symmetry

MDPI

Article

The Invariants of Dual Parallel Equidistant Ruled Surfaces

Sümeyye Gür Mazlum [1,*], Süleyman Şenyurt [2] and Luca Grilli [3]

[1] Department of Computer Technology, Gümüşhane University, 29000 Gümüşhane, Turkey
[2] Department of Mathematics, Ordu University, 52000 Ordu, Turkey
[3] Department of Economics, Management and Territory, University of Foggia, 71121 Foggia, Italy
* Correspondence: sumeyyegur@gumushane.edu.tr

Abstract: In this paper, we calculate the Gaussian curvatures of the dual spherical indicatrix curves formed on unit dual sphere by the Blaschke vectors and dual instantaneous Pfaff vectors of dual parallel equidistant ruled surfaces (DPERS) and we give the relationships between these curvatures. In addition to—in cases where the base curves of these DPERS are closed—computing the dual integral invariants of the indicatrix curves. Additionally, we show the relationships between them. Finally, we provide an example for each of these indicatrix curves.

Keywords: dual parallel equidistant ruled surfaces; integral invariants; Gaussian curvature

Citation: Gür Mazlum, S.; Şenyurt, S.; Grilli, L. The Invariants of Dual Parallel Equidistant Ruled Surfaces. *Symmetry* **2023**, *15*, 206. https://doi.org/10.3390/sym15010206

Academic Editor: Dongfang Li

Received: 11 December 2022
Revised: 2 January 2023
Accepted: 7 January 2023
Published: 10 January 2023

1. Introduction

The main sources for curves and surfaces in Euclidean space are [1–5]. The concept of ruled surfaces, formed by the movement of a line on a curve, has an important place in differential geometry. There are many studies on ruled surfaces in Euclidean and non-Euclidean spaces [6–10]. The concept of parallel equidistant ruled surfaces was first introduced by Valeontis [11]. The resources [12–19] can be examined for studies on these surfaces. Although dual numbers are a set of numbers defined by William Kingdon Clifford (1845–1879) [20], their first applications in geometry began with Eduard Study [21]. Dual numbers can be seen as similar to complex numbers, but they are important in terms of kinematics, especially in unit dual sphere applications. E. Study revealed that there is a strong relationship between a dual point on a unit dual sphere and an oriented line in the space of lines, and thus this concept gained meaning. Since kinematically correct beams are a family of axes of rotation, it is possible to examine the ruled surface of a motion over the unit dual sphere. A dual spherical curve on a dual sphere corresponds exactly to a ruled surface in the space of lines. Basic concepts in this field can be accessed from the resources [22–25]. Furthermore, some studies on dual curves or surfaces include Refs. [26–30]. In addition to geometry, studies on dual numbers and vectors are carried out in fields such as mechanical engineering, robot kinematics, physics and astronomy. The concept of motors was introduced by Clifford, but he did not examine the modeling of points, lines and planes in terms of motors. The sentence of motors is isomorphic to the sentence of dual quaternions. However, due to the geometric meaning of vectors as rotating planes, screws can be easily formulated using motors from a purely geometrical point of view. As a result, algebraic modeling of points, lines, and planes with motor algebra is easier with dual quaternions. The use of dual matrices and dual quaternions in robotics has increased in recent years [31–35]. The concept of dual numbers is studied by geometers in Euclidean spaces as well as in non-Euclidean spaces such as Lorentz–Minkowski spaces [36–40].

In Ref. [41], we defined parallel equidistant ruled surfaces (DPERS) and compared some of their geometric properties. In addition, we have shown that one of these surfaces is symmetrical to the other. In this paper, we calculate the Gaussian curvatures of the ruled surfaces corresponding in space of lines to the dual spherical indicatrix curves formed on a

unit dual sphere by the Blaschke vectors and dual instantaneous Pfaff vectors of DPERS and we give the relationships between these curvatures. Additionally, in cases where the base curves of these DPERS are closed, we compute the dual integral invariants of the indicatrix curves. We also show the relationships between them. Finally, we provide an example for each of these indicatrix curves. It is possible to combine the results of this study with the concepts of submanifold theory or studies in other spaces and obtain new results [42–47].

2. Preliminaries

The set $\mathbb{D}-Modul=\left\{\vec{M}\mid \vec{M}=\vec{m}+\varepsilon\vec{m^*},\ \vec{m},\vec{m^*}\in\mathbb{R}^3,\varepsilon=(0,1)\right\}$ is called dual vector space and the elements of this set are called dual vectors, where $\vec{m}$ and $\vec{m^*}$ are the real and dual components of the vector $\vec{M}$, respectively. The operations of addition, multiplication by scalars, inner product, norm and vector product on this set are explained in [23,41]. The set $\mathbb{K}=\{\vec{M}=\vec{m}+\varepsilon\vec{m^*}\ \mid \|\vec{M}\|=(1,0),\ \vec{m},\vec{m^*}\in\mathbb{E}^3\}$ is called a unit dual unit sphere and the elements of this set are called dual unit vectors [23]. The dual angle $\Omega=\omega+\varepsilon\omega^*$ between the unit dual vectors $\vec{M}=\vec{m}+\varepsilon\vec{m^*}$ and $\overrightarrow{M}=\overrightarrow{m}+\varepsilon\overrightarrow{m^*}$ can be calculated with any of the following two equalities:

$$\left\langle \vec{M},\overrightarrow{M}\right\rangle=\cos\Omega=\cos\omega-\varepsilon\omega^*\sin\omega,$$

$$\vec{M}\wedge\overrightarrow{M}=\sin\Omega\vec{N}=(\sin\omega+\varepsilon\omega^*\cos\omega)\vec{N},$$

where $\vec{N}=\vec{n}+\varepsilon\vec{n^*}$ is the unit dual vector, $\vec{n}$ is the distance vector between $\vec{M}$ and $\overrightarrow{M}$, $\vec{n^*}=\overrightarrow{m}\wedge\vec{n}$. Additionally, ω is the real angle between the real vectors $\vec{m}$ and $\overrightarrow{m}$ and ω^* is the shortest distance between the vectors $\vec{m}$ and $\overrightarrow{m}$, [22,23].

Theorem 1 (E. Study Mapping [21]). *A directed line in the space of lines corresponds exactly to a dual point on the dual unit sphere.*

The dual spherical curve $\vec{M}(s)=\vec{m}(s)+\varepsilon\vec{m^*}(s)$ drawn by the unit dual vector $\vec{M}$ is expressed in the space of lines by the ruled surface, see Figure 1,

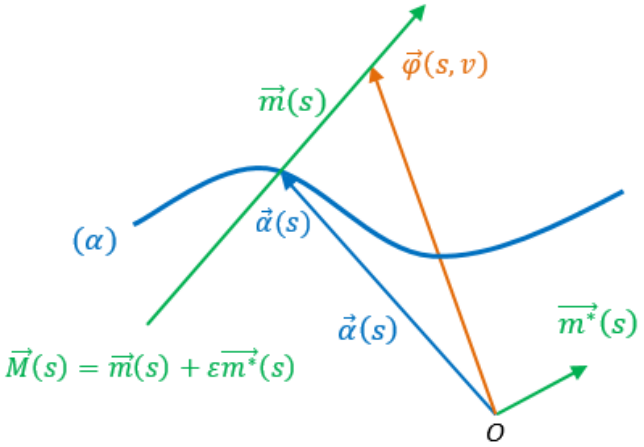

Figure 1. The dual spherical expression of the ruled surface $\overrightarrow{\varphi}(s,v)$.

$$\overrightarrow{\varphi}(s,v)=\overrightarrow{m}(s)\wedge\overrightarrow{m^*}(s)+v\overrightarrow{m}(s),\qquad \overrightarrow{m^*}(s)=\overrightarrow{\alpha}(s)\wedge\overrightarrow{m}(s), \tag{1}$$

where $\overrightarrow{\alpha}(s)$ is the base curve of the ruled surface and s is the arc lenght parameter of the curve [22,23]. That is, a dual spherical curve $\overrightarrow{M}(s)$ can be viewed as a ruled surface. The orthonormal system $\{\overrightarrow{A_1}(s) = \overrightarrow{M}(s),\ \overrightarrow{A_2}(s) = \dfrac{\overrightarrow{M'}(s)}{\left\|\overrightarrow{M'}(s)\right\|},\ \overrightarrow{A_3}(s) = \overrightarrow{A_1}(s) \wedge \overrightarrow{A_2}(s)\}$ of the ruled surface $\overrightarrow{M}(s)$ are called Blaschke frame and also the values

$$\begin{cases} \wp(s) = \rho(s) + \varepsilon\rho^*(s) = \left\|\overrightarrow{M'}(s)\right\|, \\ \Re(s) = \eta(s) + \varepsilon\eta^*(s) = \dfrac{\left(\overrightarrow{M}(s), \overrightarrow{M'}(s), \overrightarrow{M''}(s)\right)}{\wp^2(s)} \end{cases} \tag{2}$$

are called Blaschke invariants, where $\rho(s), \eta(s)$ are the reel components and $\rho^*(s), \eta^*(s)$ are dual components of these invariants [22]. Besides $\rho(s)$ and $\eta(s)$ are the curvature and the torsion of the curve $\overrightarrow{\alpha}(s)$.

The Blaschke derivative formulas are as follows:

$$\begin{cases} \overrightarrow{A_1'}(s) = \wp(s)\overrightarrow{A_2}(s), \\ \overrightarrow{A_2'}(s) = -\wp(s)\overrightarrow{A_1}(s) + \Re(s)\overrightarrow{A_3}(s), \\ \overrightarrow{A_3'}(s) = -\Re(s)\overrightarrow{A_2}(s). \end{cases} \tag{3}$$

See [41] for the real and dual Equations (2) and (3). The dual pitch length, the dual pitch angle and the drall (parameter of distribution) of the closed ruled surface $\overrightarrow{M}(s)$ are, respectively [22],

$$\begin{cases} L_M(s) = \langle\overrightarrow{d^*}(s), \overrightarrow{m}(s)\rangle + \langle\overrightarrow{d}(s), \overrightarrow{m^*}(s)\rangle, \\ \wedge_M(s) = -\langle\overrightarrow{D}(s), \overrightarrow{M}(s)\rangle, \\ P_M(s) = \dfrac{\left\langle\overrightarrow{dm}(s), \overrightarrow{dm^*}(s)\right\rangle}{\left\langle\overrightarrow{dm}(s), \overrightarrow{dm}(s)\right\rangle}, \end{cases} \tag{4}$$

where $\overrightarrow{D}(s) = \overrightarrow{A_1}(s) \oint \Re(s)ds + \overrightarrow{A_3}(s) \oint \wp(s)ds$ is the dual Steiner rotation vector and

$$\begin{cases} \overrightarrow{d}(s) = \overrightarrow{a_1}(s) \oint \eta(s)ds + \overrightarrow{a_3}(s) \oint \rho(s)ds, \\ \overrightarrow{d^*}(s) = \overrightarrow{a_1^*}(s) \oint \eta(s)ds + \overrightarrow{a_1}(s) \oint \eta^*(s)ds + \overrightarrow{a_3^*}(s) \oint \rho(s)ds + \overrightarrow{a_3}(s) \oint \rho^*(s)ds. \end{cases} \tag{5}$$

The Gaussian curvature of the ruled surface $\overrightarrow{M}(s)$ is depicted by

$$K(s) = -\left\langle S\left(\overrightarrow{E_2}(s)\right), \overrightarrow{E_1}(s)\right\rangle^2, \qquad S\left(\overrightarrow{M}(s)\right) = \overrightarrow{D_M N}(s), \tag{6}$$

where S is the shape operator, $\overrightarrow{N}(s)$ is the normal vector and $\left\{\overrightarrow{E_1}(s), \overrightarrow{E_2}(s)\right\}$ is the base vectors [14]. If a ruled surface is isomorphic to the plane (the surface is developable), the drall (or Gaussian curvature) of this surface is zero everywhere.

