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Asymptotic Behavior of the Solution to Compressible Navier–Stokes System with Temperature-Dependent Heat Conductivity in an Unbounded Domain

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Abstract: This paper concerns the one-dimensional compressible Navier–Stokes system with temperature-dependent heat conductivity in $\mathbb{R}$ with large initial data. We prove that velocity and temperature are uniformly bounded from below and above in time and space when the heat conductivity coefficient takes $\kappa = \bar{\kappa}(1+\theta^b)$ for all $b > \frac{5}{2}$. In addition, we show that the global solution is asymptotically stable as time tends to infinity.

Keywords: compressible Navier–Stokes equations; global strong solution; large-time behavior; temperature-dependent heat conductivity

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1. Introduction

This paper concerns the Cauchy problem of compressible fluids in one space dimensions. The motion of a perfect polytropic ideal heat-conducting fluids can be written in the following form [1]:

$$\begin{cases} \rho_t + (\rho u)_y = 0, \\ (\rho u)_t + (\rho u^2 + P)_y = (\mu u_y)_y, \\ (\rho(e + \frac{1}{2}u^2))_t + (\rho(e + \frac{1}{2}u^2)u + Pu)_y = (\kappa e_y)_y + (\mu u u_y)_y, \end{cases} \tag{1}$$

where $t > 0$ and $y \in \mathbb{R}$ are the time variable and spatial variable, respectively, where the unknown $\rho \geq 0$ denotes the density of the flow, u the velocity, and e the internal energy. Both pressure P and internal energy e are generally related to the density and temperature of the flow according to the equations of state: $P = P(\rho, \theta)$ and $e = e(\rho, \theta)$. Parameters $\mu = \mu(\rho, \theta)$ denote the viscosity coefficients, and $\kappa = \kappa(\rho, \theta)$ is the heat conductivity.

To solve the Cauchy problem, we transform Problem (1) into Lagrangian variables. To this end, we introduce the Lagrangian symmetry variable

$$x = \int_{y(t)}^{y} \rho(t, z) dz,$$

where $y(t)$ is the particle path satisfying $y'(t) = u(t, y(t))$. The Lagrangian version of System (1) can be written as

$$\begin{cases} v_t = u_x, & \text{(2a)} \\ u_t + P_x = (\mu \frac{u_x}{v})_x, & \text{(2b)} \\ \left(e + \frac{u^2}{2}\right)_t + (Pu)_x = \left(\kappa \frac{\theta_x}{v} + \mu \frac{u u_x}{v}\right)_x, & \text{(2c)} \\ P = R\theta / v, \quad e = c_v \theta. & \text{(2d)} \end{cases}$$

We consider a perfect gas for Navier–Stokes flow in this paper, that is,

$$P = R\frac{\theta}{v}, \quad e = c_v\theta, \tag{3}$$

where R is a positive constant, and c_v is the heat capacity of the gas. System (2) is supplemented with the following initial condition:

$$(u,\ v,\ \theta)|_{t=0} = (u_0,\ v_0,\ \theta_0), \quad x \in \mathbb{R}, \tag{4}$$

and the far-field condition:

$$\lim_{|x|\to\infty}(v(x,t), u(x,t), \theta(x,t)) = (1,0,1), \quad t > 0. \tag{5}$$

Let us review some results on System (2) in different situations. When μ and κ were constants, the existential results in bounded domains for large initial data were obtained by Kazhikhov et al. [2–4]. Regarding initial boundary value problems in bounded domains, see [2,5–9] for a thorough discussion of System (2) with initial Condition (4) and far-field Condition (5). Furthermore, the existence and uniqueness of global solutions, and the regularity are known [2,5–10]. Moreover, the asymptotic behavior of the global solution was studied as time tended to infinity; see [11–14], among others. For the Cauchy problem, the global existence of a solution was obtained by Kazhikhov [15]; then, Li [16] gave the asymptotic behavior of solutions to System (2) with initial Condition (4) and far-field Condition (5).

We could obtain compressible Navier–Stokes Equations (2) from the celebrated Boltzmann equations for monatomic gas with a slab symmetry by using the Chapman–Enskog expansion. Then, viscosity coefficient μ and heat conductivity coefficient κ are functions of density and temperature; see Chapman and Cowling [17] or Vincenti and Kruger ([18], Chapter X) for a thorough discussion of these issues. When the coefficients depended on special volume and temperature, for the one-dimensional full compressible Navier–Stokes equations of ideal polytropic gas whose viscosity coefficient and heat conductivity coefficient satisfying $\mu = \bar{\mu}h(v)\theta^b, \ \ \kappa = \bar{\kappa}h(v)\theta^b$, Liu, Yang, et al. in [19] obtained the global nonvacuum classical solutions with a smallness mechanism (i.e., $\gamma - 1$ small). Wang and Zhao in [20] obtained the global nonvacuum classical solutions with smallness assumptions for b. Later, in 2016, Wang and Zhao [21] gave the large-time behavior of the solutions under the assumptions that $Ch(v) \geq v^{l_1} + v^{-l_2}, \quad h'(v)^2 v \leq Ch(v)^3$ and that b was small enough. Duan, Guo, et al. [22] proved the existence and uniqueness of a strong global solution for ideal polytropic gas flow, with $\mu = 1 + \rho^\alpha$ and $\kappa = \theta^\beta$. Kazhikhov [15] gave frameworks when μ and κ are constants. However, if the viscosity coefficient depends on temperature, Kazhikhov's method is invalid. Li, Shu, et al. [23] proved the global existence of strong solutions to a compressible Navier–Stokes system with degenerate heat conductivity in unbounded domains. However, the asymptotic behavior of a solution with large initial data is still open.

When viscosity was a positive constant, and only heat conductivity depended on temperature, i.e.,

$$\mu = \bar{\mu}, \quad \kappa = \bar{\kappa}\theta^b, \tag{6}$$

Jenssen and Karper [24] proved the global existence of a weak solution to initial-boundary value problem (IBVP) (2) under the assumption that $b \in [0, \frac{3}{2})$; Pan and Zhang [25] extended it to $b \in [0, \infty)$. Li and Guo [1] established the global existence of strong and classical solutions to free boundary Problem (2) for $b \in [0, \infty)$, and the expanding rates of the interface were also studied. Recently, Li, Shu, et al. [23] proved the global existence of a solution to Cauchy Problem (2) for $b \in [0, \infty)$. Chen and Zhang [26] proved global existence to free boundary problems. Cai, Chen, et al. [27] obtained the asymptotic behavior of the initial boundary value problem of System (2). However, the asymptotic behavior to the Cauchy problem is still open and our focus. The research on numerical and applications in engineering to system of (2) and it's related models, see [28–31].

The mission of this paper is to establish the uniform bounds from below and above of velocity and temperature to the Cauchy problem, and the large-time behavior of strong solutions with $\kappa = \bar{\kappa}(1+\theta^b)$.

Notations:

(1) For $p \geq 1, L^p = L^p(\mathbb{R})$ denotes the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, $W^{k,p} = W^{k,p}(\mathbb{R})$ denotes the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k = W^{k,2}(\mathbb{R})$. For $k \geq 1$ and $p \geq 1$, $D^{k,p}(\mathbb{R})$ denotes the homogeneous Sobolev space, the norm of $f \in D^{k,p}(\mathbb{R})$ is $\|f^k\| \in L^p(\mathbb{R})$. $Q_T = [0,T] \times \mathbb{R}$.

(2) For the sake of simplicity, we denote various positive constants independent of time T and depending on time T with C and $C(T)$, which may be different at different occurrences.

Definition 1. *(Global strong solution) For any $(x,t) \in ([0,\infty) \times \mathbb{R})$, (v,u,θ) is called a global strong solution if*

$$\begin{cases} v - 1 \in C([0,\infty), H^1(\mathbb{R})), \\ \theta - 1 \in C([0,\infty), H^2(\mathbb{R})) \cap L^2([0,\infty), H^1(\mathbb{R})), \\ u \in L^\infty([0,\infty), H^2(\mathbb{R})) \cap L^2([0,\infty), W^{2,2}(\mathbb{R})), \\ v_t, u_t, \theta_t \in L^2([0,\infty), D^{1,2}(\mathbb{R})), \end{cases} \tag{7}$$

and (v,u,θ) satisfies both System (2) almost everywhere in $\mathbb{R} \times (0,\infty)$ and Initial Value (4) almost everywhere in $\mathbb{R}$.

The existence and uniqueness of local solution can be proven with a fixed-point theorem; see Tani [32], who proved the existence of local solution if the initial (4) and far-field Condition (5) are satisfied, and μ, κ are locally Lipschitz-continuous functions on (v,θ). As a special case of the result in [32], the following theorem gives the local existence for our problem.

Theorem 1. *Assume that μ and κ satisfy (6) for some positive constants $\bar{\mu}$ and $\bar{\kappa}$. If the initial data $(v_0,\ u_0,\ \theta_0)(x)$ are compatible with far-field Condition (5), satisfying*

$$(v_0 - 1,\ u_0,\ \theta_0 - 1)(x) \in H^1 \times H^2 \times H^2, \tag{8}$$

and there are constants $\underline{v}$, $\bar{v}$, $\underline{\theta}$, $\bar{\theta}$ such that

$$0 < \underline{v} \leq v_0(x) \leq \bar{v},\ \ 0 < \underline{\theta} \leq \theta_0(x) \leq \bar{\theta}, \tag{9}$$

then there exists a unique local strong solution $(v,u,\theta)(x,t)$ to (2) on $\mathbb{R} \times [0,T_1]$ for some $C > 0$ depending on the initial data, and T_1 satisfies

$$\begin{cases} C^{-1} \leq \theta(x,t) \leq C(T_1), \quad C^{-1} \leq v(x,t) \leq C(T_1), \\ \|(v-1,\ u,\ \theta-1)(\cdot,t)\|^2_{H^1(\mathbb{R})} + \int_0^t \|(v-1,\ u,\ \theta-1)(\cdot,s)\|^2_{H^1(\mathbb{R})} ds \leq C(T_1), \\ \|(u,\ \theta-1)(\cdot,t)\|^2_{H^2(\mathbb{R})} + \int_0^t \|(v_{xt},\ u_{xt},\ u_{xx}\,\theta_{xt},\ \theta_{xx})(\cdot,s)\|^2_{L^2(\mathbb{R})} ds \leq C(T_1). \end{cases} \tag{10}$$

if the initial data further satisfy

$$v_0(x) \in C^{1+\alpha},\ \ u_0(x) \in C^{2+\alpha},\ \ \theta_0 \in C^{2+\alpha}, \tag{11}$$

then $v \in C^{1+\alpha,\frac{\alpha}{2}}(\mathbb{R} \times [0,T_1])$, $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T_1])$, and $\theta \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T_1])$.

