symmetry

MDPI

Article

(β, γ)-Skew QC Codes with Derivation over a Semi-Local Ring

Mohammad Ashraf [1], Amal S. Alali [2], Mohd Asim [1] and Ghulam Mohammad [1,*]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: mohdghulam202@gmail.com

Abstract: In this article, we consider a semi-local ring $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$, where $u^2 = u$, $q = p^s$ and p is a prime number. We define a multiplication $yb = \beta(b)y + \gamma(b)$, where β is an automorphism and γ is a β-derivation on $\mathbf{S}$ so that $\mathbf{S}[y; \beta, \gamma]$ becomes a non-commutative ring which is known as skew polynomial ring. We give the characterization of $\mathbf{S}[y; \beta, \gamma]$ and obtain the most striking results that are better than previous findings. We also determine the structural properties of 1-generator skew cyclic and skew-quasi cyclic codes. Further, We demonstrate remarkable results of the above-mentioned codes over $\mathbf{S}$. Finally, we find the duality of skew cyclic and skew-quasi cyclic codes using a symmetric inner product. These codes are further generalized to double skew cyclic and skew quasi cyclic codes and a table of optimal codes is calculated by MAGMA software.

Keywords: skew polynomial ring; skew cyclic codes; skew QC codes; Gray map

MSC: 94B05; 94B15; 94B60

Citation: Ashraf, M.; Alali, A.S.; Asim, M.; Mohammad, G. (β, γ)-Skew QC Codes with Derivation over a Semi-Local Ring. *Symmetry* **2023**, *15*, 225. https://doi.org/10.3390/sym15010225

Academic Editors: Juan Alberto Rodríguez Velázquez and Mariano Torrisi

Received: 7 November 2022
Revised: 22 December 2022
Accepted: 29 December 2022
Published: 13 January 2023

1. Introduction

Over finite fields, error-correcting codes were first investigated but as time passed, more generic structures were examined and implemented. The study of codes over rings have attracted a lot of interest from many researchers, especially after a landmark paper [1].

Cyclic codes form an important family of linear codes. Skew quasi cyclic codes are also an immediate and important generalization of cyclic codes. Due to their rich algebraic structures, several researchers have studied them over finite fields. Recently, they have also been studied over some finite rings and many good codes have been obtained in this class. Some authors studied other generalizations of cyclic codes, such as double cyclic codes, additive codes etc., over rings and obtained some good codes, see for references [2].

In 1933, Ore [3] provided the principal results of a general non-commutative polynomial theory. In 1967, Smits [4] defined a multiplication as $ax = (a\sigma)x + a\bar{\delta}$ in a ring $\mathbf{K}[x, \sigma, \bar{\delta}]$, where $\mathbf{K}$ is a ring, σ is an endomorphism on $\mathbf{K}$ and $\bar{\delta}$ is a σ-derivation on $\mathbf{K}$. Under the above multiplication, this $\mathbf{K}[x, \sigma, \bar{\delta}]$ forms a non-commutative ring and is called a skew-polynomial ring. Abualrub et al. [5] and Bhaintwal [6] devised skew quasi-cyclic codes for various ring types. The major reason for investigating codes in this context is because polynomials in skew polynomial rings have multiple factorizations and so have more ideals than in the commutative ring. However, all of this effort is constrained by the requirement that the order of the automorphism be a factor of the code length.

In 2012, Jitman et al. [7] constructed skew cyclic codes by considering the skew polynomial ring with a coefficient from $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $u^2 = 0$, a finite chain ring. In 2013, Boulagouaz et al. [8] introduced the notion of $[f(t), \sigma, \delta]$-codes for $f(t) \in \mathbf{A}[f(t), \sigma, \delta]$, where A, σ and δ are a ring, an automorphism and a σ-derivation of $\mathbf{A}$. These codes were the generalization of the θ-codes as introduced by Boucher et al. [9]. Ashraf et al. [10] studied skew cyclic codes over a semi-local ring and proved that the Gray image of skew cyclic codes of length n over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $u^2 = 1$ is a skew 2-quasi cyclic codes of length $2n$ over $\mathbb{F}_{p^m}$. Further, Ashraf et al. [11] determined the structure of skew cyclic codes over $\mathbb{F}_3 + u\mathbb{F}_3$, where $u^2 = 1$ and found that skew cyclic codes are equivalent to either cyclic codes or quasi cyclic codes.

In 2018, Bhaintwal et al. [12] provided a class of skew cyclic codes over $\mathbf{R} = \mathbb{Z}_4 + u\mathbb{Z}_4$ with derivation. They found some new good codes over $\mathbb{Z}_4$ utilizing the Gray map and the residue codes of these codes. The discovered codes had been reported and added to the database of $\mathbb{Z}_4$-codes. Later, Fanghui et al. [13] studied the skew cyclic and skew quasi-cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$, where $u^2 = 1$. Most of the work on codes is over a commutative structure. However, recently, the authors of [2,12,14–18] have taken a keen interest in the study of codes in the setting of skew polynomial rings.

In the present article, we consider a semi-local ring $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$, where $u^2 = u$, $q = p^s$ and p is a prime number. We give the characterization of the skew polynomial ring $\mathbf{S}[y;\beta,\gamma]$, where β is an automorphism of $\mathbf{S}$ and γ is a β-derivation on $\mathbf{S}$. We obtain minimal polynomials that generate skew cyclic and skew-quasi cyclic codes. We also find some more results that describe these codes to double skew cyclic and skew quasi cyclic codes. Further, we give some examples to support our main results and find optimal codes which are given in Table 1.

This paper is organized as follows: In Section 2, we give some basic definitions and properties of ring $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$, where $u^2 = u$, $q = p^s$ and p is prime. We also define Gray maps and give some results that are very useful in proving our main results. In Section 3, we give some results on skew cyclic codes. Finally, we calculate a table of optimal codes. In Section 4, we study skew-quasi cyclic codes and find the minimal generating polynomials. In Section 5, we find the duality of skew cyclic and skew-quasi cyclic codes. In Section 6, we give some examples to support our main results and a table of optimal codes. Finally, Section 7, brings the article to an end.

2. Preliminaries

Let $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$ be a commutative ring, where $u^2 = u$, $q = p^s$ and p is a prime. Moreover, $\mathbf{S}$ is isomorphic to the ring $\mathbb{F}_q[u]/\langle u^2 - u\rangle$. Any element $z \in \mathbf{S}$ can be written as $z = c + ud$ for all $c, d \in \mathbb{F}_q$. The maximal ideals of $\mathbf{S}$ are $\langle u\rangle$ and $\langle 1 - u\rangle$. Therefore, $\mathbf{S}$ is a semi-local ring.

