_1 \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} \widehat{f}(x,y)(\xi_j) \Bigg) \Bigg] \\ & - \lambda_2 \int_0^t y(s)ds + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \widehat{g}(x,y)(s)ds, \ t \in [0,T], \end{aligned} \tag{14}$$

with

$$\widehat{f}(x,y)(t) = f\Big(t, x(t), R^{(\delta_q,\cdots,\delta_1)} x(t), y(t)\Big), \ \ \widehat{g}(x,y)(t) = g\Big(t, x(t), y(t), R^{(\zeta_p,\cdots,\zeta_1)} y(t)\Big).$$

In the sequel, we use the following notations:

$$
\begin{aligned}
\Lambda_2 &:= \Gamma\left(2+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}\right), \qquad \Lambda_3 := \Gamma\left(2+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}\right), \\
\Lambda_4 &:= \Gamma\left(1+\sum_{l=1}^{\lceil \frac{q}{2} \rceil} \delta_{2l-1}\right), \qquad \Lambda_5 := \Gamma\left(1+\sum_{l=1}^{\lceil \frac{p}{2} \rceil} \zeta_{2l-1}\right), \\
\phi_1 &:= \frac{|B||\lambda_1|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
&\quad + \frac{|C||\lambda_1|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} + |\lambda_1|T, \\
\phi_2 &:= \frac{|C||\lambda_2|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
&\quad + \frac{|B||\lambda_2|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)}, \\
\phi_3 &:= \frac{|B|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
&\quad + \frac{|C|T^{\alpha_1+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_1+1)} + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}, \\
\phi_4 &:= \frac{|C|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
&\quad + \frac{|B|T^{\alpha_2+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_2+1)}, \\
\omega_1 &:= \frac{|A||\lambda_1|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
&\quad + \frac{|D||\lambda_1|T^{\gamma_2}}{|\Delta|\Gamma(\gamma_2)}, \\
\omega_2 &:= \frac{|D|\lambda_2|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
&\quad + \frac{|A|\lambda_2|T^{\gamma_2}}{|\Delta|\Gamma(\gamma_2)} + |\lambda_2|T, \\
\omega_3 &:= \frac{|A|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
&\quad + \frac{|D|T^{\alpha_1+\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Gamma(\alpha_1+1)}, \\
\omega_4 &:= \frac{|D|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1+\sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
&\quad + \frac{|A|T^{\alpha_2+\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Gamma(\alpha_2+1)} + \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)}, \\
\Theta_1 &:= \frac{1}{\Lambda_4} \prod_{k=1}^{\lfloor \frac{q}{2} \rfloor} \left(\sum_{l=1}^{k} \delta_{2l-1}\right)^{-\delta_{2k}} \times T^{\sum_{l=1}^{\lceil \frac{q}{2} \rceil} \delta_{2l-1}},
\end{aligned} \tag{15}
$$

$$\Theta_2 \quad := \quad \frac{1}{\Lambda_5}\prod_{k=1}^{\lfloor \frac{p}{2} \rfloor}\left(\sum_{l=1}^{k}\zeta_{2l-1}\right)^{-\zeta_{2k}} \times T^{\sum_{l=1}^{\lceil \frac{p}{2} \rceil}\zeta_{2l-1}}.$$

Now we prove our first existence result for the system (1) with the aid of the Laray–Schauder alternative [45].

Theorem 1. *Assume that $f, g \in C([0,T]\times\mathbb{R}^3, \mathbb{R})$ and satisfy the following condition:*
(H_1) $\exists$ constants $x_i, y_i \geq 0$, $i = 1,2,3$ and $x_0, y_0 > 0$ for all $u, v, w \in \mathbb{R}$, such that

$$\begin{aligned} |f(t,u,v,w)| &\leq x_0 + x_1|u| + x_2|v| + x_3|w|, \\ |g(t,u,v,w)| &\leq y_0 + y_1|u| + y_2|v| + y_3|w|. \end{aligned}$$

Then, the system (1) admits at least one solution on $[0,T]$, provided that

$$(\phi_3+\omega_3)x_1 + (\phi_3+\omega_3)\Theta_1 x_2 + (\phi_4+\omega_4)y_1 + \phi_1 + \omega_1 < 1, \tag{16}$$

and

$$(\phi_3+\omega_3)x_3 + (\phi_4+\omega_4)y_2 + (\phi_4+\omega_4)\Theta_2 y_3 + \phi_2 + \omega_2 < 1, \tag{17}$$

where ϕ_i, ω_i $(i = 1,2,3,4)$ and Θ_1, Θ_2 are given in (16).

Proof. Observe that continuity of the operator G follows from that of f and g. Now, let us verify the hypotheses of the Laray–Schauder alternative [45]. Firstly, it will be established that G maps bounded sets into bounded sets in $\mathcal{J}\times\mathcal{J}$. Let us consider a bounded set $B_r := \{(x,y) \in \mathcal{J}\times\mathcal{J} : \|(x,y)\| \leq r\}$. Then, for any $(x,y) \in B_r$, $\exists$ $M_1, M_2 > 0$ such that $|\widehat{f}(x,y)(t)| < M_1$ and $|\widehat{g}(x,y)(t)| < M_2$. In consequence, we have

$$\begin{aligned} |G_1(x,y)(t)| \leq{}& \frac{t^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)}\Bigg[|C|\bigg(|\lambda_1|\int_0^T |x(s)|ds + \frac{1}{\Gamma(\alpha_1)}\int_0^T (T-s)^{\alpha_1-1}|\widehat{f}(x,y)(s)|ds \\ &+|\lambda_2|\sum_{i=1}^{m}|\varepsilon_i|R^{(\mu_\rho,\cdots,\mu_1+1)}|y(\eta_i)| + \sum_{i=1}^{m}|\varepsilon_i|R^{(\mu_\rho,\cdots,\mu_1+1)}|\widehat{g}(x,y)(\eta_i)|\bigg) \\ &+|B|\bigg(|\lambda_2|\int_0^T |y(s)|ds + \frac{1}{\Gamma(\alpha_2)}\int_0^T (T-s)^{\alpha_2-1}|\widehat{g}(x,y)(s)|ds \\ &+|\lambda_1|\sum_{j=1}^{n}|\theta_j|R^{(\nu_\rho,\cdots,\nu_\rho+1)}|x(\xi_j)| + \sum_{j=1}^{n}|\theta_j|R^{(\nu_\rho,\cdots,\nu_\rho+1)}|\widehat{f}(x,y)(\xi_j)|\bigg)\Bigg] \\ &+|\lambda_1|\int_0^t |x(s)|ds + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|\widehat{f}(x,y)(s)|ds \\ \leq{}& \frac{|C||\lambda_1|\|x\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} + \frac{|C|M_1 T^{\alpha_1+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_1+1)} \\ &+\frac{|C||\lambda_2|\|y\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil}\mu_{2l-1}} \\ &+\frac{|C|M_2 T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil}\mu_{2l-1}} \\ &+\frac{|B||\lambda_2|\|y\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} + \frac{|B|M_2 T^{\alpha_2+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_2+1)} \\ &+\frac{|B||\lambda_1|\|x\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil}\nu_{2l-1}} \\ &+\frac{|B|M_1 T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil}\nu_{2l-1}} \end{aligned}$$