Thanks to this local existence result, the existence of a global solution is established by extending the local solution with the help of the global a priori estimates stated in (12) (see Theorem 2). It is clear that (12) is sufficient to extend the local strong solution to global one by a standard continuity argument.

The following are the main results of this paper. Some uniform estimate results and the large-time behavior of the solutions are obtained when the heat conductivity coefficient is in nondegenerate form with the temperature.

Theorem 2. *Assume that the initial data (v_0, u_0, θ_0) satisfy (8), (9), $\kappa = \bar{\kappa}(1+\theta^b)$ (nondegenerate case) for $b \in (\frac{5}{2}, \infty)$. Let $(v,\ u,\ \theta)$ be a solution to (2)–(4) together with far-field Condition (5). For any $T > 0$, there exists a unique global strong solution $(v,\ u,\ \theta)$ satisfying*

$$\begin{cases} \|(v-1,\ u,\ \theta-1)(\cdot,t)\|^2_{H^1(\mathbb{R})} + \int_0^t \|(v-1,\ u,\ \theta-1)(\cdot,s)\|^2_{H^1(\mathbb{R})} ds \le C, \\ \|(u,\ \theta-1)(\cdot,t)\|^2_{H^2(\mathbb{R})} + \int_0^t \|(v_{xt}, v_{xx},\ u_{xt},\ u_{xx},\ \theta_{xt},\ \theta_{xx})(\cdot,s)\|^2_{L^2(\mathbb{R})} ds \le C. \end{cases} \tag{12}$$

Moreover, there exists a positive constant C depending only on $\mu, \kappa, R, c_v, \underline{v}, \underline{\theta}$, and the initial value; the following uniform estimate holds

$$C^{-1} \le \theta(x,t) \le C, \quad C^{-1} \le v(x,t) \le C, \tag{13}$$

and large-time behavior is obtained

$$\lim_{t\to\infty} (\|(v-1,\ u,\ \theta-1)(t)\|_{L^p(\mathbb{R})} + \|(v_x,\ u_x,\ \theta_x)(t)\|_{L^2(\mathbb{R})}) = 0, \tag{14}$$

for any $p \in (2, \infty]$.

Remark 1. *Jiang [12] and Li [16] proved the results in Theorem 2 when κ was a constant. Jiang obtained the positive upper and lower bounds of $v(x,t)$, and Li proved that $\theta(x,t)$ was bounded from below and above, and the solution was asymptotically stable as time tended to infinity for large initial data.*

Remark 2. *The global existence of a solution and large time were obtained in Theorem 2 for $\kappa = \bar{\kappa}(1+\theta^b)$. The global existence for $\kappa = \bar{\kappa}\theta^b$ could also be obtained, but Large Time (14) failed for this case in our method.*

We now outline the main ideas and difficulties in our problem compared to previous results. The existence of strong solutions can be easily obtained due to pioneering works, e.g., Tani [32], Kazhikhov [4], and Jesssen and Karper [24]. For the large-time behavior of such a solution, obtaining the uniform positive lower and upper bounds of $v(x,t)$ and $\theta(x,t)$ is a great challenge due to the strong nonlinearity of $\kappa = \bar{\kappa}(1+\theta^b)$. Jiang obtained uniform positive lower and upper bounds on $v(x,t)$ in [12] with a decent localized version of the expression for $v(x,t)$ when κ was a constant. Li and Liang deduced the uniform positive lower and upper bounds on temperature $\theta(x,t)$ in [16] with a smart test function method. However, there methods could not be applied to our case, since it is difficult to obtain the uniform bounds of the high-order estimate ($\|\theta_x\|_{L^2(\mathbb{R})}$), and bounds of θ from below and above. To overcome such a difficulty, motivated by [1,25], we obtained the high-order estimate $Y(t) = \sup_{0\le t\le T} \|\theta^b\theta_x\|_{L^2}$ with an iterative method. The crucial techniques of proofs in [25] could not be adapted directly here since their arguments depend on bounded domain and boundary conditions that were different from ours, and we could not obtain $L^p(p \ge 1)$ norm of $\theta - 1$ under the far-field condition in this paper with an unbounded domain. In combination with the above methods in the literature, we discuss it withg a space separation technique and iterative method that could obtain the global existence of a solution. Then, combining the lower bound of the temperature when $t \in [0, T_0]$ induced by the comparison principle and the lower bound when $t \in (T_0, \infty)$ obtained from a well-designed continuation argument for some suitable fixed $T_0 \in [0, \infty)$, the positive pointwise boundedness of $\theta(x,t)$ from below and above independent of time and the large-time decay behavior of solutions could be obtained.

The rest of the paper is organized as follows. In Section 2, we give some a priori estimates, and prove the uniform positive lower and upper bounds of $v(x,t)$ independent of time. In Section 3, on the basis of the local existence of the solutions and the a priori estimates in Section 2, we prove the global existence of solution with a standard continuity argument. In Section 4, we give the asymptotic behavior of the global solution for $\kappa = \bar{\kappa}(1+\theta^b)$.

2. A Priori Estimates

In this section, we perform a sequence of estimates. We proved that volume $v(x,t)$ was pointwise bounded from below and above independent of time. This is a key step in the proof of both the global existence and asymptotic behavior of the solution. Assume that $(v,\ u,\ \theta)(x,t)$ is the unique strong solution of (2), defined on $\mathbb{R}\times[0,T_0]$ for some $T_0>0$.

Lemma 1. *There are positive constants e_0 and C independent of T, such that*

$$\sup_{0\le t<\infty}\int_{\mathbb{R}}\left(\frac{1}{2}u^2+R(v-\ln v-1)+c_v(\theta-\ln\theta-1)\right)dx$$
$$+\bar{\mu}\int_0^\infty\int_{\mathbb{R}}\frac{u_x^2}{v\theta}dxdt+\bar{\kappa}\int_0^\infty\int_{\mathbb{R}}\frac{(1+\theta^b)\theta_x^2}{\theta^2 v}dxdt\le e_0, \tag{15}$$

Let $\Omega_M(t)=\{x\in\mathbb{R}|\theta(x,t)\ge M>1\}$; we derive from (15) that

- $\int_{\Omega_M}|\theta-1|dx\le C\int_{\Omega_M}(\theta-\ln\theta-1)dx\le C,$
- $\int_{\mathbb{R}/\Omega_M}|\theta-1|^2dx\le C\int_{\mathbb{R}/\Omega_M}(\theta-\ln\theta-1)dx\le C.$

Proof. By using Equation (2c) and a far-field condition, we obtain after a straightforward calculation that

$$c_v\theta_t+R\frac{u_x\theta}{v}=\bar{\kappa}\left(\frac{(1+\theta^b)\theta_x}{v}\right)_x+\bar{\mu}\frac{(u_x)^2}{v}. \tag{16}$$

Multiplying (2a) by $R(1-v^{-1})$, (2b) by u, (16) by $(1-\theta^{-1})$, and adding them together, we obtain

$$\left(\frac{1}{2}u^2+R(v-\ln v-1)+c_v(\theta-\ln\theta-1)\right)_t+\bar{\mu}\frac{u_x^2}{v}+\bar{\kappa}\frac{(1+\theta^b)\theta_x^2}{\theta^2 v}$$
$$=\bar{\mu}(\frac{uu_x}{v})_x-R(\frac{u\theta}{v})_x+Ru_x+\bar{\kappa}\left((1-\theta^{-1})\frac{(1+\theta^b)\theta_x^2}{v}\right)_x. \tag{17}$$

Using Taylor's theorem, (8) and Sobolev's imbedding theorem ($H^1\hookrightarrow L^\infty$), we have

$$\int_{\mathbb{R}}\left(\frac{1}{2}u_0^2+R(v_0-\ln v_0-1)+c_v(\theta_0-\ln\theta_0-1)\right)dx\le C(1+\|(v_0-1,u_0,\theta_0-1)\|_{H^1}^2).$$

Integrating (17) over $\mathbb{R}$ and using far-field Condition (5) obtains (15). The proof of Lemma 1 is finished. □

For some positive integer k, let $\phi\in W^{1,\infty}(\mathbb{R})$ be defined by

$$\phi=\begin{cases}1, & x\le k+1;\\ k+2-x, & k+1\le x\le k+2;\\ 0, & x\ge k+2.\end{cases}$$

For simplicity, we denote $\Omega_k := (k, k+1]$ for Cauchy problem. Then, the bounds of $v(x,t)$ dependent on T can be obtained. We prove the pointwise bounds on a specific volume into two parts, when $t \in [0,T]$ and $t \in (T, \infty)$ for some suitable T.

Lemma 2. *There exists a positive constant C, such that*

$$C^{-1}(T) \le v(x,t), \tag{18}$$

for all $(x,t) \in \mathbb{R} \times [0,T]$.

