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On the Enhanced New Qualitative Results of Nonlinear Integro-Differential Equations

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Abstract: In this article, a class of scalar nonlinear integro-differential equations of first order with fading memory is investigated. For the considered fading memory problem, we discuss the effects of the memory over all the values of the parameter in the kernel of the equations. Using the Lyapunov–Krasovskiĭ functional method, we give various sufficient conditions of stability, asymptotic stability, uniform stability of zero solution, convergence and boundedness, and square integrability of nonzero solutions in relation to the considered scalar nonlinear integro-differential equations for various cases. As the novel contributions of this article, the new scalar nonlinear integro-differential equation with the fading memory is firstly investigated in the literature, and seven theorems, which have novel sufficient qualitative conditions, are provided on the qualitative behaviors of solutions called boundedness, convergence, stability, integrability, asymptotic stability and uniform stability of solutions. The novel outcomes and originality of this article are that the considered integro-differential equations are new mathematical models, they include former mathematical models in relation to the mathematical models of this paper as well as the given main seven qualitative results are also new. The outcomes of this paper enhance some present results and provide new contributions to the relevant literature. The results of the article have complementary properties for the symmetry of integro-differential equations.

Keywords: nonlinear; integro-differential equations; stability; convergence; integrability; boundedness; Lyapunov–Krasovskiĭ functional

MSC: 34D05; 34K20; 45J05

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1. Introduction

In recent years, integro-differential equations (IDEs), fractional IDEs and qualitative theory of these equations have been broken into the field of mathematical analysis, both at the theoretical level and at the level of its applications. In essence, the qualitative theory of IDEs is a mathematical analysis tool that can occur in numerous scientific fields such as fluid mechanics, viscoelasticity, physics, artificial neural networks, biology, medicine, competition between tumor cells and the immune system, chemistry, dynamical systems, noise term phenomenon, the scattered electromagnetic fields from resistive strips and an RLC circuit, signal processing, entropy theory, and so on. This is why qualitative theory of the mentioned equations have become a focus of international academic research, and many researchers have adopted them in their new studies, see the books of Burton [1], Dutta and Cavalcanti [2], Grigoriev et al. [3], Jerri [4], Lakshmikantham et al. [5], Wazwaz [6], and the references of these books and the paper of Zhou and Meleshko [7], and Zhou et al. [8].

As for some related papers in relation to certain nonlinear IDEs, Alahmadi et al. [9] studied the qualitative properties of solutions to the nonlinear scalar IDEs of the form of

$$\frac{dy}{dt} = A(t)y + \int_0^t C(t,s)h(y(s))ds + p(t), y(0) = y_0.$$

In the proofs of Alahmadi et al. [9], the authors applied an Lyapunov–Krasovskiĭ functional (LKF) and the Laplace transform to prove the boundedness and stability results of ([9], Theorems 2.2, 2.4).

More recently, Berezansky and Braverman [10] constructed new and very interesting exponential stability criteria for the delay IDE

$$\frac{dx}{dt} = \sum_{k=1}^{m} a_k(t)x(h_k(t)) + \int_{g(t)}^{t} K(t,s)x(s))ds, t \in [0,\infty), x \in \mathbb{R}.$$

The criteria of Berezansky and Braverman [10] depend on the boundedness of solutions.

Next, recently, Berezansky et al. [11] investigated the uniform exponential stability of the linear delay vector IDE

$$\frac{dx}{dt} = \sum_{k=1}^{m} A_k(t)x(h_k(t)) + \sum_{k=1}^{l} \int_{g_k(t)}^{t} P_k(t,s)x(s))ds, t \in [0,\infty), x \in \mathbb{R}^n,$$

using on an a priori estimation of solutions, a Bohl–Perron-type result, and utilization of the matrix measure.

In Berezansky and Domoshnitsky [12], benefiting from the Bohl–Perron theorem, the authors constructed explicit criteria for uniform exponential stability for the following class of linear scalar IDEs of second-order:

$$\frac{d^2x}{dt^2} + \int_{g(t)}^{t} G(t,s)x'(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = 0.$$

More recently, Tunç and Tunç [13] took into account the nonlinear delay IDE of second order with infinite delay:

$$\begin{aligned} &x'' + a(t)F(t,x,x') + b(t)G(x,x') + c(t)H(x') + d(t)Q(x) \\ &+ \int_{-\infty}^{t} \exp(-(t-s))U(s,x'(s))\mathrm{d}s = E(t,x,x'). \end{aligned}$$

In [13], new sufficient qualitative conditions on global asymptotic stability, boundedness and integrability of solutions for this scalar nonlinear delay IDE of second order with infinite delay were constructed, and some interesting qualitative results were obtained.

