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Soliton Waves with the (3+1)-Dimensional Kadomtsev–Petviashvili–Boussinesq Equation in Water Wave Dynamics

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Abstract: We examined the (3+1)-dimensional Kadomtsev–Petviashvili–Boussinesq (KP-B) equation, which arises not only in fluid dynamics, superfluids, physics, and plasma physics but also in the construction of connections between the hydrodynamic and optical model fields. Moreover, unlike the Kadomtsev–Petviashvili equation (KPE), the KP-B equation allows the modeling of waves traveling in both directions and does not require the zero-mass assumption, which is necessary for many scientific applications. Considering these properties enables researchers to obtain more precise results in many physics and engineering applications, especially in research on the dynamics of water waves. We used the modified extended tanh function method (METFM) and Kudryashov's method, which are easily applicable, do not require further mathematical manipulations, and give effective results to investigate the physical properties of the KP-B equation and its soliton solutions. As the output of the work, we obtained some new singular soliton solutions to the governed equation and simulated them with 3D and 2D graphs for the reader to understand clearly. These results and graphs describe the single and singular soliton properties of the (3+1)-dimensional KP-B equation that have not been studied and presented in the literature before, and the methods can also help in obtaining the solution to the evolution equations and understanding wave propagation in water wave dynamics.

Keywords: Kadomtsev–Petviashvili–Boussinesq equation; Kudryashov method; modified extended tanh function; soliton solution

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1. Introduction

In recent years, many researchers have been interested in the physical properties of nonlinear waves. Among them, analytical solutions of nonlinear evolution equations (NLEEs), a class of nonlinear waves, are of great importance for understanding and explaining real-life problems. As important as they are, analytical solutions to these problems are often an equally troublesome process. Due to the developments in computers and software, there have also been many developments in symbolic computing such as Wolfram Mathematica, Maple, Matlab, and other symbolic programs. In this way, it can be said that the calculations and accuracy are at a very good level. Moreover, the analytical solutions of NLEEs play very important role in exposing the mechanisms in various fields, namely nonlinear optics, optical fibers, communication, data transfer, heat conduction, physics, plasma physics, chemical kinematics, quantum mechanics, mathematical biology, genetics, geochemistry, dynamic, fluid mechanics, oceanography, neural networks, wave propagation, control theory, and so on, see [1–8]. In our literature review, in recent years, many researchers and academics have proposed and developed a number of strategies, techniques, and methods to understand and analytically solve the NL phenomenon. These methods can be listed as follows: the modified $\tan(\varphi/2)$-expansion [9], variational principles [10], the extended rational sin–cos and sinh–cosh methods, He's variational approach and the Painleve technique [11], the residual power series [12], Lie symmetry analysis [13], the modified subequation extended method [14], improved F-expansion [15], the new compound Riccati equations rational expansion method and Fan's subequation method [16], the Kudryashov method [17–19], the generalized Kudryashov method [20], the extended Kudryashov technique [21], the Jacobi elliptic function [22–24], the Bernoulli subequation function [25,26], the first integral method [27], the projective Riccati F-expansion [28], the Sinh Gordon Expansion (ShGEM) [29], the Sinh Gordon extended Expansion (ShGEEM) [30], the modified extended tanh function (METFM) [31], the F-expansion [32], the Improved Hirota Bilinear method [33], the Backlund transformation [34], Stable Optical Solitons for the Higher-Order Non-Kerr NLSE via the Modified Simple Equation Method [35], the modified simple equation method [36], the ansatz technique [37], the rational exponential function [38], the (G/G)-expansion method [39], the modified trial equation method [40], Riccati equation mapping [41], the Lie group approach [42], and some recent approaches, which are the modified (g'), the modified (g'/g^2), and the generalized simple (w/g)-expansion methods [43], an addendum to Kudryashov's method [44], the Cole–Hopf transformation, which offers many different solutions as well as being an effective method [45], and newly introduced to the literature, the direct mapping method [46].

The (3 + 1)-dimensional KP-B equation that we examined in our study is given as [4,47–49]:

$$B_{xxxy} + 3\left(B_x B_y\right)_x + \left(B_x + B_y\right)_t + B_{tt} - B_{zz} = 0. \tag{1}$$

In (1), $B(x, y, z, t)$ is a real valued potential function, which describes the height of the wave. Here, x, y, and z are independent spatial variables, and t is the temporal variable. Equation (1) is an extended form of the integrable KPE. If the term B_{tt} is omitted, (1) degenerates to following (3+1)-dimensional generalized KPE [50],

$$B_{xxxy} + 3\left(B_x B_y\right)_x + \left(B_x + B_y\right)_t - B_{zz} = 0, \tag{2}$$

and another type of the (3+1)-dimensional generalized KPE [51],

$$B_{xxxy} + 3\left(B_x B_y\right)_x + \left(B_x + B_y + B_z\right)_t - B_{zz} = 0. \tag{3}$$

In addition, both the KPEs in (2) and (3) are not only related to fluid dynamics, superfluids, physics, and plasma physics, but there are also connections between hydrodynamic and optical models [52]. KPE is widely used in many fields, such as Bose–Einstein condensation [53], nonlinear optics, water wave dynamics, ferromagnetics, physics, plasma physics, and surface oceanic waves, for example, investigations of nonlinear ion acoustic waves in magnetized dusty plasma [54], acoustic waves studies in an elastic circular rod [55], nonlinear wave propagation in a fluid and elastic tube covering turbulence, small bubbles [56,57], the traveling of tsunami waves in the inhomogeneous zone on the bottom of the ocean [58,59], and so on. Moreover, by using the KPE it is possible to investigate quasi one dimensional shallow water waves, if the effects of the viscosity and surface tension are very small or negligible [60]. Therefore, although it is thought that Equation (1) can be obtained as a result of adding the term B_{tt} to Equation (2), in a sense, it is also possible to evaluate it as a combination of Equation (2) and the following generalized Boussinesq ((4)) or generalized KP-B equation-like equation in (5) [61–63],

$$B_{xxxy} + 3\left(B_x B_y\right)_x + B_{tt} - B_{zz} = 0, \tag{4}$$

$$B_{xxxy} + 3\left(B_x B_y\right)_x + \left(B_x + B_y\right)_t + B_{tt} - B_{zz} = 0. \tag{5}$$

To be clear, if we consider (4), the presented model in (1) not only allows for the modeling of waves traveling in both directions (both left and right going waves), but (1) also does not require the zero mass assumption that is required during many applications of the KPE, and it allows more precise results to be obtained in many physics and engineering applications.

With the aim of establishing many physical models, researchers have studied the KP-B equation and have obtained many results about the soliton behavior of the KP-B equation, for example, by using Lie symmetry reductions and direct integration by invoking the (G'/G) expansion method [42], one and two soliton solutions and multi-soliton solutions [4,47–63], by giving only one type of exponential solution. Multiple-order rogue waves for the generalized (2+1)-dimensional KPE were studied in [64,65], the nonautonomous and mixed lump–stripe soliton solutions of a variable-coefficient KPE were studied in [66], the solitary, shock, and and singular wave solutions of the (3+1)-dimensional KPE and the generalized Boussinesq equations were investigated using the solitary wave method in [67], and the (3+1)-dimensional generalized nonlinear evolution equation of shallow water waves was investigated in [68]; in addition (3+1)-dimensional KPE and its integrability, multiple-solitons, breathers, and lump waves were presented in [69], multi-order rogue wave solutions of the generalized (3+1)-dimensional KPE through its bilinear form and symbolic computation were given in, and localized nonlinear waves on spatiotemporally controllable backgrounds of a KP-B model in water waves were studied in [70].