Proof. For any $x \in \Omega_k$, we have the following local representation via Lemma 1:

$$\int_k^{k+1} [(v - \ln v - 1) + (\theta - \ln\theta - 1)]dx \le e_0, \tag{19}$$

which, together with Jensen's inequality, yields

$$\alpha_1 \le \int_k^{k+1} v(x,t)dx \le \alpha_2,\ \alpha_1 \le \int_k^{k+1} \theta(x,t)dx \le \alpha_2, \tag{20}$$

where $0 < \alpha_1 < 1 < \alpha_2$ are two roots of

$$x - \ln x - 1 = e_0.$$

Moreover, it follows from (19) that, for any $t > 0$, there exists some $b_k(t) \in [k, k+1]$, such that

$$(v - \ln v - 1) + (\theta - \ln\theta - 1)(b_k(t), t) \le e_0,$$

which implies

$$\alpha_1 \le v(b_k(t), t) \le \alpha_2,\ \alpha_1 \le \theta(b_k(t), t) \le \alpha_2. \tag{21}$$

Letting $\sigma \triangleq \frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v} = \bar{\mu}(\ln v)_t - \frac{R\theta}{v}$, we write (2b) as

$$u_t = \sigma_x. \tag{22}$$

Multiplying (2b) by ϕ gives

$$[\phi u]_t = \left[\left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right)\phi\right]_x - \phi_x\left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right).$$

Integrating over (x, ∞) $(x \in \Omega_k)$ with respect to x and recalling (2a) and the definition of ϕ, we have

$$\begin{aligned} -\int_x^\infty [\phi u]_t dy &= \left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right) + \int_x^\infty \left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right)\phi_x dy \\ &= \bar{\mu}[\ln v]_t - \frac{R\theta}{v} - \int_{k+1}^{k+2}\left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right)dy, \quad x \in \Omega_k. \end{aligned} \tag{23}$$

Furthermore, integrating over $[0,t]$, one has

$$\begin{aligned} \int_x^\infty (u_0 - u)\phi dy = \bar{\mu}(\ln v - \ln v_0) - \int_0^t \frac{R\theta}{v} ds \\ - \int_0^t \int_{k+1}^{k+2} \left(\frac{\bar{\mu}u_x}{v} - \frac{R\theta}{v}\right)dyds, \quad x \in \Omega_k. \end{aligned} \tag{24}$$

Denote

$$B(x,t) = v_0 \exp\left\{\frac{1}{\bar{\mu}}\int_x^\infty (u_0(y) - u(y,t))\phi(y)dy\right\},$$
$$Y(t) = \exp\left\{\frac{1}{\bar{\mu}}\int_0^t\int_{k+1}^{k+2}\left[\bar{\mu}\frac{u_x}{v} - R\frac{\theta}{v}\right]dyds\right\}. \tag{25}$$

Taking the exponential on both sides of (24), the following relation appears:

$$\frac{1}{B(x,t)Y(t)} = \frac{1}{v(x,t)}\exp\left\{\int_0^t \frac{R\theta}{\bar{\mu}v}ds\right\}, \quad x \in \Omega_k, \ \ t \geq 0. \tag{26}$$

Multiplying (26) by $\frac{R\theta(x,t)}{\bar{\mu}}$ and integrating over $(0,t)$, we infer

$$\exp\left\{\frac{R}{\bar{\mu}}\int_0^t \frac{\theta(x,s)}{v(x,s)}ds\right\} = 1 + \frac{R}{\bar{\mu}}\int_0^t \frac{\theta(x,s)}{B(x,s)Y(s)}ds.$$

Substituting the above identity into (26), we obtain

$$v(x,t) = B(x,t)Y(t) + \frac{R}{\bar{\mu}}\int_0^t \frac{B(x,t)Y(t)}{B(x,s)Y(s)}\theta(x,s)ds, \quad x \in \Omega_k, \ \ t \geq 0. \tag{27}$$

Since

$$\left|\int_x^\infty (u(y,t) - u_0(y))\phi(y)dy\right| \leq \left(\int_{\Omega_k} u^2 dy\right)^{\frac{1}{2}} + \left(\int_{\Omega_k} u_0^2 dy\right)^{\frac{1}{2}} \leq C(e_0),$$

we deduce from (25)

$$C^{-1}(e_0) \leq B(x,t) \leq C(e_0). \tag{28}$$

Moreover, integrating (27) with respect to x over $[k, k+1]$ gives

$$\frac{1}{Y(t)}\int_k^{k+1} v(x,t)dx = \int_k^{k+1} B(x,t)\left(1 + \int_0^t \frac{\theta(x,\tau)}{B(x,\tau)Y(\tau)}d\tau\right)dx.$$

Hence, we have

$$\frac{1}{Y(t)} \leq C(\alpha_1) + C(\alpha_1, \alpha_2, e_0)\int_0^t \frac{1}{Y(\tau)}d\tau, \tag{29}$$

where (15), (20), and (28) were used, as well as the following simple fact:

$$\int_k^{k+1} \frac{\theta(x,\tau)B(x,\tau)}{B(x,t)}dx \leq C(e_0)\int_k^{k+1} \theta(x,\tau)dx \leq C(\alpha_2, e_0). \tag{30}$$

Applying the Grönwall's inequality to (29) gives

$$\frac{1}{Y(t)} \leq C(T, \alpha_1, \alpha_2, e_0), \tag{31}$$

which implies that, for any positive integer k and $(x,t) \in [k, k+1] \times [0,T]$, from (27), we have

$$C^{-1}(T, \alpha_1, \alpha_2, e_0) \leq v(x,t). \tag{32}$$

From (29), there exists a suitable constant $N > 0$, such that $v(x,t) \geq C^{-1}(T)$ when $k < N$. Combining the fact $v(x,t) \to 1$ as $|x| \to \infty$, one has $v(x,t) \geq 1 - \epsilon$ when $k \geq N$ and ϵ small enough. So, the bounds of $v(x,t)$ from below are obtained in $\mathbb{R} \times [0,T]$. □

Lemma 3. *For any $t \in [0,T]$, and positive constants $C(T)$, it holds that*

$$\theta \geq C(T). \tag{33}$$

Proof. Let $\bar{\theta} = \frac{1}{\theta}$, and rewrite Equation (16) as follows:

$$-\frac{\bar{\theta}_t}{\bar{\theta}^2} = -\frac{Ru_x}{c_v v\bar{\theta}} - \frac{1}{c_v}\left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{v\bar{\theta}^2}\right)_x + \frac{\bar{\mu}u_x^2}{c_v v}. \tag{34}$$

So,

$$\begin{aligned}
\bar{\theta}_t &= \frac{R\bar{\theta}u_x}{c_v v} + \frac{\bar{\theta}^2}{c_v}\left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{v\bar{\theta}^2}\right)_x - \frac{\bar{\mu}\bar{\theta}^2u_x^2}{c_v v} \\
&= \left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{c_v v}\right)_x - \frac{2\kappa(\bar{\theta})\bar{\theta}_x^2}{c_v v\bar{\theta}} - \frac{\bar{\mu}\bar{\theta}^2u_x^2}{c_v v} + \frac{R\bar{\theta}u_x}{c_v v} \\
&= \left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{c_v v}\right)_x - \frac{2\kappa(\bar{\theta})\bar{\theta}_x^2}{c_v v\bar{\theta}} - \frac{\bar{\mu}\bar{\theta}^2}{c_v v}\left(u_x^2 - \frac{Ru_x}{\bar{\mu}\bar{\theta}}\right) \\
&= \left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{c_v v}\right)_x - \frac{2\kappa(\bar{\theta})\bar{\theta}_x^2}{c_v v\bar{\theta}} - \frac{\bar{\mu}\bar{\theta}^2}{c_v v}\left(u_x - \frac{R}{2\bar{\mu}\bar{\theta}}\right)^2 + \frac{R^2}{4\bar{\mu}c_v v},
\end{aligned} \tag{35}$$

which implies

$$\bar{\theta}_t \leq \left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{c_v v}\right)_x + \frac{R^2}{4\bar{\mu}c_v v} \leq \left(\frac{\kappa(\bar{\theta})\bar{\theta}_x}{c_v v}\right)_x + C(T). \tag{36}$$

Define the operator $L := -\frac{\partial}{\partial_t} + \left(\frac{\kappa(\cdot)}{v}(\cdot)_x\right)_x$ and

$$\begin{cases} L\tilde{\theta} < 0 \quad \text{on} \quad [0,\infty)\times\Omega, \\ \tilde{\theta}\mid_{t=0} \geq 0 \quad \text{on} \quad \Omega, \\ \tilde{\theta}\mid_{x\to\infty} \geq 0 \quad \text{on} \quad [0,\infty), \end{cases} \tag{37}$$

where $\tilde{\theta}(x,t) = C(T)t + \max_{\bar{\Omega}} \bar{\theta}_0 - \bar{\theta}(x,t)$; then, with the comparison theorem, we obtain

$$\min_{(x,t)\in\bar{Q}_T} \tilde{\theta}(x,t) \geq 0,$$

which implies

$$\theta(x,t) \geq \frac{1}{C(T)t + \max_{\bar{\Omega}} \bar{\theta}_0} \geq C(T). \tag{38}$$

This completes the proof of Lemma 3. □

Now, in order to obtain the uniform upper and lower bounds of $v(x,t)$, we first show the exponential decay of $Y(t)$, and use Representation (27) to obtain the following uniform bounds on $v(x,t)$.