$$
\begin{aligned}
&+|\lambda_1|\|x\|T+\frac{M_1T^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
\leq\ & r(\phi_1+\phi_2)+M_1\phi_3+M_2\phi_4,
\end{aligned}
$$

which leads to

$$
\|G_1(x,y)\|\leq r(\phi_1+\phi_2)+M_1\phi_3+M_2\phi_4. \tag{18}
$$

Likewise, we can obtain

$$
\begin{aligned}
|G_2(x,y)(t)|\ \leq\ & \frac{|D||\lambda_1|\|x\|T^{\gamma_2}}{|\Delta|\Gamma(\gamma_2)}+\frac{|D|M_1T^{\alpha_1+\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Gamma(\alpha_1+1)} \\
&+\frac{|D|\lambda_2|\|y\|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}}\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}} \\
&+\frac{|D|M_2T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}}\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}} \\
&+\frac{|A|\lambda_2|\|y\|T^{\gamma_2}}{|\Delta|\Gamma(\gamma_2)}+\frac{|A|M_2T^{\alpha_2+\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Gamma(\alpha_2+1)} \\
&+\frac{|A||\lambda_1|\|x\|T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}} \\
&+\frac{|A|M_1T^{\gamma_2-1}}{|\Delta|\Gamma(\gamma_2)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}} \\
&+|\lambda_2|\|y\|T+\frac{M_2T^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
\leq\ & r(\omega_1+\omega_2)+M_1\omega_3+M_2\omega_4,
\end{aligned}
$$

which implies that

$$
\|G_2(x,y)\|\leq r(\omega_1+\omega_2)+M_1\omega_3+M_2\omega_4. \tag{19}
$$

From (18) and (19), it follows that

$$
\|G(x,y)\|\leq r(\phi_1+\phi_2+\omega_1+\omega_2)+M_1(\phi_3+\omega_3)+M_2(\phi_4+\omega_4),
$$

which shows that the operator G is uniformly bounded.

Next, it will be established that the operator G is equicontinuous. Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x,y\in B_r$. Then,

$$
\begin{aligned}
&|G_1(x,y)(t_2)-G_1(x,y)(t_1)| \\
&\leq\frac{(t_2^{\gamma_1-1}-t_1^{\gamma_1-1})}{|\Delta|\Gamma(\gamma_1)}\Bigg(|C||\lambda_1|rT+\frac{|C|M_1T^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
&+\frac{|C||\lambda_2|r}{\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2l}}\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}} \\
&+\frac{|C|M_2}{\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2l}}\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}}+|B||\lambda_2|rT \\
&+\frac{|B|M_2T^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{|B||\lambda_1r}{\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}} \\
&+\frac{|B|M_1}{\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}}\Bigg)
\end{aligned}
$$

$$+|\lambda_1|r(t_2-t_1)+\frac{M_2}{\Gamma(\alpha_1+1)}(2(t_2-t_1)^{\alpha_1}+t_2^{\alpha_1}-t_1^{\alpha_1})\to 0,$$

as $t_2 \to t_1$ independently of x and y. Analogously, we can obtain

$$|G_2(x,y)(t_2)-G_2(x,y)(t_1)|\to 0,$$

as $t_2 \to t_1$, independently of x and y. Hence, GB_r is an equicontinuous set that implies that GB_r is relatively compact, and therefore, G is completely continuous by the Arzelá–Ascoli theorem.

Finally, we verify that the set $K := \{(x,y) \in \mathcal{J} \times \mathcal{J} : (x,y) = \theta G(x,y),\ 0 \le \theta \le 1\}$ is bounded. Let $(x,y) \in K$, then $(x,y) = \theta G(x,y)$. For any $t \in [0,T]$, we have

$$x(t)=\theta G_1(x,y)(t), \text{ and } y(t)=\theta G_2(x,y)(t).$$

In consequence, we obtain

$$\begin{aligned}
|x(t)| &\le |G_1(x,y)(t)| \\
&\le \|x\|\phi_1+\|y\|\phi_2+(x_0+x_1\|x\|+x_2\|x\||R^{(\delta_q,\cdots,\delta_1)}(1)|+x_3\|y\|)\phi_3 \\
&\quad +(y_0+y_1\|x\|+y_2\|y\|+y_3\|y\||R^{(\zeta_p,\cdots,\zeta_1)}(1)|)\phi_4 \\
&\le \|x\|\phi_1+\|y\|\phi_2+(x_0+x_1\|x\|+x_2\Theta_1\|x\|+x_3\|y\|)\phi_3 \\
&\quad +(y_0+y_1\|x\|+y_2\|y\|+y_3\Theta_2\|y\|)\phi_4,
\end{aligned}$$

$$\begin{aligned}
|y(t)| &\le |G_1(x,y)(t)| \\
&\le \|x\|\omega_1+\|y\|\omega_2+(x_0+x_1\|x\|+x_2\Theta_1\|x\|+x_3\|y\|)\omega_3 \\
&\quad +(y_0+y_1\|x\|+y_2\|y\|+y_3\Theta_2\|y\|)\omega_4.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|x\|+\|y\| &\le [(\phi_3+\omega_3)x_1+(\phi_3+\omega_3)\Theta_1x_2+(\phi_4+\omega_4)y_1+\phi_1+\omega_1]\|x\| \\
&\quad +[(\phi_3+\omega_3)x_3+(\phi_4+\omega_4)y_2+(\phi_4+\omega_4)\Theta_2y_3+\phi_2+\omega_2]\|y\| \\
&\quad +(\phi_3+\omega_3)x_0+(\phi_4+\omega_4)y_0,
\end{aligned}$$

which yields

$$\|(x,y)\| \le \frac{(\phi_3+\omega_3)x_0+(\phi_4+\omega_4)y_0}{A^*},$$

where $A^* = \min\{1-(\phi_3+\omega_3)x_1+(\phi_3+\omega_3)\Theta_1x_2+(\phi_4+\omega_4)y_1+\phi_1+\omega_1, 1-(\phi_3+\omega_3)x_3+(\phi_4+\omega_4)y_2+(\phi_4+\omega_4)\Theta_2y_3+\phi_2+\omega_2\}$. Thus, K is bounded. In consequence, by the Laray–Schauder alternative, we deduce that the problem (1) admits at least one solution on $[0,T]$, which completes the proof. □

Now, we prove our second existence result with the aid of a fixed point theorem due to Krasnosel'skiǐ's [46].

Theorem 2. *Assume that*

(H_2) $\exists\, l_1, l_2 > 0$, *such that for all* $t \in [0,T]$ *and* $u_i, v_i, w_i \in \mathbb{R},\ i = 1,2,$

$$\begin{aligned}
|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| &\le l_1(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|) \\
|g(t,u_1,v_1,w_1)-g(t,u_2,v_2,w_2)| &\le l_2(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).
\end{aligned}$$

(H_3) *There exist* $R_1, R_2 > 0$, *such that* $|f(t,u,v,w)| \le R_1$ *and* $|g(t,u,v,w)| \le R_2$, *for all* $t \in [0,T]$ *and* $u,v,w \in \mathbb{R}$.