Our motivation is that there has not been any previous study for both the (3+1)-KP-B equation and the different soliton types of the singular solutions of this equation. In addition, singular soliton solutions play an important role in the multi-soliton solutions of the KP-B equation.

Therefore, our main purpose in this article is to search for and obtain single-wave and singular solutions of the KP-B equation that will help researchers working in the field.

2. Mathematical Analysis and Obtaining a Nonlinear Ordinary Differential Form

Let we remember (1) and take into account the following transformation:

$$B = B(x, y, z, t) = B(\zeta), \zeta = x + \alpha y + \beta z - \omega\, t, \tag{6}$$

in which α, β, and ω are real values. Considering Equations (6) and (1) together, the result is the following equation:

$$\left(\omega^2 - \omega(\alpha+1) - \beta^2\right)\frac{\mathrm{d}B(\zeta)}{\mathrm{d}\zeta} + 3\alpha\left(\frac{\mathrm{d}B(\zeta)}{\mathrm{d}\zeta}\right)^2 + \alpha\frac{\mathrm{d}^3B(\zeta)}{\mathrm{d}\zeta^3} = 0. \tag{7}$$

Accepting $\frac{\mathrm{d}B(\zeta)}{\mathrm{d}\zeta} = \Gamma(\zeta)$, we derive the nonlinear ordinary differential form of (1).

$$\left(\omega^2 - \omega(\alpha+1) - \beta^2\right)\Gamma(\zeta) + 3\alpha(\Gamma(\zeta))^2 + \alpha\frac{\mathrm{d}^2\Gamma(\zeta)}{\mathrm{d}\zeta^2} = 0. \tag{8}$$

In (8), using the balance rule between the terms $\Gamma''(\zeta)$ and $\Gamma^2(\zeta)$, we reach $m = 2$, and that m is called the balancing constant.

3. Quick View of the Methods and Implementation of the KP-B Equation

3.1. Modified Extended tanh Function Method

In order to achieve the solution of (8), we propose the solution in the following truncated series:

$$\Gamma(\zeta) = A_0 + \sum_{i=1}^{m} A_i\kappa^i(\zeta) + \sum_{i=1}^{m}\frac{B_i}{\kappa^i(\zeta)}, \tag{9}$$

where $A_0, \ldots, A_m, B_1, \ldots, B_m$ are real constants (A_m and B_m should not be zero, simultaneously), m is a positive integer balancing constant, which was calculated as $m = 2$; so, (9) takes the following form:

$$\Gamma(\zeta) = A_0 + A_1\kappa(\zeta) + A_2\kappa^2(\zeta) + B_1\kappa^{-1}(\zeta) + B_2\kappa^{-2}(\zeta), \tag{10}$$

where $\kappa(\zeta)$ fulfills the following equation:

$$\frac{d\kappa(\zeta)}{d\zeta} - w + [\kappa(\zeta)]^2 = 0, \tag{11}$$

in which w is a real constant, and Equation (11) produces the following solutions:

$$\kappa(\zeta) = \begin{cases} -\sqrt{-w}\tan(\sqrt{-w}(\zeta+\zeta_0)); w<0, \\ \sqrt{-w}\cot(\sqrt{-w}(\zeta+\zeta_0)); w<0, \\ 1/(\zeta+\zeta_0); w=0, \\ \sqrt{w}\tanh(\sqrt{w}(\zeta+\zeta_0)); w>0, \\ \sqrt{w}\coth(\sqrt{w}(\zeta+\zeta_0)); w>0, \end{cases} \tag{12}$$

where w and ζ_0 are real free parameters. Substitution of (10) and (11) into (8) results in the polynomial in powers of $\kappa(\zeta)$. Applying algebraic polynomial operations to the coefficients of $\kappa(\zeta)$ will produce the following algebraic equation system:

$$\begin{aligned}
&\kappa^{-4}(\zeta): 3\alpha B_2(2w^2+B_2)=0,\\
&\kappa^{-3}(\zeta): 2\alpha B_1(w^2+3B_2)=0,\\
&\kappa^{-2}(\zeta): ((-8w-\omega+6A_0)\alpha-\beta^2+\omega^2-\omega)B_2+3\alpha {B_1}^2=0,\\
&\kappa^{-1}(\zeta): ((-2w-\omega+6A_0)\alpha-\beta^2+\omega^2-\omega)B_1+6\alpha A_1B_2=0,\\
&\kappa^{0}(\zeta): (3{A_0}^2-A_0\omega+(2w^2+6B_2)A_2+6A_1B_1+2B_2)\alpha-A_0(\beta^2-\omega^2+\omega)=0,\\
&\kappa^{1}(\zeta): ((-2w-\omega+6A_0)\alpha-\beta^2+\omega^2-\omega)A_1+6\alpha B_1A_2=0,\\
&\kappa^{2}(\zeta): ((-8w-\omega+6A_0)\alpha-\beta^2+\omega^2-\omega)A_2+3\alpha {A_1}^2=0,\\
&\kappa^{3}(\zeta): 2\alpha A_1(3A_2+1)=0,\\
&\kappa^{4}(\zeta): 3\alpha A_2(A_2+2)=0.
\end{aligned} \tag{13}$$

In (13) $A_0, A_1, A_2, B_1, B_2, \omega, \alpha, \beta$, and w are unknowns to be calculated. The solution of this system yields the possible solution sets as follows:

$$CSet_1 = \Big\{ w = \frac{\omega^2+(-\alpha-1)\omega-\beta^2}{4\alpha}, A_0 = \frac{\omega^2+(-\alpha-1)\omega-\beta^2}{6\alpha}, A_1=0, A_2=-2, B_1=0, B_2=0 \Big\},$$

$$CSet_2 = \Big\{ w = -\frac{\sqrt{2}}{2}\sqrt{-B_2}, \omega = \frac{\alpha}{2}+\frac{1}{2}+\frac{1}{2}\sqrt{8\alpha\sqrt{2}\sqrt{-B_2}+\alpha^2+4\beta^2+2\alpha+1}, A_0 = -\sqrt{2}\sqrt{-B_2}, A_1=0, A_2=0, B_1=0, B_2=B_2 \Big\},$$

$$CSet_3 = \Big\{ \omega = \frac{\alpha+1}{2}+\frac{1}{2}\sqrt{\alpha^2+(16w+2)\alpha+4\beta^2+1}, A_0=\frac{2w}{3}, A_1=0, A_2=-2, B_1=0, B_2=0 \Big\},$$

$$CSet_4 = \Big\{ w = \frac{\omega^2+(-\alpha-1)\omega-\beta^2}{4\alpha}, A_0 = \frac{\omega^2+(-\alpha-1)\omega-\beta^2}{6\alpha}, A_1=0, A_2=0, B_1=0, B_2=-\frac{(\alpha\omega+\beta^2-\omega^2+\omega)^2}{8\alpha^2} \Big\},$$

$$CSet_5 = \Big\{ w = \frac{\sqrt{2}}{2}\sqrt{-B_2}, \omega = \frac{\alpha+1}{2} - \frac{1}{2}\sqrt{32\alpha\sqrt{2}\sqrt{-B_2} + \alpha^2 + 4\beta^2 + 2\alpha + 1},$$
$$A_0 = -\frac{2\sqrt{2}}{3}\sqrt{-B_2}, A_1 = 0, A_2 = -2, B_1 = 0, B_2 = B_2 \Big\},$$

$$CSet_6 = \Big\{ \omega = \frac{\alpha+1}{2} + \frac{1}{2}\sqrt{\alpha^2 + (64w+2)\alpha + 4\beta^2 + 1}, A_0 = -\frac{4w}{3}, A_1 = 0, A_2 = -2,$$
$$B_1 = 0, B_2 = -2w^2 \Big\}.$$

Selecting any solution set and solution function given in (12) for $\zeta_0 = 0$, then substituting them together into (10) by considering $B(\zeta) = \int \Gamma(\zeta) d\zeta$ and (6), the solutions of (1) are achieved as in (14)–(18).