The definition and properties of dual parallel equidistant ruled surfaces (DPERS) are also explained in [41]. In this section, the relations that will be used in the continuation

of the study will be given. Let $\{\overrightarrow{A_1}(s),\ \overrightarrow{A_2}(s),\ \overrightarrow{A_3}(s)\}$ and $\{\overrightarrow{B_1}(\bar{s}),\ \overrightarrow{B_2}(\bar{s}),\ \overrightarrow{B_3}(\bar{s})\}$ be the Blaschke frames of the base curves of DPERS,

$$\begin{cases} \overrightarrow{\Psi}(s,v) = \overrightarrow{a_1}(s) \wedge \overrightarrow{a_1^*}(s) + v\overrightarrow{a_1}(s), & \overrightarrow{a_1^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{a_1}(s), \\ \overrightarrow{\overline{\Psi}}(\bar{s},\bar{v}) = \overrightarrow{b_1}(\bar{s}) \wedge \overrightarrow{b_1^*}(\bar{s}) + \bar{v}\overrightarrow{b_1}(\bar{s}), & \overrightarrow{b_1^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{b_1}(\bar{s}), \end{cases}$$

respectively, where $\beta(\bar{s})$ is the base curve of the ruled surface $\overrightarrow{\overline{\Psi}}$ and $\bar{s}$ is the arc length parameter of the curve. There are the following relations between these vectors:

$$\begin{cases} \overrightarrow{B_1}(\bar{s}) = \overrightarrow{b_1}(\bar{s}) + \varepsilon\overrightarrow{b_1^*}(\bar{s}) = \overrightarrow{a_1}(s) + \varepsilon(\overrightarrow{a_1^*}(s) + r(s)\overrightarrow{a_2}(s) - z(s)\overrightarrow{a_3}(s)), \\ \overrightarrow{B_2}(\bar{s}) = \overrightarrow{b_2}(\bar{s}) + \varepsilon\overrightarrow{b_2^*}(\bar{s}) = \overrightarrow{a_2}(s) + \varepsilon(\overrightarrow{a_2^*}(s) - r(s)\overrightarrow{a_1}(s) + \phi^*(s)\overrightarrow{a_3}(s)), \\ \overrightarrow{B_3}(\bar{s}) = \overrightarrow{b_3}(\bar{s}) + \varepsilon\overrightarrow{b_3^*}(\bar{s}) = \overrightarrow{a_3}(s) + \varepsilon(\overrightarrow{a_3^*}(s) + z(s)\overrightarrow{a_1}(s) - \phi^*(s)\overrightarrow{a_2}(s)), \end{cases} \tag{7}$$

where $\overrightarrow{a_i}(s)$ and $\overrightarrow{b_i}(\bar{s})$ are the real and dual components of the vectors $\overrightarrow{A_i}(s)$ and $\overrightarrow{B_i}(\bar{s})$, [41]. Additionally, $\phi^*(s)$, $z(s)$, $r(s)$ are the perpendicular projection distances on the unit vectors $\overrightarrow{u_1}(s)$, $\overrightarrow{u_2}(s)$, $\overrightarrow{u_3}(s)$ of the vector $\overrightarrow{\beta}(\bar{s}) - \overrightarrow{\alpha}(s)$, respectively [11]. The relationships between the Blaschke invariants $\wp(s) = \rho(s) + \varepsilon\rho^*(s)$, $\Re(s) = \eta(s) + \varepsilon\eta^*(s)$ and $\overline{\wp}(\bar{s}) = \overline{\rho}(\bar{s}) + \varepsilon\overline{\rho}^*(\bar{s})$, $\overline{\Re}(\bar{s}) = \overline{\eta}(\bar{s}) + \varepsilon\overline{\eta}^*(\bar{s})$ of DPERS are

$$\begin{cases} \overline{\rho}(\bar{s}) = \dfrac{\rho(s)}{1 - z(s)\rho(s)}, & \overline{\rho}^*(\bar{s}) = \rho^*(s) = 0, \\ \overline{\eta}(\bar{s}) = \dfrac{\eta(s)}{1 - z(s)p(s)}, & \overline{\eta}^*(\bar{s}) = \eta^*(s) = 1, \end{cases} \tag{8}$$

respectively [41]. The arc length between the points corresponding to the striction curves of DPERS is [41]

$$\phi^*(s) = \frac{r(s)\eta(s) - z'(s)}{\rho(s)}. \tag{9}$$

The relationship between arc-length parameters of the base curves of DPERS is [41]

$$\frac{d\bar{s}}{ds} = 1 - z(s)\rho(s). \tag{10}$$

The relationship between the striction curves $\overrightarrow{\Upsilon}(s)$ and $\overrightarrow{\overline{\Upsilon}}(\bar{s})$ of DPERS are

$$\overrightarrow{\overline{\Upsilon}}(\bar{s}) = \overrightarrow{\Upsilon}(s) + \left(\frac{r(s)\eta(s) - z'(s)}{\rho(s)} \right)\overrightarrow{a_1}(s) + z(s)\overrightarrow{a_2}(s) + r(s)\overrightarrow{a_3}(s). \tag{11}$$

The relationships between the real and dual components of the dual instantaneous Pfaff vectors $\overrightarrow{C}(s) = \overrightarrow{c}(s) + \varepsilon\overrightarrow{c^*}(s)$ and $\overrightarrow{\overline{C}}(\bar{s}) = \overrightarrow{\bar{c}}(\bar{s}) + \varepsilon\overrightarrow{\bar{c}^*}(\bar{s})$ of DPERS are [41]

$$\begin{cases} \overrightarrow{c^*}(s) = \sin\omega\overrightarrow{a_1^*}(s) + \cos\omega\overrightarrow{a_3^*}(s), \\ \overrightarrow{\bar{c}^*}(\bar{s}) = \overrightarrow{c^*}(s) + z(s)\cos\omega\overrightarrow{a_1}(s) + (r(s)\sin\omega - \phi^*(s)\cos\omega)\overrightarrow{a_2}(s) \\ \qquad\qquad - z(s)\sin\omega\overrightarrow{a_3}(s). \end{cases} \tag{12}$$

The real and dual components of the dual Steiner vectors $\overrightarrow{D}(s) = \overrightarrow{d}(s) + \varepsilon\overrightarrow{d^*}(s)$ and $\overrightarrow{\overline{D}}(\overline{s}) = \overrightarrow{\overline{d}}(\overline{s}) + \varepsilon\overrightarrow{\overline{d}^*}(\overline{s})$ of DPERS are [41]

$$\begin{cases} \overrightarrow{d}(s) = \overrightarrow{\overline{d}}(\overline{s}) = \overrightarrow{a_1}(s)\oint \eta(s)ds + \overrightarrow{a_3}(s)\oint \rho(s)ds, \\ \overrightarrow{d^*}(s) = \overrightarrow{a_1^*}(s)\oint \eta(s)ds + \overrightarrow{a_1}(s)\oint ds + \overrightarrow{a_3^*}(s)\oint \rho(s)ds, \\ \overrightarrow{\overline{d}^*}(\overline{s}) = \overrightarrow{d^*}(s) + \left(-\oint z(s)\rho(s)ds + z(s)\oint \rho(s)ds\right)\overrightarrow{a_1}(s) \\ \qquad + \left(r(s)\oint \eta(s)ds - \phi^*(s)\oint \rho(s)ds\right)\overrightarrow{a_2}(s) - z(s)\overrightarrow{a_3}\oint \eta(s)ds. \end{cases} \tag{13}$$

The relationships between the real and dual components of the dual angles $\Omega = \omega + \varepsilon\omega^*$ and $\overline{\Omega} = \overline{\omega} + \varepsilon\overline{\omega}^*$ between the vectors $\overrightarrow{A_3}(s)$,$\overrightarrow{W}(s)$ and $\overrightarrow{B_3}(\overline{s})$, $\overrightarrow{\overline{W}}(\overline{s})$ of DPERS are

$$\cos\overline{\omega} = \cos\omega = \frac{\rho(s)}{\sqrt{\rho^2(s)+\eta^2(s)}}, \quad \sin\overline{\omega} = \sin\omega = \frac{\eta(s)}{\sqrt{\rho^2(s)+\eta^2(s)}}, \tag{14}$$

respectively [41].

3. The Integral Invariants of DPERS

In this section, we will compute the integral invariants of the closed ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by the dual tangent, dual principal normal, dual binormal and intantaneous Pfaff vectors of the base curves of DPERS. While the next operations are performed, for shortness, the Equation (11) will be written as follows:

$$\overrightarrow{\overline{\Upsilon}}(\overline{s}) = \overrightarrow{\Upsilon}(s) + \overrightarrow{R}(s), \tag{15}$$

where

$$\overrightarrow{R}(s) = \left(\frac{r(s)\eta(s) - z'(s)}{\rho(s)}\right)\overrightarrow{a_1}(s) + z(s)\overrightarrow{a_2}(s) + r(s)\overrightarrow{a_3}(s).$$

3.1. The Relationships between the Integral Invariants of the Closed Ruled Surfaces Formed by the Dual Tangent Vectors of DPERS

Let $\overrightarrow{A_1}(s) = \overrightarrow{a_1}(s) + \varepsilon\overrightarrow{a_1^*}(s)$ and $\overrightarrow{B_1}(\overline{s}) = \overrightarrow{b_1}(\overline{s}) + \varepsilon\overrightarrow{b_1^*}(\overline{s})$ be the dual tangent vectors of the base curves of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\overline{\Psi}}(\overline{s},\overline{v})$, respectively. The parametric Equations of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by these vectors on the unit dual sphere are written as follows (Figure 2):

$$\begin{cases} \overrightarrow{\Psi_{A_1}}(s,v) = \overrightarrow{a_1}(s) \wedge \overrightarrow{a_1^*}(s) + v\overrightarrow{a_1}(s), \quad \overrightarrow{a_1^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{a_1}(s), \\ \overrightarrow{\overline{\Psi}_{B_1}}(\overline{s},\overline{v}) = \overrightarrow{b_1}(\overline{s}) \wedge \overrightarrow{b_1^*}(\overline{s}) + \overline{v}\overrightarrow{b_1}(\overline{s}), \quad \overrightarrow{b_1^*}(\overline{s}) = \overrightarrow{\beta}(\overline{s}) \wedge \overrightarrow{b_1}(\overline{s}). \end{cases} \tag{16}$$

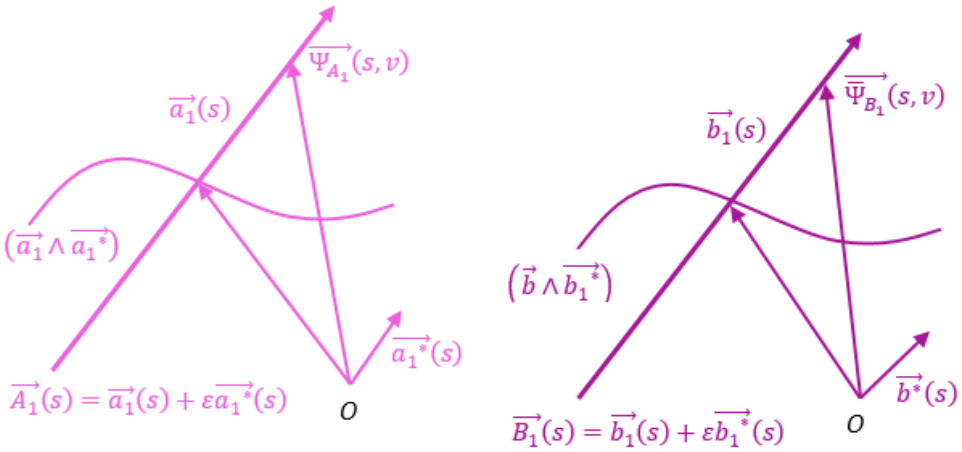

Figure 2. The dual spherical expressions of ruled surfaces formed by dual tangent vectors of DPERS.

