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Analysis and Numerical Simulation of System of Fractional Partial Differential Equations with Non-Singular Kernel Operators

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Abstract: The exact solution to fractional-order partial differential equations is usually quite difficult to achieve. Semi-analytical or numerical methods are thought to be suitable options for dealing with such complex problems. To elaborate on this concept, we used the decomposition method along with natural transformation to discover the solution to a system of fractional-order partial differential equations. Using certain examples, the efficacy of the proposed technique is demonstrated. The exact and approximate solutions were shown to be in close contact in the graphical representation of the obtained results. We also examine whether the proposed method can achieve a quick convergence with a minimal number of calculations. The present approaches are also used to calculate solutions in various fractional orders. It has been proven that fractional-order solutions converge to integer-order solutions to problems. The current technique can be modified for various fractional-order problems due to its simple and straightforward implementation.

Keywords: Adomian decomposition method; natural transform; Caputo–Fabrizio (CF) and Atangana–Baleanu Caputo operator (ABC); fractional-order coupled systems

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1. Introduction

Fractional analysis has been found to have numerous applications in many fields of science over the last few decades. Experiments have shown that fractional-order derivatives have good agreement with experimental data or real phenomena in many physical phenomena compared to derivatives with integer order. For example, the fractional-order derivative better distinguishes memory, understands the impacts of genetics on material characteristics, and processes internal friction [1–4]. Fractional calculus is currently an essential tool for describing numerous processes in physics, chemistry, engineering, and other sciences. Recent applications of fractional calculus in several fields have gained the attention of numerous scholars, and many discoveries have been made [5–7]. These facts have influenced many disciplines of science, with numerous applications in a variety of fields, such as the fractional-order time-delay system [8], the fractional Drinfeld–Sokolov–Wilson equation [9], time-fractional Swift–Hohenberg equations [10], the time-fractional Newell–Whitehead–Segel equation [11], fractional diffusion and the fractional Buck master's equation [12], fractal vehicular traffic flow [13], the time-fractional Belousov–Zhabotinskii reaction [14], fractional calculus and the dynamic system [15,16], the fractional model for the dynamics of Hepatitis B virus [17], the fractional model for tuberculosis [18], anomalous transport in disordered systems [19], the diffusion of biological populations [20], the fractional-order sliding mode-based extremum seeking control of a class of nonlinear systems [21], percolation in porous media [22], fractional-order regularized long-wave models [23], the fractional-order pine wilt disease model [24], time-fractional Klein–Gordon equations [25], fractional-order diffusion equations in a plasma and fluids [26], the time-fractional Burgers equation [27], the time-fractional Schrödinger equation [28], and so on [29–32].

Fractional partial differential equations (FPDEs) are the most common mathematical tools used to model numerous physical aspects in fields such as engineering, physics, and other social sciences. Many applications of engineering and science, such as fluid dynamics, biology, material sciences, chemical kinetics, chemistry, and many other physical processes, use simulations in the form of FPDE systems [33–37]. In biomechanics and engineering, coupled systems of fractional-order partial differential equations (PDEs) are frequently used. When describing the electrical activity of the heart in biomechanics, many implementations of coupled PDEs may develop [38–40]. Modeling other biological and physical engineering issues, such as a system with a continuous stirring boiler container and a series plug flow container [41,42], yields comparable results. Different applications can be employed in physics; for example, coupled fractional-order partial differential equations can be used to model the dynamic forces of multi-deformable objects coupled with typical light fractional-order discrete continuous surfaces [43]. Coupled PDE techniques are also used in the simulation of a number of important gravitational and electromagnetic problems [44,45]. The fractional differential equation is a helpful tool for representing nonlinear events in scientific and engineering models. In applied mathematics and engineering, partial differential equations, particularly nonlinear ones, have been utilized to simulate a wide range of scientific phenomena.

Fractional-order partial differential equations (FPDEs) allowed researchers to recognize and model a wide range of significant and real-world physical issues in parallel with their work in the physical sciences. It has always been claimed how important it is to obtain approximations for them using either numerical or analytical methods. Because of this, symmetry analysis is a fantastic tool for comprehending partial differential equations, especially when looking at equations generated from mathematical concepts connected to accounting. Despite the notion that symmetry is the foundation of nature, the bulk of observations in the natural world lacks it. A clever technique for disguising symmetry is to provide unanticipated symmetry-breaking events. The two categories are finite and infinitesimal symmetry. There are two types of discrete and continuous finite symmetries. Natural symmetries like parity and temporal inversion are discrete, while space is a continuous transformation. Mathematicians have always been fascinated by patterns.

Many mathematicians and physicists have recently introduced and developed new numerical and analytical approaches to obtain solutions and describe the physical behavior of a variety of differential and integral equations with integer or fractional-order characterizing real-world processes. Furthermore, various approaches have been presented in the literature, with the Adomian decomposition method (ADM) being the most popular due to its efficiency and accuracy [46]. ADM has been successfully and effectively used to investigate problems that have occurred in science and technology without linearization or perturbation. ADM also consumes more time and a large amount of computer memory for computational effort. As a result, the combination of this method with existing transform methods is certain. To meet these needs, Rawashdeh and Maitama developed the FNDM [47,48], which is a combination of the ADM and the natural transform technique (NTM). Because FNDM is an improved form of ADM, and it will save time and effort by reducing computations. It also does not require linearization, discretization, or perturbation.