In Bohner et al. [14], the following scalar nonlinear IDE with Caputo fractional derivative, which have multiple kernels and constant time delays, was considered:

$${}^{C}_{t_o}D^q_t x(t) = -a(t)f(t,x) - g(x) + \sum_{k=1}^{n} \int_{t-\tau_k}^{t} C_k(t,s)f_k(s,x(s))ds + p(t,x).$$

In [14], it was discussed various kind of stability, boundedness at infinity and boundedness of solutions using the Lyapunov–Razumikhin method.

Tunç and Tunç [15] considered the following scalar generalized Caputo proportional fractional derivative IDE with multiple nonlinear kernels and constant time delays

$$\left({}_{t_o}^{C}D_t^{q,\rho}x\right)(t) = -f(x(t)) - a(t)g(x(t)) - h(t,x(t)) + \sum_{n=1}^{N}\int_{t-h_n}^{t} C_n(t,s,x(t),x(s))g_n(s,x(s))ds.$$

In [15], for this equation, qualitative behaviors of solutions were provided by ([15], Theorems 1, 2), and ([15], Examples 1–4). These theorems have sufficient conditions to guarantee the various kind of stability and boundedness of solutions. The technique allowed to find the proofs of [15] involves the updated Lyapunov-Razumikhin method.

Crisci et al. [16] dealt with the IDE of second order

$$x'' + \phi(t,x') + f(x) = \int_0^t k(t,t-s)y(t-s)ds.$$

In Crisci et al. [16], the authors gave sufficient conditions for the asymptotic stability of the zero solution of this equation using the LKF approach.

Zhou and Meleshko [7] took into consideration a linear thermoviscoelastic model of homogeneous aging materials with memory. In Zhou and Meleshko [7], the invariant solutions of the corresponding system of IDEs were constructed by means of the group analysis method, which relies on symmetries of this system.

As a key reference paper for this work, Burton and Somolinos [17] considered the IDEs of the form

$$\frac{dx}{dt} = F\left(t, x(t), \int_0^t C(at-s)x(s)ds\right), \tag{1}$$

where $a \in \mathbb{R}$ and satisfies $0 < a < \infty$. The integral in the IDE (1) stands for the memory of past positions of the solution x. It is also assumed that $\int_0^\infty |C(t)|dt < \infty$. Hence, the IDE (1) is a fading memory problem. Burton and Somolinos [17] investigated the effects of this memory over various values of a. Diverse properties of solutions emerge when a varies. Burton and Somolinos [17] developed a technique that handles this diversity in a unified way. The technique used in Burton and Somolinos [17] is based on the LKF method, where it is needed to define or construct suitable LKF(s) for the qualitative studies of the problems under study.

To show the ideas most clearly, Burton and Somolinos [17] first focused on the nonlinear scalar IDE of the form

$$\frac{dx}{dt} = -h(t)x - b(t)x^3 + \int_0^t C(at-s)x(s)ds. \tag{2}$$

In Burton and Somolinos [17], the following qualitative properties of IDE (2) were investigated, respectively:

(1) Boundedness and convergence of solutions of the IDE (2) when $a > 1$, (see, [17], Theorem 1);
(2) The stability of zero solution of the IDE (2) when $a > 1$, (see, [17], Theorem 2);
(3) The asymptotic stability of zero solution of the IDE (2) when $a > 1$, (see, [17], Theorem 3);
(4) The uniform stability of the zero solution of the IDE (2) when $a > 1$, (see, [17], Theorem 4);
(5) The uniform asymptotic stability of zero solution of the IDE (2) when $a > 1$, (see, [17], Theorem 5);
(6) The stability of the zero solution of the IDE (2) when $0 < a < 1$, (see, [17], Theorem 6);

(7) The integrability of square of solutions of the IDE (2) when $0 < a < 1$, (see, [17], Theorem 7);
(8) The asymptotic stability of zero solution of the IDE (2) when $0 < a < 1$, (see, [17] Theorem 8).

The results mentioned above, i.e., Theorems 1–8 of Burton and Somolinos [17], have very suitable sufficient and interesting conditions. These results were proved by using suitable LKFs.

Motivated by the results of Burton and Somolinos ([17], Theorems 1–8), we consider the following nonlinear IDE of the form

$$\frac{dx}{dt} = -h(t)f(x) - b(t)g(x) + \int_0^t C(at-s)\ell(s,x(s))ds, \tag{3}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+ = [0,\infty)$, $C \in L^1[0,\infty)$, C is the space of all continuous functions, $h \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in C(\mathbb{R}^+,\mathbb{R}^+)$, f, $g \in C(\mathbb{R},\mathbb{R})$ and $\ell \in C(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$. Hence, the existence of the solutions of the IDE (3) holds.