The Gray map can be defined as

$$\Phi : \mathbf{S} \longrightarrow \mathbb{F}_q^2$$

$$c + ud \longmapsto (d, c + d).$$

The above Gray map can be extended as

$$\Phi : \mathbf{S}^n \longrightarrow \mathbb{F}_q^{2n}$$

$$(c_0 + ud_0, \dots, c_{n-1} + ud_{n-1}) \longmapsto (d_0, \dots, d_{n-1}, c_0 + d_0, \dots, c_{n-1} + d_{n-1}).$$

The Gray weight of v is defined as

$$w_G(v) = w_H(\Phi(v)).$$

A non-void $C \subset \mathbf{S}^n$ is a linear code over $\mathbf{S}$ of length n if it is $\mathbf{S}$-submodule of $\mathbf{S}^n$. The Gray distance of two distinct codewords can be written as

$$d_G(v_1, v_2) = w_G(v_1 - v_2) = d_H(\Phi(v_1), \Phi(v_2)).$$

It is obvious that, if C is a linear code, then $d_G(C) = \min\{w_G(v) \mid 0 \neq v, v \in C\}$. The Gray map Φ is a weight preserving rule from $\mathbf{S}^n$ (Gray weight) to $\mathbb{F}_q^{2n}$ (Hamming distance).

Let us define a map $g : U \longrightarrow U$, where U is a ring with unity. Then, g is called a (β, α)-derivation on U if it satisfies $g(c + d) = g(c) + g(d)$ and $g(cd) = \beta(c)g(d) + g(c)\alpha(d)$ for all $c, d \in U$, where α and β are any two automorphisms of U. A map $\gamma : U \longrightarrow U$ is called a β-derivation of U if it is a (β, I)-derivation, where I denotes the identity automorphism of U. Now, the ring automorphism $\beta : \mathbf{S} \longrightarrow \mathbf{S}$ of $\mathbf{S}$ is defined as follows:

$$c + du \longmapsto c^p + ud^p.$$

We can easily verify that s is the order of β. Let us define $\mathbf{S}^\beta \subseteq \mathbf{S}$, a subring of $\mathbf{S}$, as $\mathbf{S}^\beta = \{a \mid \beta(a) = a, \forall\, a \in \mathbf{S}\}$.

Define the inner β-derivation of $\mathbf{S}$ as follows:

$$\gamma : \mathbf{S} \longrightarrow \mathbf{S}$$

$$c + ud \longmapsto (1+u)\{\beta(c+ud) - (c+ud)\}.$$

Clearly, $\gamma(a) = 0$ for all $a \in \mathbf{S}^\beta$. The maps β and γ are described in [12].

If an automorphism β and inner β-derivation γ of $\mathbf{S}$ are given, the set $\mathbf{S}[y;\beta,\gamma] = \{c_0 + c_1 y + \cdots + c_{n-1}y^{n-1} \mid c_j \in \mathbf{S} \text{ for all } j = 0,1,\ldots,n-1\}$ with usual addition of polynomials and the multiplication defined by the rule:

$$yb = \beta(b)y + \gamma(b) \tag{1}$$

for all $b \in \mathbf{S}$ forms a ring. The ring $\mathbf{S}[y;\beta,\gamma]$ is termed as a skew polynomial ring over $\mathbf{S}$. The ring $\mathbf{S}[y;\beta,\gamma]$ is non-commutative under the multiplication defined in (1) unless β is the identity. For a skew polynomial ring, the center of $\mathbf{S}[y;\beta,\gamma]$ can be defined by the set

$$Z(\mathbf{S}[y;\beta,\gamma]) = \{g(y) \mid g(y)\cdot a(y) = a(y)\cdot g(y) \text{ for all } a(y) \in \mathbf{S}[y;\beta,\gamma]\}.$$

Any element $g(y) \in Z(\mathbf{S}[y;\beta,\gamma])$ is called a central element.

Lemma 1 ([12])**.** *If $q = 4$, then for any element $a \in \mathbb{F}_q$*

$$y^n a = \begin{cases} (\beta(a)y + \gamma(\beta(a))y^{n-1}, & \text{if } n \text{ is odd} \\ \beta^n(a)y^n, & \text{if } n \text{ is even.} \end{cases}$$

The skew polynomial ring $\mathbf{S}[y;\beta,\gamma]$ is neither left nor right ideal. Obviously, the right division algorithm can be stated as, if for any $f(y), g(y) \in \mathbf{S}[y;\beta,\gamma]$, where $g(y)$ has unit as its leading coefficient, then we can find $q(y),\ r(y) \in \mathbf{S}[y;\beta,\gamma]$ such that

$$f(y) = q(y)g(y) + r(y),$$

where $r(y) = 0$ or $deg(r(y)) < deg(g(y))$. Similarly, the left division algorithm can be defined. Therefore, a central element is a right as well as a left divisor.

Definition 1. *Let $a(y),\ b(y) \in \mathbf{S}$. A polynomial $d(y)$ is said to be a greatest common right (left) divisor (gcrd) of $a(y)$ and $b(y)$ if*

(i) $d(y)$ is the right (left) divisor of $a(y)$ and $b(y)$.
(ii) If $e(y)$ is the right (left) divisor of $a(y)$ and $b(y)$, then $e(y)$ is the right (left) divisor of $d(y)$.

Definition 2. *A least degree polynomial $h(y) \in \mathbf{S}$ is said to be the least common right (left) multiple of $a(y), b(y) \in \mathbf{S}$ if $a(y)$ and $b(y)$ are right (left) divisors of $h(y)$.*

Two polynomials $a(y),\ b(y) \in \mathbf{S}$ are right (left) coprime if, for any $f(y),\ g(y) \in \mathbf{S}$ such that $a(y)f(y) + b(y)g(y) = 1$.

By using the characteristic of the Gray map Φ in [19], we can easily obtain the following results.

Lemma 2. *Let C be a linear code with the parameter $[n, M, d_G]$ over R. Then $\Phi(C)$ is a $[2n, M, d_L]$-linear code over $\mathbb{F}_q$, where $d_G = d_L$.*

3. (β, γ)-Skew Cyclic Codes

In 2013, Boulagouaz [8] defined a concept of the structure of $(f(y), \beta, \gamma)$ cyclic codes for a monic polynomial $f(y) \in \mathbf{S}[y; \beta, \gamma]$, where $\mathbf{S}$ is a ring, β is an automorphism and γ is a β-derivation of $\mathbf{S}$, respectively. In 2018, Sharma et al. [12] gave the characterization of skew-cyclic codes over the ring $\mathbf{S} = \mathbb{Z}_4 + u\mathbb{Z}_4$ with $u^2 = 1$ and obtained some important structural properties. In 2021, Fanghui Ma et al. gave another description of skew-cyclic codes over $\mathbf{S} = \mathbb{Z}_4 + u\mathbb{Z}_4$ that is distinct from [12]. Now, we study the (β, γ) skew-cyclic codes of length n over $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$, where $u^2 = u$, $q = p^s$ and p is a prime number.