Many authors have recently examined the projected technique to interpret solutions to various nonlinear problems due to its efficacy and reliability [49–51]. Because the considered approach allows us to consider an initial guess and the equation type of linear sub-problems, complex nonlinear differential equations can be investigated using a simple procedure. The unique feature of FNDM is that it uses a simple algorithm to discover the solution described by the Adomian polynomial, and it enables rapid convergence in the achieved solution for the nonlinear part. To solve fractional dynamical systems, we use the natural transform decomposition method (NTDM) in combination with two alternative fractional derivatives. The current approach is found to be very effective for the solution of systems of fractional differential equations. The numerical results of the suggested method are compared with the exact solutions to the problems. The comparisons show a sufficient degree of accuracy.

2. Basic Definitions

In this section, we present some main definitions and notations that will be used in this study.

Definition 1. [52] *The fractional Riemann–Liouville integral operator is defined as:*

$$I^{\kappa}j(\varphi)=\frac{1}{\Gamma(\kappa)}\int_0^{\varphi}(\varphi-\nu)^{\kappa-1}j(\nu)d\nu,\quad \kappa>0,\varphi>0 \tag{1}$$

and $I^0 j(\varphi)=j(\varphi)$.

Definition 2. [52] *The fractional Caputo's derivative of* $j(\varphi)$ *is given as:*

$$D_{\varphi}^{\kappa}j(\varphi)=I^{m-\kappa}D^{m}j(\varphi)=\frac{1}{m-\kappa}\int_0^{\varphi}(\varphi-\nu)^{m-\kappa-1}j^{(m)}(\nu)d\nu \tag{2}$$

for $m-1<\kappa\leq m, m\in\mathbb{N}, \varphi>0, j\in C_{\nu}^{m}$ *and* $\nu\geq -1$.

Definition 3. [52] *The fractional CF derivative of* $j(\varphi)$ *is defined as:*

$$D_{\varphi}^{\kappa}j(\varphi)=\frac{F(\kappa)}{1-\kappa}\int_0^{\varphi}\exp\left(\frac{-\kappa(\varphi-\nu)}{1-\kappa}\right)D(j(\nu))d\nu \tag{3}$$

with $0<\kappa<1$ *and* $F(\kappa)$ *is a normalization function with* $F(0)=F(1)=1$.

Definition 4. [52] *The fractional ABC derivative of* $j(\varphi)$ *is defined as:*

$$D_{\varphi}^{\kappa}j(\varphi)=\frac{B(\kappa)}{1-\kappa}\int_0^{\varphi}E_{\kappa}\left(\frac{-\kappa(\varphi-\nu)}{1-\kappa}\right)D(j(\nu))d\nu \tag{4}$$

with $0<\kappa<1, B(\kappa)$ *is normalization function and*

$$E_{\kappa}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\kappa+1)}$$

represents the Mittag–Leffler function.

Definition 5. *The natural transform (NT) of a function* $\mathbb{X}(\delta)$ *is stated as:*

$$\mathbf{N}\{\mathbb{X}(\delta)\}=\mathcal{U}(\xi,\vartheta)=\int_{-\infty}^{\infty}e^{-\xi\delta}\mathbb{X}(\vartheta\delta)d\delta,\quad \xi,\vartheta\in(-\infty,\infty) \tag{5}$$

and, for $\delta\in(0,\infty)$, *the NT of* $\mathbb{X}(\delta)$ *is defined as:*

$$\mathbf{N}\{\mathbb{X}(\delta)H(\delta)\}=\mathbf{N}^{+}\{\mathbb{X}(\delta)\}=\mathcal{U}^{+}(\xi,\vartheta)=\int_0^{\infty}e^{-\xi\delta}\mathbb{X}(\vartheta\delta)d\delta,\quad \xi,\vartheta\in(0,\infty), \tag{6}$$

where $H(\delta)$ *is the Heaviside function.*

Definition 6. *The inverse NT of a function* $\mathbb{X}(\xi,\vartheta)$ *is stated as:*

$$\mathbf{N}^{-1}\{\mathcal{U}(\xi,\vartheta)\}=\mathbb{X}(\delta) \tag{7}$$

for all $\delta \geq 0$.

Lemma 1. *Suppose* $\mathcal{U}_1(\xi,\vartheta)$ *and* $\mathcal{U}_2(\xi,\vartheta)$ *are NT of* $\mathbb{X}_1(\delta)$ *and* $\mathbb{X}_2(\delta)$*, then*

$$\begin{aligned}\mathbf{N}\{c_1\mathbb{X}_1(\delta)+c_2\mathbb{X}_2(\delta)\} &= c_1\mathbf{N}\{\mathbb{X}_1(\delta)\}+c_2\mathbf{N}\{\mathbb{X}_2(\delta)\}\\ &= c_1\mathcal{U}_1(\xi,\vartheta)+c_2\mathcal{U}_2(\xi,\vartheta)\end{aligned} \tag{8}$$

with c_1 *and* c_2 *are constants.*

Lemma 2. *Suppose* $\mathbb{X}_1(\xi,\vartheta)$ *and* $\mathbb{X}_2(\xi,\vartheta)$ *are the inverse NT of* $\mathbb{X}_1(\delta)$ *and* $\mathbb{X}_2(\delta)$*, then*

$$\begin{aligned}\mathbf{N}^{-1}\{c_1\mathcal{U}_1(\xi,\vartheta)+c_2\mathcal{U}_2(\xi,\vartheta)\} &= c_1\mathbf{N}^{-1}\{\mathcal{U}_1(\xi,\vartheta)\}+c_2\mathbf{N}^{-1}\{\mathcal{U}_2(\xi,\vartheta)\}\\ &= c_1\mathbb{X}_1(\delta)+c_2\mathbb{X}_2(\delta)\end{aligned} \tag{9}$$

with c_1 *and* c_2 *constants.*

Definition 7. [52] *In the Caputo manner, the NT of* $D_\delta^\kappa\mathbb{X}(\delta)$ *is defined as:*

$$\mathbf{N}\{D_\delta^\kappa\mathbb{X}(\delta)\} = \left(\frac{\xi}{\vartheta}\right)^\kappa\left(\mathbf{N}\{\mathbb{X}(\delta)\}-\left(\frac{1}{\xi}\right)\mathbb{X}(0)\right). \tag{10}$$