The Lyapunov–Krasovskiĭ functional method is a well-known method to investigate the qualitative behaviors of solutions of functional differential equations without prior information of solutions. This method is very effective to discuss the qualitative behaviors of functional differential equations subject to the definition or construction of suitable Lyapunov–Krasovskiĭ functionals, which provide meaningful qualitative results for the problems under study. However, in the relevant literature, there is no general method to construct a suitable Lyapunov–Krasovskiĭ functional for linear and nonlinear functional differential equations. Construction of a Lyapunov–Krasovskiĭ functional is an open problem in the literature by this time. In this paper, we define suitable Lyapunov–Krasovskiĭ functionals to investigate the qualitative behaviors of solutions of the functional integro-differential equations of this paper. This is one of the motivations for using the Lyapunov–Krasovskiĭ functional method.

The aim of this article is to generalize and improve the outcomes of Burton and Somolinos [17] for the nonlinear IDE (3) according to the range of the positive constant "a" and provide new contributions to the results mentioned above, and that can be found in [18–35] and the present literature. Hence, we have new contributions to the qualitative theory of IDEs. We reach to the outcome of this article by defining new LKFs, which are different from those in the references of this article and that can be seen in relation to the relevant literature. Next, it can be seen that the IDE (3) is different from that in [18–35]. We should mention that the papers [18–35] include very interesting results on the fundamental behaviors of certain IDEs with or without delay. Our paper includes complementary results for the outcomes of [1–6,9–35] and that in the references of these sources. These are the new contributions of the outcomes of this paper in relation to the qualitative theory of IDEs.

2. Improved New Qualitative Results

In this section, first, we establish sufficient conditions, which have essential roles in the proofs of Theorems 1–7.

Before presenting the improved and new qualitative results of this article, we establish some new conditions to discuss the qualitative properties of the IDE (3). Hence, we assume that the following conditions hold throughout the article.

Conditions

(A1) $a,\ b_0 \in \mathbb{R}$, $a > 1$, $b(t) \geq b_0 \geq 0$, $h(t) > 0$, and $C \in L^1[0,\infty)$.
(A2) We have positive constants f_0, g_0, $\ell_0 \in \mathbb{R}$ such that

$$f(0) = g(0) = \ell(t,0) = 0,$$

$$(\operatorname{sgn} x)f(x) \geq f_0|x| \geq 0 \text{ for all } x \neq 0, x \in \mathbb{R},$$

$$(\operatorname{sgn} x)g^3(x) \geq g_0^3|x|^3 \geq 0 \text{ for all } x \neq 0, x \in \mathbb{R}, \text{ and}$$
$$|\ell(t,x)| \leq \ell_0|x| \text{ for all } t \in \mathbb{R}^+, x \neq 0, x \in \mathbb{R}.$$

(A3) We have positive constants g_0 and ℓ_0 from (A2) and a positive constant $\bar{f}_0$ such that

$$f(0) = g(0) = \ell(t,0) = 0,$$

$$xf(x) \geq \bar{f}_0 x^2 \geq 0 \text{ for all } x \neq 0, x \in \mathbb{R},$$
$$xg^3(x) \geq g_0^3 x^4 \text{ for all } x \neq 0, x \in \mathbb{R},$$
$$|\ell(t,x)| \leq \ell_0|x| \text{ for all } t \in \mathbb{R}^+,\ x \neq 0,\ x \in \mathbb{R}\ .$$

(A4) We have positive constants f_0 and ℓ_0 from (A2) such that

$$2f_0 h(t) - \int_{(a-1)t}^{at} |C(u)|du - a^{-1}\ell_0^2 \int_{(a-1)t}^{\infty} |C(u)|du \geq 0,$$

and

$$\int_0^{t_0}\int_{at_0-s}^{\infty} |C(u)|duds < \infty.$$

(A5) $a,\ b_0 \in \mathbb{R}, 0 < a < 1, b(t) \geq b_0 \geq 0, h(t) > 0$, and $C \in L^1[0,\infty)$.
(A6) We have positive constants f_0 and ℓ_0 from (A2) and α_0 such that

$$2f_0 h(t) - \int_{(a-1)t}^{at} |C(u)|du - a^{-1}\ell_0^2 \int_{(a-1)t}^{\infty} |C(u)|du \geq \alpha_0,$$

and

$$\int_0^{t_0}\int_{at_0-s}^{\infty} |C(u)|duds < \infty.$$

As the first new result of this article, the boundedness and convergence solutions is given in the following theorem.

Theorem 1. *Every solution of the IDE (3) is bounded and tends to zero if (A1) and (A2) hold.*

Proof. Since $|\ell(t,x)| \leq \ell_0|x|$ and $C \in L^1[0,\infty)$, then we have a positive constant C_0 such that

$$\begin{aligned}\int_0^t C(at-s)\ell(s,x(s))ds \leq& \int_0^t |C(at-s)|\,|\ell(s,x(s))|ds \\ \leq& \ell_0 \int_0^t |C(at-s)|ds \leq \ell_0 C_0.\end{aligned}$$

Let $t_0 \geq 0$ and $\phi : [0,t_0] \to \mathbb{R}$ with $|\phi(t)| < H$, where $-b_0 g_0^3 H^3 + \ell_0 C_0 H < 0$. Then, $|x(t,t_0,\phi)| < H$ for all $t > t_0$. To complete the proof, by the way of the contradiction, we assume that $|x(t)| < H$ for $t_0 \leq t < t_1$ and $|x(t_1)| = H$.