Lemma 4. *There exists a positive constant C independent of t, such that*

$$C^{-1} \leq v(x,t) \leq C, \qquad \forall x \in \mathbb{R}, \quad t \geq 0. \tag{39}$$

Proof. By using (28), one can first prove, by repeating the argument used in [12], the following estimates:

$$-\int_s^t \inf_{[k+1,k+2]} \theta(s,\cdot)ds \leq C - \frac{t-s}{C}$$

for all $0 \leq s \leq t \leq T$. Then, one can obtain, with Jenssen's inequality,

$$\begin{aligned}
&\int_s^t \int_{k+1}^{k+2} \left[\bar{\mu}\frac{u_x}{v} - R\frac{\theta}{v}\right] dyds \\
\leq\ & C\int_s^t \int_{k+1}^{k+2} \frac{u_x^2}{\theta v} dyds - \frac{R}{2}\int_s^t \int_{k+1}^{k+2} \frac{\theta}{v} dyds \\
\leq\ & C - \frac{R}{2}\int_s^t \inf_{[k+1,k+2]} \theta(s,\cdot) \int_{k+1}^{k+2} \frac{1}{v} dyds \\
\leq\ & C - \frac{R}{2}\int_s^t \inf_{[k+1,k+2]} \theta(s,\cdot) \left(\int_{k+1}^{k+2} v dy\right)^{-1} ds \\
\leq\ & C - \frac{R}{2\alpha_2}\int_s^t \inf_{[k+1,k+2]} \theta(s,\cdot) ds \\
\leq\ & C - \frac{t-s}{C}, \quad 0 \leq s \leq t \leq T.
\end{aligned} \tag{40}$$

Recalling the definition of $Y(t)$ yields that

$$Y(t) \leq Ce^{-Ct}, \quad \frac{Y(t)}{Y(s)} \leq Ce^{-C(t-s)}, \quad \forall t \geq 0, \ \ t \geq s \geq 0. \tag{41}$$

On the other hand, for a point $b_k(t) \in [k, k+1]$ via Lemma 1 implies that

$$\begin{aligned}
\left|\theta^{\frac{1}{2}}(x,t) - \theta^{\frac{1}{2}}(b_k(t),t)\right| &\leq \frac{1}{2}\int_{b_k(t)}^x \theta^{-\frac{1}{2}}|\theta_x| dx \\
&\leq \frac{1}{2}\left(\int_k^{k+1} \frac{\theta_x^2}{v\theta^2} dx\right)^{\frac{1}{2}} \left(\int_k^{k+1} v\theta dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\alpha_2}}{2}\left(\int_{\Omega_k} \frac{\theta_x^2}{v\theta^2} dx\right)^{\frac{1}{2}} \max_{[k,k+1]} v^{1/2}(\cdot,t),
\end{aligned}$$

$k = 0, \pm 1, \cdots$. By using Young's inequality, we have

$$\begin{aligned}
&C(\alpha_1, b) - C(\alpha_2)\int_{\Omega_k} \frac{\theta_x^2}{v\theta^2} dx \max_{\bar{\Omega}_k} v(\cdot,t) \leq \theta(x,t) \\
&\leq C(\alpha_2)(\alpha_2, b) + C\int_{\Omega_k} \frac{\theta_x^2}{v\theta^2} dx \max_{\bar{\Omega}_k} v(\cdot,t).
\end{aligned} \tag{42}$$

Hence, inserting (28), (41), and (42) into (28), one has

$$v(x,t) \leq Ce^{-Ct} + C\int_0^t e^{-C(t-s)} \left(C(\alpha_2, b) + C\int_{\Omega_k} \frac{\theta_x^2}{v\theta^2} dx \max_{\bar{\Omega}_k} v(\cdot,t)\right) ds,$$

Applying Gronwall's inequality and (15), we conclude

$$v(x,t) \leq C, \qquad \forall x \in \Omega_k, \quad t \geq 0. \tag{43}$$

Now, we prove the lower bound of $v(x,t)$ independent with T. The proof is divided into two parts, when $t \in [0, T_0]$ and $t \in (T_0, \infty)$, for some suitable fixed $T_0 \in [0, \infty)$. When $t \in [0, T_0]$, we know that $v(x,t) \geq C(T)$ via Lemma 2. Regarding $t \in (T_0, \infty)$, integrating (27) over $[k+1, k+2]$, using the estimate in Lemma 3, we have

$$\alpha_1 \leq Ce^{-Ct} + C\int_0^t \frac{Y(t)}{Y(s)} ds. \tag{44}$$

It follows from (20), (28), and (41)–(44) that

$$
\begin{aligned}
v(x,t) &\geq C_4 \int_0^t \frac{Y(t)}{Y(s)}(C_5 - C_6 \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dx) ds \\
&\geq C_7 - C_8 e^{-Ct} - C_9 \left(\int_0^{t/2} + \int_{t/2}^t \right) e^{-C(t-s)} \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dxds \\
&\geq C_{10} - C_{11} e^{-Ct} - C_9 e^{-(Ct)/2} \int_0^{t/2} \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dxds - C_9 \int_{t/2}^t \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dxds \\
&\geq C_{10} - \sum_{i=1}^3 J_i \\
&\geq C_{12}.
\end{aligned}
\tag{45}
$$

Obviously, via (15), we have $J_1, J_2 \to 0$ as $t \to \infty$. We also have

$$
J_3 = C_9 \int_0^t \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dxds - C_9 \int_0^{t/2} \int_{\mathbb{R}} \frac{\theta_x^2}{v\theta^2} dxds \to 0 \qquad t \to \infty.
$$

So, we can take a large enough T_0 to ensure that $C_{12} > 0$.

Therefore, by combining (43), (29), and (45), one has

$$
C^{-1}(k) \leq v(x,t) \leq C(k),
$$

Due to the far-field condition, we obtain (39). This completes the proof of Lemma 4 (i.e., part of (13) in Theorem 2). □

3. Proof of Global Existence

In this section, we apply the results obtained in Section 2 to prove Theorem 2. Motivated by [1], we give the estimate on $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (1+\theta^{2b})\theta_x^2 dx$, which is the key step in the proof of Theorem 2.

In our case, nonlinearity κ on θ requires further attention on the control of θ. For this purpose, one of the main ingredients in this paper is the following lemma-refined estimates on temperature. In order to obtain the higher-order estimates to the solution, we follow the framework introduced in [1], and define the following two functionals

$$
Z(t) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}} u_{xx}^2(x,t) dx, \quad Y(t) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}} (1+\theta^{2b})\theta_x^2(x,t) dx. \tag{46}
$$

These two functionals are useful in depicting the tangled relations of the higher order and upper bound of θ. First, we have that the following lemma holds.

Lemma 5. *For some positive constant, we have*

$$
\sup_{\mathbb{R} \times [0,T]} \theta \leq C + CY^{\frac{1}{2b+3-\delta}}, \tag{47}
$$

$$
\sup_{\mathbb{R} \times [0,T]} |u_x| \leq C + CZ^{\frac{3}{8}}. \tag{48}
$$

for any $(x,t) \in Q_T$.

Proof. For a small constant δ, via the Gagliardo–Nirenberg inequality, we infer that

$$\begin{aligned}
&\sup_{\mathbb{R}} |\theta - 1|^{2b+3-\delta} = \sup_{\mathbb{R}} \Big(-\int_x^\infty \left(|\theta-1|^{2b+3-\delta}\right)_x dy \Big) \\
&\leq C \int_{\mathbb{R}} |\theta-1|^{2b+2-\delta} |\theta_x| dx \\
&\leq C \int_{\mathbb{R}} (1+\theta^{2b}) \theta_x^2 dx + C \int_{\Omega_M \cup \mathbb{R}/\Omega_M} |\theta-1|^{2b+4-2\delta} dx \\
&\leq C \int_{\mathbb{R}} (1+\theta^{2b}) \theta_x^2 dx + C \sup_{\mathbb{R}} |\theta-1|^{2b+3-2\delta} \int_{\Omega_M} |\theta-1| dx \\
&\quad + C \sup_{\mathbb{R}} |\theta-1|^{2b+2-2\delta} \int_{\mathbb{R}/\Omega_M} |\theta-1|^2 dx \\
&\leq C + \epsilon \sup_{\mathbb{R}} |\theta-1|^{2b+3-\delta} + CY, \qquad (49)
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{\mathbb{R}} \theta \leq \sup_{\mathbb{R}} |\theta-1| + 1 \\
&\leq C + CY^{\frac{1}{2b+3-\delta}}, \qquad (50)
\end{aligned}$$

where we used the fact that $|\theta-1|^{2b} \leq C(1+\theta^{2b})$, Lemma 1 and Young inequality.

Regarding (48), we have

$$\begin{aligned}
\sup_{\mathbb{R}} |u_x|^2 &\leq \int_{\mathbb{R}} u_x^2 dx + 2\int_{\mathbb{R}} |u_x u_{xx}| dx \\
&\leq \Big(\int_{\mathbb{R}} u^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} u_{xx}^2 dx \Big)^{\frac{1}{2}} + 2 \Big(\int_{\mathbb{R}} u_x^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} u_{xx}^2 dx \Big)^{\frac{1}{2}} \\
&\leq CZ^{\frac{1}{2}} + 2 \Big(\int_{\mathbb{R}} u^2 dx \Big)^{\frac{1}{4}} \Big(\int_{\mathbb{R}} u_{xx}^2 dx \Big)^{\frac{1}{4}} \Big(\int_{\mathbb{R}} u_{xx}^2 dx \Big)^{\frac{1}{2}} \\
&\leq C + CZ^{\frac{3}{4}}.
\end{aligned}$$

This completes the proof of Lemma 5. □

In addition, we have the following key estimate.