Definition 8. [52] *In the CF manner, the NT of* $D_\delta^\kappa\mathbb{X}(\delta)$ *is defined as:*

$$\mathbf{N}\{D_\delta^\kappa\mathbb{X}(\delta)\} = \frac{1}{1-\kappa+\kappa\left(\frac{\vartheta}{\xi}\right)}\left(\mathbf{N}\{\mathbb{X}(\delta)\}-\left(\frac{1}{\xi}\right)\mathbb{X}(0)\right). \tag{11}$$

Definition 9. [52] *In ABC manner, the NT of* $D_\delta^\kappa\mathbb{X}(\delta)$ *is defined as:*

$$\mathbf{N}\{D_\delta^\kappa\mathbb{X}(\delta)\} = \frac{M[\kappa]}{1-\kappa+\kappa\left(\frac{\vartheta}{\xi}\right)^\kappa}\left(\mathbf{N}\{\mathbb{X}(\delta)\}-\left(\frac{1}{\xi}\right)\mathbb{X}(0)\right) \tag{12}$$

with $M[\kappa]$ *representing a normalization function.*

Definition 10. *The inverse natural transform* $\mathbf{N}^{-1}$ *is stated as*

$$\mathbf{N}^{-1}\{\mathcal{U}(\xi,\vartheta)\} = \mathbb{X}(\delta) = \lim_{T\to\infty}\frac{1}{2\pi\imath}\int_{\sigma-\imath T}^{\sigma+\imath T} e^{\frac{\xi\delta}{\vartheta}}\mathcal{U}(\xi,\vartheta)d\xi. \tag{13}$$

3. Methodology

In this part, we give some background about the nature of the proposed technique.

$$D_\delta^\kappa\mathbb{X}(\wp,\delta) = \mathcal{L}(\mathbb{X}(\wp,\delta))+N(\mathbb{X}(\wp,\delta))+h(\wp,\delta) = M(\wp,\delta) \tag{14}$$

with the initial condition

$$\mathbb{X}(\wp,0) = \phi(\wp) \tag{15}$$

where $\mathcal{L}, N$ are the linear and nonlinear differential operators and $h(\wp,\delta)$ is the source term.

3.1. Case I ($NTDM_{CF}$)

By applying the CF fractional derivative in connection with the NT, (14) may be expressed as

$$\frac{1}{p(\kappa,\vartheta,\xi)}\left(\mathbf{N}\{\mathbb{X}(\wp,\delta)\}-\frac{\phi(\wp)}{\xi}\right) = \mathbf{N}\{M(\wp,\delta)\} \tag{16}$$

with

$$p(\kappa,\vartheta,\xi) = 1-\kappa+\kappa\left(\frac{\vartheta}{\xi}\right). \tag{17}$$

After we use the inverse natural transform, then we have

$$\mathbb{X}(\wp,\delta) = \mathbf{N}^{-1}\left\{\frac{\phi(\wp)}{\xi} + p(\kappa,\vartheta,\xi)\mathbf{N}\{M(\wp,\delta)\}\right\}. \tag{18}$$

Assume that the unknown function $\mathbb{X}(\wp,\delta)$ has the following solution in the infinite series form:

$$\mathbb{X}(\wp,\delta) = \sum_{i=0}^{\infty}\mathbb{X}_i(\wp,\delta) \tag{19}$$

and the decomposition of $N(\mathbb{X}(\wp,\delta))$ is stated as

$$N(\mathbb{X}(\wp,\delta)) = \sum_{i=0}^{\infty} A_i(\mathbb{X}_0,\ldots,\mathbb{X}_i). \tag{20}$$

By means of the Adomian polynomials, the nonlinear terms are calculated as

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}N\left(t,\sum_{k=0}^{n}\varepsilon^k\mathbb{X}_k\right)\Bigg|_{\varepsilon=0}.$$

Substituting (19) and (20) into (18) gives

$$\begin{aligned}\sum_{i=0}^{\infty}\mathbb{X}_i(\wp,\delta) = {}& \mathbf{N}^{-1}\left\{\frac{\phi(\wp)}{\xi} + p(\kappa,\vartheta,\xi)\mathbf{N}\{h(\wp,\delta)\}\right\}\\ &+\mathbf{N}^{-1}\left\{p(\kappa,\vartheta,\xi)\mathbf{N}\left\{\sum_{i=0}^{\infty}\mathcal{L}(\mathbb{X}_i(\wp,\delta))+A_\delta\right\}\right\}.\end{aligned} \tag{21}$$

From (21), we have

$$\begin{aligned}\mathbb{X}_0^{CF}(\wp,\delta) &= \mathbf{N}^{-1}\left\{\frac{\phi(\wp)}{\xi} + p(\kappa,\vartheta,\xi)\mathbf{N}\{h(\wp,\delta)\}\right\},\\ \mathbb{X}_1^{CF}(\wp,\delta) &= \mathbf{N}^{-1}\{p(\kappa,\vartheta,\xi)\mathbf{N}\{\mathcal{L}(\mathbb{X}_0(\wp,\delta))+A_0\}\},\\ &\vdots\\ \mathbb{X}_{l+1}^{CF}(\wp,\delta) &= \mathbf{N}^{-1}\{p(\kappa,\vartheta,\xi)\mathbf{N}\{\mathcal{L}(\mathbb{X}_l(\wp,\delta))+A_l\}\}\end{aligned} \tag{22}$$

for $l \in \mathbb{N}$.