Case-1: for $\kappa(\zeta) = -W_2 \tan(W_2\zeta)$:

$$\begin{aligned} B_1(x,y,z,t) =& A_0\zeta - \frac{A_1}{2}\ln\Big(1 + \Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big)^2\Big) - A_2 w \tan\Big(\sqrt{-w}\zeta\Big)\frac{1}{\sqrt{-w}} \\ &+ A_2 w \arctan\Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big)\frac{1}{\sqrt{-w}} + \frac{B_1}{w}\ln\Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big) \\ &- \frac{B_1}{2w}\ln\Big(1 + \Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big)^2\Big) + \frac{B_2}{w}\frac{1}{\sqrt{-w}}\Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big)^{-1} \\ &+ \frac{B_2}{w\sqrt{-w}}\arctan\Big(\tan\Big(\sqrt{-w}\zeta\Big)\Big), \end{aligned} \tag{14}$$

or $\kappa(\zeta) = W_2 \cot(W_2\zeta)$:

$$\begin{aligned} B_2(x,y,z,t) =& A_0\zeta - \frac{A_1}{2}\ln\Big(\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big)^2 + 1\Big) + \frac{B_1}{w}\ln\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big) \\ &- \frac{B_1}{2w}\ln\Big(\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big)^2 + 1\Big) + A_2 w \cot\Big(\sqrt{-w}\zeta\Big)\frac{1}{\sqrt{-w}} \\ &- \frac{A_2 w\pi}{2\sqrt{-w}} + \frac{A_2 w}{\sqrt{-w}}\operatorname{arccot}\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big) \\ &- \frac{B_2\pi}{2w\sqrt{-w}} + \frac{B_2}{w\sqrt{-w}}\operatorname{arccot}\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big) \\ &- \frac{B_2}{w\sqrt{-w}}\Big(\cot\Big(\sqrt{-w}\zeta\Big)\Big)^{-1}. \end{aligned} \tag{15}$$

Case-2: $\kappa(\zeta) = W_1 \tanh(W_1\zeta)$:

$$\begin{aligned} B_3(x,y,z,t) =& A_0\zeta - \frac{A_1}{2}\ln(\tanh\psi - 1) - \frac{A_1}{2}\ln(\tanh\psi + 1) \\ &- A_2\sqrt{w}\tanh\psi - \frac{A_2}{2}\sqrt{w}\ln(\tanh\psi - 1) + \frac{A_2}{2}\sqrt{w}\ln(\tanh\psi + 1) \\ &+ \frac{B_1}{w}\ln(\tanh\psi) - \frac{B_1}{2w}\ln(\tanh\psi - 1) - \frac{B_1}{2w}\ln(\tanh\psi + 1) \\ &- \frac{B_2}{2}\ln(\tanh\psi - 1)w^{-\frac{3}{2}} - B_2 w^{-\frac{3}{2}}(\tanh\psi)^{-1} + \frac{B_2}{2}\ln(\tanh\psi + 1)w^{-\frac{3}{2}}, \end{aligned} \tag{16}$$

or $\kappa(\zeta) = W_1 \coth(W_1\zeta)$:

$$\begin{aligned} B_4(x,y,z,t) =& A_0\zeta - \frac{A_1}{2}\ln(\coth\psi - 1) \\ & - \frac{A_1}{2}\ln(\coth\psi + 1) - A_2\sqrt{w}\coth\psi - \frac{A_2}{2}\sqrt{w}\ln(\coth\psi - 1) \\ & + \frac{A_2}{2}\sqrt{w}\ln(\coth\psi + 1) - \frac{B_1}{2w}\ln(\coth\psi - 1) + \frac{B_1}{w}\ln(\coth\psi) \\ & - \frac{B_1}{2w}\ln(\coth\psi + 1) - \frac{B_2}{2}\ln(\coth\psi - 1)w^{-\frac{3}{2}} \\ & + \frac{B_2}{2}\ln(\coth\psi + 1)w^{-\frac{3}{2}} - B_2 w^{-\frac{3}{2}}(\coth\psi)^{-1}, \end{aligned} \tag{17}$$

where $\psi = W_1\zeta$, $W_1 = \sqrt{\omega}$, and $W_2 = \sqrt{-\omega}$.

Case-3: $\kappa(\zeta) = 1/\zeta$:

$$B_5(x,y,z,t) = A_0\zeta + A_1\ln\zeta - \frac{A_2}{\zeta} + \frac{B_1(\zeta)^2}{2} + \frac{B_2(\zeta)^3}{3}. \tag{18}$$

In (14)–(18), $\zeta = x + \alpha y + \beta z - \omega t$.

3.2. Kudryashov Method

In order to derive the solution of (8), we suggest the solution in the following form:

$$\Gamma(\zeta) = \sum_{i=0}^{m} a_i\kappa^i(\zeta), \tag{19}$$

where $a_m \neq 0$, and in (19), $a_0, ..., a_m$ are real constants, and m is also a balancing constant. From the previous section, remembering that $m = 2$, (19) turns into the following form:

$$\Gamma(\zeta) = a_0 + a_1\kappa(\zeta) + a_2\kappa^2(\zeta), \tag{20}$$

where $\kappa(\zeta)$ satisfies the following equation [71]:

$$\frac{d\kappa(\zeta)}{d\zeta} - \delta\kappa(\zeta)(\kappa(\zeta) - 1) = 0, \tag{21}$$

where δ is a nonzero real free parameter, and (21) gives the following solution:

$$\kappa(\zeta) = \frac{1}{1 \mp \eta e^{-\delta\zeta}}, \tag{22}$$

in which η is a nonzero arbitrary real constant to be found later. Unlike its general usage, we take the expression in (22) as follows.