Lemma 6. *For a positive constant C, $0 \leq t \leq T$ and $b \geq 2$, it holds that*

$$\int_0^T \sup_{\mathbb{R}} |\theta - 1|^{b+1} dt \leq C \qquad (51)$$

Proof. Through the Gagliardo–Nirenberg inequality and Lemma 1, we infer that

$$\begin{aligned}
&\int_0^T \sup_{\mathbb{R}} |\theta-1|^{b+1}dt = \int_0^T \sup_{\mathbb{R}}(|\theta-1|^{\frac{b+1}{2}})^2 dt \\
&\le \int_0^T \left(\int_{\mathbb{R}} |\theta-1|^{b+1}dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\theta-1|^{b-1}\theta_x^2 dx\right)^{\frac{1}{2}} dt \\
&\le \int_0^T \left(\int_{\Omega_M} |\theta-1|^{b+1}dx\right)^{\frac{1}{2}} \sup_{\mathbb{R}} |\theta-1|^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\theta-1|^{b-2}\theta_x^2 dx\right)^{\frac{1}{2}} dt \\
&\quad + \int_0^T \left(\int_{\mathbb{R}/\Omega_M} |\theta-1|^{b+1}dx\right)^{\frac{1}{2}} \sup_{\mathbb{R}} |\theta-1| \left(\int_{\mathbb{R}} |\theta-1|^{b-3}\theta_x^2 dx\right)^{\frac{1}{2}} dt \\
&\le \varepsilon \int_0^T \sup_{\mathbb{R}} |\theta-1|^{b+1}dt + C\int_{Q_T} |\theta-1|^{b-2}\theta_x^2 dxdt + C\int_{Q_T} |\theta-1|^{b-3}\theta_x^2 dxdt \\
&\le \varepsilon \int_0^T \sup_{\mathbb{R}} |\theta-1|^{b+1}dt + C\int_{Q_T} \frac{(1+\theta^b)}{\theta^2}\theta_x^2 dxdt \\
&\le \varepsilon \int_0^T \sup_{\mathbb{R}} |\theta-1|^{b+1}dt + C.
\end{aligned} \tag{52}$$

Here, we use the fact that, for $b \ge 2$ and $\kappa = \bar{\kappa}(1+\theta^b)$, we have

$$\int_{Q_T} \theta^{b-2}\theta_x^2 dxdt \le \int_{Q_T} \frac{1+\theta^b}{\theta^2} dxdt,$$

and for $b \ge 1$

$$\int_{Q_T} \theta^{b-3}\theta_x^2 dxdt \le C\int_{Q_T} \frac{\theta_x^2}{\theta^2}dxdt + C\int_{Q_T} \frac{\theta^b\theta_x^2}{\theta^2}dxdt.$$

It yields that

$$\int_0^T \sup_{\mathbb{R}} |\theta-1|^{b+1}dt \le C. \tag{53}$$

Then, the proof of Lemma 6 is completed. □

The following lemma gives estimates on the L^2 norm of v_x.

Lemma 7. *For any $t \in [0,T]$, there exists a constant C independent of time; it holds that*

$$\sup_{0\le t\le T} \int_{\mathbb{R}} v_x^2 dx + \int_{Q_T} \theta v_x^2 dxdt \le C + CY^{\frac{1}{2b+3-\delta}}. \tag{54}$$

Proof. First, integrating (2b) multiplied by $\frac{v_x}{v}$ over $\mathbb{R}$, we obtain that, after using (2a),

$$\begin{aligned}
&\frac{\bar{\mu}}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{v_x^2}{v^2}dx = R\int_{\mathbb{R}} \left(\frac{\theta}{v}\right)_x \frac{v_x}{v}dx + \int_{\mathbb{R}} u_t\frac{v_x}{v}dx \\
&= R\int_{\mathbb{R}} \frac{\theta_x v_x}{v^2}dx - R\int_{\mathbb{R}} \frac{\theta v_x^2}{v^3}dx + \frac{d}{dt}\int_{\mathbb{R}} \frac{u v_x}{v}dx + \int_{\mathbb{R}} \frac{u_x^2}{v}dx \\
&\le C\int_{\mathbb{R}} \frac{\theta_x^2}{v\theta}dx - \frac{R}{2}\int_{\mathbb{R}} \frac{\theta v_x^2}{v^3}dx + \frac{d}{dt}\int_{\mathbb{R}} \frac{u v_x}{v}dx + \int_{\mathbb{R}} \frac{u_x^2}{v}dx,
\end{aligned} \tag{55}$$

which together with (15), which yields

$$\int_{\mathbb{R}} \frac{v_x^2}{v^2}dx + \int_{Q_T} \frac{\theta v_x^2}{v^3}dxdt \le C + \sup_{Q_T}\theta \int_{Q_T} \frac{u_x^2}{v\theta}dxdt \le C + CY^{\frac{1}{2b+3-\delta}}.$$

□

We have the following relationship between the high-order estimates on $u(x,t)$ and Y.

Lemma 8. *For any $t\in[0,T]$, we have following estimate:*

$$\sup_{0\le t\le T}\int_{\mathbb{R}} u_x^2dx+\int_{Q_T}u_{xx}^2dxdt\le C+CY^{\frac{3}{2b+3-\delta}}. \tag{56}$$

Proof. We rewrite the momentum equation into the following form:

$$u_t-\frac{\bar{\mu}u_{xx}}{v}=-\frac{\bar{\mu}u_xv_x}{v^2}-\frac{R\theta_x}{v}+\frac{R\theta v_x}{v^2}. \tag{57}$$

Multiplying (57) by u_{xx}, and integrating in x over $\mathbb{R}$, one has

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\mathbb{R}} u_x^2dx+\int_{Q_T}\frac{\bar{\mu}u_{xx}^2}{v}dxdt\\
&\le C+\left|\int_{Q_T}\frac{\bar{\mu}u_xv_x}{v^2}u_{xx}dxdt\right|+\left|\int_{Q_T}\frac{R\theta_x}{v}u_{xx}dxdt\right|+\left|\int_{Q_T}\frac{R\theta v_x}{v^2}u_{xx}dxdt\right|\\
&\le C+\frac{\bar{\mu}}{4}\int_{Q_T}\frac{u_{xx}^2}{v}dxdt+C\int_{Q_T}(u_x^2v_x^2+\theta_x^2+\theta^2v_x^2)dxdt.
\end{aligned} \tag{58}$$

According to Equation (2a) and the far-field conditions, one has

$$\begin{aligned}
\int_0^T\sup_{\mathbb{R}}u_x^2dt &= 2\int_0^T\sup_{\mathbb{R}}\int_{-\infty}^x u_xu_{xx}dydt\\
&\le C\int_0^T\left(\int_{\mathbb{R}}u_x^2dx\right)^{\frac12}\left(\int_{\mathbb{R}}u_{xx}^2dx\right)^{\frac12}dt.
\end{aligned}$$

Hence, via (15), Young's inequality, and the uniform boundedness of $v(x,t)$, we have the term

$$\begin{aligned}
\int_{Q_T}u_x^2v_x^2dxdt &\le \int_0^T\sup_{\mathbb{R}}u_x^2dt\sup_{0\le t\le T}\int_{\mathbb{R}}v_x^2dx\\
&\le C\int_0^T\left(\int_{\mathbb{R}}u_x^2dx\right)^{\frac12}\left(\int_{\mathbb{R}}u_{xx}^2dx\right)^{\frac12}dt\sup_{0\le t\le T}\int_{\mathbb{R}}v_x^2dx\\
&\le \varepsilon\int_{Q_T}\frac{u_{xx}^2}{v}dxdt+C\int_{Q_T}u_x^2dxdt\left(\sup_{0\le t\le T}\int_{\mathbb{R}}v_x^2dx\right)^2\\
&\le \varepsilon\int_{Q_T}\frac{u_{xx}^2}{v}dxdt+CY^{\frac{3}{2b+3-\delta}}+C.
\end{aligned}$$

For the other two terms, we have the following estimate

$$\begin{aligned}
\int_{Q_T}(\theta_x^2+\theta^2v_x^2)dxdt &\le \sup_{Q_T}\theta^2\int_{Q_T}\frac{\theta_x^2}{v\theta}dxdt+\sup_{Q_T}\theta\int_{Q_T}\theta v_x^2dxdt\\
&\le C\sup_{Q_T}\theta^2+C\sup_{Q_T}\theta Y^{\frac{1}{2b+3-\delta}}\\
&\le CY^{\frac{2}{2b+3-\delta}}+C,
\end{aligned}$$

Here, we use (15) and Lemma 7. Then, (58) shows that Lemma 8 holds. □

The relation of Y and Z is given in the following lemma.