In this manner, the solution of (14), in terms of $NTDM_{CF}$, is obtained by putting (22) into (19)

$$\mathbb{X}^{CF}(\wp,\delta) = \mathbb{X}_0^{CF}(\wp,\delta) + \mathbb{X}_1^{CF}(\wp,\delta) + \mathbb{X}_2^{CF}(\wp,\delta) + \cdots. \tag{23}$$

3.2. *Case II* $(NTDM_{ABC})$

By applying the CF fractional derivative in connection with the NT, (14) may be expressed as

$$\frac{1}{q(\kappa,\vartheta,\xi)}\left(\mathbf{N}\{\mathbb{X}(\wp,\delta)\} - \frac{\phi(\wp)}{\xi}\right) = \mathbf{N}\{M(\wp,\delta)\} \tag{24}$$

with

$$q(\kappa,\vartheta,\xi) = \frac{1-\kappa+\kappa\left(\frac{\vartheta}{\xi}\right)^{\kappa}}{B(\kappa)}. \tag{25}$$

After we use the inverse natural transform, then

$$\mathbb{X}(\wp,\delta) = \mathbf{N}^{-1}\left\{\frac{\phi(\wp)}{\xi} + q(\kappa,\vartheta,\xi)\mathbf{N}\{M(\wp,\delta)\}\right\}. \tag{26}$$

In terms of the Adomian decomposition, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\mathbb{X}_i(\wp,\delta) = \mathbf{N}^{-1}&\left\{\frac{\phi(\wp)}{\xi} + q(\kappa,\vartheta,\xi)\mathbf{N}\{h(\wp,\delta)\}\right\}\\ &+ \mathbf{N}^{-1}\left\{q(\kappa,\vartheta,\xi)\mathbf{N}\left\{\sum_{i=0}^{\infty}\mathcal{L}(\mathbb{X}_i(\wp,\delta)) + A_\delta\right\}\right\}.\end{aligned} \tag{27}$$

From (21), we have

$$\begin{aligned}\mathbb{X}_0^{ABC}(\wp,\delta) &= \mathbf{N}^{-1}\left\{\frac{\phi(\wp)}{\xi} + q(\kappa,\vartheta,\xi)\mathbf{N}\{h(\wp,\delta)\}\right\},\\ \mathbb{X}_1^{ABC}(\wp,\delta) &= \mathbf{N}^{-1}\{q(\kappa,\vartheta,\xi)\mathbf{N}\{\mathcal{L}(\mathbb{X}_0(\wp,\delta)) + A_0\}\},\\ &\vdots\\ \mathbb{X}_{l+1}^{ABC}(\wp,\delta) &= \mathbf{N}^{-1}\{q(\kappa,\vartheta,\xi)\mathbf{N}\{\mathcal{L}(\mathbb{X}_l(\wp,\delta)) + A_l\}\}\end{aligned} \tag{28}$$

for $l \in \mathbb{N}$.

In this manner, the solution of (14), in terms of $NTDM_{ABC}$, is

$$\mathbb{X}^{ABC}(\wp,\delta) = \mathbb{X}_0^{ABC}(\wp,\delta) + \mathbb{X}_1^{ABC}(\wp,\delta) + \mathbb{X}_2^{ABC}(\wp,\delta) + \cdots. \tag{29}$$

4. Convergence Analysis

In this section, we discuss the uniqueness and convergence of the $NTDM_{CF}$ and $NTDM_{ABC}$.

The proof of the following Theorems are given in [53].

Theorem 1. *Suppose that* $|\mathcal{L}(\mathbb{X}) - \mathcal{L}(\mathbb{X}^*)| < \gamma_1|\mathbb{X} - \mathbb{X}^*|$ *and* $|N(\mathbb{X}) - N(\mathbb{X}^*)| < \gamma_2|\mathbb{X} - \mathbb{X}^*|$, *where* $\mathbb{X} := \mathbb{X}(\mu,\delta)$ *and* $\mathbb{X}^* := \mathbb{X}^*(\mu,\delta)$ *are two different function values,* γ_1, γ_2 *are Lipschitz constants and* $\mathcal{L}, N$ *are the operators defined in* (14). *Then, the problem* (14) *has a unique solution for* $NTDM_{CF}$, *when* $0 < (\gamma_1 + \gamma_2)(1 - \kappa + \kappa\delta) < 1$ *for all* δ.

Theorem 2. *Under the same hypothesis as in Theorem 1, the problem* (14) *has a unique solution for* $NTDM_{ABC}$, *when* $0 < (\gamma_1 + \gamma_2)\left(1 - \kappa + \kappa\frac{\delta^\kappa}{\Gamma(\kappa+1)}\right) < 1$ *for all* δ.