By considering

$$sinh(x) = \frac{e^x - e^{-x}}{2},\ cosh(x) = \frac{e^x + e^{-x}}{2},\ e^x = sinh(x) + cosh(x),\ \eta = 1, \tag{23}$$

one can easily obtain the following two forms:

$$\kappa(\zeta) = \frac{1}{2}\left[1 - \tanh\left(\frac{\delta}{2}\zeta\right)\right], \tag{24}$$

$$\kappa(\zeta) = \frac{1}{2}\left[1 - \coth\left(\frac{\delta}{2}\zeta\right)\right]. \tag{25}$$

Substitution of (20) and (21) into (8) provides us a polynomial in powers of $\kappa(\zeta)$. Applying polynomial operations based on $\kappa(\zeta)$, we obtain the following algebraic equation system:

$$\begin{aligned}
&\kappa^0(\zeta): -a_0\alpha\omega + 3\alpha {a_0}^2 - a_0\beta^2 + a_0\omega^2 - a_0\omega = 0,\\
&\kappa^1(\zeta): \alpha\delta^2 a_1 - \alpha\omega a_1 + 6\alpha a_0 a_1 - \beta^2 a_1 + \omega^2 a_1 - \omega a_1 = 0,\\
&\kappa^2(\zeta): -3\alpha\delta^2 a_1 + 4\alpha\delta^2 a_2 - \alpha\omega a_2 + 6\alpha a_0 a_2\\
&\qquad + 3\alpha {a_1}^2 - \beta^2 a_2 + \omega^2 a_2 - \omega a_2 = 0,\\
&\kappa^3(\zeta): 2\alpha\delta^2 a_1 - 10\alpha\delta^2 a_2 + 6\alpha a_1 a_2 = 0,\\
&\kappa^4(\zeta): 6\alpha\delta^2 a_2 + 3\alpha {a_2}^2 = 0.
\end{aligned} \tag{26}$$

In (26) $a_0, a_1, a_2, \alpha, \beta, \omega$, and δ are real values to be found. The solution of this system gives us the following solution sets:

$$CSet_7 = \left\{\delta = \frac{1}{2}\sqrt{-2a_2}, \alpha = -\frac{2\beta^2 - 2\omega^2 + 2\omega}{-a_2 + 2\omega}, a_0 = \frac{a_2}{6}, a_1 = -a_2, a_2 = a_2\right\},$$

$$CSet_8 = \left\{\beta = -\sqrt{\alpha\,\delta^2 - \alpha\,\omega + \omega^2 - \omega}, a_0 = 0, a_1 = 2\delta^2, a_2 = -2\delta^2\right\}.$$

Selecting any solution set and then substituting it into (20) by considering $B(\zeta) = \int \Gamma(\zeta)d\zeta$ and (6), the solution of NLPDE (1) is found, as given in (27) and (28). We consider Equation (24):

$$\begin{aligned}
B_{6,1}(x,y,z,t) =& a_0\vartheta + \frac{a_1\vartheta}{2} + \frac{a_1}{2\delta}\ln\left(\tanh\left(\frac{\delta\vartheta}{2}\right) - 1\right) + \frac{a_1}{2\delta}\ln\left(\tanh\left(\frac{\vartheta}{2}\right) + 1\right)\\
&- \frac{a_2}{2\delta}\tanh\left(\frac{\delta\vartheta}{2}\right) + \frac{a_2}{\delta}\ln\left(\tanh\left(\frac{\delta\vartheta}{2}\right) + 1\right),
\end{aligned} \tag{27}$$

and we consider Equation (25):

$$\begin{aligned}
B_{6,2}(x,y,z,t) =& a_0\vartheta + \frac{a_1\vartheta}{2} + \frac{a_1}{2\delta}\ln\left(\coth\left(\frac{\delta\vartheta}{2}\right) - 1\right) + \frac{a_1}{2\delta}\ln\left(\coth\left(\frac{\vartheta}{2}\right) + 1\right)\\
&- \frac{a_2}{2\delta}\coth\left(\frac{\delta\vartheta}{2}\right) + \frac{a_2}{\delta}\ln\left(\coth\left(\frac{\delta\vartheta}{2}\right) + 1\right),
\end{aligned} \tag{28}$$

where $\vartheta = (\alpha y + \beta z - \omega t + x)$.

4. Results and Discussion

In this part, some of the graphical illustrations are presented for the soliton solution functions given in (14), (15), (17), (27), and (28). The simulations of these solutions are given in order, from Figures 1–8, according to the appropriate selection of parameters. Although we obtained many solution sets from the solution of (13) and (26) according to different parameter values, in order to not occupy unnecessary pages, only some selected solution sets of $Cset_1$,...,$Cset_8$ are used for graphical representations.

Figures 1–8 generally show the graphical representations of the singular solutions of the KP-B equation. Although the graphs generally express the singular wave representation, there are still some differences between these representations. Figure 1 represents the periodic singular solution graph. The values taken by the function are given in different signs on the left and right where the singularity occurs. In Figure 2, there is a graph of the periodic singular soliton solution. However, if we pay attention to this graph, the wave takes a step-like appearance at the point where the singularity occurs, just before diverging to negative or positive infinity. In Figure 3, there is a graph of a periodic singular solution. In this graph, the wave has a bending point on an axis that can be considered a vertical symmetry point. Then, in a sense, the concavity diverges to different infinities after its direction changes. Figure 4 is a graph that gives a different image among these graphs. It represents a singular behavior, which can be called a smooth kink, albeit partially. The graph of Figure 5 is yet another singular graph, and it gives an image of the soliton here with inclined wings to the left and right. The singular soliton graphs in Figures 6 and 7 are representations of step-shaped physical formations. Although the soliton given with the Figure 8 graph is similar to the graphical expression given in Figure 3, there is a vertical distance between its left and right skirts. In other words, although the singular soliton graphs represented by Figures 3 and 8 are categorically the same, they do not represent exactly the same physical behavior. In this sense, in any soliton representation of the problem modeling a physical event, even if the same type of soliton is formed, the physical representations of such solitons can express different physical phenomena. All solution functions obtained in the study have also been confirmed by careful examination to satisfy (1), which is the nonlinear form of the main problem.

When the METFM and Kudryashov methods we used in the article are taken into account, it is easily seen that these methods are easy to apply, effective, and give direct results. However, it is important to emphasize the following: what is important here is the researcher's purpose and the method one chooses for this purpose. When evaluated from this point of view, these two methods, which are widely used and effective, were chosen in our study, especially as we wanted to work on single wave and singular solutions. Therefore, it is also possible to obtain different types of solitons by using methods other than those mentioned here. For example, in the article [43], there were various and different soliton solutions obtained by using modified (g'), the modified (g'/g^2), and the generalized simple (w/g)-expansion methods. Similarly, it is obvious that Wang's direct algebraic method [46], which has just been introduced to the literature, is an effective method based on an easily applicable auxiliary equation and can give different singular solutions. In addition, it seems possible that this method could be applied to the equation in our study, similar to the different types of solitons obtained from the problems using the Cole–Hopf transformation [45]. It is possible to add other techniques, such as the Hirota bilinear form [33,47], to examine multiwave solutions or soliton types such as breather, lump, and rouge. In addition, it would be appropriate to emphasize two points. All of the solution functions obtained in the study were determined to satisfy the main equation Equation (1). In our study, both the solution functions we obtained and the graphic representations we presented do not have any contradictions with the studies in this field and the generally accepted concepts in this field.

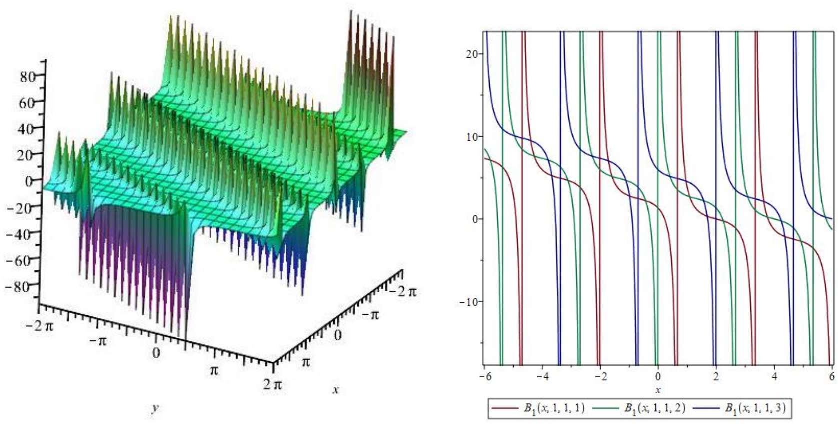

Figure 1. 3D and 2D projections of $B_1(x, y, z, t)$ in (14), selecting the $CSet_1$ and $\alpha = -0.5, \beta = 0.5, \omega = 2$, and $z = t = 1$.