Definition 3 ([20,21])**.** *Let $\mathbf{S}$ be a ring with unity. A pseudo-linear map $H : R^n \longrightarrow \mathbf{S}^n$ is an additive map defined by*

$$H(u) = \beta(u)M + \gamma(u), \tag{2}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbf{S}^n$, $\beta(u) = (\beta(u_1), \beta(u_2), \ldots, \beta(u_n))$, M is a $n \times n$ matrix over $\mathbf{S}$ and $\gamma(u) = (\gamma(u_1), \gamma(u_2), \ldots, \gamma(u_n))$. If $\gamma = 0$, then H is known as a semi-linear transformation.

Definition 4. *Let γ be a β-derivation on $\mathbf{S}$, where β is an automorphism of S. A non-subset $C \subset \mathbf{S}^n$ is said to be a (β, γ)-skew cyclic code over $\mathbf{S}$ of length n if*

(i) C is an $\mathbf{S}$-submodule of $\mathbf{S}^n$;
(ii) $H(C)$ contained in C

and H is defined in (2) with

$$M = \begin{pmatrix} 0 & 1 & 0\cdots & 0 \\ 0 & 0 & 1\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots & 1 \\ 1 & 0 & 0\cdots & 0 \end{pmatrix} \tag{3}$$

such that $H(C)$ can be written as $H(C) = \{H(w) \mid w \in C\}$.

Suppose that $\mathbf{S}_n = \mathbf{S}[y; \beta, \gamma]/\langle y^n - 1\rangle$. Let $w(y) + \langle y^n - 1\rangle \in S_n$ and for any $a(y) \in \mathbf{S}[y; \beta, \gamma]$, define the product from left as

$$a(y)(w(y) + \langle y^n - 1\rangle) = a(y)w(y) + \langle y^n - 1\rangle.$$

S_n forms a left $\mathbf{S}[y; \beta, \gamma]$ module under the above definition. We can express every codeword $w \in C$ as $w = (w_0, w_1, \ldots, w_{n-1})$ of a (β, γ)-skew cyclic code C by a polynomial $w(y) = (w_0 + w_1 y + \cdots + w_{n-1}y^{n-1})$. We obtain that $yw(y)$ corresponds to the codeword $H(c)$.

Lemma 3. *If $u(y) = (u_0 + u_1 y + \cdots + u_{n-1}y^{n-1}) \in \mathbf{S}[y; \beta, \gamma]/\langle y^n - 1\rangle$ represents the word $u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbf{S}^n$, then $yu(y)$ can be represented by the word $(\beta(u_{n-1}) + \gamma(u_0), \beta(u_0) + \gamma(u_1), \ldots, \beta(u_{n-2}) + \gamma(u_{n-1})) \in \mathbf{S}^n$.*

Proof. In order to prove the given statement, we just calculate the value of $yu(y)$, i.e.,

$$\begin{aligned} yu(y) &= y\left(\sum_{i=0}^{n-1} u_i y^i\right) = \left(\sum_{i=0}^{n-1} y u_i y^i\right) \\ &= \left(\sum_{i=0}^{n-1} (\beta(u_i)y + \gamma(u_i))y^i\right) \\ &= \beta(u_0)y + \gamma(u_0) + \beta(u_1)^2 + \gamma(u_1)y + \beta(u_2)y^3 + (u_2)y^2 + \cdots + \beta(u_{n-1}) \\ &\quad + \gamma(u_{n-1})y^{n-1} \\ &= (\beta(u_{n-1}) + \gamma(u_0)) + (\beta(u_0) + \gamma(u_1))y + \cdots + (\beta(u_{n-2} + \gamma(u_{n-1}))y^{n-1}. \end{aligned}$$

Hence, $yu(y)$ can be represented by the word

$$(\beta(u_{n-1})+\gamma(u_0),\beta(u_0)+\gamma(u_1),\ldots,\beta(u_{n-2}+\gamma(u_{n-1}))\in \mathbf{S}^n.$$

□

Lemma 4. *A code C of length n over $\mathbf{S}$ is a (β,γ)-skew cyclic code if and only if C is an $\mathbf{S}[y;\beta,\gamma]$-submodule of $\mathbf{S}_n$.*

Proof. Let C be a (β,γ)-skew cyclic code of length n over $\mathbf{S}$. Then, for any $c(y)\in C$, the (β,γ)-cyclic shift $yc(y)$ also belongs to C by Lemma 5. Therefore, $y^ic(y)$ also belongs to C for all $i\in N$. It follows that, for any $a(y)\in \mathbf{S}[y;\beta,\gamma]$ and $c(y)\in C$ their product $a(y)c(y)$ also belongs to C. This implies that C is a submodule of S_n. The converse part is directly followed by the definition. □

Theorem 1. *Let $C=\langle p(y)\rangle$, where $p(y)$ is monic polynomial of degree $n-k$. Then C, a (β,γ)-skew cyclic code of length n over $\mathbf{S}$ is a $\mathbf{S}$-free code with rank k if and only if $p(y)$ is a right divisor of y^n-1.*

Proof. Suppose that $p(y)|y^n-1$ and $y^n-1=p(y)q(y)$. We notice that $q(y)$ must be a monic polynomial of degree k. Let $q(y)=y^k+b_{k-1}y^{k-1}+\cdots+b_0$. Now, we have to show that the set $S=\{p(y),yp(y),\ldots,y^{k-1}p(y)\}$ forms a basis for C. Clearly, $span(S)\subseteq C$. For the reverse inclusion, let any codeword $c(y)\in C$ be of the form $c(y)=h(y)p(y)$, for some $h(y)\in\mathbf{S}[y;\beta,\gamma]$. Since $p(y)q(y)=0$ in $\mathbf{S}_n$, $y^kp(y)$ can be expressed as a linear combination in S. It follows that $y^rp(y)$ is also true, whenever $r>k$. Hence, $h(y)p(y)\in span(S)$ for any polynomial $h(y)\in\mathbf{S}[y;\beta,\gamma]$. Therefore, S is the spanning set for C.

Now, we have to show that S is linearly independent. Let us consider $h(y)p(y)=0$, where $deg(h(y))<k$. This means that $y^n-1|h(y)p(y)$, i.e., $h(y)p(y)=r(y)(y^n-1)$, for some polynomial $r(y)\in\mathbf{S}[y;\beta,\gamma]$. However, since $p(y)$ is monic

$$n-k\le deg(h(y)p(y))\le n-1$$

and for the same reason, $deg(r(y)(y^n-1)\ge n$. This implies that $h(y)$ must be a zero polynomial, hence S is linearly independent. Finally, we get C as a R-free code with rank k.