Lemma 9. *For any $(x,t) \in Q_T$ and positive constant C, we have*

$$Y + \int_{Q_T} (1+\theta^b)\theta_t^2 dxdt \leq C + \varepsilon_1 Z. \tag{59}$$

Proof. Define $K(v,\theta) = \frac{\theta}{v} + \frac{\theta^{1+b}}{(1+b)v}$. Multiplying (16) by K_t, integrating over Q_T and by parts, we have

$$\int_{Q_T} \theta_t K_t dxdt + \frac{\bar{\kappa}}{c_v} \int_{Q_T} \frac{(1+\theta^b)\theta_x}{v} K_{xt} dxdt = \int_{Q_T} \left(\frac{\bar{\mu} u_x^2}{c_v v} - \frac{R\theta u_x}{c_v v} \right) K_t dxdt. \tag{60}$$

Then, we compute

$$K_t = \frac{(1+\theta^b)\theta_t}{v} - \frac{\theta v_t}{v^2} - \frac{\theta^{1+b} v_t}{(1+b)v^2},$$

$$K_x = \frac{(1+\theta^b)\theta_x}{v} - \frac{\theta v_x}{v^2} - \frac{\theta^{1+b} v_x}{(1+b)v^2},$$

$$\begin{aligned} K_{xt} &= \left(\frac{(1+\theta^b)\theta_x}{v}\right)_t + \frac{2\theta v_x u_x}{v^3} + \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} \\ &\quad - \frac{(1+\theta^b)\theta_t v_x}{v^2} - \frac{\theta u_{xx}}{v^2} - \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \\ &\doteq \left(\frac{(1+\theta^b)\theta_x}{v}\right)_t + \tilde{K}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{(1+\theta^{2b})\theta_x^2}{v^2} dx + \int_{Q_T} \frac{(1+\theta^b)\theta_t^2}{v} dxdt \\ &\leq C + \int_{Q_T} \frac{\theta^{1+b}\theta_t u_x}{(1+b)v^2} + \frac{\theta\theta_t u_x}{v^2} dxdt - \frac{\bar{\kappa}}{c_v} \int_{Q_T} \frac{(1+\theta^b)\theta_x}{v} \tilde{K} dxdt \qquad (61) \\ &\quad + \int_{Q_T} \left(\frac{\bar{\mu} u_x^2}{v} - \frac{R\theta u_x}{v} \right) K_t dxdt \\ &\doteq C + \sum_{i=1}^{3} I_i. \qquad (62) \end{aligned}$$

Next, we give the estimate on $I_1,\ I_2,\ I_3$. Using Lemmas 1 and 5, the first term can be estimated as follows:

$$\begin{aligned} I_1 &= \int_{Q_T} \frac{\theta^{1+b}\theta_t u_x}{(1+b)v^2} + \frac{\theta\theta_t u_x}{v^2} dxdt \\ &\leq \varepsilon \int_{Q_T} (1+\theta^b)\theta_t^2 dxdt + C \int_{Q_T} (\theta^{b+2} + \theta^2) u_x^2 dxdt \\ &\leq \varepsilon \int_{Q_T} (1+\theta^b)\theta_t^2 dxdt + \sup_{Q_T} (\theta^{b+3} + \theta^3) \int_{Q_T} \frac{u_x^2}{\theta} dxdt \\ &\leq \varepsilon \int_{Q_T} (1+\theta^b)\theta_t^2 dxdt + CY^{\frac{b+3}{2b+3-\delta}} + C. \end{aligned} \tag{63}$$

Then, we turn to estimate I_2. We divide the proof into three parts through the lemmas proved in Section 2, Lemmas 5–8, and the interpolation inequality.

$$\begin{aligned} I_2 &\le C\int_{Q_T}\Big((\theta+\theta^{1+2b})|\theta_x u_x v_x| + (1+\theta^{2b})|\theta_x\theta_t v_x| \\ &\quad +(\theta+\theta^{1+2b})|\theta_x u_{xx}|\Big)dxdt \doteq \sum_{i=1}^{3} I_{2j}. \end{aligned} \tag{64}$$

We now give the estimate on I_{2j} $(j=1,2,3)$ term by term. For term I_{21}, when $b\ge 1$, one has

$$\begin{aligned} I_{21} &\le C\int_{Q_T}\Big((\theta+\theta^{1+2b})|\theta_x u_x v_x|dxdt \\ &\le \sup_{Q_T}|u_x|\Big[\Big(\int_{Q_T}\frac{\theta^3\theta_x^2}{\theta^2}dxdt\Big)^{\frac12}\Big(\int_{Q_T}\theta v_x^2 dxdt\Big)^{\frac12} \\ &\quad +\Big(\int_{Q_T}\theta^{3b}\theta_x^2 dxdt\Big)^{\frac12}\Big(\int_{Q_T}\theta^{2+b}v_x^2 dxdt\Big)^{\frac12}\Big] \\ &\le C(1+Z^{\frac38})\Big[Y^{\frac{2}{2b+3-\delta}} + \sup_{Q_T}\theta^{b+1}\Big(\int_{Q_T}\theta^{2+b}v_x^2 dxdt\Big)^{\frac12}\Big] \\ &\le C(1+Z^{\frac38})\Big[Y^{\frac{2}{2b+3-\delta}} + \sup_{Q_T}\theta^{b+1}\Big(\int_0^T\sup_{\mathbb{R}}|\theta-1|^{b+1}\int_{\mathbb{R}}\theta v_x^2 dxdt + \\ &\quad +\int_{Q_T}\theta v_x^2 dxdt\Big)^{\frac12}\Big] \\ &\le C(1+Z^{\frac38})\Big[Y^{\frac{2}{2b+3-\delta}} + \sup_{Q_T}\theta^{b+1}\Big(C\sup_{Q_T}\theta\int_{\mathbb{R}}v_x^2 dx + +Y^{\frac{1}{2b+3-\delta}}\Big)^{\frac12}\Big] \\ &\le C(1+Z^{\frac38})\Big[C+CY^{\frac{b+2}{2b+3-\delta}}\Big] \\ &\le C+CZ^{\frac{3(2b+3-\delta)}{8(b+1-\delta)}}+\epsilon Y \le C+CZ^{\frac{21-3\delta}{24-8\delta}}+\epsilon Y \\ &\le C+\varepsilon Z+\varepsilon Y \end{aligned}$$

Here, we use (15), (47), (48), (51), (54), and Young's inequality.
Regarding I_{22},

$$\begin{aligned} I_{22} &\le \int_{Q_T}(1+\theta^{2b})|\theta_x\theta_t v_x|dxdt \\ &\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt + C\sup_{Q_T}(1+\theta^b)\int_0^T\sup_{\mathbb{R}}(\theta_x^2+\theta^{2b}\theta_x^2)dt\int_{\mathbb{R}}v_x^2 dx \\ &\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt + C\sup_{Q_T}(1+\theta^b)\int_0^T\sup_{\mathbb{R}}(\theta_x^2+\theta^{2b}\theta_x^2)dtY^{\frac{1}{2b+3-\delta}} \\ &\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt + C(1+Y^{\frac{b+1}{2b+3-\delta}})\int_0^T\sup_{\mathbb{R}}(1+\theta^{2b})\theta_x^2 dt. \end{aligned} \tag{65}$$

For the last term in the right-hand side of (65), we have the following estimate:

$$\begin{aligned}
&\int_0^T \sup_{\mathbb{R}}(1+\theta^{2b})\theta_x^2 dt \\
&\le \int_{Q_T}(1+\theta^{2b})\theta_x^2 dxdt + C\int_{Q_T}\frac{(1+\theta^b)\theta_x}{v}\left|\left(\frac{(1+\theta^b)\theta_x}{v}\right)_x\right|dxdt \\
&\le C + C\sup_{Q_T}\theta^{b+2} + C\left((1+\sup_{Q_T}\theta^2)\int_{Q_T}\theta^{b-2}\theta_x^2 dxdt\right)^{\frac{1}{2}} \times \\
&\quad \left(\int_{Q_T}(1+\theta^b)(\theta_t^2+u_x^4+\theta^2 u_x^2)dxdt\right)^{\frac{1}{2}} \\
&\le C + CY^{\frac{b+2}{2b+3-\delta}} + C(1+Y^{\frac{1}{2b+3-\delta}})\left(\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt + \int_{Q_T}(1+\theta^b)u_x^4 dxdt\right. \\
&\quad \left. +(\theta^2+\sup_{Q_T}\theta^{b+2})\int_{Q_T}u_x^2 dxdt\right)^{\frac{1}{2}} \\
&\le C + CY^{\frac{b+2}{2b+3-\delta}} + Y^{\frac{b+4}{2(2b+3-\delta)}} + \epsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt \\
&\quad +C(1+Y^{\frac{1}{2b+3-\delta}})\left(\int_{Q_T}(1+\theta^b)u_x^4 dxdt\right)^{\frac{1}{2}}.
\end{aligned} \tag{66}$$

Then, using (56) and the Gagliardo–Nirenberg inequality, we infer that

$$\begin{aligned}
&(1+\sup_{Q_T}\theta^{\frac{b}{2}})\left(\int_{Q_T}u_x^4 dxdt\right)^{\frac{1}{2}} \\
\le\ & C(1+Y^{\frac{b}{2(2b+3-\delta)}})\left(\int_0^T \|u^2\|_{L^\infty}\|u_{xx}\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\
\le\ & C(1+Y^{\frac{b}{2(2b+3-\delta)}})\left(\int_0^T \int_{\mathbb{R}} u|u_x|dx \int_{\mathbb{R}} u_{xx}^2 dxdt\right)^{\frac{1}{2}} \\
\le\ & C(1+Y^{\frac{b}{2(2b+3-\delta)}})\left(\int_0^T \left(\int_{\mathbb{R}} u^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{2}} \int_{\mathbb{R}} u_{xx}^2 dxdt\right)^{\frac{1}{2}} \\
\le\ & C(1+Y^{\frac{b}{2(2b+3-\delta)}})\sup_{0\le t\le T}\left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{4}}\left(\int_0^T\int_{\mathbb{R}} u_{xx}^2 dxdt\right)^{\frac{1}{2}} \\
\le\ & C(1+Y^{\frac{b}{2(2b+3-\delta)}})Y^{\frac{3}{4(2b+3-\delta)}}Y^{\frac{3}{2(2b+3-\delta)}} \\
\le\ & CY^{\frac{2b+9}{4(2b+3-\delta)}} + C.
\end{aligned}$$

Then, we can obtain that

$$\begin{aligned}
I_{22} \ \le\ & C + CY^{\frac{b+2}{2b+3-\delta}} + Y^{\frac{b+5}{2(2b+3-\delta)}} + CY^{\frac{6b+17}{4(2b+3-\delta)}} + \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt \\
\le\ & C + \varepsilon Z + \varepsilon Y + \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2 dxdt,
\end{aligned} \tag{67}$$

Here, we use the fact that if $b > \frac{5}{2}$; then, $\frac{6b+17}{4(2b+3-\delta)} < 1$.