Then, the problem (1) *has at least one solution on* $[0, T]$, *if*

$$\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} l_1(\Omega_1+1) + \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)} l_2(\Omega_2+1) < 1, \ \ \phi_1+\phi_2 < 1, \ \ \omega_1+\omega_2 < 1, \tag{20}$$

where $\Omega_1 = 1+\Theta_1$ *and* $\Omega_2 = 1+\Theta_2$ *and* $\phi_i, \omega_i, \Theta_i$ $(i = 1, 2)$ *are given in* (16).

Proof. In order to satisfy the hypotheses of Krasnosel'skiĭ's fixed point theorem [46], let us split the operator G, defined in (12), into four operators as

$$\begin{aligned}
G_{11}(x,y)(t) &= \frac{t^{\gamma_1-1}}{\Delta\Gamma(\gamma_1)}\Bigg[C\Bigg(\lambda_1\int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)}\int_0^T (T-s)^{\alpha_1-1}\widehat{f}(x,y)(s)ds \\
&\quad -\lambda_2\sum_{i=1}^m \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^m \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)}\widehat{g}(x,y)(\eta_i)\Bigg) \\
&\quad +B\Bigg(\lambda_2\int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)}\int_0^T (T-s)^{\alpha_2-1}\widehat{g}(x,y)(s)ds \\
&\quad -\lambda_1\sum_{j=1}^n \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^n \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)}\widehat{f}(x,y)(\xi_j)\Bigg)\Bigg] \\
&\quad -\lambda_1\int_0^t x(s)ds, \ t\in[0,T], \\
G_{12}(x,y)(t) &= \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\widehat{f}(x,y)(s)ds, \ t\in[0,T], \\
G_{21}(x,y)(t) &= \frac{t^{\gamma_2-1}}{\Delta\Gamma(\gamma_2)}\Bigg[D\Bigg(\lambda_1\int_0^T x(s)ds - \frac{1}{\Gamma(\alpha_1)}\int_0^T (T-s)^{\alpha_1-1}\widehat{f}(x,y)(s)ds \\
&\quad -\lambda_2\sum_{i=1}^m \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)} y(\eta_i) + \sum_{i=1}^m \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1+1)}\widehat{g}(x,y)(\eta_i)\Bigg) \\
&\quad +A\Bigg(\lambda_2\int_0^T y(s)ds - \frac{1}{\Gamma(\alpha_2)}\int_0^T (T-s)^{\alpha_2-1}\widehat{g}(x,y)(s)ds \\
&\quad -\lambda_1\sum_{j=1}^n \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)} x(\xi_j) + \sum_{j=1}^n \theta_j R^{(\nu_\rho,\cdots,\nu_\rho+1)}\widehat{f}(x,y)(\xi_j)\Bigg)\Bigg] \\
&\quad -\lambda_2\int_0^t y(s)ds, \ t\in[0,T], \\
G_{22}(x,y)(t) &= \frac{1}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}\widehat{g}(x,y)(s)ds \ t\in[0,T].
\end{aligned}$$

Notice that $G_1(x,y)(t) = G_{11}(x,y)(t) + G_{12}(x,y)(t)$ and $G_2(x,y)(t) = G_{21}(x,y)(t) + G_{22}(x,y)(t)$. Let $B_\epsilon = \{(x,y) \in \mathcal{J}\times\mathcal{J} : \|(x,y)\| \le \epsilon\}$ be a convex, closed, and bounded subset of the Banach apace $\mathcal{J}\times\mathcal{J}$, where

$$\epsilon = \max\left\{\frac{R_1\phi_3+R_2\phi_4}{1-(\phi_1+\phi_2)}, \frac{R_1\omega_3+R_2\omega_4}{1-(\omega_1+\omega_2)}\right\},$$

ϕ_i, ω_i $(i = 1, 2, 3, 4)$ are given in (16).

In our first step, we show that $GB_\epsilon \subset B_\epsilon$. Let (x,y), $(w,z) \in B_\epsilon$. As in the proof of Theorem 1, one can obtain the following estimates:

$$|G_{11}(x,y)(t) + G_{12}(w,z)(t)| \le \epsilon(\phi_1+\phi_2) + R_1\phi_3 + R_2\phi_4 \le \epsilon,$$

and

$$|G_{21}(x,y)(t) + G_{22}(w,z)(t)| \le \epsilon(\omega_1+\omega_2) + R_1\omega_3 + R_2\omega_4 \le \epsilon,$$

which imply that $GB_\epsilon \subset B_\epsilon$. Next, we show that (G_{11}, G_{21}) is continuous and compact. Note that continuity of G_{11} and G_{21} follows from that of f and g. For any $(x,y) \in B_\epsilon$, we have

$$
\begin{aligned}
|G_{11}(x,y)(t)| \leq\ & \frac{|C||\lambda_1|\|x\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} + \frac{|C|R_1 T^{\alpha_1+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_1+1)} \\
& + \frac{|C||\lambda_2|\|y\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1 + \sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
& + \frac{|C|R_2 T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2} \sum_{i=1}^{m} |\varepsilon_i| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1 + \sum_{l=1}^{k} \mu_{2l-1}\right)^{-\mu_{2k}} \times \eta_i^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \mu_{2l-1}} \\
& + \frac{|B||\lambda_2|\|y\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} + \frac{|B|R_2 T^{\alpha_2+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_2+1)} \\
& + \frac{|B||\lambda_1|\|x\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1 + \sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
& + \frac{|B|R_1 T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3} \sum_{j=1}^{n} |\theta_j| \prod_{k=1}^{\lfloor \frac{\varrho}{2} \rfloor} \left(1 + \sum_{l=1}^{k} \nu_{2l-1}\right)^{-\nu_{2k}} \times \xi_j^{1+\sum_{l=1}^{\lceil \frac{\varrho}{2} \rceil} \nu_{2l-1}} \\
& + |\lambda_1|\|x\|T \\
\leq\ & \epsilon(\phi_1+\phi_2) + R_1\left(\phi_3 - \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}\right) + R_2\phi_4 = I^*.
\end{aligned}
$$

In a similar manner, one can obtain

$$
|G_{21}(x,y)(t)| \leq \epsilon(\omega_1+\omega_2) + R_1\left(\omega_3 - \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}\right) + R_2\omega_4 = L^*.
$$

Consequently, we have $\|(G_{11}, G_{21})(x,y)\| \leq I^* + L^*$. Hence $G_{11}B_\epsilon$ and $G_{21}B_\epsilon$ are uniformly bounded. Next, we verify that $G_{11}B_\epsilon$ and $G_{21}B_\epsilon$ are equicontinuous. Similar to the arguments used in proving equicontinuity of the operators G_1 and G_2 in Theorem 1, it is easy to find that $|G_{21}(x,y)(t_2) - G_{21}(x,y)(t_1)| \to 0$ as $t_1 \to t_2$, independently of $(x,y) \in B_\epsilon$. Hence, $(G_{11}, G_{21})B_\epsilon$ is equicontinuous.