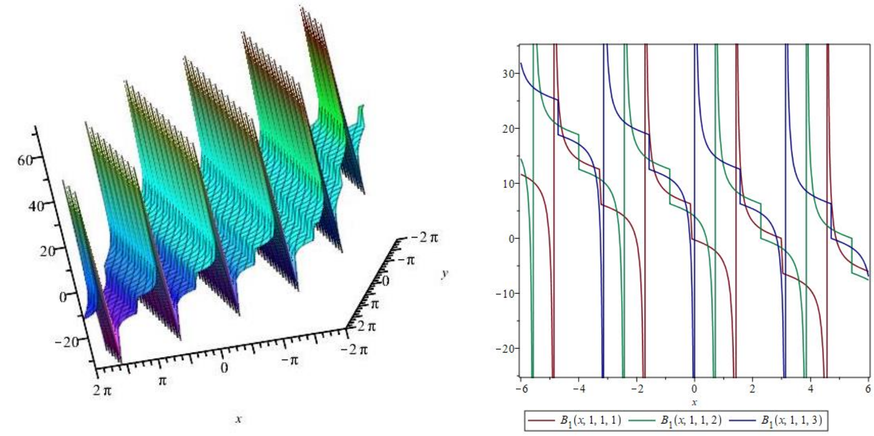

Figure 2. 3D and 2D depictions of $B_1(x, y, z, t)$ in (14), selecting the $CSet_2$ and $B_2 = -2, \alpha = 0.5, \beta = 0.5$, and $z = t = 1$.

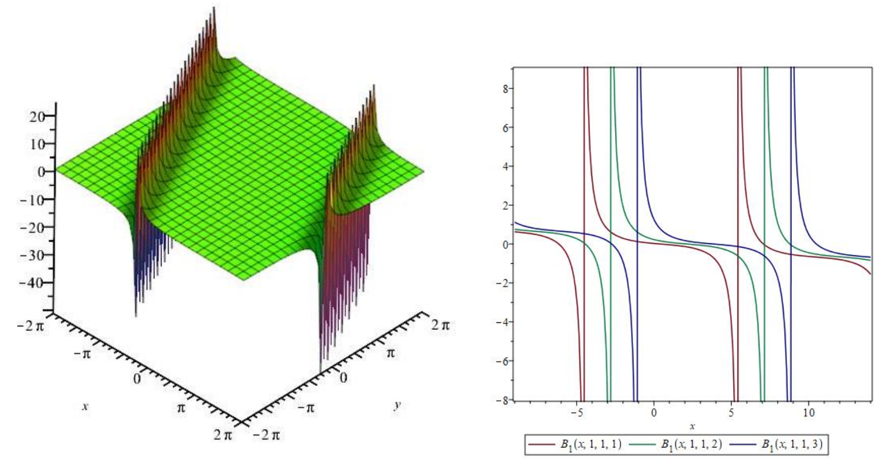

Figure 3. 3D and 2D silhouettes of $B_1(x, y, z, t)$ in (14), selecting the $CSet_3$ and $w = -0.1, \alpha = 0.5, \beta = 0.75$, and $z = t = 1$.

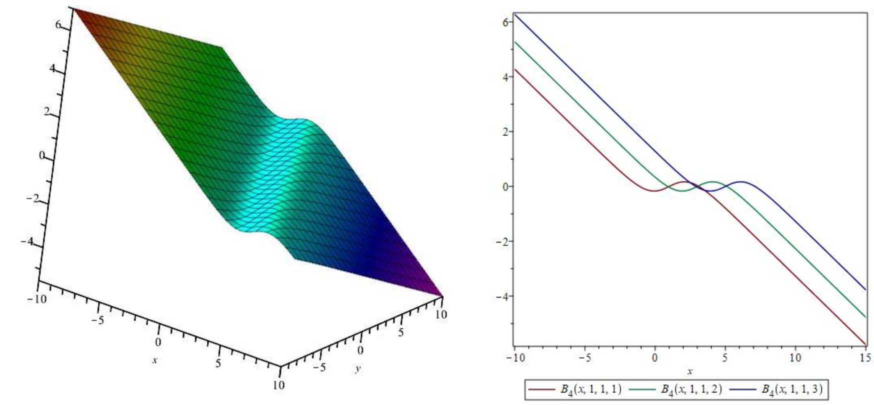

Figure 4. 3D and 2D views of $B_4(x, y, z, t)$ in (17), selecting the $CSet_4$ and $\alpha = 0.5, \beta = 0.5, \omega = 2$, and $z = t = 1$.

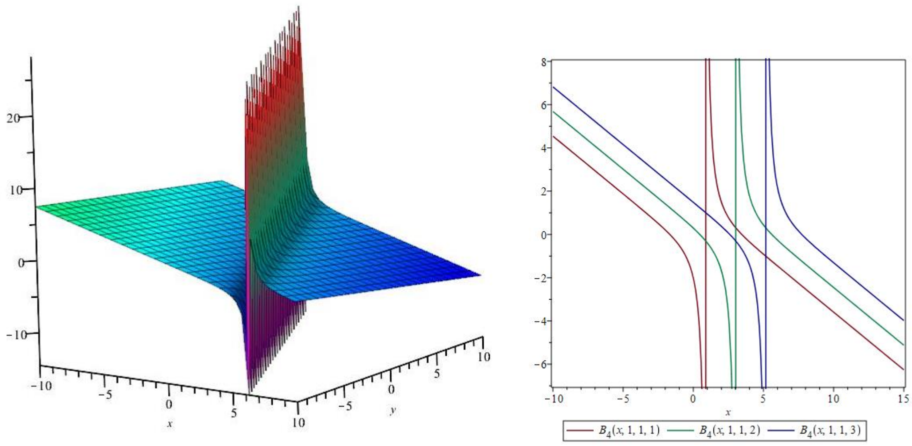

Figure 5. 3D and 2D presentations of $B_4(x, y, z, t)$ in (17), selecting the $CSet_5$ for the selected parameters $w = 0.1, \alpha = 0.5, \beta = 0.75$, and $z = t = 1$.

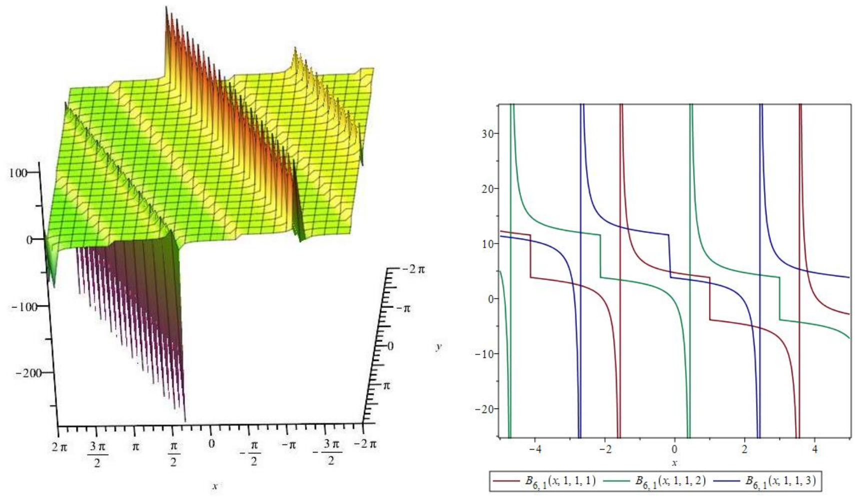

Figure 6. 3D and 2D plots of $B_{6,1}(x, y, z, t)$ in (27), selecting the $CSet_6$ and $\alpha = 0.5, \beta = 0.75, \omega = 2$, and $z = t = 1$.