Conversely, suppose that $C=\langle p(y)\rangle$ is free of rank k. Clearly, $span(S)\subseteq C$. On the other hand, S is linearly independent. Thus, $|C|=|span(S)|$ and $C=span(S)$. Now, let us consider $y^kp(y)\in C=span(S)$. Then $y^kp(y)=h(y)p(y)$ for some polynomial $h(y)$ of degree less than or equal to $k-1$. Finally, we have $(y^k-h(y))p(y)=0$ in $\mathbf{S}_n$. The left side is a monic polynomial of degree n which is divisible by y^n-1. Therefore, it must be equal to y^n-1. Hence, $p(y)|y^n-1$. □

Lemma 5. *Let $y^n-1=q(y)p(y)$ in $\mathbf{S}[y;\beta,\gamma]$ and let C be a free (β,γ)-skew cyclic code generated by $p(y)$. If $f(y)$ and $q(y)$ are right co-prime, then $C=\langle f(y)p(y)\rangle$.*

Proof. It is clear that $\langle f(y)p(y)\rangle\subseteq C$. Since $f(y)$ and $q(y)$ are right co-prime, there exist polynomials $a(y),b(y)\in\mathbf{S}[y;\beta,\gamma]$ such that

$$a(y)f(y)+b(y)q(y)=1.$$

Multiplying by $p(y)$ on both sides, we get

$$a(y)f(y)p(y)+b(y)q(y)p(y)=p(y)$$

or

$$a(y)f(y)p(y)+b(y)(y^n-1)=p(y).$$

Finally, it can be written as $a(y)f(y)p(y) = p(y)$ in $\mathbf{S}_n$. This implies that $p(y) \in \langle f(y)p(y)\rangle$. Thus, $C = \langle f(y)p(y)\rangle$.

Let $C = \langle p(y)\rangle$ be a (β, γ)-skew cyclic code of length n over $\mathbf{S}$, where $p(y) = p_0 + p_1 y + \cdots + p_{n-k}y^{n-k}$ is right divisor of $y^n - 1$, where $g_{n-k} = 1$. From Theorem 1, we find that C is a free left $\mathbf{S}[y; \beta, \gamma]$-submodule of dimension $n - deg(p(y))$. According to [8], the generator matrix G of C can be written as

$$G = \begin{pmatrix} p \\ H(p) \\ H^2(p) \\ \vdots \\ H^{k-1}(p) \end{pmatrix},$$

where $p = (p_0, p_1, \ldots p_{n-k})$ is the corresponding codeword of $p(y)$. $\square$

4. (β, γ)-Skew Quasi (QC) Codes

In this section, we will give the introduction of (β, γ)-skew quasi cyclic codes over $\mathbf{S}$ and structural properties of (β, γ)-skew QC codes with 1-generator.

Definition 5. *An automorphism β and β-derivationγ are defined on $\mathbf{S}$. A non-void subset C of $\mathbf{S}^N$ is known as a (β, γ)-skew quasi cyclic code of length $n\ell$, where $N = n\ell$ and index ℓ if it satisfies the conditions given below,*

(i) C is an $\mathbf{S}$-submodule of $\mathbf{S}^N$;

(ii) If $w = (w_0, w_1, \ldots, w_{n-1}) \in C$,

then

$$H_\ell(w) = \beta(w)M + \gamma(w) \in C,$$

where $w_i = w_{i,0}, w_{i,1}, \ldots, w_{i,\ell-1})$ for $i = 0, 1, \ldots, n-1$ and M is given as in (3), $\beta(w) = (\beta(w_0), \beta(w_1), \ldots, \beta(w_{n-1}))$, $\gamma(w) = (\gamma(w_0), \gamma(w_1), \ldots, \gamma(w_{n-1}))$, $\beta(w_i) = (\beta(w_{i,0}), \beta(w_{i,1}), \ldots, \beta(w_{i,\ell-1}))$ and $\gamma(w_i) = (\gamma(w_{i,0}), \gamma(w_{i,1}), \ldots, \gamma(w_{i,\ell-1}))$.

Now consider the ring $\mathbf{S}_n^\ell = (\mathbf{S}[y; \beta, \gamma]/\langle y^n - 1\rangle)^\ell$ as a left $\mathbf{S}_n = \mathbf{S}[y; \beta, \gamma]/\langle y^n - 1\rangle$-module, whereas the product from the left is expressed as $f(y)(p_1(y), p_2(y), \ldots, p_\ell(y)) = (f(y)p_1(y), f(y)p_2(y), \ldots, f(y)p_\ell(y))$. Define a map $\psi : \mathbf{S}^{n\ell} \longrightarrow \mathbf{S}_n^\ell$ such that

$$\psi(w) = (w_0(y), w_1(y), \ldots, w_{\ell-1}(y)),$$

where $w = (w_{0,0}, \ldots, w_{0,\ell-1}, w_{1,0}, \ldots, w_{1,\ell-1}, \ldots, w_{n-1,0}, \ldots, w_{n-1,\ell-1}) \in \mathbf{S}^{n\ell}$, $w_j(y) = \sum_{i=0}^{n-1} w_{i,j}y^i \in \mathbf{S}_n$. We can easily show that ψ is a module isomorphism.

Theorem 2. *Let $C \subset \mathbf{S}^{n\ell}$. Then C is a (β, γ)-skew quasi cyclic code of length $n\ell$ with index ℓ if and only if $\psi(C)$ is a left $\mathbf{S}_n$-submodule of $\mathbf{S}_n^\ell$.*

Proof. The proof is directly followed by Theorem 3 of [13]. $\square$

The (β, γ)-skew QC code of length $n\ell$ with index ℓ over $\mathbf{S}$ generated by t elements $p_1(y), p_2(y), \ldots, p_t(y)$ is a left $\mathbf{S}_n$ submodule of $\mathbf{S}_n^\ell$ and is called a t-generator (β, γ)-skew QC code of length $n\ell$ and index ℓ. For $t = 1$, C is called the 1-generator (β, γ)-skew QC code. The 1-generator (β, γ)-skew cyclic code C is generated by $p(y) = (p_1(y), p_2(y), \ldots, p_\ell(y))$, which has the form

$$C = \{f(y)p(y) = (f(y)p_1(y), f(y)p_2(y), \ldots, f(y)p_t(y)) \mid f(y) \in \mathbf{S}_n\}.$$

Now, let us define a map

$$\pi_i : \mathbf{S}_n^{\ell} \longrightarrow \mathbf{S}_n, \quad i = 1, 2, \ldots, n$$

$$(f_1(y), f_2(y), \ldots, f_\ell(y)) \longmapsto f_i(y).$$

The map π_i is the module homomorphism.