Finally, it will be established that the operator (G_{12}, G_{22}) is a contraction. For any $t \in [0,T]$ and each pair of elements $(x_1,y_1),(x_2,y_2) \in \mathcal{J} \times \mathcal{J}$, it follows by using the condition (H_2) that

$$
\begin{aligned}
& |\widehat{f}(x_1,y_1)(t) - \widehat{f}(x_2,y_2)(t)| \\
& = |f\left(t, x_1(t), R^{(\delta_q,\cdots,\delta_1)}x_1(t), y_1(t)\right) - f\left(t, x_2(t), R^{(\delta_q,\cdots,\delta_1)}x_2(t), y_2(t)\right)| \\
& \leq l_1(|x_1(t)-x_2(t)| + |R^{(\delta_q,\cdots,\delta_1)}x_1(t) - R^{(\delta_q,\cdots,\delta_1)}x_2(t)| + |y_1(t)-y_2(t)|) \\
& \leq l_1(\|x_1-x_2\| + \|x_1-x_2\|R^{(\delta_q,\cdots,\delta_1)}(1) + \|y_1-y_2\|) \\
& \leq l_1(\|x_1-x_2\|(1+R^{(\delta_q,\cdots,\delta_1)}(1)) + \|y_1-y_2\|) \\
& \leq l_1\Omega_1\|x_1-x_2\| + l_1\|y_1-y_2\|,
\end{aligned}
$$

and

$$
\begin{aligned}
& |\widehat{g}(x_1,y_1)(t) - \widehat{g}(x_2,y_2)(t)| \\
& = |g\left(t, x_1(t), y_1(t), R^{(\zeta_p,\cdots,\zeta_1)}y_1(t)\right) - g(t, x_2(t), y_2(t), R^{(\zeta_p,\cdots,\zeta_1)}y_2(t)| \\
& \leq l_2(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)| + |R^{(\zeta_q,\cdots,\zeta_1)}y_1(t) - R^{(\zeta_q,\cdots,\zeta_1)}y_2(t)|) \\
& \leq l_2(\|x_1-x_2\| + \|y_1-y_2\| + \|y_1-y_2\|R^{(\zeta_q,\cdots,\zeta_1)}(1)) \\
& \leq l_2(\|x_1-x_2\| + \|y_1-y_2\|(1+R^{(\zeta_q,\cdots,\zeta_1)}(1)))
\end{aligned}
$$

$$\leq l_2\|x_1 - x_2\| + l_2\Omega_2\|y_1 - y_2\|,$$

where $\Omega_1 = 1 + \Theta_1$ and $\Omega_2 = 1 + \Theta_2$. In consequence, we obtain

$$\|G_{12}(x_1, y_1) - G_{12}(x_2, y_2)\| \leq \frac{T^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(l_1\Omega_1 + l_1)(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

and

$$\|G_{22}(x_1, y_1) - G_{22}(x_2, y_2)\| \leq \frac{T^{\alpha_2}}{\Gamma(\alpha_2 + 1)}(l_2\Omega_2 + l_2)(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

which imply that

$$\begin{aligned}
&\|(G_{12}, G_{22})(x_1, y_1) - (G_{12}, G_{22})(x_2, y_2)\| \\
&\leq \frac{T^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(l_1\Omega_1 + l_1)(\|x_1 - x_2\| + \|y_1 - y_2\|) \\
&\quad + \frac{T^{\alpha_2}}{\Gamma(\alpha_2 + 1)}(l_2\Omega_2 + l_2)(\|x_1 - x_2\| + \|y_1 - y_2\|) \\
&= \left(\frac{T^{\alpha_1}}{\Gamma(\alpha_1 + 1)}(l_1\Omega_1 + l_1) + \frac{T^{\alpha_2}}{\Gamma(\alpha_2 + 1)}(l_2\Omega_2 + l_2)\right)(\|x_1 - x_2\| + \|y_1 - y_2\|).
\end{aligned}$$

Therefore, by the assumption (20), (G_{12}, G_{22}) is a contraction. In view of the foregoing arguments, we note that the hypotheses of Krasnosel'skiĭ's fixed point theorem [46] are satisfied. Therefore, the problem (1) has at least one solution on $[0, T]$. This finishes the proof. □

Lastly, the uniqueness of solutions for the problem (1) is established by means of Banach's fixed point theorem [47].

Theorem 3. *Let $f, g \in C([0, T] \times \mathbb{R}^3, \mathbb{R})$ satisfy the assumption (H_2). Then, the system (1) has a unique solution on $[0, T]$, if*

$$\alpha_1 + \aleph_2 + \hbar_1 + \hbar_2 < 1, \tag{21}$$

where

$$\alpha_1 := \phi_1 + \Omega_1 l_1\phi_3 + l_2\phi_4,\ \aleph_2 := \phi_2 + l_1\phi_3 + \Omega_2 l_2\phi_4,$$
$$\hbar_1 := \omega_1 + l_2\omega_4 + \Omega_1 l_1\omega_3,\ \hbar_2 := \omega_2 + l_2\Omega_2\omega_4 + l_1\omega_3.$$

Proof. For the fixed point problem $(x, y)(t) = G(x, y)(t)$ equivalent to the system (1), we show that the operator G has a unique fixed point on $[0, T]$ by means of the Banach's fixed point theorem [47], where the operator G is defined in (12). Let us consider a bounded, closed, and convex subset of $\mathcal{J} \times \mathcal{J}$ defined by

$$B_\varepsilon := \{(x, y) \in \mathcal{J} \times \mathcal{J} : \|(x, y)\| \leq \varepsilon\},$$

where

$$\varepsilon \geq \frac{\bar{M}(\phi_3 + \omega_3) + \bar{N}(\phi_4 + \omega_4)}{1 - (\aleph_1 + \aleph_2 + \hbar_1 + \hbar_2)}, \tag{22}$$

$\sup_{t\in[0,T]} = |f(t, 0, 0, 0)| := \bar{M} < \infty$, and $\sup_{t\in[0,T]} = |g(t, 0, 0, 0)| := \bar{N} < \infty$. For all $(x, y) \in B_\varepsilon,\ t \in [0, T]$, by using (H_2), we have

$$\begin{aligned}
|\widehat{f}(x, y)(t)| &= \left|f\left(t, x(t), R^{(\delta_q, \cdots, \delta_1)}x(t), y(t)\right)\right| \\
&\leq \left|f\left(t, x(t), R^{(\delta_q, \cdots, \delta_1)}x(t), y(t)\right) - f(t, 0, 0, 0)\right| + |f(t, 0, 0, 0)| \\
&\leq l_1(|x(t)| + |R^{(\delta_q, \cdots, \delta_1)}x(t)| + |y(t)|) + \bar{M}
\end{aligned}$$

$$
\begin{aligned}
&\leq l_1(\|x\|(1+R^{(\delta_q,\cdots,\delta_1)}(1))+\|y\|)+\bar{M} \\
&\leq l_1\Omega_1\|x\|+l_1\|y\|+\bar{M} \\
&\leq (\Omega_1+1)\varepsilon l_1+\bar{M}.
\end{aligned}
$$

Similarly, we one can obtain

$$|\widehat{g}(x,y)(t)| = |g\Big(t,x(t),y(t),R^{(\zeta_p,\cdots,\zeta_1)}y(t)\Big)| \leq (1+\Omega_2)\varepsilon l_2+\bar{N}.$$