Let us consider the Lyapunov function (LF) $V(x) = |x|$ to prove every solution of the IDE (3) is bounded and tends to zero. Differentiating this LF along solutions of the IDE (3), we obtain

$$\begin{aligned}\frac{d}{dt}V(x(t)) =& \dot{x}(t)\,\mathrm{sgn}(x(t+0))\\ =& \left[-h(t)f(x) - b(t)g^3(x) + \int\limits_0^t C(at-s)\ell(s,x(s))ds\right]\mathrm{sgn}(x(t+0))\\ \le& -h(t)|f(x)| - b(t)|g(x)|^3 + \int\limits_0^t |C(at-s)|\,|\ell(s,x(s))|ds\\ \le& -f_0h(t)|x| - b(t)g_0^3|x|^3 + \ell_0\int\limits_0^t |C(at-s)|\,|x(s)|ds.\end{aligned}$$

Hence, for $t \in [t_0, t_1]$ we have

$$\begin{aligned}\frac{d}{dt}V(x(t_1)) \le& -f_0h(t_1)|x(t_1)| - b(t_1)g_0^3|x(t_1)|^3 + \ell_0\int\limits_0^{t_1} |C(at-s)|\,|x(s)|ds\\ \le& -b_0g_0^3H^3 + \ell_0C_0H < 0,\end{aligned}$$

which is a contradict to $|x(t_1)| = H$. This completes the proof of the boundedness of the solution.

For the coming step, we prove that $x(t)$ tends to zero. By the way of contradiction, if $x(t)$ does not tend to zero, then there exists a nonzero constant A, (assume $A > 0$) and a sequence (t_n) such that if $t_n \to \infty$, then $x(t_n) \to A$. Consider the integration

$$\int\limits_0^t |C(at-s)|ds.$$

Let $at - s = v$. Next, we have $-ds = dv$. Hence,

$$\int\limits_0^t |C(at-s)|ds = \int\limits_{(a-1)t}^{at} |C(v)|dv \to 0.$$

Thus, there is a positive constant T such that $t \ge T$ implies that

$$-b_0g_0^3A^3 + 2H\ell_0\int\limits_0^t |C(at-s)|ds < -\alpha$$

for some positive constant α. Hence, for $t \ge T$, we have $\frac{d}{dt}V(x(t)) \le -\beta < 0$ for some positive constant β whenever $|x(t)| \ge A - \delta$ for some $\delta > 0$. Next, we cannot have $|x(t)| \ge A - \delta$ for all large t or $V(x) \to -\infty$. However, there is a $t_2 > t_1$ with $|x(t_2)| = A - \delta$ and $|x(t)| < A - \delta$ on $[t_1, t_2)$. This result is a contradiction just as before. As a result, the constant A does not exist and the solution tends to zero. The proof of Theorem 1 is completed. □

As the second new result of this article, the stability and boundedenss result of this article is given in the following theorem.

Theorem 2. *The zero solution of the IDE (3) is stable, and nonzero solutions of the IDE (3) are bounded if (A1), (A3) and (A4) hold.*

Proof. We define the LKF

$$W(t,x_t) = x^2(t) + a^{-1}\gamma \int\limits_0^t \int\limits_{at-s}^{\infty} |C(u)| du \ell^2(s,x(s))ds, \tag{4}$$

where $a > 1$ and γ is an arbitrary positive constant to be chosen in the proof.

Easily, we see that the LKF $W(t,x_t)$ in (4) is positive definite.

Differentiating the LFK (4) along solutions of the IDE (3), we obtain

$$\begin{aligned}
\frac{d}{dt}W(t,x_t) =& 2x\left[-h(t)f(x) - b(t)g^3(x) + \int\limits_0^t C(at-s)\ell(s,x(s))ds\right] \\
&+ a^{-1}\gamma \int\limits_{(a-1)t}^{\infty} |C(u)| du \ell^2(t,x(t)) - \gamma \int\limits_0^t |C(at-s)| \ell^2(s,x(s))ds \\
\leq& -2f_0 h(t)x^2 - 2b(t)xg^3(x) + 2|x| \int\limits_0^t |C(at-s)||\ell(s,x(s))|ds \\
&+ a^{-1}\gamma \int\limits_{(a-1)t}^{\infty} |C(u)| du \ell^2(t,x(t)) - \gamma \int\limits_0^t |C(at-s)| \ell^2(s,x(s))ds \\
\leq& -2f_0 h(t)x^2 - 2g_0^3 b(t)x^4 + x^2 \int\limits_0^t |C(at-s)|ds \\
&+ \int\limits_0^t |C(at-s)| \left|\ell^2(s,x(s))\right| ds + a^{-1}\gamma \int\limits_{(a-1)t}^{\infty} |C(u)| du \ell^2(t,x(t)) \\
&- \gamma \int\limits_0^t |C(at-s)| \ell^2(s,x(s))ds.
\end{aligned}$$