Since C is the left S_n-submodule of $\mathbf{S}_n^{\ell}$, then $\pi_i(c) = c_i$ is the left $\mathbf{S}[y;\beta,\gamma]$-submodule of $\mathbf{S}_n$. Thus, $C_i = \langle p(y) \rangle$. If $p(y)$ is the monic right divisor of $y^n - 1$ such that $y^n - 1 = h_i(y)p_i(y)$ and if the polynomial $g_i(y)$ and $h_i(y)$ are the right co-prime for all $i = 1, 2, \ldots, \ell$, then from Lemma 5, any generator C has the form

$$p(y) = (g_1(y)p_1(y), g_2(y)p_2(y), \ldots, g_\ell(y)p_\ell(y)).$$

Theorem 3. *Let C be a 1-generator (β,γ)-skew QC code of length $n\ell$ and index ℓ over $\mathbf{S}$ generated by $p(y) = (g_1(y)p_1(y), g_2(y)p_2(y), \cdots, g_\ell(y)p_\ell(y))$, where $p_i(y)$ is the monic polynomial and $gcrd(g_i(y), y^n - 1|p_i(y)) = 1$, $i = 1, 2, \ldots, \ell$. Then, C is a free skew QC code with the rank $n - deg(q(y))$ and has a basis $S = \left\{p(y), yp(y), \ldots, y^{n-deg(q(y))-1}p(y)\right\}$, where $q(y) = gcld(p(y), y^n - 1)$.*

Proof. Let $S = \left\{p(y), yp(y), \ldots, y^{n-deg(q(y))-1}p(y)\right\}$. We need to prove that S generates C. Since $q(y) = gcld(p(y), y^n - 1)$, there is a monic polynomial $h(y)$ such that $y^n - 1 = q(y)h(y)$. Let $deg(q(y)) = k$. Then, $deg(h(y)) = n - k$. Any element of C can be written as $b(y) = a(y)p(y)$, where $a(y) \in \mathbf{S}_n$. If $deg(a(y)) < n - k$, then S generates C, otherwise, using the division algorithm, there exist two polynomials $c(y)$ and $d(y)$ such that $a(y) = c(y)h(y) + d(y)$, where $d(y) = 0$ or $deg(d(y)) < n - k$. It is clear that $q(y) = gcld(p(y), y^n - 1)$. The polynomial $y^n - 1 \in Z(\mathbf{S}[y;\beta,\gamma])$ because n is even then by Lemma 5, we have

$$y^n - 1 = p(y)h(y) = h(y)p(y).$$

Then, it can be written as

$$h(y)p(y) \equiv 0 \mod (y^n - 1),$$

which implies that $c(y) \equiv d(y)p(y) \mod (y^n - 1)$. Since $deg(d(y)) = n - k$, we get S generating C.

Now, we have to prove that S is linearly independent. So, let there be a polynomial $a(y) = \sum_{i=0}^{n-k-1} a_i y^i$ such that $a(y)p(y) = 0$, where $a_i \in \mathbf{S}$ for all $0 \le i \le n - k - 1$. Then, $a(y)g_i(y)p_i(y) = 0$ for all $1 \le i \le l$, which implies that $y^n - 1$ is a divisor of $a(y)g_i(y)p_i(y)$ for all i. Hence, $y^n - 1$ is divisor of $gcld(a(y)g_1(y)p_1(y), a(y)g_2(y)p_2(y), \cdots a(y)g_l(y)p_l(y), a(y)(y^n - 1)) = a(y)p(y)$. Since $deg(a(y)p(y)) = n - 1 < deg(y^n - 1)$, $a(y)$ must be a zero polynomial. This implies that S is linearly independent. Hence, S forms a basis for C. □

5. Dual of (β,γ)-Skew Cyclic and Skew QC Codes under Symmetric Inner Product

Let C be a (β,γ)-skew QC code with index ℓ and length $N = n\ell$ over $\mathbf{S}$. Let $u = (x_0, x_1, \ldots, x_{n-1}) \in \mathbf{S}^{n\ell}$ and $v = (y_0, y_1, \ldots, y_{n-1}) \in \mathbf{S}^{n\ell}$, where $x_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,\ell-1}) \in \mathbf{S}^{\ell}$ and $y_i = (y_{i,0}, y_{i,1}, \ldots, y_{i,\ell-1}) \in \mathbf{S}^{\ell}$ for all $i = 0, 1, \ldots, n - 1$. Then, the inner product is represented by

$$\langle u, v \rangle = \sum_{i=0}^{n-1} x_i \cdot y_i = \sum_{i=0}^{n-1} \sum_{j=0}^{\ell-1} x_{ij} y_{ij}.$$

It can be easily seen that $\langle u, v \rangle = \langle u, v \rangle$, that is, it is symmetric.

According to the inner product of u and v, the dual code of C is determined as

$$C^{\perp} = \{v \in \mathbf{S}^{n\ell} \mid \langle u, v \rangle = 0, \forall\, u \in C\}.$$

Lemma 6. *Let C be a (β,γ)-skew cyclic code of length n over $\mathbf{S}$ generated by a monic right divisor $g(y)$ of y^n-1. Then, $u(y)\in\mathbf{S}_n$ is in C if and only if $u(y)h(y)=0$ in $\mathbf{S}_n$, where $y^n-1=h(y)p(y)$.*

Proof. Suppose that $u(y)\in C$. Then it can be written as $u(y)=a(y)p(y)$ for some $a(y)\in\mathbf{S}_n$. So, $u(y)h(y)=a(y)p(y)h(y)=0$ in S_n by Lemma 5. Let $u(y)h(y)=0$ in S_n for some $u(y)\in\mathbf{S}_n$. Then there exists $r(y)\in\mathbf{S}[y;\beta,\gamma]$ such that $u(y)h(y)=r(y)(y^n-1)=r(y)h(y)p(y)$. This implies that $u(y)=r(y)p(y)$. Hence the result. □

Theorem 4. *Let $C=\langle p(y)\rangle$ be a principally generated (β,γ)- skew cyclic code of length n over $\mathbf{S}$ such that $y^n-1=h(y)p(y)$ for some $h(y)=a_0+a_1y+\cdots+a_ky^k\in\mathbf{S}[y;\beta,\gamma]$, where k is odd, then the matrix*

$$H=\begin{pmatrix} a_k & \beta(a_{k-1})+\gamma(a_k) & a_{k-1} & \cdots & \beta(a_0)+\gamma(a_1) & \cdots & 0 & 0 \\ 0 & \beta(a_k) & a_{k-1} & \cdots & a_0 & \gamma(a_0) & \cdots & 0 \\ 0 & 0 & a_k & a_{k-2} & \beta(a_{k-3})+\gamma(a_{k-2}) & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_k & \beta(a_{k-1})+\gamma(a_k) & \cdots & a_1 & \beta(a_0)+\gamma(a_1) \end{pmatrix}$$

is a parity check matrix for C.