Next, we show that $GB_\varepsilon \subset B_\varepsilon$. As in the proof of Theorem 1, it follows by the above inequalities that

$$|G_1(x,y)(t)| \leq \varepsilon(\phi_1+\phi_2)+((l_1\Omega_1+l_1)\varepsilon+\bar{M})\phi_3+((l_2+l_2\Omega_2)\varepsilon+\bar{N})\phi_4 \leq \varepsilon,$$

$$|G_2(x,y)(t)| \leq \varepsilon(\omega_1+\omega_2)+((l_1\Omega_1+l_1)\varepsilon+\bar{M})\omega_3+(l_2+(l_2\Omega_2)\varepsilon+\bar{N})\omega_4 \leq \varepsilon.$$

Hence,

$$
\begin{aligned}
\|G(x,y)\| \leq\ & \varepsilon(\phi_1+\phi_2)+((l_1\Omega_1+l_1)\varepsilon+\bar{M})\phi_3+((l_2+l_2\Omega_2)\varepsilon+\bar{N})\phi_4 \\
&+\varepsilon(\omega_1+\omega_2)+((l_1\Omega_1+l_1)\varepsilon+\bar{M})\omega_3+(l_2+(l_2\Omega_2)\varepsilon+\bar{N})\omega_4 \leq \varepsilon,
\end{aligned}
$$

which implies that $GB_\varepsilon \subset B_\varepsilon$.

Next, we want to show that the operator $G : \mathcal{J}\times\mathcal{J} \to \mathcal{J}\times\mathcal{J}$ is a contraction. For any $(x_1,y_2),(x_2,y_2) \in \mathcal{J}\times\mathcal{J}$ and for each $t \in [0,T]$, we obtain

$$
\begin{aligned}
&|G_1(x_1,y_1)(t)-G_1(x_2,y_2)(t)| \\
&\leq \frac{t^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)}\Bigg[|C|\Bigg(|\lambda_1|\int_0^T|x_1(s)-x_2(s)|ds \\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^T(T-s)^{\alpha_1-1}|\widehat{f}(x_1,y_1)(s)-\widehat{f}(x_2,y_2)(s)|ds \\
&+|\lambda_2|\sum_{i=1}^m|\varepsilon_i|R^{(\mu_\rho,\cdots,\mu_1+1)}|y_1(\eta_i)-y_2(\eta_i)| \\
&+\sum_{i=1}^m|\varepsilon_i|R^{(\mu_\rho,\cdots,\mu_1+1)}|\widehat{g}(x_1,y_1)(\eta_i)-\widehat{g}(x_2,y_2)(\eta_i)|\Bigg) \\
&+|B|\Bigg(|\lambda_2|\int_0^T|y_1(s)-y_2(s)|ds \\
&+\frac{1}{\Gamma(\alpha_2)}\int_0^T(T-s)^{\alpha_2-1}|\widehat{g}(x_1,y_1)(s)-\widehat{g}(x_2,y_2)(s)|ds \\
&+|\lambda_1|\sum_{j=1}^n|\theta_j|R^{(\nu_\rho,\cdots,\nu_\rho+1)}|x_1(\xi_j)-x_2(\xi_j)| \\
&+\sum_{j=1}^n|\theta_j|R^{(\nu_\rho,\cdots,\nu_\rho+1)}|\widehat{f}(x_1,y_1)(\xi_j)-\widehat{f}(x_2,y_2)(\xi_j)|\Bigg)\Bigg] \\
&+|\lambda_1|\int_0^t|x_1(s)-x_2(s)|ds \\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}|\widehat{f}(x_1,y_1)(s)-\widehat{f}(x_2,y_2)(s)|ds \\
&\leq \frac{|C||\lambda_1|\|x_1-x_2\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)} \\
&+(\Omega_1 l_1\|x_1-x_2\|+l_1|y_1-y_2\|)\frac{|C|T^{\alpha_1+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_1+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|C||\lambda_2|\|y_1-y_2\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}}\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}}\\
&+(l_2\|x_1-x_2\|+l_2\Omega_2\|y_1-y_2\|)\frac{|C|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_2}\sum_{i=1}^{m}|\varepsilon_i|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\mu_{2l-1}\right)^{-\mu_{2k}}\\
&\times\eta_i^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\mu_{2l-1}}+\frac{|B||\lambda_2|\|y_1-y_2\|T^{\gamma_1}}{|\Delta|\Gamma(\gamma_1)}\\
&+(l_2\|x_1-x_2\|+l_2\Omega_2\|y_1-y_2\|)\frac{|B|T^{\alpha_2+\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Gamma(\alpha_2+1)}\\
&+\frac{|B||\lambda_1|\|x_1-x_2\|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}}\\
&+(l_1\Omega_1\|x_1-x_2\|+l_1\|y_1-y_2\|)\frac{|B|T^{\gamma_1-1}}{|\Delta|\Gamma(\gamma_1)\Lambda_3}\sum_{j=1}^{n}|\theta_j|\prod_{k=1}^{\lfloor\frac{\varrho}{2}\rfloor}\left(1+\sum_{l=1}^{k}\nu_{2l-1}\right)^{-\nu_{2k}}\\
&\times\xi_j^{1+\sum_{l=1}^{\lceil\frac{\varrho}{2}\rceil}\nu_{2l-1}}+|\lambda_1|\|x_1-x_2\|T+(l_1\Omega_1\|x_1-x_2\|+l_1\|y_1-y_2\|)\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}\\
\leq\;&\phi_1\|x_1-x_2\|+\phi_2\|y_1-y_2\|+(\Omega_1 l_1\|x_1-x_2\|+l_1\|y_1-y_2\|)\phi_3\\
&+(l_2\|x_1-x_2\|+l_2\Omega_2\|y_1-y_2\|)\phi_4\\
\leq\;&(\phi_1+\Omega_1 l_1\phi_3+l_2\phi_4)\|x_1-x_2\|+(\phi_2+l_1\phi_3+\Omega_2 l_2\phi_4)\|y_1-y_2\|\\
\leq\;&\aleph_1\|x_1-x_2\|+\aleph_2\|y_1-y_2\|\leq(\aleph_1+\aleph_2)(\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}
$$

Therefore, we obtain

$$\|G_1(x_1,y_1)-G_1(x_2,y_2)\|\leq(\aleph_1+\aleph_2)(\|x_1-x_2\|+\|y_1-y_2\|). \tag{23}$$

Similarly, one can obtain

$$\|G_2(x_1,y_1)-G_2(x_2,y_2)\|\leq(\hbar_1+\hbar_2)(\|x_1-x_2\|+\|y_1-y_2\|). \tag{24}$$

From (23) and (24), we have

$$\|G(x_1,y_1)-G(x_2,y_2)\|\leq(\aleph_1+\aleph_2+\hbar_1+\hbar_2)(\|x_1-x_2\|+\|y_1-y_2\|),$$

which, by the condition (21), shows that G is a contraction. Hence, the conclusion of Banach's fixed point theorem [47] implies that the problem (1) has a unique solution on $[0,T]$. The proof is complete. □