Theorem 2. *There are the following relationships between the dual pitch lengths, the dual pitch angles and the dralls of the closed ruled surfaces formed by the dual tangent vectors* $\overrightarrow{A_1}(s)$ *and* $\overrightarrow{B_1}(\bar{s})$ *of DPERS, respectively:*

$$\begin{cases} L_{B_1}(\bar{s}) = L_{A_1}(s) + A_1(s), & A_1(s) = \oint\limits_{(\overrightarrow{R})} ds - \oint\limits_{(\overrightarrow{\Upsilon})} z(s)\rho(s)ds, \\ \Lambda_{B_1}(\bar{s}) = \Lambda_{A_1}(s) + A_2(s), & A_2(s) = -\oint\limits_{(\overrightarrow{R})} \eta(s)ds + \varepsilon - \oint\limits_{(\overrightarrow{\Upsilon})} z(s)\rho(s)ds, \\ P_{B_1}(\bar{s}) = P_{A_1}(s) = 0. \end{cases}$$

Proof. From the Equations (4), (8) and (13), the pitch lengths of the ruled surfaces formed by $\overrightarrow{A_1}(s)$ and $\overrightarrow{B_1}(\bar{s})$ are found as

$$L_{A_1}(s) = \oint\limits_{(\overrightarrow{\Upsilon})} ds, \quad L_{B_1}(\bar{s}) = \oint\limits_{(\overrightarrow{\Upsilon})} d\bar{s},$$

respectively. By using the Equations (10) and (15), the Equation $L_{B_1}(\bar{s})$ is stated as follows more clearly:

$$L_{B_1}(\bar{s}) = \oint\limits_{(\overrightarrow{\Upsilon})} ds + \oint\limits_{(\overrightarrow{R})} ds - \oint\limits_{(\overrightarrow{\Upsilon})} z(s)\rho(s)ds. \tag{17}$$

Additionally, from the Equations (4), (8) and (13), the dual pitch angles of the ruled surfaces formed by $\overrightarrow{A_1}(s)$ and $\overrightarrow{B_1}(\bar{s})$ are obtained as

$$\Lambda_{A_1}(s) = -\oint\limits_{(\overrightarrow{\Upsilon})} \eta(s)ds - \varepsilon \oint\limits_{(\overrightarrow{\Upsilon})} ds,$$

$$\Lambda_{B_1}(\bar{s}) = -\oint\limits_{(\overrightarrow{\Upsilon})} \overline{\eta}(\bar{s})d\bar{s} - \varepsilon \oint\limits_{(\overrightarrow{\Upsilon})} d\bar{s}.$$

By using the Equations (10) and (15), if the Equation $\Lambda_{B_1}(\bar{s})$ is stated more clearly,

$$\Lambda_{B_1}(\bar{s}) = -\oint_{(\vec{Y})} \eta(s)ds - \varepsilon \oint_{(\vec{Y})} ds - \oint_{(\vec{R})} \eta(s)ds - \varepsilon \oint_{(\vec{R})} ds - \varepsilon + \oint_{(\vec{\bar{Y}})} z(s)\rho(s)ds \tag{18}$$

is attained. Moreover, from the Equations (3) and (4), the dralls of the ruled surfaces formed by $\overrightarrow{A_1}(s)$ and $\overrightarrow{B_1}(\bar{s})$ are obtained as

$$P_{B_1}(\bar{s}) = P_{A_1}(s) = 0. \tag{19}$$

Thus, from the Equations (17)–(19), the proof is completed. □

Corollary 1. *The closed ruled surfaces formed by the dual tangent vectors $\overrightarrow{A_1}(s)$ and $\overrightarrow{B_1}(\bar{s})$ of DPERS are always isomorphic to the plane.*

3.2. The Relationships between the Integral Invariants of the Closed Ruled Surfaces Formed by the Dual Principal Normal Vectors of DPERS

Let $\overrightarrow{A_2}(s) = \overrightarrow{a_2}(s) + \varepsilon\overrightarrow{a_2^*}(s)$ and $\overrightarrow{B_2}(\bar{s}) = \overrightarrow{b_2}(\bar{s}) + \varepsilon\overrightarrow{b_2^*}(\bar{s})$ be the dual principal normal vectors of the base curves of DPERS $\overrightarrow{\Psi}(s,v)$ and $\overrightarrow{\bar{\Psi}}(\bar{s},\bar{v})$, respectively. The parametric Equations of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by these vectors on the unit dual sphere are written as follows:

$$\begin{cases} \overrightarrow{\Psi_{A_2}}(s,v) = \overrightarrow{a_2}(s) \wedge \overrightarrow{a_2^*}(s) + v\overrightarrow{a_2}(s), \quad \overrightarrow{a_2^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{a_2}(s), \\ \overrightarrow{\bar{\Psi}_{B_2}}(\bar{s},\bar{v}) = \overrightarrow{b_2}(\bar{s}) \wedge \overrightarrow{b_2^*}(\bar{s}) + \bar{v}\overrightarrow{b_2}(\bar{s}), \quad \overrightarrow{b_2^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{b_2}(\bar{s}). \end{cases} \tag{20}$$

Theorem 3. *There are the following relationships between the dual pitch lengths, the dual pitch angles and the dralls of the closed ruled surfaces formed by the dual principal normal vectors $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(\bar{s})$ of DPERS, respectively:*

$$\begin{cases} L_{B_2}(\bar{s}) = L_{A_2}(s) = 0, \\ \Lambda_{B_2}(\bar{s}) = \Lambda_{A_2}(s) = 0, \\ P_{B_2}(\bar{s}) = (1 - z(s)\rho(s))P_{A_2}(s). \end{cases}$$

Proof. From the Equations (4) and (13), the pitch lengths of the ruled surfaces formed by $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(\bar{s})$ are found as

$$L_{B_2}(\bar{s}) = L_{A_2}(s) = 0. \tag{21}$$

Additionally, from the Equations (4) and (13), the dual pitch angles of the ruled surfaces formed by $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(\bar{s})$ are obtained as

$$\Lambda_{B_2}(\bar{s}) = \Lambda_{A_2}(s) = 0. \tag{22}$$

Moreover, from the Equations (3), (4), (8) and (10), the dralls of the ruled surfaces formed by $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(\bar{s})$ are found as

$$\begin{aligned} P_{A_2}(s) &= \frac{\eta(s)}{\rho^2(s) + \eta^2(s)}, \\ P_{B_2}(\bar{s}) &= \frac{\eta(s) - z(s)\rho(s)\eta(s)}{\rho^2(s) + \eta^2(s)}. \end{aligned} \tag{23}$$

Thus, from the Equations (21), (22) and (23), the proof is completed. □

Corollary 2. *The closed ruled surfaces formed by the dual principal normal vectors* $\overrightarrow{A_2}(s)$ *and* $\overrightarrow{B_2}(\bar{s})$ *of DPERS are isomorphic to the plane if and only if* $\eta(s) = 0$.

3.3. The Relationships between the Integral Invariants of the Closed Ruled Surfaces Formed by the Dual Binormal Vectors of DPERS

Let $\overrightarrow{A_3}(s) = \overrightarrow{a_3}(s) + \varepsilon\overrightarrow{a_3^*}(s)$ and $\overrightarrow{B_3}(\bar{s}) = \overrightarrow{b_3}(\bar{s}) + \varepsilon\overrightarrow{b_3^*}(\bar{s})$ be the dual binormal vectors of the base curves DPERS $\overrightarrow{\Psi}(s, v)$ and $\overrightarrow{\overline{\Psi}}(\bar{s}, \bar{v})$, respectively. The parametric Equations of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by these vectors on the unit dual sphere are written as follows:

$$\begin{cases} \overrightarrow{\Psi_{A_3}}(s, v) = \overrightarrow{a_3}(s) \wedge \overrightarrow{a_3^*}(s) + v\overrightarrow{a_3}(s), \quad \overrightarrow{a_3^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{a_3}(s), \\ \overrightarrow{\overline{\Psi}_{B_3}}(\bar{s}, \bar{v}) = \overrightarrow{b_3}(\bar{s}) \wedge \overrightarrow{b_3^*}(\bar{s}) + \bar{v}\overrightarrow{b_3}(\bar{s}), \quad \overrightarrow{b_3^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{b_3}(\bar{s}). \end{cases} \tag{24}$$

Theorem 4. *There are the following relationships between the dual pitch lengths, the dual pitch angle and the dralls of the closed ruled surfaces formed by the dual binormal vectors* $\overrightarrow{A_3}(s)$ *and* $\overrightarrow{B_3}(\bar{s})$ *of DPERS, respectively:*

$$\begin{cases} L_{B_3}(\bar{s}) = L_{A_3}(s) = 0, \\ \Lambda_{B_3}(\bar{s}) = \Lambda_{A_3}(s) + A_3(s), \qquad A_3(s) = -\oint\limits_{(\overrightarrow{R})} \rho(s)ds, \\ P_{B_3}(\bar{s}) = (1 - z(s)\rho(s))P_{A_3}(s). \end{cases}$$

Proof. From the Equations (4) and (13), the pitch lengths of the ruled surfaces formed by $\overrightarrow{A_3}(s)$ and $\overrightarrow{B_3}(\bar{s})$ are found as

$$L_{B_3}(\bar{s}) = L_{A_3}(s) = 0. \tag{25}$$

Additionally, from the Equations (4), (8) and (13), the dual pitch angles of the ruled surfaces formed by $\overrightarrow{A_3}(s)$ and $\overrightarrow{B_3}(\bar{s})$ are obtained as

$$\Lambda_{A_3}(s) = -\oint\limits_{(\overrightarrow{Y})} \rho(s)ds, \qquad \Lambda_{B_3}(\bar{s}) = -\oint\limits_{(\overrightarrow{\bar{Y}})} \bar{\rho}(\bar{s})d\bar{s}\,.$$

By using the Equations (10) and (15), if the Equation $\Lambda_{B_3}(\bar{s})$ is stated more clearly,

$$\Lambda_{B_3}(\bar{s}) = -\oint\limits_{(\overrightarrow{Y})} \rho(s)ds - \oint\limits_{(\overrightarrow{R})} \rho(s)ds \tag{26}$$

is attained. Moreover, from the Equations (3), (4), (8) and (10), the dralls of the ruled surfaces formed by $\overrightarrow{A_3}(s)$ and $\overrightarrow{B_3}(\bar{s})$ are obtained as

$$P_{A_3}(s) = \frac{1}{\eta(s)}, \qquad P_{B_3}(\bar{s}) = \frac{1 - z(s)\rho(s)}{\eta(s)}. \tag{27}$$

Thus, from the Equations (25)–(27), the proof is completed. □

Corollary 3. *The closed ruled surfaces formed by the dual binormal vectors* $\overrightarrow{A_3}(s)$ *and* $\overrightarrow{B_3}(\bar{s})$ *of DPERS are never isomorphic to the plane.*

3.4. The Relationships between the Integral Invariants of the Closed Ruled Surfaces Formed by the Dual Instantaneous Pfaff Vectors of DPERS

The parametric Equations of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by the unit dual vectors $\overrightarrow{C}(s)$ and $\overrightarrow{\overline{C}}(\bar{s})$ in the direction of the dual instantaneous Pfaff vectors of DPERS on the dual unit sphere are written as follows:

$$\begin{cases} \overrightarrow{\Psi_C}(s,v) = \overrightarrow{c}(s) \wedge \overrightarrow{c^*}(s) + v\,\overrightarrow{c}(s), \quad \overrightarrow{c^*}(s) = \overrightarrow{\alpha}(s) \wedge \overrightarrow{c}(s), \\ \overrightarrow{\overline{\Psi}_{\overline{C}}}(\bar{s},\bar{v}) = \overrightarrow{\overline{c}}(\bar{s}) \wedge \overrightarrow{\overline{c}^*}(\bar{s}) + v\,\overrightarrow{\overline{c}}(\bar{s}), \quad \overrightarrow{\overline{c}^*}(\bar{s}) = \overrightarrow{\beta}(\bar{s}) \wedge \overrightarrow{\overline{c}}(\bar{s}). \end{cases} \tag{28}$$

Theorem 5. *There are the following relationships between the dual pitch lengths, the dual pitch angles and the dralls of the closed ruled surfaces formed by the dual instantaneous Pfaff vectors* $\overrightarrow{C}(s)$ *and* $\overrightarrow{\overline{C}}(\bar{s})$ *of DPERS, respectively:*

$$\begin{cases} L_{\overline{C}}(\bar{s}) = L_C(s) + A_4(s), & A_4(s) = \sin\omega \oint\limits_{(\overrightarrow{R})} ds - \sin\omega \oint\limits_{(\overrightarrow{\overline{Y}})} z(s)\rho(s)ds, \\ \Lambda_{\overline{C}}(\bar{s}) = \Lambda_C(s) + A_5(s), & A_5(s) = -\cos\omega \oint\limits_{(\overrightarrow{R})} \rho(s)ds - \sin\omega \oint\limits_{(\overrightarrow{R})} \Re(s)ds \\ & \qquad + \varepsilon\sin\omega \oint\limits_{(\overrightarrow{\overline{Y}})} z(s)\rho(s)ds, \\ P_{\overline{C}}(\bar{s}) = P_C(s) = 0. \end{cases}$$

Proof. From the Equations (4), (12) and (13), the pitch lengths of the ruled surfaces formed by the vectors $\overrightarrow{C}(s)$ and $\overrightarrow{\overline{C}}(\bar{s})$ are obtained as

$$L_C(s) = \sin\omega \oint\limits_{(\overrightarrow{Y})} ds = \sin\omega L_{A_1}(s),$$

$$L_{\overline{C}}(\bar{s}) = \sin\overline{\omega} \oint\limits_{(\overrightarrow{\overline{Y}})} d\bar{s} = \sin\overline{\omega} L_{B_1}(\bar{s}) \ ,$$

respectively. By using the Equations (10), (14) and (15), the Equation $L_{\overline{C}}(\bar{s})$ is stated as follows more clearly,

$$L_{\overline{C}}(\bar{s}) = \sin\omega \oint\limits_{(\overrightarrow{Y})} ds + \sin\omega \oint\limits_{(\overrightarrow{R})} ds - \sin\omega \oint\limits_{(\overrightarrow{\overline{Y}})} z(s)\rho(s)ds. \tag{29}$$