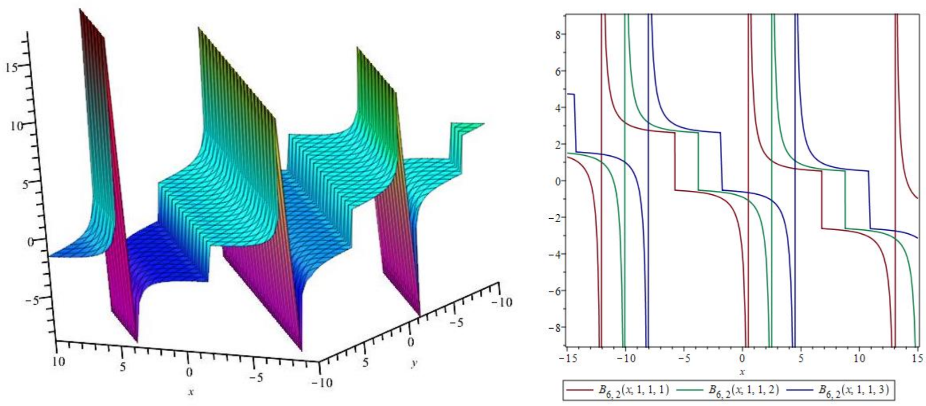

Figure 7. 3D and 2D portraits of $B_{6,2}(x, y, z, t)$ in (28), selecting the $CSet_7$ and $a_2 = 0.5, \beta = 0.5, \omega = 2$, and $z = t = 1$.

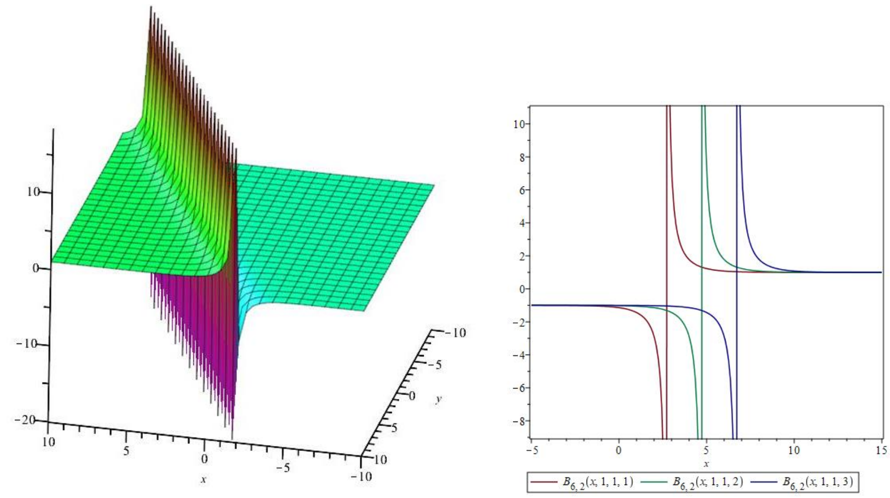

Figure 8. 3D and 2D portraits of $B_{6,2}(x,y,z,t)$ in (28), selecting the $CSet_8$ and $\delta = 1, \alpha = 0.5, \omega = 2$, and $z = t = 1$.

5. Conclusions

In this study, we examined in detail the KP-B equation that widely appears in the models of physical phenomena, such as fluid dynamics, superfluids, physics, plasma physics, and hydrodynamic and optical models. Moreover, the KP-B equation provides for the modeling of waves traveling in both directions and more accurate approximations than the KPE. We applied two well-known analytical methods, the METFM function and the Kudryashov method. For this purpose, we obtained the solution functions by taking the Riccati equation as given in (11), while applying the modified extended tanh expansion method and by taking the solution functions as hyperbolic given in (24) and (25) instead of the exponential form in the application of the Kudryashov function method. The scientific contributions of this study are to show that different singular solutions of the multi-soliton solutions of the KP-B equation can be obtained with two easy-to-apply and effective methods. We obtained and exhibited a plethora of solutions, such as anti-kink, singular, periodic, and singular periodic. In this study, we covered some topics that can be studied prospectively, such as detailed examinations of the step-shaped formations observed in singular soliton solutions, examining the equations of different fractional forms of this structure, and investigating the effects of the results obtained from single-wave solutions on multiwave solutions and different soliton types. In addition to effective and different methods, such as the Hirota and Cole–Hopf, we aim to obtain further different solutions to the problem examined here, with new methods such as Wang's direct algebraic method. In addition, considering that the single-wave and singular solutions of the KP-B equation have a significant effect on obtaining multi-soliton solutions, we believe that the results obtained in the study will be useful for research in this field.