4. Examples

Example 1. *Consider a system of nonlinear Hilfer iterated-integro differential equations with iterated fractional integral boundary conditions given by*

$$\begin{cases} ({}^{H}D^{\frac{3}{2},\frac{1}{2}}x)(t)+\frac{1}{7}({}^{H}D^{\frac{1}{2},\frac{1}{2}}x)(t)=f\Big(t,x(t),R^{(\frac{3}{25},\frac{1}{23},\frac{2}{25},\frac{1}{17})}x(t),y(t)\Big),\\ ({}^{H}D^{\frac{4}{3},\frac{5}{6}}y)(t)+\frac{1}{5}({}^{H}D^{\frac{1}{3},\frac{5}{6}}y)(t)=g\Big(t,x(t),y(t),R^{(\frac{2}{23},\frac{1}{22},\frac{4}{27},\frac{2}{29},\frac{1}{28},\frac{2}{25})}y(t)\Big),\\ x(0)=0,\ x\left(\frac{34}{25}\right)=R^{(\frac{6}{5},\frac{1}{4},1,\frac{1}{2},2)}y\left(\frac{1}{2}\right)+\frac{2}{3}R^{(\frac{6}{5},\frac{1}{4},1,\frac{1}{2},2)}y\left(\frac{2}{3}\right)\\ \qquad +\frac{1}{5}R^{(\frac{6}{5},\frac{1}{4},1,\frac{1}{2},2)}y\left(\frac{1}{5}\right)+\frac{6}{5}R^{(\frac{6}{5},\frac{1}{4},1,\frac{1}{2},2)}y(1)+\frac{1}{8}R^{(\frac{6}{5},\frac{1}{4},1,\frac{1}{2},2)}y\left(\frac{3}{4}\right),\\ y(0)=0,\ y\left(\frac{34}{25}\right)=\frac{2}{15}R^{(\frac{3}{2},\frac{5}{8},\frac{1}{2},\frac{2}{3},1)}y\left(\frac{5}{14}\right)+\frac{4}{5}R^{(\frac{3}{2},\frac{5}{8},\frac{1}{2},\frac{2}{3},1)}y\left(\frac{5}{7}\right)\\ \qquad +\frac{3}{7}R^{(\frac{3}{2},\frac{5}{8},\frac{1}{2},\frac{2}{3},1)}y\left(\frac{5}{13}\right)+\frac{1}{2}R^{(\frac{3}{2},\frac{5}{8},\frac{1}{2},\frac{2}{3},1)}y\left(\frac{2}{5}\right). \end{cases} \tag{25}$$

Here, $\alpha_1 = 3/2$, $\alpha_2 = 4/3$, $\beta_1 = 1/2$, $\beta_2 = 5/6$, $T = 34/25$, $\lambda_1 = 1/17$, $\lambda_2 = 1/5$, $\delta_1 = 1/17$, $\delta_2 = 2/25$, $\delta_3 = 1/23$, $\delta_4 = 3/25$, $\zeta_1 = 2/25$, $\zeta_2 = 1/28$, $\zeta_3 = 2/29$, $\zeta_4 = 4/27$, $\zeta_5 = 1/22$, $\zeta_6 = 2/23$, $\mu_1 = 2$, $\mu_2 = 1/2$, $\mu_3 = 1$, $\mu_4 = 1/4$, $\mu_5 = 6/5$, $\nu_1 = 1$, $\nu_2 = 2/3$, $\nu_3 = 1/2$, $\nu_4 = 5/8$, $\nu_5 = 3/2$, $\theta_1 = 2/15$, $\theta_2 = 4/5$, $\theta_3 = 3/7$, $\theta_4 = 1/2$, $\varepsilon_1 = 1$, $\varepsilon_2 = 2/3$, $\varepsilon_3 = 1/5$, $\varepsilon_4 = 6/5$, $\varepsilon_5 = 1/8$, $\eta_1 = 1/2$, $\eta_2 = 2/3$, $\eta_3 = 1/5$, $\eta_4 = 1$, $\eta_5 = 3/4$, $\xi_1 = 5/14$, $\xi_2 = 5/7$, $\xi_3 = 5/13$, $\xi_4 = 2/5$. Using the given data, it is found that $\gamma_1 = 1.75$, $\gamma_2 = 1.8889$, $A \approx 1.3703$, $B \approx 0.004$, $C \approx 1.3719$, $D \approx 0.0172$, $\Delta \approx 1.8799$, $\phi_1 \approx 0.3886$, $\phi_2 \approx 0.0014$, $\phi_3 \approx 2.3862$, $\phi_4 \approx 0.0044$, $\omega_1 \approx 0.0025$, $\omega_2 \approx 0.5013$,$\omega_3 \approx 0.0184$, $\omega_4 \approx 2.5311$, $\Theta_1 \approx 1.7906$ and $\Theta_2 \approx 1.8766$.

(i) For the illustration of Theorem 1, we consider

$$\begin{aligned} &f(t,u,v,w)\\ &= \frac{2^{-t}}{t+3}\left(\frac{1+\cos^2 \pi t}{2}\right)+\frac{e^{-u^2}|u|}{6}+\frac{1}{6\sqrt{t^2+36}}\frac{|v|^{25}}{(2+v^{24})}+\frac{\sin|w|}{8(t^2+1)}+\frac{1}{9}, \end{aligned} \tag{26}$$

$$\begin{aligned} &g(t,u,v,w)\\ &= \frac{4e^{-u^2}}{5}\left(\frac{|u|}{1+|u|}\right)+\frac{|v|\sin t}{3\sqrt{9+t^2}}+\frac{w^4}{5(1+|w|^3)}+\frac{(1-t)u^2}{36(1+|u|)}. \end{aligned} \tag{27}$$

It easy to see that

$$\begin{aligned} |f(t,u,v,w)| &\le \frac{4}{9}+\frac{|u|}{6}+\frac{|v|}{36}+\frac{|w|}{8},\\ |g(t,u,v,w)| &\le \frac{4}{5}+\frac{|u|}{36}+\frac{|v|}{9}+\frac{|w|}{5}. \end{aligned}$$

Note that (H_1) is satisfied with $x_0 = 4/9$, $x_1 = 1/6$, $x_2 = 1/36$, $x_3 = 1/8$, $y_0 = 4/5$, $y_1 = 1/36$, $y_2 = 1/9$ and $y_3 = 1/5$. Moreover, $(\phi_3+\omega_3)x_1+(\phi_3+\omega_3)\Theta_1 x_2+(\phi_4+\omega_4)y_1+\phi_1+\omega_1 \approx 0.9819$ and $(\phi_3+\omega_3)x_3+(\phi_4+\omega_4)y_2+(\phi_4+\omega_4)\Theta_2 y_3+\phi_2+\omega_2 \approx 0.9091$. As all the conditions of Theorem 1 are satisfied, its conclusion implies that the coupled systems (25) with f and g given by (26) and (27), respectively, has at least one solution on $[0, 34/25]$.