On the other hand, from the Equations (4), (12) and (13), the dual pitch angles of the ruled surfaces formed by $\overrightarrow{C}(s)$ and $\overrightarrow{\overline{C}}(\bar{s})$ are obtained as follows:

$$\Lambda_C(s) = -\cos\omega \oint\limits_{(\overrightarrow{Y})} \rho(s)ds - \sin\omega \oint\limits_{(\overrightarrow{Y})} (\eta(s) + \varepsilon)ds,$$

$$\Lambda_{\overline{C}}(\bar{s}) = -\cos\overline{\omega} \oint_{\left(\overrightarrow{\overline{Y}}\right)} \overline{\rho}(\bar{s})d\bar{s} - \sin\overline{\omega} \oint_{\left(\overrightarrow{\overline{Y}}\right)} (\overline{\eta}(\bar{s}) + \varepsilon)d\bar{s}.$$

By using the Equations (8), (10), (14) and (15), if the Equation $\Lambda_{\overline{C}}(\bar{s})$ is stated more clearly,

$$\begin{aligned}\Lambda_{\overline{C}}(\bar{s}) = &-\cos\omega \oint_{\left(\overrightarrow{Y}\right)} \rho(s)ds - \sin\omega \oint_{\left(\overrightarrow{Y}\right)} \Re(s)ds - \cos\omega \oint_{\left(\overrightarrow{R}\right)} \rho(s)ds \\ &- \sin\omega \oint_{\left(\overrightarrow{R}\right)} \Re(s)ds + \varepsilon\sin\omega \oint_{\left(\overrightarrow{Y}\right)} z(s)\rho(s)ds\end{aligned} \tag{30}$$

is attained. Moreover, if the vectors $\overrightarrow{c}(s)$, $\overrightarrow{c^*}(s)$, $\overrightarrow{\overline{c}}(\bar{s})$, $\overrightarrow{\overline{c}^*}(\bar{s})$ in the Equation (12) are differentiated, we get

$$\begin{cases} \overrightarrow{dc}(s) = \overrightarrow{d\overline{c}}(\bar{s}) = \omega'(s)\cos\omega\overrightarrow{a_1}(s) - \omega'(s)\sin\omega\overrightarrow{a_3}(s), \\ \overrightarrow{dc^*}(s) = \omega'(s)\cos\omega\overrightarrow{a_1^*}(s) - \cos\omega\overrightarrow{a_2}(s) - \omega'(s)\sin\omega\overrightarrow{a_3^*}(s), \\ \overrightarrow{d\overline{c}^*}(\bar{s}) = \overrightarrow{dc^*}(s) \\ \qquad + [z'(s)\cos\omega - z(s)\omega'(s)\sin\omega - r(s)\rho(s)\sin\omega + \phi^*(s)\rho(s)\cos\omega]\overrightarrow{a_1}(s) \\ \qquad + [z(s)\rho(s)\cos\omega + r'(s)\sin\omega + r(s)\omega'(s)\cos\omega \\ \qquad + z(s)\eta(s)\sin\omega + \phi^*(s)\omega'(s)\sin\omega]\overrightarrow{a_2}(s) \\ \qquad + [r(s)\eta(s)\sin\omega - z'(s)\sin\omega - z(s)\omega'(s)\cos\omega - \phi^*(s)\eta(s)\cos\omega]\overrightarrow{a_3}(s). \end{cases} \tag{31}$$

The vectors of the Equation (31) are substituted in Equation (4); we have

$$\left\langle \overrightarrow{dc}(s), \overrightarrow{dc^*}(s) \right\rangle = -\omega'^2(s)\cos\omega\sin\omega\left(\langle\overrightarrow{a_1}(s), \overrightarrow{a_3^*}(s)\rangle + \langle\overrightarrow{a_3}(s), \overrightarrow{a_1^*}(s)\rangle\right) = 0,$$

$$\left\langle \overrightarrow{d\overline{c}}(\bar{s}), \overrightarrow{d\overline{c}^*}(\bar{s}) \right\rangle = \omega'(s)(z'(s) + (\rho(s)\cos\omega + \eta(s)\sin\omega)(\phi^*(s)\cos\omega - r(s)\sin\omega)).$$

And so, from the Equation (9), we get

$$\left\langle \overrightarrow{d\overline{c}}(\bar{s}), \overrightarrow{d\overline{c}^*}(\bar{s}) \right\rangle = \omega'(s)z'(s)\left(1 - \frac{\rho(s)\cos\omega + \eta(s)\sin\omega}{\sqrt{\rho^2(s) + \eta^2(s)}}\right) = 0.$$

Thus, the dralls of the ruled surfaces formed by the vectors $\overrightarrow{C}(s)$ and $\overrightarrow{\overline{C}}(\bar{s})$ are obtained as follows:

$$P_{\overline{C}}(\bar{s}) = P_C(s) = 0, \tag{32}$$

respectively. Thus, from the Equations (29), (30) and (32), the proof is completed. □

Corollary 4. *The closed ruled surfaces formed by the dual instantaneous Pfaff vectors* $\overrightarrow{C}(s)$ *and* $\overrightarrow{\overline{C}}(\bar{s})$ *of DPERS are always isomorphic to the plane.*

4. The Gaussian Curvatures of DPERS

In this section, we will compute the Gaussian curvatures of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed by the dual tangent, dual principal normal, dual binormal and dual instantaneous Pfaff vectors of the base curves of DPERS.

4.1. The Relationship between the Gaussian Curvatures of the Ruled Surfaces Formed by the Dual Tangent Vectors of DPERS

Theorem 6. *There is the following relationship between the Gaussian curvatures of the ruled surfaces formed by the dual tangent vectors* $\overrightarrow{A_1}(s)$ *and* $\overrightarrow{B_1}(\bar{s})$ *of DPERS:*

$$K_{B_1}(\bar{s}) = K_{A_1}(s) = 0.$$

Proof. If the ruled surface $\overrightarrow{\Psi_{A_1}}(s,v)$ in the Equation (16) is derived according to the parameters s and v,

$$\begin{cases} \left(\overrightarrow{\Psi_{A_1}}\right)_v = \overrightarrow{a_1}(s), \\ \left(\overrightarrow{\Psi_{A_1}}\right)_s = -\rho(s)\langle \overrightarrow{a_2}(s), \overrightarrow{\alpha}(s)\rangle \overrightarrow{a_1}(s) + \rho(s)\left(v - \langle \overrightarrow{a_1}(s), \overrightarrow{\alpha}(s)\rangle\right)\overrightarrow{a_2}(s) \end{cases} \tag{33}$$

are obtained. If the vectors $\left(\overrightarrow{\Psi_{A_1}}\right)_v$ and $\left(\overrightarrow{\Psi_{A_1}}\right)_s$ in Equation (33) are performed the inner product operation,

$$\left\langle \left(\overrightarrow{\Psi_{A_1}}\right)_v, \left(\overrightarrow{\Psi_{A_1}}\right)_s \right\rangle = -\rho(s)\langle \overrightarrow{a_2}(s), \overrightarrow{\alpha}(s)\rangle$$

is found. Since this inner product is non zero, the system $\left\{\left(\overrightarrow{\Psi_{A_1}}\right)_v, \left(\overrightarrow{\Psi_{A_1}}\right)_s\right\}$ is not an orthogonal system. So now, let us perform the Gram Schmitd 's method on these vectors. Firstly, the following vectors are obtained:

$$\overrightarrow{(X_{A_1})_1}(s) = \left(\overrightarrow{\Psi_{A_1}}\right)_v, \quad \overrightarrow{(X_{A_1})_2}(s) = \left(\overrightarrow{\Psi_{A_1}}\right)_s,$$

$$\overrightarrow{(Y_{A_1})_1}(s) = \overrightarrow{a_1}(s), \quad \overrightarrow{(Y_{A_1})_2}(s) = \rho(s)\left(v - \langle \overrightarrow{a_1}(s), \overrightarrow{\alpha}(s)\rangle\right)\overrightarrow{a_2}(s). \tag{34}$$

Thus, if the vectors $\overrightarrow{(Y_{A_1})_1}(s)$ and $\overrightarrow{(Y_{A_1})_2}(s)$ in Equation (34) are rendered orthonormal,

$$\overrightarrow{(E_{A_1})_1}(s) = \overrightarrow{a_1}(s), \quad \overrightarrow{(E_{A_1})_2}(s) = \overrightarrow{a_2}(s) \tag{35}$$

are attained. Then, if the vectors $\overrightarrow{(E_{A_1})_1}(s)$ and $\overrightarrow{(E_{A_1})_2}(s)$ in Equation (35) are performed the vectoral product operation, the normal vector $\overrightarrow{N_{A_1}}(s)$ of the ruled surface $\overrightarrow{\Psi_{A_1}}(s,v)$ is found as follows:

$$\overrightarrow{N_{A_1}}(s) = \overrightarrow{a_3}(s). \tag{36}$$

Let $S_{A_1}(s)$ and $K_{A_1}(s)$ be the shape operator and the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{A_1}}(s,v)$, respectively. From the Equation (6),

$$K_{A_1}(s) = -\left(\left\langle S_{A_1}\left(\overrightarrow{(E_{A_1})_2}(s)\right), \overrightarrow{(E_{A_1})_1}(s)\right\rangle\right)^2 \tag{37}$$

is written. Here, from Equations (6) and (36),

$$S_{A_1}\left(\overrightarrow{(E_{A_1})_2}(s)\right) = -\frac{\eta(s)}{\rho(s)\left(v - \langle \overrightarrow{a_1}(s), \overrightarrow{\alpha}(s)\rangle\right)}\overrightarrow{a_2}(s). \tag{38}$$

If the Equations (35) and (38) are substituted in the Equation (37), the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{A_1}}(s,v)$ is obtained as follows:

$$K_{A_1}(s) = 0. \tag{39}$$

Likewise, the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{B_1}}(\bar{s},\bar{v})$ is found as follows:

$$K_{B_1}(\bar{s}) = 0. \tag{40}$$

Thus, from the Equations (39) and (40), the proof is completed. □

Corollary 5. *The ruled surfaces formed by the dual tangent vectors $\overrightarrow{A_1}(s)$ and $\overrightarrow{B_1}(\bar{s})$ of DPERS are always isomorphic to the plane.*

4.2. The Relationship Between the Gaussian Curvatures of the Ruled Surfaces Formed by the Dual Principal Normal Vectors of DPERS

Theorem 7. *There is the following relationship between the Gaussian curvatures of the ruled surfaces formed by the dual principal normal vectors $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(\bar{s})$ of DPERS:*

$$\frac{1 - z(s)\rho(s)}{\sqrt{-K_{B_2}(\bar{s})}} = \frac{1}{\sqrt{-K_{A_2}(s)}} + z^2(s)\eta(s) - 2m(s)z(s)\eta(s),$$

here $m(s) = v - \langle \overrightarrow{a_2}(s), \overrightarrow{\alpha}(s) \rangle$ and $\eta(s) \neq 0$.