Let $\gamma = 1$. Then,

$$\begin{aligned}
\frac{d}{dt}W(t,x_t) \leq& -2f_0 h(t)x^2 - 2g_0^3 b(t)x^4 + x^2 \int\limits_0^t |C(at-s)|ds \\
&+ a^{-1} \int\limits_{(a-1)t}^{\infty} |C(u)| du \ell^2(t,x(t)) \\
\leq& -\left[2f_0 h(t) - \int\limits_0^t |C(at-s)|ds - a^{-1}\ell_0^2 \int\limits_{(a-1)t}^{\infty} |C(u)| du\right] x^2 \\
&- 2g_0^3 b(t)x^4 \leq 0
\end{aligned} \tag{5}$$

by (A4).

Since from (5) we have

$$\frac{d}{dt}W(t,x_t) \leq 0,$$

then the LKF (4) is a decreasing functional, i.e.,

$$W(t,x_t) \leq W(t_0,\phi(t_0)) \;\text{ for all } t \geq t_0.$$

Let $t_0 \geq 0$ and ε be given. Then, for an undetermined $\delta > 0$ and $|\phi| < \delta$, from the definition of the LKF $W(t, x_t)$ (4) and the last inequality, we can write

$$x^2(t, t_0\phi) \leq W(t, x_t) \leq W(t_0, \phi(t_0))$$
$$\leq \delta^2 + a^{-1}\ell_0^2\delta^2 \int_0^{t_0} \int_{at_0-s}^{\infty} |C(u)|duds = K,\ K > 0.$$

Hence,

$$|x(t, t_0\phi)| \leq \sqrt{K},\ t \geq t_0,$$

and

$$x^2(t, t_0\phi) \leq \delta^2 + a^{-1}\ell_0^2\delta^2 \int_0^{t_0} \int_{at_0-s}^{\infty} |C(u)|duds < \varepsilon^2$$

provided that

$$\delta^2 < \frac{\varepsilon^2}{1 + a^{-1}\ell_0^2 \int_0^{t_0} \int_{at_0-s}^{\infty} |C(u)|duds}.$$

This result completes the proofs of the boundedness and stability results of Theorem 2.

As the third new result, the asymptotic stability result of this article is given in the following theorem.

In the next result, we explain how the fading memory can be used to verify asymptotic stability and to affect the derivative of LKF $W(t, x_t)$ (4), hence the kernel of the LKF (4). It follows here that if the term $b(t)x^n$ changes, then the conditions on the kernel C of the IDE (3) also changes as mentioned in the introduction. □

Theorem 3. *Let (A1) hold with $b(t) > 0$, $\int_0^{\infty} b(s)ds = \infty$, and let (A3), (A6) hold.*

Let also B and ℓ be positive constants such that for each $T > 0$ if $t > T$, then

$$\int_T^t \left\{ \int_{at-s}^{\infty} |C(u)|du \right\}^2 b^{-1}(s)ds < \ell_0^{-4}B. \tag{6}$$

Then, the zero solution of the IDE (3) is asymptotically stable.

Proof. In Theorem 2, it has been already proved that the zero solution of the IDE (3) is stable and the LKF has negative derivative in (5).

Let $x(t)$ be a solution with $|x(t)| < 1$. It follows that $b(t)x^4(t) \in L^1[0, \infty)$. We will verify that $W(t, x_t) = W(t) \to 0$. If it does not, then there is a $\mu > 0$ such that

$$2\mu < W(t) \leq x^2(t) + a^{-1}\ell_0^2 \int_0^t \int_{at-s}^{\infty} |C(u)|dux^2(s)ds.$$