Regarding I_{23},

$$
\begin{aligned}
I_{23} &\le C\int_{Q_T}(\theta+\theta^{1+2b})|\theta_x u_{xx}|dxdt\\
&\le \left(\int_{Q_T}u_{xx}^2dxdt\right)^{\frac{1}{2}}\left(\int_{Q_T}\theta^2\theta_x^2dxdt\right)^{\frac{1}{2}}\\
&\quad+\left(\int_{Q_T}\theta^{1+b}u_{xx}^2dxdt\right)^{\frac{1}{2}}\left(\int_{Q_T}\theta^{3b+1}\theta_x^2dxdt\right)^{\frac{1}{2}}\\
&\le C+CY^{\frac{7}{2(2b+3-\delta)}}+\sup_{Q_T}\theta^{\frac{b+1}{2}}\left(\int_{Q_T}u_{xx}^2dxdt\right)^{\frac{1}{2}}\sup_{Q_T}\theta^{\frac{2b+3}{2}}\left(\int_{Q_T}\theta^{b-2}\theta_x^2dxdt\right)^{\frac{1}{2}}\\
&\le C+CY^{\frac{7}{2(2b+3-\delta)}}+CY^{\frac{3b+7}{2(2b+3-\delta)}}\le C+\varepsilon Z+\varepsilon Y.
\end{aligned}
$$

Here, we use the fact that $\frac{7}{2(2b+3-\delta)}<\frac{3b+7}{2(2b+3-\delta)}<1$, if $b>1$.
Then, we give the estimate on I_3,

$$
\begin{aligned}
I_3 &= \int_{Q_T}\left(\frac{Ru_x^2}{c_v v}-\frac{\bar{\mu}\theta u_x}{c_v v}\right)\left(\frac{(1+\theta^b)\theta_t}{v}-\frac{\theta v_t}{v^2}-\frac{\theta^{1+b}v_t}{(1+b)v^2}\right)dxdt\\
&\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2dxdt+C\int_{Q_T}\big((1+\theta^b)|u_x|^4+\theta u_x^3\\
&\quad+\theta^{1+b}u_x^3+(1+\theta^{1+b})\theta u_x^2+\theta^2u_x^2+\theta^{2+b}u_x^2\big)dxdt\\
&\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2dxdt+C(1+\sup_{Q_T}\theta^b)\int_{Q_T}u_x^4dxdt\\
&\quad+C\sup_{Q_T}\theta^{b+2}\sup_{Q_T}|u_x|\int_{Q_T}\frac{u_x^2}{\theta}dxdt+C(1+\sup_{Q_T}\theta^{b+3})\int_{Q_T}\frac{u_x^2}{\theta}dxdt\\
&\le \varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2dxdt+CY^{\frac{4b+18}{4(2b+3-\delta)}}+CY^{\frac{b+2}{2b+3-\delta}}Z^{\frac{3}{8}}+CY^{\frac{b+3}{2b+3-\delta}}+C\\
&\le C+\varepsilon Z+\varepsilon Y+\varepsilon\int_{Q_T}(1+\theta^b)\theta_t^2dxdt.
\end{aligned}
$$

Here, we use the fact that $\frac{4b+18}{4(2b+3-\delta)}<1$, if $b>\frac{3}{2}$ and $\frac{b+2}{2b+3-\delta}<\frac{5}{8}$, if $b>\frac{1}{2}$.
Adding the estimations of I_1, I_2, I_3 and taking a suitable δ, it holds that

$$
Y+\int_{Q_T}(1+\theta^b)\theta_t^2dxdt\le C+\varepsilon_1 Z.
$$

This completes the proof of Lemma 9. □

Next, we give the uniform boundedness of z.

Lemma 10. *For any $b\in(\frac{5}{2},\infty)$ and $(x,t)\in Q_T$, it holds that*

$$
\sup_{[0,T]}\int_{\mathbb{R}}u_t^2dx+\int_{Q_T}u_{xt}^2dxdt\le C+\varepsilon_2 Z, \tag{68}
$$

$$
Z\le C,\ \ Y\le C. \tag{69}
$$

Proof. Differentiating (2b) with respect to t, multiplying by u_t, and integrating over Q_T yields

$$\begin{aligned}
&\int_{\mathbb{R}} u_t^2 dx + \int_{Q_T} \frac{u_{xt}^2}{v} dxdt \le C + \int_{Q_T} \left(\frac{\bar{\mu} u_x^2}{v^2} + \frac{R\theta_t}{v} - \frac{\theta u_x}{v^2} \right) u_{xt} dxdt + C \\
&\le \varepsilon \int_{Q_T} \frac{u_{xt}^2}{v} dxdt + C \int_{Q_T} (u_x^4 + \theta_t^2 + u_x^2 \theta^2) dxdt + C \\
&\le \varepsilon \int_{Q_T} \frac{u_{xt}^2}{v} dxdt + \int_{Q_T} \theta_t^2 dxdt \\
&\quad + C(\sup_{Q_T} u_x^2 \theta + \sup_{Q_T} \theta^3) \int_{Q_T} \frac{u_x^2}{\theta} dxdt + C \\
&\le \varepsilon \int_{Q_T} \frac{u_{xt}^2}{v} dxdt + C Z^{\frac{3}{4}} Y^{\frac{1}{2b+3-\delta}} + C Y^{\frac{3}{2b+3-\delta}} + C \\
&\le \varepsilon \int_{Q_T} \frac{u_{xt}^2}{v} dxdt + C + \varepsilon_2 Z,
\end{aligned} \tag{70}$$

Here, we use (59) and the fact that $\frac{1}{2b+3-\delta} < \frac{1}{4}$, if $b > \frac{1}{2}$. This completes the proof of (68).

Rewrite Equation (2b) as

$$\frac{\bar{\mu} u_{xx}}{v} = u_t + \left(\frac{R\theta}{v} \right)_x + \frac{\bar{\mu} u_x v_x}{v^2},$$

which implies that

$$\begin{aligned}
Z &\le C \sup_{0 \le t \le T} \left(\int_{\mathbb{R}} u_t^2 dx + \int_{\mathbb{R}} u_x^2 v_x^2 dx + \int_{\mathbb{R}} \theta_x^2 dx + \int_{\mathbb{R}} \theta^2 v_x^2 dx \right) \\
&\le C \left(1 + \varepsilon_2 Z + \sup_{Q_T} (u_x^2 + \theta^2) \int_{\mathbb{R}} v_x^2 dx + \int_{\mathbb{R}} \theta_x^2 dx \right) \\
&\le C \left(1 + \varepsilon_2 Z + Z^{\frac{3}{4}} Y^{\frac{1}{2b+3-\delta}} + Y \right) \le C + \varepsilon_3 Z.
\end{aligned} \tag{71}$$

Substituting (59) into (71), we obtain

$$Z \le C, \quad Y \le C. \tag{72}$$

This completes the proof. □

By Lemmas 5–10, we have the following high-order estimates.

Lemma 11. *Via the estimations above, for any $b \in (\frac{5}{2}, \infty)$ and $(x,t) \in Q_T$, it holds*

$$\sup_{Q_T} \theta \le C, \quad \sup_{Q_T} |u_x| \le C, \quad \int_{Q_T} (1 + \theta^b) \theta_t^2 dxdt \le C, \tag{73}$$

$$\sup_{0 \le t \le T} \int_{\mathbb{R}} (u_t^2 + v_t^2) dx + \int_{Q_T} u_{xx}^2 dxdt + \int_{Q_T} (u_{xt}^2 + v_{xt}^2) dxdt \le C. \tag{74}$$

It remains to obtain the $L^2(\mathbb{R})$-norm bound of θ_t, $L^2(Q_T)$-norm bound of θ_{xt} and θ_{xx}.

Lemma 12. *For any $b \in (\frac{5}{2}, \infty)$ and $(x,t) \in Q_T$, it holds that*

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\theta_t^2dx + \int_{Q_T}\theta_{xt}^2dxdt + \int_{Q_T}\theta_{xx}^2dxdt \le C. \tag{75}$$

Proof. Differentiating the temperature equation with respect to t, multiplying the result equation by θ_t, and integrating over Q_T using (73) and (74) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\theta_t^2dx + \int_{\mathbb{R}}(1+\theta^b)\theta_{xt}^2dx \\
&\le C\int_{\mathbb{R}}|\theta_{xt}|(|\theta_t||u_x| + |u_x|^2 + |u_{xt}| + |\theta_t||\theta_x| + |\theta_x||u_x|)dx \\
&\le \varepsilon\int_{\mathbb{R}}\theta_{xt}^2dx + C\int_{\mathbb{R}}(\theta_x^2+u_x^2)\theta_t^2dx + C\int_{\mathbb{R}}\theta_x^2u_x^2dx + C\int_{\mathbb{R}}(u_x^2+\theta_t^2+u_{xt}^2)dx \\
&\le \varepsilon\int_{\mathbb{R}}\theta_{xt}^2dx + C(\sup_{\mathbb{R}}u_x^2 + \sup_{\mathbb{R}}\theta_x^2)\int_{\mathbb{R}}\theta_t^2dx + C\int_{\mathbb{R}}(\theta_t^2+u_{xt}^2)dx + C.
\end{aligned} \tag{76}$$

Combining this with (66) and (72) , we have

$$\int_0^T\sup_{\mathbb{R}}\theta_x^2dt \le C.$$

Then, applying the Gronwall inequality on (76) yields

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\theta_t^2dx + \int_{Q_T}(1+\theta^b)\theta_{xt}^2dxdt \le C. \tag{77}$$

Next, we rewrite the temperature equation as follows:

$$\frac{\bar{\kappa}(1+\theta^b)\theta_{xx}}{v} = \theta_t + R\frac{\theta u_x}{v} - \bar{\mu}\frac{u_x^2}{v} - \frac{b\bar{\kappa}\theta^{b-1}\theta_x^2}{v} + \frac{\bar{\kappa}\theta^b\theta_x v_x}{v^2},$$

which gives

$$\begin{aligned}
&\int_{\mathbb{R}}(1+\theta^{2b})\theta_{xx}^2dx \le C\int_{\mathbb{R}}(\theta_t^2+u_x^2+u_x^4+\theta_x^4+\theta_x^2v_x^2)dx \\
&\le C + C\sup_{\mathbb{R}}\theta_x^2 \le C + C\int_{\mathbb{R}}|\theta_x\theta_{xx}|dx \\
&\le C + C\left(\int_{\mathbb{R}}\theta_{xx}^2dx\right)^{\frac{1}{2}},
\end{aligned} \tag{78}$$

and implies

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\theta_{xx}^2dx \le C.$$

We complete the proof of Lemma 12. □

It is clear that we carried out all estimates in (12) of Theorem 2. Then, Theorem 2 follows via the standard procedures. We omitted the details.