Proof. If the ruled surface $\overrightarrow{\Psi_{A_2}}(s,v)$ in the Equation (20) is derived according to the parameters s and v,

$$\begin{cases} \left(\overrightarrow{\Psi_{A_2}}\right)_v = \overrightarrow{a_2}(s), \\ \left(\overrightarrow{\Psi_{A_2}}\right)_s = (1 - m(s)\rho(s))\overrightarrow{a_1}(s) + \left(\rho(s)\langle \overrightarrow{a_1}(s), \overrightarrow{\alpha}(s) \rangle - \eta(s)\langle \overrightarrow{a_3}(s), \overrightarrow{\alpha}(s) \rangle\right)\overrightarrow{a_2}(s) \\ \qquad\qquad + m(s)\eta(s)\overrightarrow{a_3}(s) \end{cases} \tag{41}$$

are obtained, here $m(s) = v - \langle \overrightarrow{a_2}(s), \overrightarrow{\alpha}(s) \rangle$. If the vectors $\left(\overrightarrow{\Psi_{A_2}}\right)_v$ and $\left(\overrightarrow{\Psi_{A_2}}\right)_s$ in the Equation (41) are performed, the inner product operation,

$$\left\langle \left(\overrightarrow{\Psi_{A_2}}\right)_v, \left(\overrightarrow{\Psi_{A_2}}\right)_s \right\rangle = \rho(s)\langle \overrightarrow{a_1}(s), \overrightarrow{\alpha}(s) \rangle - \eta(s)\langle \overrightarrow{a_3}(s), \overrightarrow{\alpha}(s) \rangle$$

is found. Since this inner product is non zero, the system $\left\{ \left(\overrightarrow{\Psi_{A_2}}\right)_v, \left(\overrightarrow{\Psi_{A_2}}\right)_s \right\}$ is not an orthogonal system. So now, let us perform the Gram Schmitd 's method on these vectors. Firstly, the following vectors are obtained:

$$\begin{gathered} \overrightarrow{(X_{A_2})_1}(s) = \left(\overrightarrow{\Psi_{A_2}}\right)_v, \qquad \overrightarrow{(X_{A_2})_2}(s) = \left(\overrightarrow{\Psi_{A_2}}\right)_s, \\ \overrightarrow{(Y_{A_2})_1}(s) = \overrightarrow{a_2}(s), \qquad \overrightarrow{(Y_{A_2})_2}(s) = (1 - m(s)\rho(s))\overrightarrow{a_1}(s) + m(s)\eta(s)\overrightarrow{a_3}(s). \end{gathered} \tag{42}$$

Thus, if the vectors $\overrightarrow{(Y_{A_2})_1}(s)$ and $\overrightarrow{(Y_{A_2})_2}(s)$ in Equation (42) are rendered orthonormal,

$$\begin{cases} \overrightarrow{(E_{A_2})_1}(s) = \overrightarrow{a_2}(s), \\ \overrightarrow{(E_{A_2})_2}(s) = \dfrac{(1 - m(s)\rho(s))\overrightarrow{a_1}(s) + m(s)\eta(s)\overrightarrow{a_3}(s)}{\sqrt{(1 - m(s)\rho(s))^2 + m^2(s)\eta^2(s)}} \end{cases} \tag{43}$$

are attained. Then, if the vectors $\overrightarrow{(E_{A_2})_1}(s)$ and $\overrightarrow{(E_{A_2})_2}(s)$ in Equation (43) are performed, the vectoral product operation, the normal vector $\overrightarrow{N_{A_2}}(s)$ of the ruled surface $\overrightarrow{\Psi_{A_2}}(s,v)$, is found as follows:

$$\overrightarrow{N_{A_2}}(s) = \frac{m(s)\eta(s)\overrightarrow{a_1}(s) - (1 - m(s)\rho(s))\overrightarrow{a_3}(s)}{\sqrt{(1 - m(s)\rho(s))^2 + m^2(s)\eta^2(s)}}. \tag{44}$$

Let $S_{A_2}(s)$ and $K_{A_2}(s)$ be the shape operator and the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{A_2}}(s,v)$, respectively. From the Equation (6),

$$K_{A_2}(s) = -\left(\left\langle S_{A_2}\left(\overrightarrow{(E_{A_2})_2}(s)\right), \overrightarrow{(E_{A_2})_1}(s)\right\rangle\right)^2 \tag{45}$$

is written. From Equations (6) and (44),

$$\begin{aligned} S_{A_2}\left(\overrightarrow{(E_{A_2})_2}(s)\right) &= \frac{d}{ds}\left(\frac{m(s)\eta(s)}{b(s)}\right)\frac{1}{b(s)}\overrightarrow{a_1}(s) + \frac{\eta(s)}{b^2(s)}\overrightarrow{a_2}(s) \\ &\quad -\frac{d}{ds}\left(\frac{1 - m(s)\rho(s)}{b(s)}\right)\frac{1}{b(s)}\overrightarrow{a_3}(s), \end{aligned} \tag{46}$$

where $b(s) = \left\|\overrightarrow{(Y_{A_2})_2}(s)\right\| = \sqrt{(1 - m(s)\rho(s))^2 + m^2(s)\eta^2(s)}$. If the Equations (43) and (46) are substituted in the Equation (45), the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{A_2}}(s,v)$ is obtained as follows:

$$K_{A_2}(s) = -\left(\frac{\eta(s)}{\left[1 - \rho(s)(v - \langle\overrightarrow{a_2}(s), \overrightarrow{\alpha}(s)\rangle)\right]^2 + \left[\eta(s)(v - \langle\overrightarrow{a_2}(s), \overrightarrow{\alpha}(s)\rangle)\right]^2}\right)^2. \tag{47}$$

Likewise, the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{B_2}}(\bar{s},\bar{v})$ is found as follows:

$$K_{B_2}(\bar{s}) = -\left(\frac{\bar{\eta}(\bar{s})}{\left[1 - \bar{\rho}(\bar{s})\left(\bar{v} - \left\langle\overrightarrow{b_2}(\bar{s}), \overrightarrow{\beta}(\bar{s})\right\rangle\right)\right]^2 + \left[\bar{\eta}(\bar{s})\left(\bar{v} - \left\langle\overrightarrow{b_2}(\bar{s}), \overrightarrow{\beta}(\bar{s})\right\rangle\right)\right]^2}\right)^2, \tag{48}$$

where $\overrightarrow{\beta}(\bar{s}) = \overrightarrow{\alpha}(s) + \phi^*(s)\overrightarrow{a_1}(s) + z(s)\overrightarrow{a_2}(s) + r(s)\overrightarrow{a_3}(s)$ [41]. If we take $\bar{v} = v$ and the Equations (10) and (8) are substituted in Equation (48), we obtain

$$K_{B_2}(\bar{s}) = -\left(\frac{\eta(s) - z(s)\rho(s)\eta(s)}{\left((1 - m(s)\rho(s))^2 + m^2(s)\eta^2(s) + z^2(s)\eta^2(s) - 2m(s)z(s)\eta^2(s)\right)}\right)^2. \tag{49}$$

Since $\eta(s) \neq 0$, from the Equations (47) and (49), we can write

$$\frac{1}{\sqrt{-K_{B_2}(\bar{s})}} = \frac{1}{(1 - z(s)\rho(s))\sqrt{-K_{A_2}(s)}} + z^2(s)\eta(s) - \frac{2m(s)z(s)\eta(s)}{1 - z(s)\rho(s)}. \tag{50}$$

Thus, the proof is completed. □

Corollary 6. *The ruled surfaces formed by the dual principal normal vectors $\overrightarrow{A_2}(s)$ and $\overrightarrow{B_2}(s)$ of DPERS are isomorphic to the plane if and only if $\eta(s) = 0$.*

4.3. The Relationship between the Gaussian Curvatures of the Ruled Surfaces Formed by the Dual Binormal Vectors of DPERS

Theorem 8. *There is the following relationship between the dual Gaussian curvatures of the ruled surfaces formed by the dual unit binormal vectors* $\overrightarrow{A_3}(s)$ *and* $\overrightarrow{B_3}(\bar{s})$ *of DPERS:*

$$\frac{1-z(s)\rho(s)}{\sqrt{-K_{B_3}(\bar{s})}}=\frac{1}{\sqrt{-K_{A_3}(s)}}+\frac{[2n(s)r(s)+r^2(s)]\eta^2(s)-2z(s)\rho(s)+z^2(s)\rho^2(s)}{\eta(s)},$$

where $n(s)=\langle\overrightarrow{a_3}(s),\overrightarrow{\alpha}(s)\rangle-v$ *and* $\eta(s)\neq 0$.

Proof. If the ruled surface $\overrightarrow{\Psi_{A_3}}(s,v)$ in the Equation (24) is derived according to the parameters s and v,

$$\begin{cases}\left(\overrightarrow{\Psi_{A_3}}\right)_v=\overrightarrow{a_3}(s),\\ \left(\overrightarrow{\Psi_{A_3}}\right)_s=\overrightarrow{a_1}(s)+\eta(s)n(s)\overrightarrow{a_2}(s)+\eta(s)\langle\overrightarrow{a_2}(s),\overrightarrow{\alpha}(s)\rangle\overrightarrow{a_3}(s)\end{cases}\tag{51}$$

are obtained. If the vectors $\left(\overrightarrow{\Psi_{A_3}}\right)_v$ and $\left(\overrightarrow{\Psi_{A_3}}\right)_s$ in the Equation (51) are performed, the inner product operation,

$$\left\langle\left(\overrightarrow{\Psi_{A_3}}\right)_v,\left(\overrightarrow{\Psi_{A_3}}\right)_s\right\rangle=\eta(s)\langle\overrightarrow{a_2}(s),\overrightarrow{\alpha}(s)\rangle$$

is obtained. Since this inner product is non zero, the system $\left\{\left(\overrightarrow{\Psi_{A_3}}\right)_v,\left(\overrightarrow{\Psi_{A_3}}\right)_s\right\}$ is not an orthogonal system. So now, let us perform the Gram Schmitd 's method on these vectors. Firstly, the following vectors are obtained:

$$\overrightarrow{(X_{A_3})_1}(s)=\left(\overrightarrow{\Psi_{A_3}}\right)_v,\qquad\overrightarrow{(X_{A_3})_2}(s)=\left(\overrightarrow{\Psi_{A_3}}\right)_s,$$

$$\overrightarrow{(Y_{A_3})_1}(s)=\overrightarrow{a_3}(s),\qquad\overrightarrow{(Y_{A_3})_2}(s)=\overrightarrow{a_1}(s)+\eta(s)n(s)\overrightarrow{a_2}(s).\tag{52}$$

Thus, if the vectors $\overrightarrow{(Y_{A_3})_1}(s)$ and $\overrightarrow{(Y_{A_3})_2}(s)$ in the Equation (52) are rendered orthonormal,

$$\begin{cases}\overrightarrow{(E_{A_3})_1}(s)=\overrightarrow{a_3}(s),\\ \overrightarrow{(E_{A_3})_2}(s)=\dfrac{\overrightarrow{a_1}(s)+\eta(s)n(s)\overrightarrow{a_2}(s)}{\sqrt{1+\eta^2(s)n^2(s)}}\end{cases}\tag{53}$$

are attained. Then, if the vectors $\overrightarrow{(E_{A_3})_1}(s)$ and $\overrightarrow{(E_{A_3})_2}(s)$ in the Equation (53) are performed, the vectorial product operation, the normal vector $\overrightarrow{N_{A_3}}(s)$ of the ruled surface $\overrightarrow{\Psi_{A_3}}(s,v)$, is found as follows:

$$\overrightarrow{N_{A_3}}(s)=\frac{-\eta(s)n(s)\overrightarrow{a_1}(s)+\overrightarrow{a_2}(s)}{\sqrt{1+\eta^2(s)n^2(s)}}.\tag{54}$$

Let $S_{A_3}(s)$ and $K_{A_3}(s)$ be the shape operator and the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{A_3}}(s,v)$, respectively. From the Equation (6),

$$K_{A_3}(s)=-\left(\left\langle S_{A_3}\left(\overrightarrow{(E_{A_3})_2}(s)\right),\overrightarrow{(E_{A_3})_1}(s)\right\rangle\right)^2\tag{55}$$

is written. From Equations (6) and (54), we get

$$\begin{aligned} S_{A_3}\left(\overrightarrow{(E_{A_3})_2}(s)\right) = &-\left[\frac{d}{ds}\left(\frac{\eta(s)n(s)}{e(s)}\right)+\frac{\rho(s)}{e(s)}\right]\frac{1}{e(s)}\overrightarrow{a_1}(s) \\ &+\left[\frac{d}{ds}\left(\frac{1}{e(s)}\right)-\frac{\rho(s)\eta(s)(\langle\overrightarrow{a_3}(s),\overrightarrow{\alpha}(s)\rangle-v)}{e(s)}\right]\frac{1}{e(s)}\overrightarrow{a_2}(s) \\ &+\frac{\eta(s)}{e^2(s)}\overrightarrow{a_3}(s), \end{aligned} \tag{56}$$

where $e(s)=\left\|\overrightarrow{(Y_{A_3})_2}(s)\right\|=\sqrt{1+\eta^2(s)(\langle\overrightarrow{a_3}(s),\overrightarrow{\alpha}(s)\rangle-v)^2}$. If the Equations (53) and (56) are substituted in the Equation (55), the Gauss curvature of the ruled surface $\overrightarrow{\Psi_{A_3}}(s,v)$ is obtained as follows:

$$K_{A_3}(s)=-\left(\frac{\eta(s)}{1+\eta^2(s)(\langle\overrightarrow{a_3}(s),\overrightarrow{\alpha}(s)\rangle-v)^2}\right)^2. \tag{57}$$

Likewise, the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{B_3}}(\bar{s},\bar{v})$ is found as follows:

$$K_{B_3}(\bar{s})=-\left(\frac{\overline{\eta}(\bar{s})}{1+\overline{\eta}^2(\bar{s})\left(\left\langle\overrightarrow{b_3}(\bar{s}),\overrightarrow{\beta}(\bar{s})\right\rangle-\overline{v}\right)^2}\right)^2. \tag{58}$$