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References

1. Wazwaz, A.M. *Partial Differential Equations and Solitary Waves Theory*; Springer: Berlin/Heidelberg, Germnay, 2009. [CrossRef]
2. Moroşanu, G. *Nonlinear Evolution Equations and Applications*; Mathematics and its Applications (East European Series); D. Reidel Publishing Co.: Dordrecht, The Netherlands, 1988; Volume 26, p. xii+340. With a preface by Viorel Barbu, Translated from the Romanian by the author.
3. Ablowitz, M.J.; Barone, V.; Lillo, S.D.; Sommacal, M. Traveling Waves in Elastic Rods with Arbitrary Curvature and Torsion. *J. Nonlinear Sci.* **2012**, *22*, 1013–1040. [CrossRef]
4. Paul, G.C.; Eti, F.Z.; Kumar, D. Dynamical analysis of lump, lump-triangular periodic, predictable rogue and breather wave solutions to the (3+1)-dimensional gKP–Boussinesq equation. *Results Phys.* **2020**, *19*, 103525. [CrossRef]
5. Heimburg, T.; Jackson, A.D. On soliton propagation in biomembranes and nerves. *Proc. Natl. Acad. Sci. USA* **2005**, *102*, 9790–9795. [CrossRef] [PubMed]
6. Wang, K.J. A new fractional nonlinear singular heat conduction model for the human head considering the effect of febrifuge. *Eur. Phys. J. Plus* **2020**, *135*, 871. [CrossRef]
7. Cinar, M.; Onder, I.; Secer, A.; Yusuf, A.; Sulaiman, T.A.; Bayram, M.; Aydin, H. The analytical solutions of Zoomeron equation via extended rational sin-cos and sinh-cosh methods. *Phys. Scr.* **2021**, *96*, 094002. [CrossRef]
8. Yusuf, A.; Sulaiman, T.A.; Khalil, E.; Bayram, M.; Ahmad, H. Construction of multi-wave complexiton solutions of the Kadomtsev-Petviashvili equation via two efficient analyzing techniques. *Results Phys.* **2021**, *21*, 103775. [CrossRef]
9. Zahran, E.H.M. Traveling Wave Solutions of Nonlinear Evolution Equations via Modified $exp(-\varphi(\xi))$-Expansion Method. *J. Comput. Theor. Nanosci.* **2015**, *12*, 5716–5724. [CrossRef]
10. Wang, K.J.; Wang, K.L. Variational principles for fractal whitham–broer–kaup equations in shallow water. *Fractals* **2021**, *29*, 2150028. [CrossRef]
11. Asjad, M.I.; Ullah, N.; Rehman, H.U.; Inc, M. Construction of optical solitons of magneto-optic waveguides with anti-cubic law nonlinearity. *Opt. Quantum Electron.* **2021**, *53*. [CrossRef]
12. Al-Smadi, M.; Arqub, O.A.; Hadid, S. Approximate solutions of nonlinear fractional Kundu-Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method. *Phys. Scr.* **2020**, *95*, 105205. [CrossRef]
13. Baleanu, D.; Inc, M.; Yusuf, A.; Aliyu, A.I. Lie symmetry analysis, exact solutions and conservation laws for the time fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *59*, 222–234. [CrossRef]
14. Yépez-Martínez, H.; Rezazadeh, H.; Souleymanou, A.; Mukam, S.P.T.; Eslami, M.; Kuetche, V.K.; Bekir, A. The extended modified method applied to optical solitons solutions in birefringent fibers with weak nonlocal nonlinearity and four wave mixing. *Chin. J. Phys.* **2019**, *58*, 137–150. [CrossRef]
15. Akbar, M.A.; Ali, N.H.M. The improved F-expansion method with Riccati equation and its applications in mathematical physics. *Cogent Math.* **2017**, *4*, 1282577. [CrossRef]
16. Zedan, H. Applications of the New Compound Riccati Equations Rational Expansion Method and Fan's Subequation Method for the Davey-Stewartson Equations. *Bound. Value Probl.* **2010**, *2010*, 915721. [CrossRef]
17. Kudryashov, N.A. One method for finding exact solutions of nonlinear differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 2248–2253. [CrossRef]
18. Kudryashov, N.A. Logistic function as solution of many nonlinear differential equations. *Appl. Math. Model.* **2015**, *39*, 5733–5742. [CrossRef]
19. Kudryashov, N.A. Method for finding highly dispersive optical solitons of nonlinear differential equations. *Optik* **2020**, *206*, 163550. [CrossRef]
20. Al-Nowehy, A.G. Generalized Kudryashov method and general Exp_a-function method for solving a higher order nonlinear Schrödinger equation. *J. Space Explor.* **2017**, *6*, 1–26.
21. Zayed, E.M.; Alngar, M.E.; El-Horbaty, M.; Biswas, A.; Alshomrani, A.S.; Khan, S.; Ekici, M.; Triki, H. Optical solitons in fiber Bragg gratings having Kerr law of refractive index with extended Kudryashov's method and new extended auxiliary equation approach. *Chin. J. Phys.* **2020**, *66*, 187–205. [CrossRef]

22. Ghanbari, B.; Baleanu, D. New Solutions of Gardner's Equation Using Two Analytical Methods. *Front. Phys.* **2019**, *7*. [CrossRef]
23. Kaewta, S.; Sirisubtawee, S.; Khansai, N. Explicit Exact Solutions of the (2+1)-Dimensional Integro-Differential Jaulent–Miodek Evolution Equation Using the Reliable Methods. *Int. J. Math. Math. Sci.* **2020**, *2020*, 1–19. [CrossRef]
24. Biswas, A.; Ekici, M.; Sonmezoglu, A.; Belic, M.R. Highly dispersive optical solitons with kerr law nonlinearity by extended Jacobi's elliptic function expansion. *Optik* **2019**, *183*, 395–400. [CrossRef]
25. Baskonus, H.M.; Ercan, M. Extraction Complex Properties of the Nonlinear Modified Alpha Equation. *Fractal Fract.* **2021**, *5*, 6. [CrossRef]
26. Bulut, H.; Isik, H.A.; Sulaiman, T.A. On Some Complex Aspects of the (2+1)-dimensional Broer-Kaup-Kupershmidt System. *ITM Web Conf.* **2017**, *13*, 01019. [CrossRef]
27. Ilie, M.; Biazar, J.; Ayati, Z. The first integral method for solving some conformable fractional differential equations. *Opt. Quantum Electron.* **2018**, *50*, 55. [CrossRef]
28. Inc, M.; Aliyu, A.I.; Yusuf, A.; Baleanu, D.; Nuray, E. Complexiton and solitary wave solutions of the coupled nonlinear Maccari's system using two integration schemes. *Mod. Phys. Lett. B* **2018**, *32*, 1850014. [CrossRef]
29. Kundu, P.R.; Fahim, M.R.A.; Islam, M.E.; Akbar, M.A. The sine-Gordon expansion method for higher-dimensional NLEEs and parametric analysis. *Heliyon* **2021**, *7*, e06459. [CrossRef]
30. Cattani, C.; Sulaiman, T.A.; Baskonus, H.M.; Bulut, H. Solitons in an inhomogeneous Murnaghan's rod. *Eur. Phys. J. Plus* **2018**, *133*, 228. [CrossRef]
31. Ozisik, M. On the optical soliton solution of the (1+1)- dimensional perturbed NLSE in optical nano-fibers. *Optik* **2022**, *250*, 168233. [CrossRef]
32. Filiz, A.; Ekici, M.; Sonmezoglu, A. F-Expansion Method and New Exact Solutions of the Schrödinger-KdV Equation. *Sci. World J.* **2014**, *2014*, 1–14. [CrossRef]
33. Li, L.; Duan, C.; Yu, F. An improved Hirota bilinear method and new application for a nonlocal integrable complex modified Korteweg-de Vries (MKdV) equation. *Phys. Lett. A* **2019**, *383*, 1578–1582. [CrossRef]
34. Liang, Z.; Tang, X.; Lou, S. New nonlocal symmetries and conservation laws of the (1+1)-dimensional Sine-Gordon equation. *J. Phys. Conf. Ser.* **2014**, *490*, 012032. [CrossRef]
35. Rasheed, N.M.; Al-Amr, M.O.; Az-Zo'bi, E.A.; Tashtoush, M.A.; Akinyemi, L. Stable Optical Solitons for the Higher-Order Non-Kerr NLSE via the Modified Simple Equation Method. *Mathematics* **2021**, *9*, 1986. [CrossRef]
36. Jawad, A.J.M.; Petković, M.D.; Biswas, A. Modified simple equation method for nonlinear evolution equations. *Appl. Math. Comput.* **2010**, *217*, 869–877. [CrossRef]
37. Triki, H.; Hayat, T.; Aldossary, O.; Biswas, A. 1-soliton solution of the three component system of Wu-Zhang equations. *Hacet. J. Math. Stat.* **2012**, *41*, 537–543.
38. Ghanbari, B.; Inc, M.; Yusuf, A.; Baleanu, D.; Bayram, M. Families of exact solutions of Biswas-Milovic equation by an exponential rational function method. *Tbil. Math. J.* **2020**, *13*, 39–65. [CrossRef]
39. Tripathy, A.; Sahoo, S. A novel analytical method for solving (2+1)- dimensional extended Calogero-Bogoyavlenskii-Schiff equation in plasma physics. *J. Ocean Eng. Sci.* **2021**, *6*, 405–409. [CrossRef]
40. Gepreel, K.A.; Nofal, T.A.; Al-Asmari, A.A. Abundant travelling wave solutions for nonlinear Kawahara partial differential equation using extended trial equation method. *Int. J. Comput. Math.* **2018**, *96*, 1357–1376. [CrossRef]
41. dong Zhu, S. The generalizing Riccati equation mapping method in non-linear evolution equation: Application to (2+1)-dimensional Boiti–Leon–Pempinelle equation. *Chaos Solitons Fractals* **2008**, *37*, 1335–1342. [CrossRef]
42. Khalique, C.M.; Moleleki, L.D. A (3+1)-dimensional generalized BKP-Boussinesq equation: Lie group approach. *Results Phys.* **2019**, *13*, 102239. [CrossRef]
43. Alotaibi, H. Explore Optical Solitary Wave Solutions of the kp Equation by Recent Approaches. *Crystals* **2022**, *12*, 159. [CrossRef]
44. Alotaibi, H. Traveling Wave Solutions to the Nonlinear Evolution Equation Using Expansion Method and Addendum to Kudryashov's Method. *Symmetry* **2021**, *13*, 2126. [CrossRef]
45. Wang, K.J. Diverse soliton solutions to the Fokas system via the Cole-Hopf transformation. *Optik* **2023**, *272*, 170250. [CrossRef]
46. Wang, K.J. A fast insight into the optical solitons of the generalized third-order nonlinear Schrödinger's equation. *Results Phys.* **2022**, *40*, 105872. [CrossRef]
47. Wazwaz, A.M.; El-Tantawy, S.A. Solving the (3+1)-dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota's method. *Nonlinear Dyn.* **2017**, *88*, 3017–3021. [CrossRef]
48. Yu, J.P.; Sun, Y.L. A direct Bäcklund transformation for a (3+1)-dimensional Kadomtsev–Petviashvili–Boussinesq-like equation. *Nonlinear Dyn.* **2017**, *90*, 2263–2268. [CrossRef]
49. Wang, L.; Zhou, Y.; Liu, Q.; Zhang, Q. Traveling waves of the (3+1)-Dimensıonal Kadomtsev-Petviashvili-Boussinesq Equation. *J. Appl. Anal. Comput.* **2020**, *10*, 267–281. [CrossRef]
50. Wazwaz, A.M. Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 491–495. [CrossRef]
51. Liu, J.G.; Tian, Y.; Zeng, Z.F. New exact periodic solitary-wave solutions for the new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in multi-temperature electron plasmas. *AIP Adv.* **2017**, *7*, 105013. [CrossRef]
52. Baronio, F.; Onorato, M.; Chen, S.; Trillo, S.; Kodama, Y.; Wabnitz, S. Optical-fluid dark line and X solitary waves in Kerr media. *Opt. Data Process. Storage* **2017**, *3*, 1–7. [CrossRef]