4. Proof of Asymptotic Behavior

With Lemmas 7–11 in hand, we study the asymptotic behavior as $t \to \infty$ of the solutions to System (2) with $\kappa = \bar{\kappa}(1+\theta^b)$ for $b \in (\frac{5}{2}, \infty)$ in this section. When $b = 0$, see Li [16], who deduced the uniform positive lower and upper bounds on temperature $\theta(x,t)$ with a smart test function.

The following large-time behavior of global solutions together with Lemmas 7–11 finish the proof of Theorem 2.

Lemma 13. *For all $t \in [0, \infty)$ and $b \in (\frac{5}{2}, \infty)$, it holds that*

$$C^{-1} \leq \theta(x,t) \leq C, \tag{79}$$

$$\lim_{t\to\infty} \left(\|(v-1, u, \theta - 1)(t)\|_{L^p(\mathbb{R})} + \|(v_x, u_x, \theta_x)(t)\|_{L^2(\mathbb{R})} \right) = 0, \tag{80}$$

for any $p \in (2, \infty]$.

Proof. Integrating (2b) multiplied by u_{xx} over $\mathbb{R}$ leads to

$$\begin{aligned}
&\int_0^\infty \left| \frac{d}{dt} \int_{\mathbb{R}} u_x^2 dx \right| dt + \int_0^\infty \int_{\mathbb{R}} \frac{u_{xx}^2}{v} dxdt \\
&\leq C \int_0^\infty (\sup_{\mathbb{R}} u_x^2) \int_{\mathbb{R}} v_x^2 dxdt + \sup_{Q_T} \theta \int_0^\infty \theta v_x^2 dxdt + C \int_0^\infty \int_{\mathbb{R}} \theta_x^2 dxdt \\
&\leq C \int_0^\infty \sup_{\mathbb{R}} u_x^2 dt + C \sup_{\mathbb{R}\times[0,\infty)} \theta \int_0^\infty \int_{\mathbb{R}} \frac{\theta_x^2}{\theta} dxdt + C \\
&\leq C.
\end{aligned} \tag{81}$$

Next, multiplying the temperature equation by θ_{xx} and integrating over $\mathbb{R}$ leads to

$$\begin{aligned}
&\frac{c_v}{2} \frac{d}{dt} \int_{\mathbb{R}} \theta_x^2 dx + \bar{\kappa} \int_{\mathbb{R}} \frac{(1+\theta^b)\theta_{xx}^2}{v} dx \\
&= -b\bar{\kappa} \int_{\mathbb{R}} \frac{\theta^{b-1}\theta_x^2\theta_{xx}}{v} dx + \bar{\kappa} \int_{\mathbb{R}} \frac{\theta^b \theta_x v_x \theta_{xx}}{v^2} dx - \bar{\mu} \int_{\mathbb{R}} \frac{u_x^2 \theta_{xx}}{v} dx + R \int_{\mathbb{R}} \frac{\theta u_x \theta_{xx}}{v} dx.
\end{aligned} \tag{82}$$

Using the Cauchy inequality and Sobolev interpolation inequality gives

$$\begin{aligned}
&\int_0^\infty \left| \int_{\mathbb{R}} \frac{\theta^{b-1}\theta_x^2\theta_{xx}}{v} dx + \int_{\mathbb{R}} \frac{\theta^b \theta_x v_x \theta_{xx}}{v^2} dx - \int_{\mathbb{R}} \frac{u_x^2 \theta_{xx}}{v} dx + \int_{\mathbb{R}} \frac{\theta u_x \theta_{xx}}{v} dx \right| dt \\
&\leq C \int_0^\infty \|\theta_{xx}\|_{L^2} \|\theta_x\|_{L^2} \|\theta_x\|_{L^\infty} dt + C \int_0^\infty \|\theta_{xx}\|_{L^2} \|\theta_x\|_{L^\infty} \|v_x\|_{L^2} dt \\
&\quad + C \int_0^\infty \|\theta_{xx}\|_{L^2} \|u_x\|_{L^\infty} \|u_x\|_{L^2} dt + C \int_0^\infty \|\theta_{xx}\|_{L^2} \|u_x\|_{L^2} dt \\
&\leq C \int_0^\infty \|\theta_{xx}\|_{L^2} (\|\theta_{xx}\|_{L^2}^{\frac{1}{2}} \|\theta_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}}) dt + C \int_0^\infty \|\theta_{xx}\|_{L^2} \|u_x\|_{L^2} dt \\
&\leq \varepsilon \int_0^\infty \int_{\mathbb{R}} \theta_{xx}^2 dxdt + C.
\end{aligned} \tag{83}$$

It follows from (81) and (83) that

$$\begin{aligned}
&\int_0^\infty \left(\left| \frac{d}{dt} \|u_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \right| + \|u_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \right) dt \\
&+ \int_0^\infty \left(\left| \frac{d}{dt} \|\theta_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \right| + \|\theta_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \right) dt \leq C,
\end{aligned} \tag{84}$$

which gives

$$\lim_{t\to\infty} (\|u_x(\cdot,t)\|_{L^2(\mathbb{R})} + \|\theta_x(\cdot,t)\|_{L^2(\mathbb{R})}) = 0. \tag{85}$$

Thanks to the uniform lower and upper bounds of $v(x,t)$, and upper bounds of $\theta(x,t)$, we have

$$\begin{aligned}&\sup_{0\le t\le\infty}\int_{\mathbb{R}}(v-1)^2dx+\sup_{0\le t\le\infty}\int_{\mathbb{R}}(\theta-1)^2dx\\&\le C\sup_{0\le t\le\infty}\int_{\mathbb{R}}(v-\ln v-1)dx+C\sup_{0\le t\le\infty}\int_{\mathbb{R}}(\theta-\ln\theta-1)dx\le C.\end{aligned}\tag{86}$$

On the other hand, we have the following estimates for all $t\ge 0$:

$$\begin{aligned}\|(\theta-1)(\cdot,t)\|^2_{C(\mathbb{R})}&\le C\|(\theta-1)(\cdot,t)\|_{L^2(\mathbb{R})}\|\theta_x(\cdot,t)\|_{L^2(\mathbb{R})}\\&\le C\|\theta_x(\cdot,t)\|_{L^2(\mathbb{R})},\end{aligned}\tag{87}$$

and

$$\begin{aligned}\|(v-1)(\cdot,t)\|^2_{C(\mathbb{R})}&\le C\|(v-1)(\cdot,t)\|_{L^2(\mathbb{R})}\|v_x(\cdot,t)\|_{L^2(\mathbb{R})}\\&\le C\|v_x(\cdot,t)\|_{L^2(\mathbb{R})}.\end{aligned}\tag{88}$$

This, combined with (85), shows

$$\lim_{t\to\infty}\|\theta(\cdot,t)-1\|_{C(\mathbb{R})}=0.\tag{89}$$

Hence, there exists some $T_0>0$ such that for all $(x,t)\in\mathbb{R}\times[T_0,\infty)$

$$\frac{1}{2}\le\theta(x,t)\le\frac{3}{2}.\tag{90}$$

Lastly, it follows from the proof in Lemma 3 that there exists a constant, such that, for all $(x,t)\in\mathbb{R}\times[0,T_0]$

$$\theta(x,t)\ge C(T_0).$$

Hence, we have

$$C^{-1}\le\theta(x,t)\le C\quad\text{for all }\ t\in[0,\infty).\tag{91}$$

Then, combining with Lemma 7, one has

$$\int_0^\infty\left(\left|\frac{d}{dt}\|v_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\right|+\|v_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\right)dt\le C,\tag{92}$$

it yields that

$$\lim_{t\to\infty}\|v_x(\cdot,t)\|_{L^2(\mathbb{R})}=0.\tag{93}$$

The pointwise bounds of $v(x,t)$ and $\theta(x,t)$ from below and above independent of time were proven in Lemmas 4 and 13. The asymptotic behavior as $t\to\infty$ of the solutions was proven in Lemma 13. This completes the proof of Theorem 2. □

5. Conclusions

In this paper, we considered the Cauchy problems to a one-dimensional compressible Navier–Stokes system with temperature-dependent heat conductivity, and general large initial data and far-field conditions. We proved that velocity and temperature are uniformly bounded from below and above in time and space. Further, we proved that the global solution was asymptotically stable as time tended to infinity for $b>\frac{5}{2}$. Our approaches relied upon the maximal principle, and the iteration and energy estimate method. The conclusions in this manuscript are primitive. However, there are limitations to this conclusion because the corresponding results were not obtained when $0<b\le\frac{5}{2}$. We can further study the global well-posedness to System (2) with a viscosity coefficient and heat

conduction coefficient that are both dependent on density and temperature. However, there is a great challenge: since we could not obtain an expression for velocity as in (27), the uniform estimates to velocity and temperature were difficult to obtain with the Gronwall inequality. Therefore, there were only some small initial value conclusions in this situation.

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