If we can find a t_f such that for $t > t_f$, then

$$a^{-1}\ell_0^2 \int_0^t \int_{at-s}^{\infty} |C(u)|dux^2(s)ds < \mu. \tag{7}$$

Hence, for $t > t_f$, we find

$$2\mu < W(t) \leq x^2(t) + \mu$$

and

$$-x^4(t) < -\mu^2, -b(t)x^4(t) < -b(t)\mu^2,$$

which is a contradiction to

$$b(t)x^4(t) \in L^1[0,\infty).$$

Let us find a t_f such that for $t > t_f$ the condition (7) holds. Hence, without loss of generality, let $\mu < 1$. Next, let B be that of condition (6). Since $b(t)x^4(t) \in L^1[0,\infty)$, there would be a t_1 such that for any $T > t_1$, we get

$$\int_T^\infty b(s)x^4(s)ds < \frac{\mu^2}{4B}.$$

Fix a $T > t_1$. For all $t > T$, it is clear that

$$\begin{aligned}\int_0^t \int_{at-s}^\infty |C(u)|du\ell^2(s,x(s))ds &= \int_0^T \int_{at-s}^\infty |C(u)|du\ell^2(s,x(s))ds \\ &\quad + \int_T^t \int_{at-s}^\infty |C(u)|du\ell^2(s,x(s))ds. \qquad (8)\end{aligned}$$

Since $|x(t)| < 1$ and $|\ell(s,x(s))| \le \ell_0|x(s)|$, for the first integral of (8) we have

$$\begin{aligned}\int_0^T \int_{at-s}^\infty |C(u)|du\ell^2(s,x(s))ds &\le \ell_0^2 \int_0^T \int_{at-s}^\infty |C(u)|dux^2(s)ds \\ &\le \ell_0^2 \int_0^T \int_{at-T}^\infty |C(u)|duds \\ &= \ell_0^2 T \int_{at-T}^\infty |C(u)|du.\end{aligned}$$

Next, since $C(t) \in L^1[0,\infty)$, similarly, we get for $t > t_f$

$$\int_{at-T}^\infty |C(u)|du < \frac{\mu}{2\ell_0^2 T}.$$

Hence,

$$\int_0^T \int_{at-s}^\infty |C(u)|du\ell^2(s,x(s))ds \le \ell_0^2 T \int_{at-T}^\infty |C(u)|du < \frac{1}{2}\mu.$$

As the next step, for the second integral of (8) we have

$$\int_T^t \int_{at-s}^{\infty} |C(u)|du\ell^2(s,x(s))ds = \int_T^t \frac{\int_{at-s}^{\infty} |C(u)|du}{\sqrt{b(s)}} \sqrt{b(s)}\ell^2(s,x(s))ds$$
$$\le \ell_0^2 \int_T^t \frac{\int_{at-s}^{\infty} |C(u)|du}{\sqrt{b(s)}} \sqrt{b(s)}x^2(s)ds$$
$$\le \ell_0^2 \left\{ \int_T^t \left(\int_{at-s}^{\infty} |C(u)|du \right)^2 b^{-1}(s)ds \right\}^{\frac{1}{2}} \left(\int_T^t b(s)x^4(s)ds \right)^{\frac{1}{2}}.$$

Using

$$\int_T^t \left\{ \int_{at-s}^{\infty} |C(u)|du \right\}^2 b^{-1}(s)ds < \ell_0^{-4}B$$

and

$$\int_T^{\infty} b(s)x^4(s)ds < \frac{\mu^2}{4B},$$

we have

$$\int_T^t \int_{at-s}^{\infty} |C(u)|du\ell^2(s,x(s))ds \le \ell_0^2 \int_T^t \int_{at-s}^{\infty} |C(u)|dux^2(s)ds$$
$$\le \sqrt{B} \left(\int_T^t b(s)x^4(s)ds \right)^{\frac{1}{2}} < \frac{1}{2}\sqrt{B}\frac{\mu}{\sqrt{B}} = \frac{1}{2}\mu.$$

Putting together these two integrals, we arrive at for $t > t_f$ that

$$\int_0^t \int_{at-s}^{\infty} |C(u)|du\ell^2(s,x(s))ds < \frac{1}{2}\mu + \frac{1}{2}\mu = \mu.$$

Since $a > 1$, this result completes the proof.

As the fourth new result, the uniform stability result of this article is given in the following theorem. □

Theorem 4. *Let (A1), (A3) and (A4) hold. Suppose also that there is a positive constant B such that*

$$\int_{(a-1)t}^{at} \int_v^{\infty} |C(u)|du < B \tag{9}$$

for all $t > 0$. Then, the zero solution of the IDE (3) is uniformly stable.

Proof. For any t_0 from the condition (9), it follows that

$$\int_0^{t_0} \int_{at_0-s}^{\infty} |C(u)|duds = \int_{at_0-t_0}^{at_0} \int_v^{\infty} |C(u)|duds < B$$