Theorem 3. *Suppose* $\mathcal{L}$ *and* N *are Lipschitz functions as in Theorem 1, then the* $NTDM_{CF}$ *result of* (14) *is convergent.*

Theorem 4. *Suppose* $\mathcal{L}$ *and* N *are Lipschitz functions as in Theorem 1, then the* $NTDM_{ABC}$ *result of* (14) *is convergent.*

5. Applications

Example 1. *Let us consider the fractional PDE system*

$$\begin{aligned}D_\delta^\kappa\mathbb{X} - \mathbb{Y}_\wp + \mathbb{Y} + \mathbb{X} &= 0,\\ D_\delta^\kappa\mathbb{Y} - \mathbb{X}_\wp + \mathbb{Y} + \mathbb{X} &= 0, \quad 0 < \kappa \leq 1\end{aligned} \tag{30}$$

with the initial conditions

$$\begin{aligned}\mathbb{X}(\wp,0) &= \sinh(\wp),\\ \mathbb{Y}(\wp,0) &= \cosh(\wp).\end{aligned} \tag{31}$$

Applying the NT, we have

$$\begin{aligned}\mathbf{N}\{D_\delta^\kappa \mathbb{X}(\wp,\delta)\} &= \mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\},\\ \mathbf{N}\{D_\delta^\kappa \mathbb{Y}(\wp,\delta)]\} &= \mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\}.\end{aligned} \tag{32}$$

By using the transform property, we have

$$\begin{aligned}\frac{1}{\xi^\kappa}\mathbf{N}\{\mathbb{X}(\wp,\delta)\} - \xi^{2-\kappa}\mathbb{X}(\wp,0) &= \mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\},\\ \frac{1}{\xi^\kappa}\mathbf{N}\{\mathbb{Y}(\wp,\delta)\} - \xi^{2-\kappa}\mathbb{Y}(\wp,0) &= \mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\}.\end{aligned} \tag{33}$$

The above algorithm's simplified form is

$$\begin{aligned}\mathbf{N}\{\mathbb{X}(\wp,\delta)\} &= \xi^2\sinh(\wp) - \frac{\kappa(\xi-\kappa(\xi+\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\},\\ \mathbf{N}\{\mathbb{Y}(\wp,\delta)\} &= \xi^2\cosh(\wp) - \frac{\kappa(\xi-\kappa(\xi+\kappa))}{\xi^2}\mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\}.\end{aligned} \tag{34}$$

Using the inverse NT, we obtain

$$\begin{aligned}\mathbb{X}(\wp,\delta) &= \sinh(\wp) + \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\}\right\},\\ \mathbb{Y}(\wp,\delta) &= \cosh(\wp) + \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\}\right\}.\end{aligned} \tag{35}$$

***Solution by Means of** NDM_{CF}*

Assume that the unknown functions $\mathbb{X}(\wp,\delta)$ and $\mathbb{Y}(\wp,\delta)$ have the following solution in the infinite series form:

$$\mathbb{X}(\wp,\delta) = \sum_{l=0}^{\infty}\mathbb{X}_l(\wp,\delta) \quad \textit{and} \quad \mathbb{Y}(\wp,\delta) = \sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\delta). \tag{36}$$

Thus, (35) *can be rewritten using certain terms as*

$$\begin{aligned}\sum_{l=0}^{\infty}\mathbb{X}_{l+1}(\wp,\delta) &= \sinh(\wp) + \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\}\right\},\\ \sum_{l=0}^{\infty}\mathbb{Y}_{l+1}(\wp,\delta) &= \cosh(\wp) + \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\}\right\}.\end{aligned} \tag{37}$$

Thus, by comparing both sides of (37)*, we obtain*

$$\begin{aligned}\mathbb{X}_0(\wp,\delta) &= \sinh(\wp),\\ \mathbb{Y}_0(\wp,\delta) &= \cosh(\wp),\\ \mathbb{X}_1(\wp,\delta) &= -\cosh(\kappa(\delta-1)+1),\\ \mathbb{Y}_1(\wp,\delta) &= -\sinh(\kappa(\delta-1)+1),\\ \mathbb{X}_2(\wp,\delta) &= \sinh\left((1-\kappa)^2 + 2\kappa(1-\kappa)\delta + \frac{\kappa^2\delta^2}{2}\right),\\ \mathbb{Y}_2(\wp,\delta) &= \cosh\left((1-\kappa)^2 + 2\kappa(1-\kappa)\delta + \frac{\kappa^2\delta^2}{2}\right).\end{aligned}$$

In the same manner, the remaining $\mathbb{X}_l$ and $\mathbb{Y}_l$ $(l \geq 3)$ elements are easy to obtain. So, we describe the alternative sequences as

$$\begin{aligned}
\mathbb{X}(\wp,\delta) &= \sum_{l=0}^{\infty} \mathbb{X}_l(\wp,\delta) = \mathbb{X}_0(\wp,\delta) + \mathbb{X}_1(\wp,\delta) + \mathbb{X}_2(\wp,\delta) + \cdots, \\
&= \sinh(\wp) - \cosh(\kappa(\delta-1)+1) + \sinh\left((1-\kappa)^2 + 2\kappa(1-\kappa)\delta + \frac{\kappa^2\delta^2}{2}\right) + \cdots. \\
\mathbb{Y}(\wp,\delta) &= \sum_{l=0}^{\infty} \mathbb{Y}_l(\wp,\delta) = \mathbb{Y}_0(\wp,\delta) + \mathbb{Y}_1(\wp,\delta) + \mathbb{Y}_2(\wp,\delta) + \cdots, \\
&= \cosh(\wp) - \sinh(\kappa(\delta-1)+1) + \cosh\left((1-\kappa)^2 + 2\kappa(1-\kappa)\delta + \frac{\kappa^2\delta^2}{2}\right) + \cdots.
\end{aligned}$$