Proof. Let $g(y)\in C$. Then, by Lemma 4.1, we have $g(y)h(y)=0$ in $\mathbf{S}_n$. Therefore, the coefficients of y^k, y^{k+1},..., y^{n-1} in $(g_0+g_1y+g_2y^2+\cdots+g_{n-2}y^{n-2}+g_{n-1}y^{n-1})(a_0+a_1y+\cdots+a_{k-1}y^{k-1}+a_ky^k)$ are all zero, that is, after simplification, we get

$$\begin{aligned} g_0a_k+g_1(\beta(a_{k-1})+\gamma(a_k))+g_2a_{k-2}+\cdots+g_k(\beta(a_0)+\gamma(a_1)) &= 0 \\ g_1(\beta(a_k))+g_2a_{k-1}+g_3(\beta(a_{k-2})+(\gamma a_{k-1}))+\cdots+g_{k+1}a_0+g_{k+2}\gamma(a_0) &= 0 \\ g_2a_k+g_3(\beta(a_{k-1})+\gamma(a_k))+g_4a_{k-2}+\cdots+g_{k+1}a_1+g_{k+2}(\beta(a_0)+\gamma(a_1)) &= 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ g_{n-k-1}a_k+g_{n-k}(\beta(a_{k-1})+\gamma(a_k))+\cdots+g_{n-2}a_1+g_{n-1}(\beta(a_0)+\gamma(a_1)) &= 0. \end{aligned}$$

From these equations, we find that for any $c\in C$ and $cH^T=0$, thus $GH^T=0$. Now each row of H is orthogonal to C for every $c\in C$. So, $span(H)\subseteq C^\perp$ and also H contains square sub-matrix of order $n-k$, with a non-zero determinant, as H is a lower triangular matrix with all diagonal entries being units by Lemma 4.2. This implies that all rows of H are linearly independent. Therefore, $|span(H)|=|\mathbf{S}|^{n-k}$, $|C||C^\perp|=|\mathbf{S}|^n$ and $|C|=\mathbf{S}^k$ gives $|C^\perp|=\mathbf{S}^{n-k}$. Hence, $span(H)=C^\perp$ and so H is a parity check matrix of C. □

Theorem 5 ([22])**.** *Let C be a (β,γ)-skew cyclic code of length n over $\mathbf{S}$ and it is invariant under a pseudo-linear map $H(a)=\beta(a)M+\gamma(a)$ for all $a\in C$ of length n and M is represented in (2). Then $C^\perp$, the dual of C is a $(\beta^{-1},\gamma^{'})$-skew cyclic code over $\mathbf{S}$ and is also invariant under a pseudo-linear map $H^{'}(b)=\beta^{-1}(b)\cdot M_t+\gamma^{'}(v)$ for all $b\in CC$, where $\gamma^{'}=-\beta^{-1}\gamma$ is a (id,β^{-1})-derivation and M_t is a transpose of M.*

Theorem 6. *Let C be a (β,γ)-skew QC code with index ℓ and length $n\ell$ over $\mathbf{S}$. Then, $C^\perp$, the Euclidean dual of C, is invariant under a pseudo-linear map $H^{'}(a)=\beta^{-1}(a)\cdot M_t+\gamma^{'}(a)$ for all $a\in C$, where $C^\perp$ is a $(\beta^{-1},\gamma^{'})$-skew QC code with index ℓ and length $n\ell$ over $\mathbf{S}$ and $\gamma^{'}=-\beta^{-1}\gamma$ is a (id,β^{-1})-derivation and M_t is a transpose of M.*

Proof. Let $a=(a_0,a_1,\ldots,a_{n-1})\in R^{nl}$, $a_j=(a_{j,o},a_{j,1},\ldots,a_{j,\ell-1})\in R^\ell$ for $j=0,1,\ldots,n-1$. Applying $H^{'}$ on a, we get

$$H^{'}(a)=\beta^{-1}(a)\cdot M_t+\gamma^{'}(a),$$

where

$$\beta^{-1}(a) = (\beta^{-1}(a_0), \beta^{-1}(a_1), \ldots, \beta^{-1}(a_{n-1})), \gamma^{'}(a) = (\gamma^{'}(a_0), \gamma^{'}(a_1), \ldots, \gamma^{'}(a_{n-1})),$$

$$\beta^{-1}(a_j) = (\beta^{-1}(a_{j,0}), \beta^{-1}(a_{j,1}), \ldots, \beta^{-1}(a_{j,\ell-1})),\ \gamma^{'}(a_j) = (\gamma^{'}(a_{j,0}), \gamma^{'}(a_{j,1}), \ldots, \gamma^{'}(a_{j,\ell-1}))$$

for $j = 0, 1, \ldots, n-1$. For any $a \in C$ and $u \in C^{\perp}$, we have

$$\begin{aligned} 0 &= u \cdot H_l(a)_t \\ &= u \cdot (\beta(a) \cdot M)_t + u \cdot (\gamma(a))_t \\ &= u \cdot (\beta(a) \cdot M)_t + (\gamma(au)) - \gamma(u) \cdot (\beta(a))_t \\ &= u \cdot M_t \cdot (\beta(a))_t - \gamma(u) \cdot (\beta(a))_t \\ &= (u \cdot M_t - \gamma(u) \cdot)(\beta(a))_t. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \beta^{-1}(0) \\ &= \beta^{-1}(u \cdot M_t - \gamma(u)) \cdot)(\beta(a))_t \\ &= \beta^{-1}(u \cdot M_t - \gamma(u) \cdot)(a)_t \\ &= (\beta^{-1}(u) \cdot M_t + \gamma^{'}(u))(a)_t, \end{aligned}$$

i.e., $H^{'}(u) \cdot a_t = 0$ for all $a \in C$ and $u \in C^{\perp}$. Therefore, $H^{'}(u) \in C^{\perp}$. □

6. Example

Example 1. *Assume that $\mathbb{F}_4 = \mathbb{F}_2(w)$, $w^2 + w + 1 = 0$. Let $n = 6$*

$$y^6 - 1 = (w^2y^3 + wy^2 + y + 1)(w^2y^3 + w^2y^2 + y + 1).$$

*Define an automorphism $\beta : \mathbb{F}_4 \to \mathbb{F}_4$ by $\beta(a) = a^2$ and $\gamma : \mathbb{F}_4 \to \mathbb{F}_4$ by $\gamma(a) = \beta(a) - a$. Let $C = \langle p(y) \rangle$ be a 1-generator (β, γ) skew cyclic code of length 6 over **S** with generator $p(y) = w^2y^3 + wy^2 + y + 1$. Then, the parameters of $\Phi(C)$ are [9,13,23], that is, $\Phi(C)$ is an optimal code.*