If we take $\overline{v}=v$ and the Equations (7) and (8) are substituted in the Equation (58),

$$K_{B_3}(\bar{s})=-\left(\frac{\eta(s)-z(s)\rho(s)\eta(s)}{1+\eta^2(s)n^2(s)+(2r(s)n(s)+r^2(s))\eta^2(s)-2z(s)\rho(s)+z^2(s)\rho^2(s)}\right)^2 \tag{59}$$

is obtained. Since $\eta(s)\neq 0$, from the Equations (57) and (59), we can write

$$\frac{1}{\sqrt{-K_{B_3}(\bar{s})}}=\frac{1}{(1-z(s)\rho(s))\sqrt{-K_{A_3}(s)}}+\frac{[2n(s)r(s)+r^2(s)]\eta^2(s)-2z(s)\rho(s)+z^2(s)\rho^2(s)}{\eta(s)-z(s)\rho(s)\eta(s)}.$$

Thus, the the proof is completed. □

Corollary 7. *The ruled surfaces formed by the dual binormal vectors $\overrightarrow{A_3}(s)$ and $\overrightarrow{B_3}(\bar{s})$ of DPERS are isomorphic to the plane if and only if $\eta(s)=0$.*

4.4. The Relationship between the Gaussian Curvatures of the Ruled Surfaces Formed by the Dual Instantaneous Pfaff Vectors of DPERS

Theorem 9. *There is the following relationship between the dual Gaussian curvatures of the ruled surfaces formed by the dual instantaneous Pfaff vectors $\overrightarrow{C}(s)$ and $\overrightarrow{\overline{C}}(\bar{s})$ of DPERS:*

$$K_{\overline{C}}(\bar{s})=K_C(s)=0.$$

Proof. If the ruled surface $\overrightarrow{\Psi_C}(s,v)$ in the Equation (28) is derived according to the parameters s and v,

$$\begin{cases} \left(\overrightarrow{\Psi_C}\right)_v = \overrightarrow{c}(s) = \sin\omega\,\overrightarrow{a_1}(s) + \cos\omega\,\overrightarrow{a_3}(s), \\ \left(\overrightarrow{\Psi_C}\right)_s = \left[v\omega'(s)\cos\omega + \cos^2\omega - 2\omega'(s)\cos\omega\sin\omega\left\langle\overrightarrow{a_1}(s),\,\overrightarrow{\alpha}(s)\right\rangle\right. \\ \qquad\qquad - \omega'(s)(\cos^2\omega - \sin^2\omega)\left\langle\overrightarrow{a_3}(s),\,\overrightarrow{\alpha}(s)\right\rangle\Big]\overrightarrow{a_1}(s) \\ \qquad\qquad - \left[v\omega'(s)\sin\omega + \cos\omega\sin\omega - 2\omega'(s)\cos\omega\sin\omega\left\langle\overrightarrow{a_3}(s),\,\overrightarrow{\alpha}(s)\right\rangle\right. \\ \qquad\qquad + (\omega'(s)\cos^2\omega - \sin^2\omega)\left\langle\overrightarrow{a_1}(s),\,\overrightarrow{\alpha}(s)\right\rangle\Big]\overrightarrow{a_3}(s) \end{cases} \tag{60}$$

are obtained. If the vectors $\left(\overrightarrow{\Psi_C}\right)_v$ and $\left(\overrightarrow{\Psi_C}\right)_s$ in the Equation (60) are performed, the inner product operation,

$$\left\langle\left(\overrightarrow{\Psi_C}\right)_v, \left(\overrightarrow{\Psi_C}\right)_s\right\rangle = \omega'(s)\left(\sin\omega\left\langle\overrightarrow{a_1^*}(s), \overrightarrow{a_2}(s)\right\rangle + \cos\left\langle\overrightarrow{a_2}(s), \overrightarrow{a_3^*}(s)\right\rangle\right),$$

is found. Since this inner product is non zero, the system $\left\{\left(\overrightarrow{\Psi_C}\right)_v, \left(\overrightarrow{\Psi_C}\right)_s\right\}$ is not an orthogonal system. So now, let us perform the Gram Schmitd 's method on these vectors. Firstly, the following vectors are obtained:

$$\overrightarrow{(X_C)_1}(s) = \left(\overrightarrow{\Psi_C}\right)_v, \qquad \overrightarrow{(X_C)_2}(s) = \left(\overrightarrow{\Psi_C}\right)_s,$$

$$\begin{cases} \overrightarrow{(Y_C)_1}(s) = \sin\omega\,\overrightarrow{a_1}(s) + \cos\omega\,\overrightarrow{a_3}(s), \\ \overrightarrow{(Y_C)_2}(s) = \left[v\omega'(s)\cos\omega + \cos^2\omega - \omega'(s)\cos\omega\sin\omega\left\langle\overrightarrow{a_1}(s),\,\overrightarrow{\alpha}(s)\right\rangle\right. \\ \qquad\qquad - \omega'(s)\cos^2\omega\left\langle\overrightarrow{a_3}(s),\,\overrightarrow{\alpha}(s)\right\rangle\Big]\overrightarrow{a_1}(s) \\ \qquad\qquad - \left[v\omega'(s)\sin\omega + \cos\omega\sin\omega - \omega'(s)\cos\omega\sin\omega\left\langle\overrightarrow{a_3}(s),\,\overrightarrow{\alpha}(s)\right\rangle\right. \\ \qquad\qquad - \omega'(s)\sin^2\omega\left\langle\overrightarrow{a_1}(s),\,\overrightarrow{\alpha}(s)\right\rangle\Big]\overrightarrow{a_3}(s). \end{cases} \tag{61}$$

Thus, if the vectors $\overrightarrow{(Y_C)_1}(s)$ and $\overrightarrow{(Y_C)_2}(s)$ in the Equation (61) are rendered orthonormal,

$$\overrightarrow{(E_C)_1}(s) = \sin\omega\,\overrightarrow{a_1}(s) + \cos\omega\,\overrightarrow{a_3}(s), \qquad \overrightarrow{(E_C)_2}(s) = \frac{\overrightarrow{(Y_C)_2}(s)}{\left\|\overrightarrow{(Y_C)_2}(s)\right\|} \tag{62}$$

are obtained. Here, after a series of long operations, we have

$$\left\|\overrightarrow{(Y_C)_2}(s)\right\| = \left|\left(\cos\omega\left\langle\overrightarrow{a_3}(s),\,\overrightarrow{\alpha}(s)\right\rangle + \sin\omega\left\langle\overrightarrow{a_1}(s),\,\overrightarrow{\alpha}(s)\right\rangle - v\right)\frac{d\omega}{ds} - \cos\omega\right|. \tag{63}$$

Then, if the vectors $\overrightarrow{(E_C)_1}(s)$ and $\overrightarrow{(E_C)_2}(s)$ in the Equation (62) are performed, the vectoral product operation, the normal vector $\overrightarrow{N_C}(s)$ of the ruled surface $\overrightarrow{\Psi_C}(s,v)$, is found as follows:

$$\overrightarrow{N_C}(s) = \pm\overrightarrow{a_2}(s). \tag{64}$$

Let $S_C(s)$ and $K_C(s)$ be the shape operator and the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_C}(s,v)$, respectively. From the Equation (6),

$$K_C(s) = -\left(\left\langle S_C\left(\overrightarrow{(E_C)_2}(s)\right), \overrightarrow{(E_C)_1}(s)\right\rangle\right)^2 \tag{65}$$

is written. Here, from the Equations (6) and (64),

$$S_C\left(\overrightarrow{(E_C)_2}(s)\right) = \frac{\mp\rho(s)\overrightarrow{a_1}(s) \pm \eta(s)\overrightarrow{a_3}(s)}{\left|\omega'(s)\left(\cos\omega\langle\overrightarrow{a_3}(s), \overrightarrow{\alpha}(s)\rangle + \sin\omega\langle\overrightarrow{a_1}(s), \overrightarrow{\alpha}(s)\rangle - v\right) - \cos\omega\right|}. \tag{66}$$

If the Equations (62) and (66) are substituted in the Equation (65), the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_C}(s,v)$ is obtained as follows:

$$K_C(s) = \frac{\mp\rho(s)\sin\omega \pm \eta(s)\cos\omega}{\left|\omega'(s)\left(\cos\omega\langle\overrightarrow{a_3}(s), \overrightarrow{\alpha}(s)\rangle + \sin\omega\langle\overrightarrow{a_1}(s), \overrightarrow{\alpha}(s)\rangle - v\right) - \cos\omega\right|} = 0. \tag{67}$$

Likewise, the Gaussian curvature of the ruled surface $\overrightarrow{\Psi_{\overline{C}}}(\overline{s},\overline{v})$ is found as follows:

$$K_{\overline{C}}(\overline{s}) = 0. \tag{68}$$

Thus, the proof is completed. □

Corollary 8. *The ruled surfaces formed by the dual instantaneous Pfaff vectors* $\overrightarrow{C}(s)$ *and* $\overrightarrow{\overline{C}}(\overline{s})$ *of DPERS are always isomorphic to the plane.*

Example 1. *Let* $\overrightarrow{\Psi}(s,v)$ *and* $\overrightarrow{\overline{\Psi}}(\overline{s},v)$ *be DPERS: (*$v = \overline{v}$*)*

$$\begin{cases} \overrightarrow{\Psi}(s,v) = \left(\frac{s-2v}{2\sqrt{2}}\cos\frac{s}{\sqrt{2}} - \sin\frac{s}{\sqrt{2}},\ \cos\frac{s}{\sqrt{2}} + \frac{s-2v}{2\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{s+2v}{2\sqrt{2}}\right), \\ \overrightarrow{\overline{\Psi}}(\overline{s},v) = \left(\left(\frac{s-2v}{2\sqrt{2}} + \frac{\pi}{4}\right)\cos\frac{s}{\sqrt{2}} - \sin\frac{s}{\sqrt{2}},\ \cos\frac{s}{\sqrt{2}} + \left(\frac{s-2v}{2\sqrt{2}} + \frac{\pi}{4}\right)\sin\frac{s}{\sqrt{2}},\right. \\ \qquad\qquad \left.\frac{s+2v}{2\sqrt{2}} + \frac{\pi}{4}\right). \end{cases}$$

(**i**) *The parametric expressions of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed on the unit dual sphere by the dual tangent vectors of these DPERS are (Figure 3)*

$$\begin{cases} \overrightarrow{\Psi}(s,v) = \left(\frac{s-2v}{2\sqrt{2}}\cos\frac{s}{\sqrt{2}} - \sin\frac{s}{\sqrt{2}},\ \cos\frac{s}{\sqrt{2}} + \frac{s-2v}{2\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{s+2v}{2\sqrt{2}}\right), \\ \overrightarrow{\overline{\Psi}}(\overline{s},v) = \left(\left(\frac{s-2v}{2\sqrt{2}} + \frac{\pi}{4}\right)\cos\frac{s}{\sqrt{2}} - \sin\frac{s}{\sqrt{2}},\ \cos\frac{s}{\sqrt{2}} + \left(\frac{s-2v}{2\sqrt{2}} + \frac{\pi}{4}\right)\sin\frac{s}{\sqrt{2}},\right. \\ \qquad\qquad \left.\frac{s+2v}{2\sqrt{2}} + \frac{\pi}{4}\right). \end{cases}$$

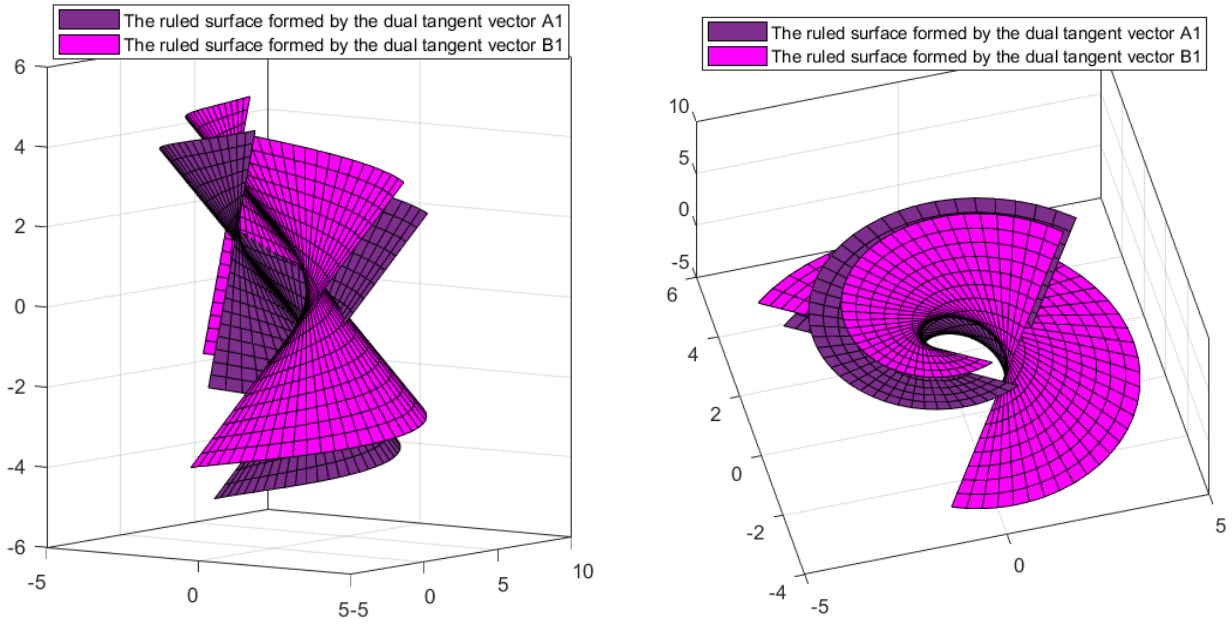

Figure 3. The ruled surfaces $\overrightarrow{\Psi_{A_1}}(s,v)$ and $\overrightarrow{\overline{\Psi}_{B_1}}(\bar{s},v)$ formed by the dual tangent vectors $\overrightarrow{A_1}$ and $\overrightarrow{B_1}$.