***Solution by Means of** NDM_{ABC}*

Assume that the unknown functions $\mathbb{X}(\wp,\delta)$ and $\mathbb{Y}(\wp,\delta)$ have the following solution in the infinite series form:

$$\mathbb{X}(\wp,\delta) = \sum_{l=0}^{\infty} \mathbb{X}_l(\wp,\delta) \quad \textit{and} \quad \mathbb{Y}(\wp,\delta) = \sum_{l=0}^{\infty} \mathbb{Y}_l(\wp,\delta). \tag{38}$$

Thus, (35) *can be rewritten using certain terms as*

$$\begin{aligned}
\sum_{l=0}^{\infty} \mathbb{X}_{l+1}(\wp,\delta) &= \sinh(\wp) - \mathbf{N}^{-1}\left\{ \frac{\vartheta^\kappa(\xi^\kappa + \kappa(\vartheta^\kappa - \xi^\kappa))}{\xi^{2\kappa}} \mathbf{N}\{\mathbb{Y}_\wp - \mathbb{Y} - \mathbb{X}\} \right\}, \\
\sum_{l=0}^{\infty} \mathbb{Y}_{l+1}(\wp,\delta) &= \cosh(\wp) - \mathbf{N}^{-1}\left\{ \frac{\vartheta^\kappa(\xi^\kappa + \kappa(\vartheta^\kappa - \xi^\kappa))}{\xi^{2\kappa}} \mathbf{N}\{\mathbb{X}_\wp - \mathbb{Y} - \mathbb{X}\} \right\}.
\end{aligned} \tag{39}$$

Thus, by comparing both sides of (39)*, we obtain*

$$\begin{aligned}
\mathbb{X}_0(\wp,\delta) &= \sinh(\wp), \\
\mathbb{Y}_0(\wp,\delta) &= \cosh(\wp), \\
\mathbb{X}_1(\wp,\delta) &= -\cosh\left(1 - \kappa + \frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right), \\
\mathbb{Y}_1(\wp,\delta) &= -\sinh\left(1 - \kappa + \frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right), \\
\mathbb{X}_2(\wp,\delta) &= \sinh\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)} + 2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)} + (1-\kappa)^2\right), \\
\mathbb{Y}_2(\wp,\delta) &= \cosh\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)} + 2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)} + (1-\kappa)^2\right).
\end{aligned}$$

In the same manner, the remaining $\mathbb{X}_l$ and $\mathbb{Y}_l$ $(l \geq 3)$ elements are easy to obtain. So, we describe the alternative sequences as:

$$\begin{aligned}
\mathbb{X}(\wp,\delta) &= \sum_{l=0}^{\infty} \mathbb{X}_l(\wp,\delta) = \mathbb{X}_0(\wp,\delta) + \mathbb{X}_1(\wp,\delta) + \mathbb{X}_2(\wp,\delta) + \cdots \\
&= \sinh(\wp) - \cosh\left(1 - \kappa + \frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right) \\
&\quad + \sinh\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)} + 2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)} + (1-\kappa)^2\right) + \cdots,
\end{aligned}$$

$$\begin{aligned}\mathbb{Y}(\wp,\delta) &= \sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\delta) = \mathbb{Y}_0(\wp,\delta)+\mathbb{Y}_1(\wp,\delta)+\mathbb{Y}_2(\wp,\delta)+\cdots \\ &= \cosh(\wp) - \sinh\left(1-\kappa+\frac{\kappa\delta^{\kappa}}{\Gamma(\kappa+1)}\right) \\ &\quad + \cosh\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)} + 2\kappa(1-\kappa)\frac{\delta^{\kappa}}{\Gamma(\kappa+1)} + (1-\kappa)^2\right) + \cdots .\end{aligned}$$

At $\kappa = 1$, the exact solution of (30) *is*

$$\begin{aligned}\mathbb{X}(\wp,\delta) &= \sinh(\wp-\delta), \\ \mathbb{Y}(\wp,\delta) &= \cosh(\wp-\delta).\end{aligned} \tag{40}$$

In Figure 1, the exact and approximate solutions, respectively, for system (30)*. In Figure 2, the approximate solution of fractional-order at $\kappa = 0.8, 0.6$ for system* (30)*. In Figure 3, approximate solution to system* (30) *at various values of κ with respect to two and three dimensional. In Tables 1 and 2 show that the absolute error obtained for various values of δ of system* (30)*.*

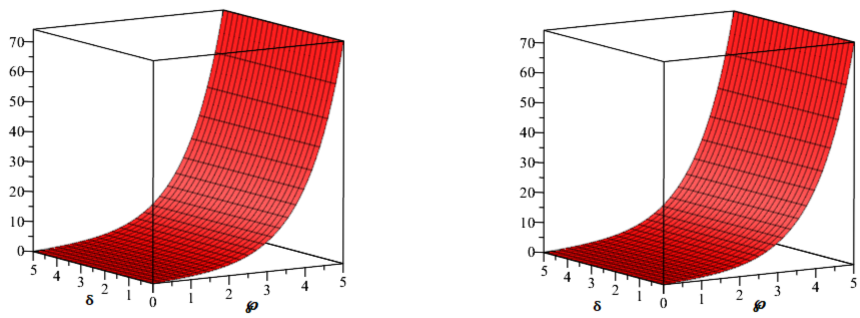

Figure 1. The exact approximate solutions, respectively, for system (30).