Example 2. *Consider that $\mathbb{F}_4 = \mathbb{F}_2(w)$, $w^2 + w + 1 = 0$. Let $n = 6$*

$$y^6 - 1 = (w^2y^3 + wy^2 + y + 1)(w^2y^3 + w^2y^2 + y + 1).$$

*Define an automorphism $\beta : \mathbb{F}_4 \to \mathbb{F}_4$ by $\beta(a) = a^2$ and $\gamma : \mathbb{F}_4 \to \mathbb{F}_4$ by $\gamma(a) = \beta(a) - a$. Let $C = \langle p(y) \rangle$ be a 1-generator (β, γ) skew cyclic code of length 6 over **S** with generator $p(y) = w^2y^3 + w^2y^2 + y + 1$. Then, the parameters of $\Phi(C)$ are [13,19,23].*

Example 3. *For $n = 8$ and $\mathbf{S} = \mathbb{F}_4 + u\mathbb{F}_4$, we have*

$$y^8 - 1 = (y^2 - 1)(y^6 + y^4 + y^2 + 1).$$

*Let C be a 1-generator (β, γ) skew QC cyclic code of length 56 with index 7 over **S** with generator $p(y) = (p_1(y), p_2(y), p_3(y), p_4(y), p_5(y), p_6(y), p_7(y))$, where $p_1(y) = p_2(y) = p_3(y) = p_4(y) = p_5(y) = p_6(y) = p_7(y) = y^6 + y^4 + y^2 + 1$. According to Theorem 3, we can find $q(y)$ as $q(y) = gcld(p(y), y^8 - 1) = y^6 + y^4 + y^2 + 1$. Then, $\{p(y), yp(y)\}$ forms a basis of C and $|C| = 16^2$. By using Theorem 2.1 of [19] $\ell d \leq d(C)$, i.e., $d(C) = 28$. Hence, C and $\Phi(C)$ are (β, γ)-skew QC codes over **S** with parameters $(56, 16^2, 28)$ and $(112, 4^4, 28)$, respectively.*

Example 4. *For $n = 5$ and $\mathbf{S} = \mathbb{F}_3 + u\mathbb{F}_3$, we have*

$$y^5 - 1 = (y - 1)(y^4 + y^3 + y^2 + y + 1).$$

Let C be a 1-generator (β, γ) skew QC cyclic code of length 30 with an index 6 over $\mathbf{S}$ with generator $p(y) = (p_1(y), p_2(y), p_3(y), p_4(y), p_5(y), p_6(y))$, where $p_1(y) = p_2(y) = p_3(y) = p_4(y) = p_5(y) = p_6(y) = y^4 + y^3 + y^2 + y + 1$. According toTheorem 3, we can find $q(y)$ as $q(y) = gcld(p(y), y^5 - 1) = y^4 + y^3 + y^2 + y + 1$. Then, $\{p(y)\}$ forms a basis of C and $|C| = 9$. By using Theorem 2.1 of [19] $\ell d \leq d(C)$, i.e., $d(C) = 30$. Hence, C and $\Phi(C)$ are (β, γ)-skew QC codes over $\mathbf{S}$ with parameters $(30, 9, 30)$ and $(60, 3^2, 30)$, respectively.

Remark 1. *In the table, we use the following factorization:*

$$
\begin{aligned}
y^8 - 1 &= (wy^2 + wy + w^2)(wy^3 + w^2y^2 + w)(w^2y^3 + y + 1) \in \mathbb{F}_4[y; \beta, \gamma] \\
y^{16} - 1 &= (wy^{15} + y^{14} + wy^{13} + y^{12} + wy^{11} + y^{10} + wy^9 + y^8 + wy^7 + y^6 + wy^5 + y^4 \\
&\quad + wy^3 + y^2 + wy + 1)(wy + 1) \in \mathbb{F}_4[y; \beta, \gamma] \\
y^{22} - 1 &= (w^2y^{21} + y^{20} + w^2y^{19} + y^{18} + w^2y^{17} + y^{16} + w^2y^{15} + y^{14} + w^2y^{13} + y^{12} \\
&\quad + w^2y^{11} + y^{10} + w^2y^9 + y^8 + w^2y^7 + y^6 + w^2y^5 + y^4 \\
&\quad + w^2y^3 + y^2 + w^2y + 1)(w^2y + 1) \in \mathbb{F}_4[y; \beta, \gamma] \\
y^{26} - 1 &= (wy^{25} + wy^{24} + wy^{23} + wy^{22} + wy^{21}21 + wy^{20} + wy^{19} + wy^{18}18 + wy^{17} \\
&\quad + wy^{16} + wy^{15} + wy^{14} + wy^{13} + wy^{12}12 + wy^{11} + wy^{10} + wy^9 + wy^8 \\
&\quad + wy^7 + wy^6 + wy^5 + wy^4 + wy^3 + wy^2 + wy + w)(wy + w^2) \in \mathbb{F}_4[y; \beta, \gamma] \\
y^{28} - 1 &= (y^{27} + wy^{26} + y^{25}25 + wy^{24} = y^{23} + wy^{22} + y^{21} + wy^{20} + y^{19} + wy^{18} \\
&\quad + y^{17} + wy^{16} + y^{15} + wy^{14} + y^{13} + wy^{12} + y^{11} + wy^{10} + y^9 + wy^8 + y^7 \\
&\quad + wy^6 + y^5 + wy^4 + y^3 + wy^2 + y + w)(y + w^2) \in \mathbb{F}_4[y; \beta, \gamma] \\
y^{40} - 1 &= (wy^{39} + w^2y^{38} + wy^{37} + w^2y^{36} + wy^{35} + w^2y^{34} + wy^{33} + w^2y^{32} + wy^{31} \\
&\quad + w^2y^{30} + wy^{29} + w^2y^{28} + wy^{27} + w^2y^{26} + wy^{25} + w^2y^{24} + wy^{23} + w^2y^{22} \\
&\quad + wy^{21} + w^2y^{20} + wy^{19} + w^2y^{18} + wy^{17} + w^2y^{16} + wy^{15} + w^2y^{14} + wy^{13} \\
&\quad + w^2y^{12} + wy^{11} + w^2y^{10} + wy^9 + w^2y^8 + wy^7 + w^2y^6 + wy^5 + w^2y^4 \\
&\quad + wy^3 + w^2y^2 + wy + w^2)(wy + w) \in \mathbb{F}_4[y; \beta, \gamma].
\end{aligned}
$$

Table 1. The list of optimal codes.