(**ii**) *The parametric expressions of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed on the unit dual sphere by the dual principal normal vectors of these DPERS are (Figure 4)*

$$\begin{cases} \overrightarrow{\Psi_{A_2}}(s,v) = \left(v\sin\dfrac{s}{\sqrt{2}}, -v\cos\dfrac{s}{\sqrt{2}}, \dfrac{s}{\sqrt{2}}\right), \\ \overrightarrow{\overline{\Psi}_{B_2}}(\bar{s},v) = \left(v\sin\dfrac{s}{\sqrt{2}}, -v\cos\dfrac{s}{\sqrt{2}}, \dfrac{s}{\sqrt{2}} + \dfrac{\pi}{2}\right). \end{cases}$$

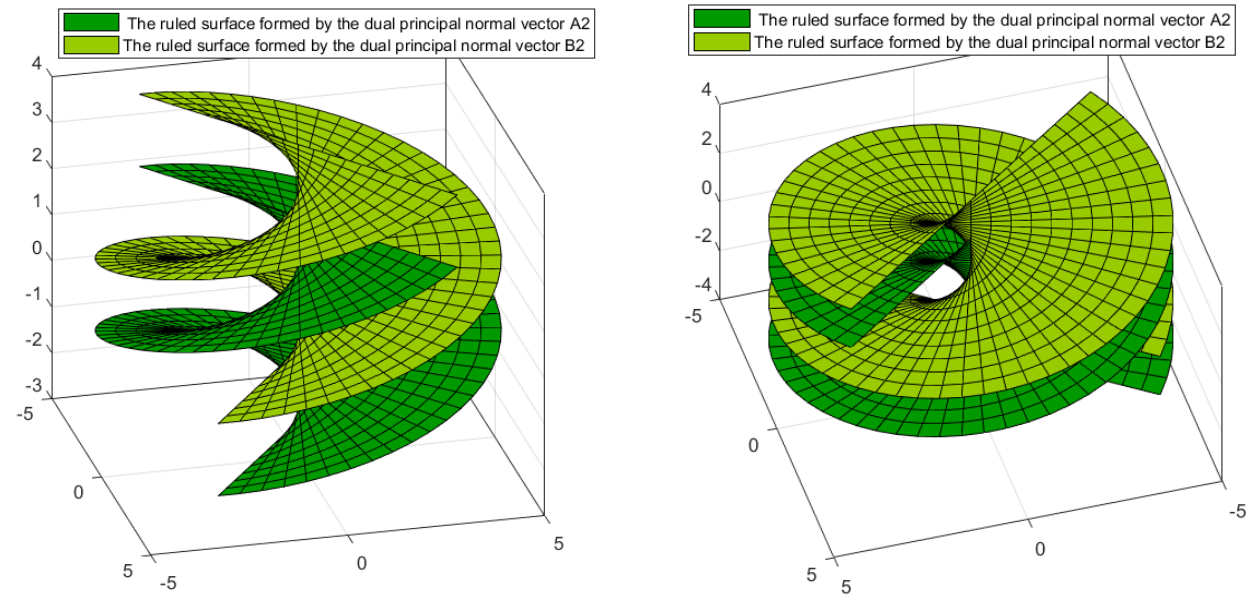

Figure 4. The ruled surfaces $\overrightarrow{\Psi_{A_2}}(s,v)$ and $\overrightarrow{\overline{\Psi}_{B_2}}(\bar{s},v)$ formed by the dual principal normal vectors $\overrightarrow{A_2}$ and $\overrightarrow{B_2}$.

(**iii**) *The parametric expressions of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed on the unit dual sphere by the dual binormal vectors of these DPERS are (Figure 5)*

$$\begin{cases} \overrightarrow{\Psi_{A_3}}(s,v) = \left(-\dfrac{s-2v}{2\sqrt{2}}\cos\dfrac{s}{\sqrt{2}} - \sin\dfrac{s}{\sqrt{2}},\ \cos\dfrac{s}{\sqrt{2}} - \dfrac{s-2v}{2\sqrt{2}}\sin\dfrac{s}{\sqrt{2}},\ \dfrac{s+2v}{2\sqrt{2}}\right), \\ \overrightarrow{\overline{\Psi}_{B_3}}(\bar{s},v) = \left(-\left(\dfrac{s-2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right)\cos\dfrac{s}{\sqrt{2}} - \sin\dfrac{s}{\sqrt{2}},\ \cos\dfrac{s}{\sqrt{2}} - \left(\dfrac{s-2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right)\sin\dfrac{s}{\sqrt{2}},\right. \\ \qquad\qquad \left.\dfrac{s+2v}{2\sqrt{2}}+\dfrac{\pi}{4}\right). \end{cases}$$

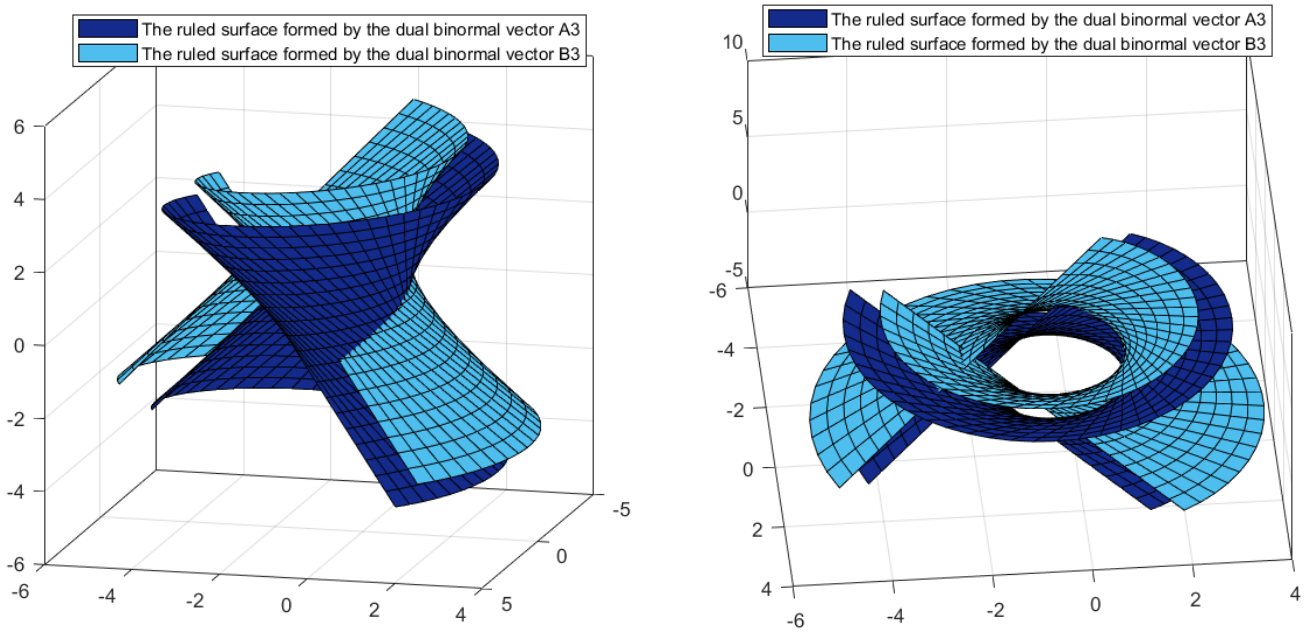

Figure 5. The ruled surfaces $\overrightarrow{\Psi_{A_3}}(s,v)$ and $\overrightarrow{\overline{\Psi}_{B_3}}(\bar{s},v)$ formed by the dual binormal vectors $\overrightarrow{A_3}$ and $\overrightarrow{B_3}$.

(**iv**) *The parametric expressions of the ruled surfaces corresponding in the space of lines to the dual indicatrix curves formed on the unit dual sphere by the dual instantaneous Pfaff vectors of these DPERS are (Figure 6)*

$$\overrightarrow{\Psi_C}(s,v) = \overrightarrow{\overline{\Psi}_C}(\bar{s},v) = \left(-\sin\frac{s}{\sqrt{2}},\ \cos\frac{s}{\sqrt{2}}, v\right).$$

For the all graphs drawn in the above example, the parameter v is taken in the range $-5:1/2:5$, and the parameter t is in the range $-\pi:\pi/20:\pi$.

The dual spherical indicatrix curves corresponding on the unit dual sphere to these ruled surfaces can be shown imaginatively as in Figure 7.

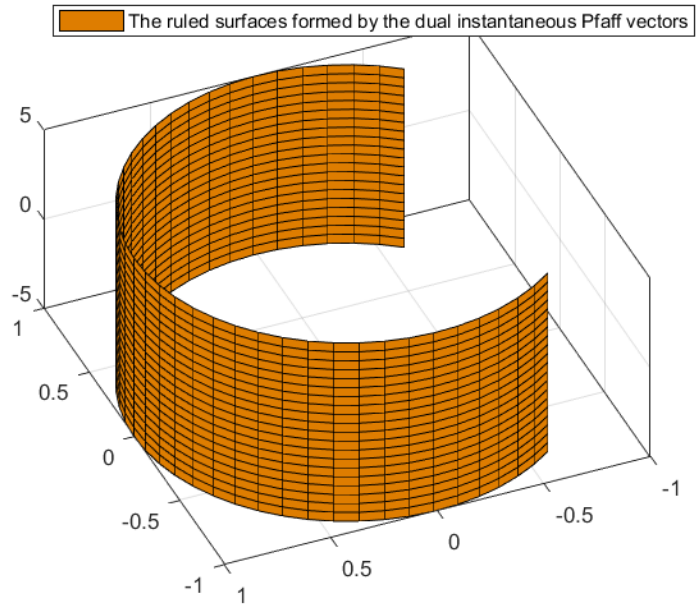

Figure 6. The ruled surfaces $\overrightarrow{\Psi_C}(s,v)$ and $\overrightarrow{\overline{\Psi}_{\overline{C}}}(\bar{s},v)$ formed by the dual instantaneous Pfaff vectors $\overrightarrow{C}$ and $\overrightarrow{\overline{C}}$.

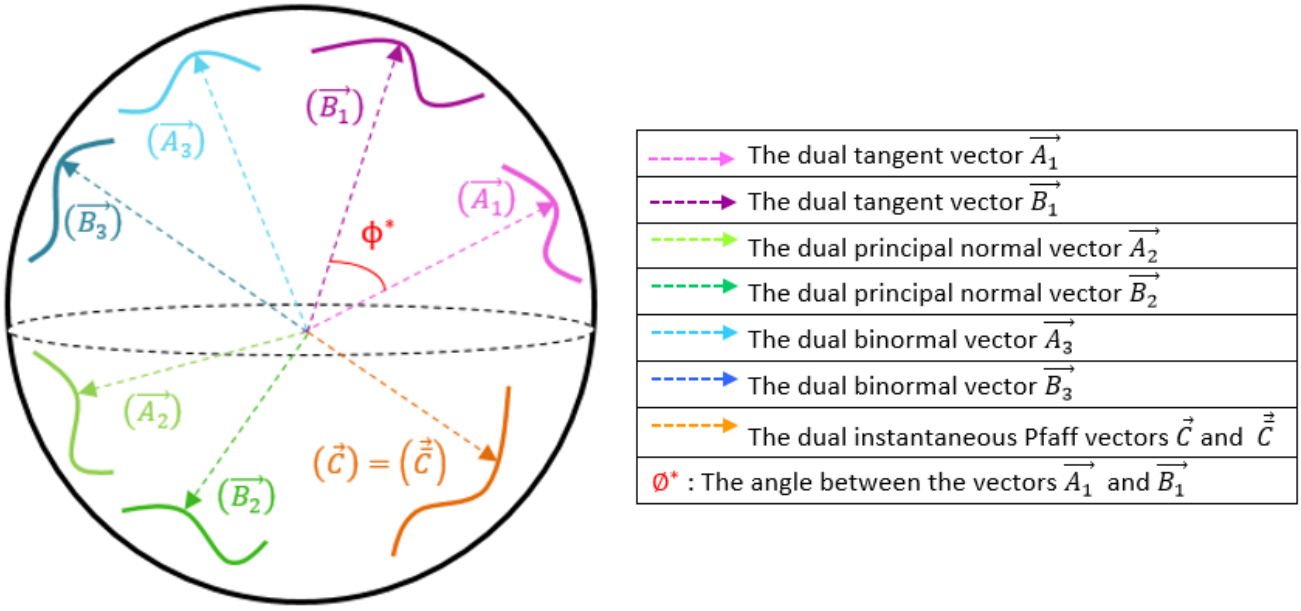

Figure 7. The dual spherical indicatrix curves corresponding to the unit dual of the ruled surfaces sphere formed by the tangent, principal normal, binormal and instantaneous Pfaff vectors of DPERS (imaginary figure).