(ii) To demonstrate the application of Theorem 2, we take

$$\begin{aligned} f(t,u,v,w) &= \frac{8\cos t}{7(9t^2+7)}\left(\frac{|u|}{1+|u|}\right)+\frac{32\arctan|v|}{(3t+14)^2}\\ &\quad +\frac{8e^{-2t^4}\sin|w|}{49}+\frac{1}{2}, \end{aligned} \tag{28}$$

$$\begin{aligned} g(t,u,v,w) &= \frac{3(\sin^2 t+1)\sin|u|}{109+\sqrt{169+t^4}}+\frac{3(\cos^2 t+1)}{2(t^2+61)}\left(\frac{|v|}{1+|v|}\right)\\ &\quad +\frac{6e^{-3t}\arctan|w|}{2^{2t}+121}+\frac{e^{-2t}}{1+t^3}. \end{aligned} \tag{29}$$

Observe that

$$\begin{aligned}|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| &\le \frac{8}{49}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),\\ |g(t,u_1,v_1,w_1)-g(t,u_2,v_2,w_2)| &\le \frac{3}{61}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).\end{aligned}$$

Clearly, (H_2) is satisfied with $l_1 = 8/49$, $l_2 = 3/61$. Moreover, the functions f and g are bounded as

$$|f(t,u,v,w)| \le \frac{81+8\pi}{98} \quad \text{and} \quad |g(t,u,v,w)| \le \frac{134+3\pi}{122}. \tag{30}$$

Additionally, we have

$$\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} l_1(\Omega_1+1) + \frac{T^{\alpha_2}}{\Gamma(\alpha_2+1)} l_2(\Omega_2+1) \approx 0.9977 < 1,$$

and

$$\phi_1+\phi_2 \approx 0.3900 < 1, \qquad \omega_1+\omega_2 \approx 0.5038 < 1.$$

Since the hypothesis of Theorem 2 is satisfied, its conclusion implies that the coupled systems (25) with f and g given by (28) and (29), respectively, has at least one solution on $[0, 34/25]$.

(iii) To explain the application of Theorem 3, we choose the functions f and g as follows

$$f(t,u,v,w) = \frac{e^{-t^2}}{2^{t^2+8}}\left(\frac{2u^2+|u|}{1+2|u|}\right) + \frac{\sin|v|}{t^2+16^2} + \frac{|w|}{128\sqrt{4+t^3}} + \frac{1}{4}, \tag{31}$$

$$\begin{aligned} g(t,u,v,w) &= \frac{\cos^2 t}{(3t+12)^2}\left(\frac{|u|}{1+|u|}\right) + \frac{2\arctan|v|}{11t+288}\\ &\quad + \frac{1}{144}\left(\frac{3w^2+4|w|}{4+3|w|}\right) + \frac{1}{4}, \end{aligned} \tag{32}$$

which satisfy the Lipschitz condition with $l_1 = 1/256$, $l_2 = 1/144$ as

$$\begin{aligned}|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| &\le \frac{1}{256}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),\\ |g(t,u_1,v_1,w_1)-g(t,u_2,v_2,w_2)| &\le \frac{1}{144}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).\end{aligned}$$

In addition, we find that $\aleph_1+\aleph_2+\hbar_1+\hbar_2 \approx 0.9977 < 1$. Hence, by the application of Theorem 3, the problem (25) with f and g given by (31) and (32), respectively, has a unique solution on $[0, 34/25]$.

Remark 1. *We observe that the functions f and g given by (28) and (29), respectively, in (ii) satisfy the Lipschitz condition with $l_1 = 8/49$ and $l_2 = 3/61$, but the uniqueness of solutions for the problem (25) does not follow as (21) is not satisfied. In fact, when $l_1 = 8/49$ and $l_2 = 3/61$, the condition (21) becomes $\aleph_1+\aleph_2+\hbar_1+\hbar_2 \approx 2.8654 > 1$.*

Example 2. *We examine the behavior of solutions to the following system by varying the values of β_1:*

$$\begin{cases} ({}^{H}D^{\frac{3}{2},\beta_1}x)(t) + ({}^{H}D^{\frac{1}{2},\beta_1}x)(t) = t^2, \qquad t \in (0,2), \\ ({}^{H}D^{\frac{4}{3},\frac{1}{5}}y)(t) + \frac{1}{5}({}^{H}D^{\frac{1}{3},\frac{1}{5}}y)(t) = t^3, \\ x(0) = 0,\ x(2) = R^{\left(\frac{4}{5},\frac{7}{10}\right)}y\left(\frac{1}{2}\right) + \frac{1}{2}R^{\left(\frac{4}{5},\frac{7}{10}\right)}y\left(\frac{2}{3}\right) + \frac{1}{5}R^{\left(\frac{4}{5},\frac{7}{10}\right)}y\left(\frac{3}{4}\right), \\ y(0) = 0,\ y(2) = \frac{1}{2}R^{\left(\frac{2}{3},\frac{1}{2}\right)}x\left(\frac{1}{5}\right) + \frac{4}{5}R^{\left(\frac{2}{3},\frac{1}{2}\right)}x\left(\frac{4}{3}\right). \end{cases} \tag{33}$$

Let us take β_1 as a parameter with $\alpha_1 = 3/2$, $\alpha_2 = 4/5$, $\beta_2 = 1/5$, $T = 2$, $\mu_1 = 7/10$, $\mu_2 = 4/5$, $\nu_1 = 1/2$, $\nu_2 = 2/3$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1/2$, $\varepsilon_3 = 1/5$, $\eta_1 = 1/2$, $\eta_2 = 2/3$, $\eta_3 = 3/4$, $\theta_1 = 1/2$, $\theta_2 = 4/5$, and $\xi_1 = 1/5$, $\xi_2 = 4/3$. By Lemma 4, the solution of the system in (33) with $h_1(t) = t^2$ and $h_2(t) = t^3$ can be rewritten as

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\, s^{5/2}ds + R\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds, \\ y(t) &= \frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds + S\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds, \end{aligned}$$

where

$$R = \frac{\mathcal{F}(\mathcal{A}-\mathcal{C}) + \mathcal{D}(\mathcal{E}-\mathcal{G})}{\mathcal{D}\mathcal{H} - \mathcal{F}\mathcal{B}},\ \ S = \frac{\mathcal{H}(\mathcal{A}-\mathcal{C}) + \mathcal{B}(\mathcal{E}-\mathcal{G})}{\mathcal{D}\mathcal{H} - \mathcal{F}\mathcal{B}},\ \ \mathcal{D}\mathcal{H} - \mathcal{F}\mathcal{B} \neq 0, \tag{34}$$

with

$$\begin{aligned} \mathcal{A} &= \int_0^T e^{-\lambda_1(T-s)}\, I^{\alpha_1}s^2 ds, \qquad \mathcal{B} = \int_0^T \frac{s^{\gamma_{11}-1}e^{-\lambda_1(T-s)}}{\Gamma(\gamma_{11})}ds, \\ \mathcal{C} &= \sum_{i=1}^{m} \varepsilon_i R^{(\mu_\rho,\cdots,\mu_1)} \int_0^{\eta_i} e^{-\lambda_2(\eta_i-s)}\, I^{\alpha_2}s^3 ds, \\ \mathcal{E} &= \int_0^T e^{-\lambda_2(T-s)}\, I^{\alpha_2}s^3 ds, \qquad \mathcal{F} = \int_0^T \frac{s^{\gamma_{12}-1}e^{-\lambda_2(T-s)}}{\Gamma(\gamma_{12})}ds, \\ \mathcal{G} &= \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_1)} \int_0^{\xi_j} e^{-\lambda_1(\xi_j-s)}\, I^{\alpha_1}s^2 ds, \\ \mathcal{H} &= \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\cdots,\nu_1)} \int_0^{\xi_j} \frac{s^{\gamma_{11}-1}e^{-\lambda_1(\xi_j-s)}}{\Gamma(\gamma_{11})}ds, \end{aligned}$$

$\gamma_{11} = \alpha_1 - 1 + (2-\alpha_1)\beta_1$, and $\gamma_{12} = \alpha_2 - 1 + (2-\alpha_2)\beta_2$. Next, we give some numerical approximations and graphs of $x(t)$ and $y(t)$ when the values of β_1 vary from 0.1 to 0.9.