n	q	Generators	k	$\Phi(C)$
8	4	$p_1(y) = wy^2 + wy + w^2$, $p_1(y) = wy^3 + w^2y^2 + w$	5	[16, 10, 3]
16	4	$p_1(y) = p_2(y) = wy + 1$	15	[32, 30, 2]
22	4	$p_1(y) = p_2(y) = w^2y + 1$	21	[44, 42, 2]
26	4	$p_1(y) = p_2(y) = wy + w^2$	24	[52, 48, 2]
28	4	$p_1(y) = p_2(y) = y + w^2$	27	[56, 54, 2]
40	4	$p_1(y) = p_2(y) = wy + w$	39	[80, 78, 2]

7. Conclusions

In the present article, the structural properties of a semi-local ring $\mathbf{S} = \mathbb{F}_q + u\mathbb{F}_q$, $u^2 = u$, $q = p^s$ with derivation have been studied. We have extended the notion of the multiplication of polynomials using automorphism and a derivation. With respect to such multiplication, the ring $\mathbf{S}[y; \beta, \gamma]$, where β is an automorphism of $\mathbf{S}$ and γ is a β-derivation on $\mathbf{S}$, forms a skew polynomial ring. Some results on the (β, γ)-skew cyclic code are demonstrated and they are very useful for determining the rank of the above-mentioned codes. Additionally, the skew-quasi cyclic codes over $\mathbf{S}$ and their properties are investigated. Further, we have obtained the duality of 1-generator skew cyclic and skew-quasi cyclic codes. Finally, some examples are given in support of our main results and a table of optimal codes is evaluated.

Author Contributions: Supervision, M.A. (Mohammad Ashraf); Conceptualization, M.A. (Mohammad Ashraf), A.S.A., M.A. (Mohd Asim) and G.M.; Methodology, M.A. (Mohammad Ashraf), A.S.A., M.A. (Mohd Asim) and G.M.; Software, M.A. (Mohd Asim); Writing-original draft, M.A. (Mohd Asim) and G.M.; Writing-review & editing, M.A. (Mohammad Ashraf), G.M. and A.S.A.; Validation, M.A. (Mohammad Ashraf), A.S.A.; Formal analysis, M.A. (Mohammad Ashraf), A.S.A. and G.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Princess Nourah bint Abdulrahman University grant number PNURSP2022R231.

Data Availability Statement: This manuscript has no associated data set.

Acknowledgments: The authors extend their appreciation to Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, for funding this research under the Researchers Supporting Project (No. PNURSP2022R231). The authors are also thankful to the anonymous referees for their fruitful comments and suggestions.

Conflicts of Interest: The authors state no conflict of interest.

References

1. Calderbank, A.R.; Hammons, A.R., Jr.; Kumar, P.V.; Sloane, N.J.A.; Solé, P. The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes. *IEEE Trans. Inf. Theory* **1994**, *40*, 301–319.
2. Siap, I.; Abualrub, T.; Aydin, N.; Seneviratne, P. Skew cyclic codes of arbitrary length. *Int. J. Inform. Coding Theory* **2011**, *2*, 10–20. [CrossRef]
3. Ore, O. Theory of non-commutative polynomials. *Ann. Math.* **1933**, *34*, 480–508. [CrossRef]
4. Smits, T.H.M. Skew polynomial rings. *Nederl. Akad. Wetensch. Proc. Ser. A 71 Indag. Math.* **1968**, *30*, 209–224. [CrossRef]
5. Abualrub, T.; Ghrayeb, A.; Aydin, N.; Siap, I. On the construction of skew quasi cyclic codes. *IEEE. Trans. Inf. Theory* **2010**, *56*, 2081–2090. [CrossRef]
6. Bhaintwal, M. Skew quasi cyclic codes over Galois rings. *Designs Codes Cryptogr.* **2012**, *62*, 85–101. [CrossRef]
7. Jitman, S.; Ling, S.; Udomkavanich, P. Skew constacyclic codes over finite chain rings. *Adv. Math. Commun.* **2012**, *6*, 29–63.
8. Boulagouaz, M.H.; Leroy, A. (σ, δ)-codes. *Adv. Math. Commun.* **2013**, *7*, 463–474.
9. Boucher, D.; Geiselmann, W.; Ulmer, F. Skew cyclic codes. *Appl. Algebra Eng. Comm.* **2007**, *18*, 379–389. [CrossRef]
10. Ashraf, M.; Mohammad, G. On skew cyclic codes over a semi-local ring. *Discrete Math. Algorithms Appl.* **2015**, *7*, 1550042 . [CrossRef]
11. Ashraf, M.; Mohammad, G. Skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$. *Asian Eur. J. Math.* **2018**, *11*, 1850072 . [CrossRef]
12. Sharma, A.; Bhaintwal, S. A class of skew-cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ with derivation. *Adv. Math. Commun.* **2018**, *12*, 723–739. [CrossRef]
13. Fanghui, M.; Jian, G.; Juan, L.; Fang-Wei, F. (σ, δ)-skew quasi-cyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$. *Cryptogr. Commun.* **2021**, *13*, 307–320.
14. Ashraf, M.; Mohammad, G. On skew cyclic codes over $\mathbb{F}_3 + v\mathbb{F}_3$. *Int. J. Inf. Coding Theory* **2014**, *2*, 218–225.
15. Aydin, N.; Karadeniz, S.; Yildiz, B. Some new binary quasi-cyclic codes from codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. *Appl. Algebra Eng. Commun. Comput.* **2013**, *24*, 355–367. [CrossRef]
16. Gursoy, F.; Siap, I.; Yildiz, B. Construction of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$. *Adv. Math. Commun.* **2014**, *3*, 313–322. [CrossRef]
17. Heijnen, P.; Tilborg, H.V.; Verhoeff, T.; Weijs, S. Some new binary quasi-cyclic codes. *IEEE Trans. Inf. Theory* **1998**, *45*, 1994–1996. [CrossRef]
18. Prakash, O.; Patel, S. (θ, δ_θ)-cyclic codes over $\mathbb{F}_q[u, v]/\langle u^2 - u, v^2 - v, uv - uv\rangle$. *Des. Codes Cryptogr.* **2022**, *90*, 2763–2781.
19. Aydin, N.; Halilovic, H. A generalization of quasi-twisted codes: Multi-twisted codes. *Finite Fields Appl.* **2017**, *45*, 96–106. [CrossRef]
20. Jacobson, N. Pseudo-linear transformations. *Ann. Math.* **1937**, *38*, 484–507. [CrossRef]
21. Leroy, A. Pseudo-linear transformation and evaluation in Ore extension. *Bull. Belg. Math. Soc. Simon Stevin* **1995**, *2*, 321–347. [CrossRef]
22. Cuitino, L.F.T.; Tironi, A. Some properties of skew codes over finite fields. *Des. Codes Cryptogr.* **2017**, *85*, 359–380. [CrossRef]
23. Ashraf, M.; Mohammad, G. On skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q$. *Tbilisi Math. J.* **2018**, *11*, 35–45. [CrossRef]