53. Klein, C.; Sparber, C.; Markowich, P. Numerical Study of Oscillatory Regimes in the Kadomtsev–Petviashvili Equation. *J. Nonlinear Sci.* **2007**, *17*, 429–470. [CrossRef]
54. Seadawy, A.; El-Rashidy, K. Dispersive solitary wave solutions of Kadomtsev-Petviashvili and modified Kadomtsev-Petviashvili dynamical equations in unmagnetized dust plasma. *Results Phys.* **2018**, *8*, 1216–1222. [CrossRef]
55. Seadawy, A.R. Ion acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili-Burgers equation in quantum plasma. *Math. Methods Appl. Sci.* **2016**, *40*, 1598–1607. [CrossRef]
56. Treumann, R.A.; Pottelette, R. Plasma Soliton Turbulence and Statistical Mechanics. In Proceedings of the Plasma Turbulence and Energetic Particles in Astrophysics, Cracow, Poland, 5–10 September 1999; pp. 167–181.
57. Qin, Y.; Liu, Y. Multiwave interaction solutions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation. *Chin. J. Phys.* **2021**, *71*, 561–573. [CrossRef]
58. Li, Y.; Mei, C.C. Modified Kadomtsev–Petviashvili equation for tsunami over irregular seabed. *Nat. Hazards* **2016**, *84*, 513–528. [CrossRef]
59. Xu, B.; Zhang, Y.; Zhang, S. Line Soliton Interactions for Shallow Ocean Waves and Novel Solutions with Peakon, Ring, Conical, Columnar, and Lump Structures Based on Fractional KP Equation. *Adv. Math. Phys.* **2021**, *2021*, 1–15. [CrossRef]
60. Irwaq, I.A.; Alquran, M.; Jaradat, I.; Baleanu, D. New dual-mode Kadomtsev–Petviashvili model with strong–weak surface tension: Analysis and application. *Adv. Differ. Equ.* **2018**, *2018*, 433. [CrossRef]
61. Wu, P.X.; Zhang, Y.F.; Yin, Q.Q.; Wang, Y. Integrability and lump-type solutions to the 3-D Kadomtsev-Petviashvili-Boussinesq-like equation. *Therm. Sci.* **2019**, *23*, 2373–2380. [CrossRef]
62. Gao, B.; Zhang, Y. Exact Solutions and Conservation Laws of the (3+1)-Dimensional B-Type Kadomstev–Petviashvili (BKP)-Boussinesq Equation. *Symmetry* **2020**, *12*, 97. [CrossRef]
63. Zhou, Q.; Pan, A.; Mirhosseini-Alizamini, S.M.; Mirzazadeh, M.; Liu, W.; Biswas, A. Group Analysis and Exact Soliton Solutions to a New (3+1)-Dimensional Generalized Kadomtsev-Petviashvili Equation in Fluid Mechanics. *Acta Phys. Pol. A* **2018**, *134*, 564–569. [CrossRef]
64. Li, L.; Xie, Y.; Mei, L. Multiple-order rogue waves for the generalized (2+1)-dimensional Kadomtsev–Petviashvili equation. *Appl. Math. Lett.* **2021**, *117*, 107079. [CrossRef]
65. Li, L.; Xie, Y. Rogue wave solutions of the generalized (3+1)-dimensional Kadomtsev–Petviashvili equation. *Chaos Solitons Fractals* **2021**, *147*, 110935. [CrossRef]
66. Wang, Y.H. Nonautonomous lump solutions for a variable–coefficient Kadomtsev–Petviashvili equation. *Appl. Math. Lett.* **2021**, *119*, 107201. [CrossRef]
67. Lu, D.; Tariq, K.; Osman, M.; Baleanu, D.; Younis, M.; Khater, M. New analytical wave structures for the (3+1)-dimensional Kadomtsev-Petviashvili and the generalized Boussinesq models and their applications. *Results Phys.* **2019**, *14*, 102491. [CrossRef]
68. Shen, Y.; Tian, B. Bilinear auto-Bäcklund transformations and soliton solutions of a (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves. *Appl. Math. Lett.* **2021**, *122*, 107301. [CrossRef]
69. Ma, Y.L.; Wazwaz, A.M.; Li, B.Q. A new (3+1)-dimensional Kadomtsev–Petviashvili equation and its integrability, multiple-solitons, breathers and lump waves. *Math. Comput. Simul.* **2021**, *187*, 505–519. [CrossRef]
70. Singh, S.; Sakkaravarthi, K.; Murugesan, K. Localized nonlinear waves on spatio-temporally controllable backgrounds for a (3+1)-dimensional Kadomtsev-Petviashvili-Boussinesq model in water waves. *Chaos Solitons Fractals* **2022**, *155*, 111652. [CrossRef]
71. Ozisik, M.; Secer, A.; Bayram, M.; Aydin, H. An encyclopedia of Kudryashov's integrability approaches applicable to optoelectronic devices. *Optik* **2022**, *265*, 169499. [CrossRef]