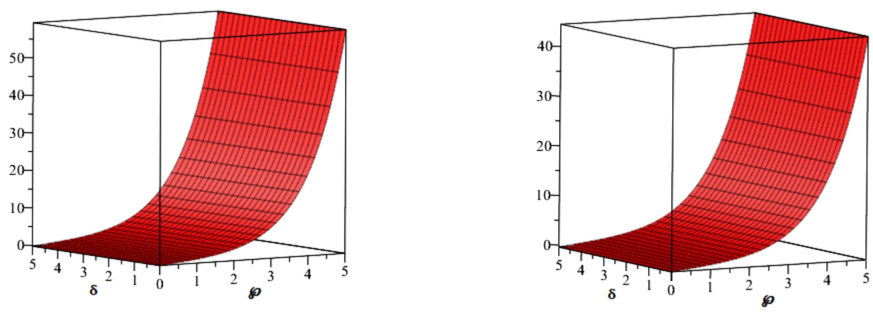

Figure 2. The approximate solution when $\kappa = 0.8, 0.6$ for system (30).

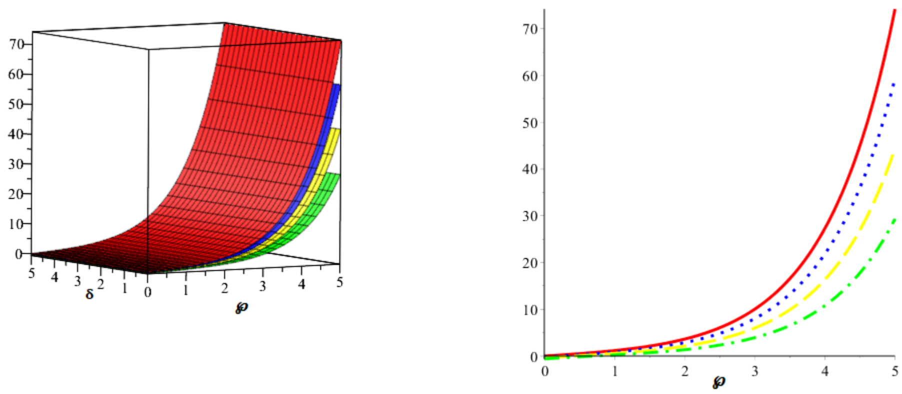

Figure 3. The approximate solution to system (30) at various values of κ.

Table 1. The absolute error obtained at different values of δ for system (30).

$(\wp,\delta)$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.4$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.6$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.8$	$(NTDM_{CF})$ at $\kappa=1$	$(NTDM_{ABC})$ at $\kappa=1$
(0.3,0.01)	$2.0926431500\times10^{-2}$	$1.0462803100\times10^{-2}$	$1.0462464000\times10^{-3}$	$1.6000000000\times10^{-9}$	$1.6000000000\times10^{-9}$
(0.5,0.02)	$2.2573729200\times10^{-2}$	$1.1286419900\times10^{-2}$	$1.1286062000\times10^{-3}$	$2.6000000000\times10^{-9}$	$2.6000000000\times10^{-9}$
(0.7,0.03)	$2.5126989900\times10^{-2}$	$1.2563000400\times10^{-2}$	$1.2562610000\times10^{-3}$	$3.8000000000\times10^{-9}$	$3.8000000000\times10^{-9}$
(0.3,0.01)	$2.0942676400\times10^{-2}$	$1.0470649800\times10^{-2}$	$1.0470109000\times10^{-3}$	$6.1000000000\times10^{-9}$	$6.1000000000\times10^{-9}$
(0.5,0.02)	$2.2591255700\times10^{-2}$	$1.1294887100\times10^{-2}$	$1.1294338000\times10^{-3}$	$1.0400000000\times10^{-8}$	$1.0400000000\times10^{-8}$
(0.7,0.03)	$2.5146501600\times10^{-2}$	$1.2572428100\times10^{-2}$	$1.2571849000\times10^{-3}$	$1.5200000000\times10^{-8}$	$1.5200000000\times10^{-8}$
(0.3,0.01)	$2.0957665900\times10^{-2}$	$1.0477912100\times10^{-2}$	$1.0477241000\times10^{-3}$	$1.3800000000\times10^{-8}$	$1.3800000000\times10^{-8}$
(0.5,0.02)	$2.2607429900\times10^{-2}$	$1.1302725900\times10^{-2}$	$1.1302079000\times10^{-3}$	$2.3400000000\times10^{-8}$	$2.3400000000\times10^{-8}$
(0.7,0.03)	$2.5164509700\times10^{-2}$	$1.2581158000\times10^{-2}$	$1.2580512000\times10^{-3}$	$3.4200000000\times10^{-8}$	$3.4200000000\times10^{-8}$
(0.3,0.01)	$2.0971849200\times10^{-2}$	$1.0484799000\times10^{-2}$	$1.0484038000\times10^{-3}$	$2.4400000000\times10^{-8}$	$2.4400000000\times10^{-8}$
(0.5,0.02)	$2.2622736500\times10^{-2}$	$1.1310161700\times10^{-2}$	$1.1309479000\times10^{-3}$	$4.1700000000\times10^{-8}$	$4.1700000000\times10^{-8}$
(0.7,0.03)	$2.5181553800\times10^{-2}$	$1.2589441000\times10^{-2}$	$1.2588810000\times10^{-3}$	$6.0700000000\times10^{-8}$	$6.0700000000\times10^{-8}$
(0.3,0.01)	$2.0985436600\times10^{-2}$	$1.0491407700\times10^{-2}$	$1.0490603000\times10^{-2}$	$3.8100000000\times10^{-8}$	$3.8100000000\times10^{-8}$
(0.5,0.02)	$2.2637402200\times10^{-2}$	$1.1317299300\times10^{-2}$	$1.1316648000\times10^{-3}$	$6.5100000000\times10^{-8}$	$6.5100000000\times10^{-8}$
(0.7,0.03)	$2.5197886200\times10^{-2}$	$1.2597393900\times10^{-2}$	$1.2596870000\times10^{-3}$	$9.4800000000\times10^{-8}$	$9.4800000000\times10^{-8}$

Table 2. The absolute error obtained for various values of δ of system (30).