Let $\varepsilon > 0$ be given and choose a δ such that $(1 + a^{-1}\ell_0^2 B)\delta^2 < \varepsilon^2$. Assume $|\phi(t)| < \delta$ on $[0, t_0]$ and let $x(t)$ be a solution $x(t, t_0, \phi)$. Then, since the LKF $W(t, x_t)$ is decreasing, then, in view of this property and the definition of the LKF $W(t, x_t)$, we can write

$$\begin{aligned} x^2(t) \le W(t, x_t) \le W(t_0, x_{t_0}) &\le \phi^2(t_0) + a^{-1} \int_0^{t_0} \int_{at_0 - s}^{\infty} |C(u)| du \ell^2(s, \phi(s)) ds \\ &\le \phi^2(t_0) + a^{-1}\ell_0^2 \int_0^{t_0} \int_{at_0 - s}^{\infty} |C(u)| du \phi^2(s) ds \\ &\le \delta^2 + a^{-1}\ell_0^2 \delta^2 \int_0^{t_0} \int_{at_0 - s}^{\infty} |C(u)| du ds \\ &\le (1 + a^{-1}\ell_0^2 B)\delta^2 < \varepsilon^2. \end{aligned}$$

Hence, we conclude that, for any $|\phi(t)| < \delta$ and any t_0, the solution $x(t, t_0, \phi)$ satisfies $|x(t)| < \varepsilon$, where δ is independent from t_0. This completes the proof. □

Remark 1. *In this article, we use the same LKF for all values of $a > 1$ and $0 < a < 1$, except Theorem 1. The proof of the uniform stability theorem, Theorem 4, is also completed by the same way. Here, for the case $a > 1$, we do not need the integrability of the function $\int_t^{\infty} |C(u)| du$ for the infinity. In this particular case, if this function is chosen as $\frac{1}{t+1}$, then the condition just mentioned holds. However, for the case $a = 1$, the integrability condition to the infinity is needed as the main requirement in the proofs. When $a \in (0, 1)$, it is not possible to find a condition for the uniform stability of zero solution. By this way, the uniform stability concept of solutions means that the motions of solutions depending upon similar initial functions, but different starting times, can be very similar to each other. This case is consistent with the rapidly fading memory when $a > 1$. However, for the case $a \in (0, 1)$, more of the memory is retained. For more details of the information, see the remarks in the paper of Burton and Somolinos [17].*

As the fifth new result of this article, the next stability result of this article is given in the following theorem.

Theorem 5. *Let (A3), (A4) and (A5) hold. Suppose also that there is a positive constant B such that*

$$\int_{(a-1)t}^{at} \int_{v}^{\infty} |C(u)| du < B$$

for all $t > 0$. Then, the zero solution of the IDE (3) is stable.

Proof. We follow the proof of Theorem 2 and define the LKF

$$W(t, x_t) = x^2(t) + a^{-1} \int_0^{t} \int_{at - s}^{\infty} |C(u)| du \ell^2(s, x(s)) ds, \, 0 < a < 1. \tag{10}$$

It is known that the given computations do not include the range of the constant a. As in Theorem 2, from the time derivative of the LKF (10) along the solutions of the IDE (3), we have

$$\frac{d}{dt} W(t, x_t) \le -g_0^3 b(t) x^4 \le -(g_0^3 b_0) x^4 \le 0.$$

The remaining part of the proof can be completed as the similar to Theorem 2. We neglect it. □

As the sixth new result of this article, the integrability result of this article is given in the following theorem.

Theorem 6. *If the conditions of Theorem 5 hold and*

$$\int_{0}^{\infty} |C(s)|ds = M > 0, \tag{11}$$

then there is a positive constant K with $M \int_{at}^{t} \ell^2(s, x(s))ds \le M\ell_0^2 \int_{at}^{t} x^2(s)ds < \ell_0^2 K.$

Proof. From Theorem 5, we concluded that the zero solution of the IDE (3) is stable. Let $x(t) = x(t, t_0, \phi)$ be a fixed solution of the IDE (3) with $|x(t)| < 1$ when $0 < a < 1$. We also have $\frac{d}{dt}W(t, x_t) \le 0$. Since the LKF $W(t, x_t)$ is decreasing, then

$$W(t, x_t) \le W(t_0, \phi(t_0)).$$

Here, $W(t_0, \phi(t_0))$ can be considered as a positive constant, say $W(t_0, \phi(t_0)) = K > 0$. This fact implies that both terms in $W(t, x_t)$ are bounded. Hence, from the LKF (10), we have

$$\int_{0}^{t} \int_{at-s}^{\infty} |C(u)|du\ell^2(s, x(s))ds \le \ell_0^2 \int_{0}^{t} \int_{at-s}^{\infty} |C(u)|dux^2(s)ds < \ell_0^2 K.$$

Next, it follows that

$$\begin{aligned} M\int_{at}^{t} \ell^2(s, x(s))ds &\le \int_{at}^{t} \int_{0}^{\infty} |C(v)|dv\ell^2(s, x(s))ds \le \ell_0^2 \int_{at}^{t} \int_{0}^{\infty} |C(v)|dvx^2(s)ds \\ &\le \ell_0^2 \int_{at}^{t} \int_{at-s}^{\infty} |C(v)|dvx^2(s)ds \\ &\le \ell_0^2 \int_{0}^{t} \int_{at-s}^{\infty} |C(v)|dvx^2(s)ds < \ell_0^2 K. \end{aligned}$$

This completes the proof of Theorem 6. □

As the last result of this article, the next asymptotic stability result of this article is given in the following theorem.