From Figure 1, we see that if the value of β_1 increases from 0.1 to 0.9, (Table 1) the corresponding graphs of $x(t)$ also increase. The lower curve occurs when $\beta_1 = 0.1$.

Table 1. The approximate solutions $x(t)$ and the values of R with varying values of β_1.

β_1	R	x(t)
0.1	-1.7587	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.7587\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.2	-1.6061	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.6061\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.3	-1.4790	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.4790\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.4	-1.3720	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.3720\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.5	-1.2810	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.2810\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.6	-1.2030	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.2030\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.7	-1.1355	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.1355\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.8	-1.0769	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.0769\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$
0.9	-1.0257	$\frac{1}{\Gamma(7/2)}\int_0^t e^{-(t-s)}\,s^{5/2}ds - 1.0257\int_0^t \frac{s^{1/2(1+\beta_1)-1}e^{-(t-s)}}{\Gamma(1/2(1+\beta_1))}ds$

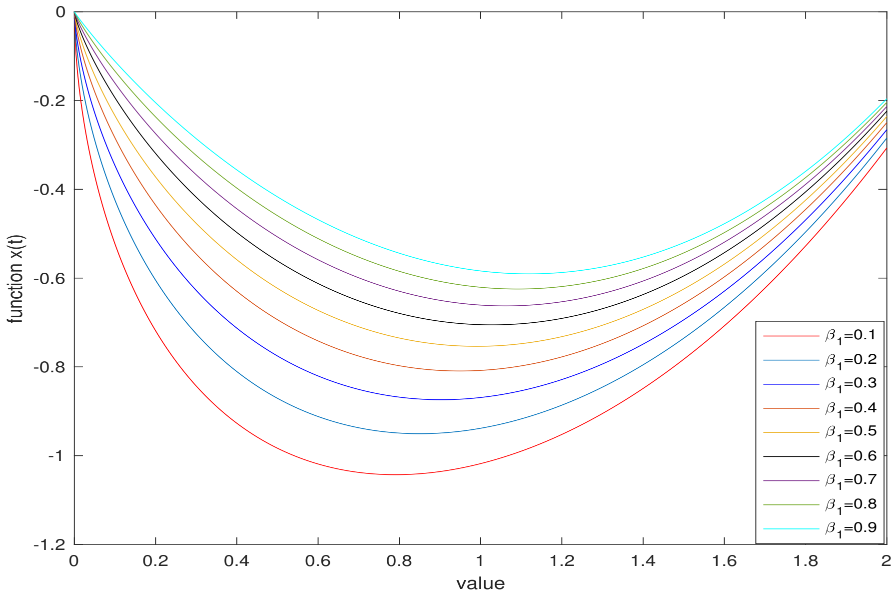

Figure 1. The graph of solutions $x(t)$ with varying values of β_1 from $\beta_1 = 0.1$ to $\beta_1 = 0.9$ and $\beta_2 = 0.2$.

In Figure 2, if the value of β_1 increases, then the value of $y(t)$ also increases (Table 2). The lower and upper bounds for the above curves correspond to $\beta_1 = 0.1$ and $\beta_1 = 0.9$, respectively.

Table 2. The approximate solutions $y(t)$ and the values of S with varying values of β_1.

β_1	S	y(t)
0.1	-1.1961	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 1.196\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.2	-1.1092	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 1.1092\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.3	-1.0360	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 1.0360\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.4	-0.9737	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.9737\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.5	-0.9200	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.9200\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.6	-0.8734	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.8734\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.7	-0.8325	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.8325\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.8	-0.7964	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.7964\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$
0.9	-0.7643	$\frac{1}{\Gamma(13/3)}\int_0^t e^{-1/5(t-s)}\, s^{10/3}ds - 0.7643\int_0^t \frac{s^{12/15-1}e^{-1/5(t-s)}}{\Gamma(12/15)}ds$

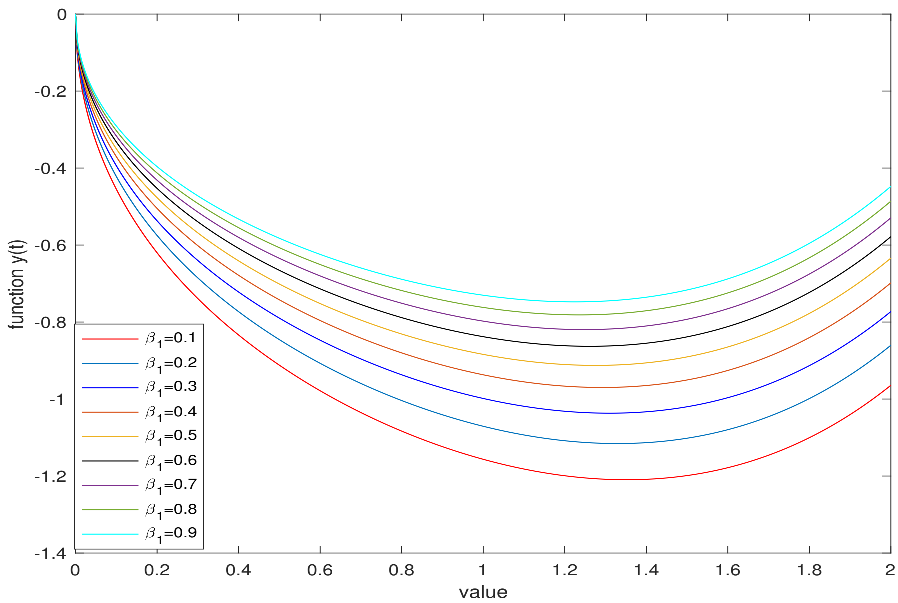

Figure 2. The graph of solutions $y(t)$ with varying values of β_1 from $\beta_1 = 0.1$ to $\beta_1 = 0.9$ and $\beta_2 = 0.2$.

5. Conclusions

In this paper, the tools of fixed point theory are successfully applied to obtain the existence criteria for solutions of a new class of boundary value problems involving coupled nonlinear Hilfer iterated-integro-differential equations, and Riemann–Liouville and Hadamard-type iterated fractional integral operators. The first two results (Theorems 3.1 and 3.2) present the different criteria for the existence of solutions to the problem at hand, while a sufficient criterion ensuring the unique solution of the given problem is accomplished in the third result. It is believed that the work established in this paper is a useful contribution to the existing literature on Hilfer-type fractional boundary value problems as it takes care of Riemann–Liouville and Caputo fractional derivative operators as special cases of the Hilfer fractional derivative operator. We have presented numerical examples to show the applicability of the obtained results by using the Matlab program. Our results are new in the given configuration and enrich the literature on the topic of nonlinear coupled Hilfer fractional differential equations equipped with nonlocal boundary conditions involving Riemann–Liouville and Hadamard-type iterated integral operators.

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