The ruled surfaces in the above example are not closed, but it is possible to find examples of closed ruled surfaces.

5. Discussion and Conclusions

In this study, the integral invariants and Gaussian curvatures of the ruled surfaces corresponding in the space of lines to the spherical indicator curves of the Blaschke vectors and the dual instantaneous Pfaff vectors on the unit dual sphere of the parallel equidistant ruled surfaces (DPERS), which were previously defined in [41] and some of their properties

given, are investigated. As a result of this examination, considering the dralls (distribution parameters), it is concluded that the closed ruled surfaces formed by dual tangent and dual instantaneous Pfaff vectors of DPERS will always be isomorphic to plane (developable), the closed ruled surfaces formed by dual principal normal vectors of DPERS will always be isomorphic to plane under a certain condition ($\eta(s) = 0$), and the closed ruled surfaces formed by dual binormal vectors of DPERS will never be isomorphic to plane. In addition, when Gaussian curvatures are examined, it is concluded that the ruled surfaces formed by dual tangent and instantaneous Pfaff vectors of DPERS will always be isomorphic to plane, while the closed ruled surfaces obtained by dual principal normal and dual binormal vectors will be isomorphic to plane under a certain condition ($\eta(s) = 0$). This study can be combined with studies on submanifold theories, singularity theories, kinematics theories, etc. or it can be studied in non-Euclidean spaces also.

Author Contributions: Conceptualization, S.G.M., S.Ş. and L.G.; methodology, S.G.M., S.Ş. and L.G.; validation, S.G.M., S.Ş. and L.G.; formal analysis, S.G.M., S.Ş. and L.G.; investigation, S.G.M.; resources, S.G.M., S.Ş. and L.G.; writing—original draft preparation, S.G.M.; writing—review and editing, S.G.M., S.Ş. and L.G.; visualization, S.G.M., S.Ş. and L.G.; supervision, S.Ş. and L.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We, the authors, gratefully and most sincerely thank the reviewers, who criticized and improved the quality of our article, for their generous comments, corrections and contributions, as well as the editor of the journal who took care of our article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hacisalioglu, H.H. *Differential Geometry-II*; Ankara University, Faculty of Science Publications: Ankara, Turkey, 1994; pp. 203–286.
2. Ozdemir, M. *Quaternions and Geometry*; Altin Nokta Press: İzmir, Turkey, 2020; pp. 150–182.
3. Sabuncuoglu, A. *Differential Geometry*; Nobel Press: Ankara, Turkey, 2006.
4. Fenchel, W. On the Differential Geometry of Closed Space Curves. *Bull. Am. Math. Soc.* **1951**, *57*, 44–54. [CrossRef]
5. Senatalar, M. *Differential Geometry (Curves and Surfaces Theory)*; Istanbul State Engineering and Architecture Academy Publications: Istanbul, Turkey, 1978; pp. 153–170.
6. Hagemann, M.; Klawitter, D.; Lordick, D. Force Driven Ruled Surfaces. *J. Geom. Graph.* **2013**, *17*, 193–204.
7. Ali, A.T.; Abdel Aziz, H.S.; Sorour, A.H. Ruled surfaces generated by some special curves in Euclidean 3-Space. *J. Egypt. Math. Soc.* **2013**, *21*, 285–294. [CrossRef]
8. Schaaf, A.; Ravani, B. Geometric Continuity of Ruled Surfaces. *Comput. Aided Geom. Des.* **1998**, *15*, 289–310. [CrossRef]
9. Hacisalihoglu, H.H. On the pitch of a closed ruled surfaces. *Mech. Mach. Theory* **1972**, *7*, 291–305. [CrossRef]
10. Senyurt, S.; Gur Mazlum, S.; Grilli, G. Gaussian curvatures of parallel ruled surfaces. *Appl. Math. Sci.* **2020**, *14*, 171–183. [CrossRef]
11. Valeontis, I. Parallel P-Äquidistante Regelflachen Manuscripta. *Mathematics* **1986**, *54*, 391–404.
12. Bektaş, M. On characterizations of general helices for ruled surfaces in the pseudo-Galilean space G_3^1-(Part-I). *J. Math. Kyoto Univ.* **2004**, *44*, 523–528.
13. Masal, M.; Kuruoglu, N. Some Characteristic Properties of the Parallel P-Equidistant Ruled Surfaces in The Euclidean Space. *Pure Appl. Math. Sci.* **1999**, *50*, 35–42.
14. Masal, M.; Kuruoglu, N. Some Characteristic Properties of the Shape Operators of Parallel p-Equidistant Ruled Surfaces. *Bull. Pure Appl. Sci.* **2000**, *19*, 361–364.
15. Masal, M.; Kuruoglu, N. Spacelike parallel pi-equidistant ruled surfaces in the Minkowski 3-space ${R_1}^3$. *Algebr. Groups Geom.* **2005**, *22*, 13–24.
16. Li, Y.; Şenyurt, S.; Özduran, A.; Canlı, D. The Characterizations of Parallel q-Equidistant Ruled Surfaces. *Symmetry* **2022**, *14*, 1879. [CrossRef]
17. As, E.; Senyurt, S. Some Characteristic Properties of Parallel z-Equidistant Ruled Surfaces. *Hindawi Publ. Corp. Math. Probl. Eng.* **2013**, *2013*, 587289.
18. Sarioglugil, A.; Senyurt, S.; Kuruoglu, N. On the Integral Invariants of the Closed Ruled Surfaces Generated by a Parallel p-Equidistant Dual Centroit Curve in the Line Space. *Hadron. J.* **2011**, *34*, 34–47.

19. Senyurt, S. Integral Invariants of Parallel P-Equidistant Ruled Surfaces Which Are Generated by Instantaneous Pfaff Vector. *Ordu Univ. Sci. Tech. J.* **2012**, *2*, 13–22.
20. Clifford, W.K. Preliminary sketch of biquaternions. *Proc. Lond. Math. Soc.* **1873**, *1*, 381–395. [CrossRef]
21. Study, E. *Geometrie der Dynamen*; Verlag Teubner: Leipzig, Germany, 1903.
22. Blaschke, W. *Differential Geometry Courses*; Istanbul University Publications: Istanbul, Turkey, 1949; pp. 332–345.
23. Hacisalihoglu, H.H. *The Motion Geometry and Quaternions Theory*; Gazi University, Faculty of Science and Literature Publications: Ankara, Turkey, 1983; pp. 3–55.
24. Hacisalihoglu, H.H. Acceleration Axes in Spatian Kinematics I. *Commun. Fac. Sci. Univ. Ank. Ser. Math. Stat.* **1971**, *20*, 1–15.
25. Muller, H.R. *Kinematics Courses*; Ankara University Press: Ankara, Turkey, 1963; pp. 240–251.
26. Bilici, M. On the Invariants of Ruled Surfaces Generated by the Dual Involute Frenet Trihedron. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2017**, *66*, 62–70.
27. Oral, S.; Kazaz, M. Characterizations for Slant Ruled Surfaces in Dual Space. *Iran. J. Sci. Technol. Trans. A Sci.* **2017**, *41*, 191–197. [CrossRef]
28. Güven, İ.A.; Kaya, S.; Hacisalihoğlu, H.H. On closed ruled surfaces concerned with dual Frenet and Bishop frames. *J. Dyn. Syst. Geom. Theor.* **2011**, *9*, 67–74. [CrossRef]
29. Saracoglu, S.; Yayli, Y. Ruled Surfaces and Dual Spherical Curves. *Acta Univ. Apulensis* **2012**, *20*, 337–354.
30. Gursoy, O. The dual angle of the closed ruled surfaces. *Mech. Mach. Theory* **1990**, *25*, 131–140. [CrossRef]
31. Sahiner, B.; Kazaz, M.; Ugurlu, H.H. A Dual Method to Study Motion of a Robot End-Effector. *J. Inform. Math. Sci.* **2018**, *10*, 247–259. [CrossRef]
32. Culmer, P.R.; Jackson, A.E.; Makower, S.; Richardson, R.; Cozens, J.A.; Levesley, M.C.; Bhakta, B.B. A control strategy for upper limb robotic rehabilitation with a dual robot system. *IEEE/ASME Trans. Mechatron.* **2009**, *15*, 575–585. [CrossRef]
33. Gu, Y.L.; Luh, J. Dual-number transformation and its applications to robotics. *IEEE J. Robot. Autom.* **1987**, *3*, 615–623.
34. Bayro-Corrochano, E.; Kähler, D. Motor algebra approach for computing the kinematics of robot manipulators. *J. Robot. Syst.* **2000**, *17*, 495–516. [CrossRef]
35. Özgür, E.; Mezouar, Y. Kinematic modeling and control of a robot arm using unit dual quaternions. *Robot. Auton. Syst.* **2016**, *77*, 66–73. [CrossRef]
36. Yayli, Y.; Çalişkan, A.; Uğurlu, H.H. The E. Study maps of circles on dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2. *Math. Proc. R. Ir. Acad.* **2002**, *102A*, 37–47. [CrossRef]
37. Şenyurt, S.; Gür, S. Timelike–spacelike involute–evolute curve couple on dual Lorentzian space. *J. Math. Comput. Sci.* **2012**, *2*, 1808–1823.
38. Bektas, Ö.; Senyurt, S. On Some Characterizations of Ruled Surface of a Closed Timelike Curve in Dual Lorentzian Space. *Adv. Appl. Clifford Algebr.* **2012**, *22*, 939–953. [CrossRef]
39. Li, Y.L.; Alluhaibi, N.; Abdel-Baky, R.A. One-parameter Lorentzian dual spherical movements and invariants of the axodes. *Symmetry* **2022**, *14*, 1930. [CrossRef]
40. Gür, S.; Şenyurt, S. Spacelike–timelike involute–evolute curve couple on dual Lorentzian space. *J. Math. Comput. Sci.* **2013**, *3*, 1054–1075.
41. Gür Mazlum, S.; Şenyurt, S.; Grilli, L. The Dual Expression of Parallel Equidistant Ruled Surfaces in Euclidean 3-Space. *Symmetry* **2022**, *14*, 1062. [CrossRef]
42. Li, Y.; Dey, S.; Pahan, S.; Ali, A. Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on Paracontact geometry. *Open Math.* **2022**, *20*, 1–20. [CrossRef]
43. Li, Y.; Ali, A.; Mofarreh, F.; Alluhaibi, N. Homology groups in warped product submanifolds in hyperbolic spaces. *J. Math.* **2021**, *2021*, 8554738. [CrossRef]
44. Yüzbaşı, Z.K.; Bektaş, M.; Yılmaz, M.Y. Compact Totally Real Minimal Submanifolds in a Bochner-Kaehler Manifold, *Univers. J. Math. Appl.* **2018**, *1*, 254–257.
45. Yılmaz, M.Y.; Bektaş, M. Curvature Inequalities between a Hessian Manifold with Constant Curvature and its Submanifolds. *Math. Sci. Appl. -Notes* **2017**, *5*, 27–33. [CrossRef]
46. Li, Y.; Gür Mazlum, S.; Şenyurt, S. The Darboux Trihedrons of Timelike Surfaces in the Lorentzian 3-Space. *Int. J. Geom. Methods Mod. Phys.* **2023**, *2350030*, 1–35. [CrossRef]
47. Chen, Z.; Li, Y.; Sarkar, S.; Dey, S.; Bhattacharyya, A. Ricci Soliton and Certain Related Metrics on a Three-Dimensional Trans-Sasakian Manifold. *Universe* **2022**, *8*, 595. [CrossRef]