Theorem 7. *Let (A3), (A5) and (A6) hold. Suppose also that*

$$\int_{-\infty}^{\infty} |C(u)|du = H < \infty.$$

Then, the zero solution of the IDE (3) is asymptotically stable.

Proof. In the proof of this theorem, we use the LKF (10). In the light of the conditions of Theorem 7, we can obtain

$$\frac{d}{dt}W(t, x_t) \le -(\alpha_0)x^2 - (g_0^3 b_0)x^4.$$

The remaining of the proof can be completed by the way of Burton and Somolinos ([17], Theorem 8). Therefore, we omit the details of the proof. □

3. Numerical Application

In this section, we present a numerical example for illustrations.

Example 1. *We consider the nonlinear scalar IDE of the form*

$$\begin{aligned}\frac{dx}{dt} = &- (1+t^2)(x+x^5) - (1+\exp(-t))(x+x^3) \\ &+ \int_0^t \frac{1}{1+(2t-s)^2}\frac{x}{1+s^4}ds.\end{aligned} \tag{12}$$

We can see that the IDE (12) is a special case of the IDE (3). It is also obvious that (6) has the zero solution. Comparing the IDEs (12) and (3) we have the following relations, respectively:

$$a = 2 > 1,$$

$$b(t) = 1 + \exp(-t) \geq 1 = b_0 > 0,$$
$$h(t) = 1 + t^2 > 0,$$
$$g(x) = x(1 + x^2),$$
$$g(0) = 0,$$
$$(\operatorname{sgn} x)g^3(x) = (\operatorname{sgn} x)x^3\left(1 + x^2\right)^3 \geq |x|^3 \geq 0,\ g_0^3 = 1,$$
$$f(x) = x + x^5,$$
$$f(0) = 0,$$
$$(\operatorname{sgn} x)f(x) = (\operatorname{sgn} x)(x + x^5) \geq |x| \geq 0,\ f_0 = 1;$$
$$\ell(t,x) = \frac{x}{1+t^4},$$
$$\ell(t,0) = 0,$$
$$|\ell(t,x)| = \frac{|x|}{1+t^4} \leq |x|,\ \ell_0 = 1,$$
$$C(at-s) = \frac{1}{1+(2t-s)^2},$$
$$\int_0^\infty |C(at-s)|ds = \int_0^\infty \frac{1}{1+(2t-s)^2}ds < \infty, \textit{i.e.}, C \in L^1[0,\infty).$$

Hence, we see that (A1) and (A2) of Theorem 1 hold. Thus, every solution of the IDE (12) is bounded and tends to zero. As for the numerical applications of Theorems 2–7, similar examples can be given.

4. Discussion

In this section, we explain shortly the contributions of Theorems 1–7 to the present literature and qualitative theory of IDEs.

(1) The nonlinear thee IDE (3) of this article is more general and includes the IDE (2). When $f(x) = x$, $g(x) = x^3$ and $\ell(s, x(s)) = x(s)$, then the IDE (3) reduces to the IDE (2). The integral of the IDE (3) represents the memory of past positions of the solution x. To the best information of the authors of this paper, the qualitative behaviors of solutions of the nonlinear the IDE (3) were not investigated when $a > 1$ and $0 < a < 1$. These are clear new contributions of this article.

(2) As for our claim that the results obtained here are more effective and convenient for tests and applications, we mean that they can be found in numerous functions as

those included in the IDE (3), which satisfy conditions (A1)–(A6) of the results of this paper. This means and implies that the results of this paper are more effective and convenient for tests and applications. Here, we studied our results theoretically; however, working on proper applications may be the subject of a future work.

5. Conclusions

This article is devoted to the qualitative study of a new class of mathematical model of nonlinear integro-differential equations of the first order. Various new and improved qualitative properties of solutions, called stability, convergence, asymptotic stability, uniform stability, boundedness and square integrability of solutions in relation to the considered integro-differential equations are investigated. Here, seven new theorems, which have sufficient conditions, are obtained in relation to the mentioned qualitative concepts. Constructing two new the Lyapunov–Krasovskiĭ functionals, the technique called Lyapunov–Krasovskiĭ method or approach is applied to prove these new results. Compared with the existing results that can be found in the present literature, our results are new, original, more general, effective and convenient for tests and applications. As future works, qualitative behaviors of fractional-order mathematical models of nonlinear integro-differential equations of this paper can be investigated in the sense of Caputo, Reimann–Liouville, Mittag–Leffler and so on. In addition, the problems of this paper can be discussed for nonlinear systems of integro-differential equations.

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