$(\wp,\delta)$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.4$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.6$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.8$	$(NTDM_{CF})$ at $\kappa=1$	$(NTDM_{ABC})$ at $\kappa=1$
(0.3,0.01)	$6.0961380000\times10^{-3}$	$3.0479510000\times10^{-3}$	$3.0478900000\times10^{-4}$	$5.0000000000\times10^{-9}$	$5.0000000000\times10^{-9}$
(0.5,0.02)	$1.0431712000\times10^{-2}$	$5.2156530000\times10^{-3}$	$5.2155300000\times10^{-4}$	$6.0000000000\times10^{-9}$	$6.0000000000\times10^{-9}$
(0.7,0.03)	$1.5185947000\times10^{-2}$	$7.5926770000\times10^{-3}$	$7.5924800000\times10^{-4}$	$6.0000000000\times10^{-9}$	$6.0000000000\times10^{-9}$
(0.3,0.01)	$6.1008850000\times10^{-3}$	$3.0502520000\times10^{-3}$	$3.0502700000\times10^{-4}$	$2.1000000000\times10^{-8}$	$2.1000000000\times10^{-8}$
(0.5,0.02)	$1.0439825000\times10^{-2}$	$5.2195790000\times10^{-3}$	$5.2194900000\times10^{-4}$	$2.3000000000\times10^{-8}$	$2.3000000000\times10^{-8}$
(0.7,0.03)	$1.5197751000\times10^{-2}$	$7.5983860000\times10^{-3}$	$7.5981800000\times10^{-4}$	$2.5000000000\times10^{-8}$	$2.5000000000\times10^{-8}$
(0.3,0.01)	$6.1052750000\times10^{-3}$	$3.0523910000\times10^{-3}$	$3.0525800000\times10^{-4}$	$4.7000000000\times10^{-8}$	$4.7000000000\times10^{-8}$
(0.5,0.02)	$1.0447321000\times10^{-2}$	$5.2232230000\times10^{-3}$	$5.2232800000\times10^{-4}$	$5.1000000000\times10^{-8}$	$5.1000000000\times10^{-8}$
(0.7,0.03)	$1.5208654000\times10^{-2}$	$7.6036820000\times10^{-3}$	$7.6036100000\times10^{-4}$	$5.6000000000\times10^{-8}$	$5.6000000000\times10^{-8}$
(0.3,0.01)	$6.1094410000\times10^{-3}$	$3.0544310000\times10^{-3}$	$3.0549000000\times10^{-4}$	$8.4000000000\times10^{-8}$	$8.4000000000\times10^{-8}$
(0.5,0.02)	$1.0454426000\times10^{-2}$	$5.2266910000\times10^{-3}$	$5.2270100000\times10^{-4}$	$9.0000000000\times10^{-8}$	$9.0000000000\times10^{-8}$
(0.7,0.03)	$1.5218983000\times10^{-2}$	$7.6087160000\times10^{-3}$	$7.6089100000\times10^{-4}$	$1.0000000000\times10^{-7}$	$1.0000000000\times10^{-7}$
(0.3,0.01)	$6.1134420000\times10^{-3}$	$3.0563990000\times10^{-3}$	$3.0572500000\times10^{-4}$	$1.3100000000\times10^{-7}$	$1.3100000000\times10^{-7}$
(0.5,0.02)	$1.0461243000\times10^{-2}$	$5.2300290000\times10^{-3}$	$5.2307200000\times10^{-4}$	$1.4100000000\times10^{-7}$	$1.4100000000\times10^{-7}$
(0.7,0.03)	$1.5228890000\times10^{-2}$	$5.2300290000\times10^{-3}$	$7.6141400000\times10^{-4}$	$1.5700000000\times10^{-7}$	$1.5700000000\times10^{-7}$

Example 2. *Let us consider the fractional PDE system*

$$
\begin{aligned}
&D_{\delta}^{\kappa}\mathbb{X}+\mathbb{Y}_{\wp}\mathbb{Z}_{\rho}-\mathbb{Y}_{\rho}\mathbb{Z}_{\wp}=\mathbb{X},\\
&D_{\delta}^{\kappa}\mathbb{Y}+\mathbb{Z}_{\wp}\mathbb{X}_{\rho}-\mathbb{X}_{\rho}\mathbb{Z}_{\wp}=\mathbb{Y},\\
&D_{\delta}^{\kappa}\mathbb{Z}+\mathbb{X}_{\wp}\mathbb{Y}_{\rho}-\mathbb{X}_{\rho}\mathbb{Y}_{\wp}=\mathbb{Z},\quad 0<\kappa\leq 1
\end{aligned}
\tag{41}
$$

with the initial conditions

$$\begin{aligned}
\mathbb{X}(\wp,\rho,0) &= \exp(\wp+\rho), \\
\mathbb{Y}(\wp,\rho,0) &= \exp(\wp-\rho), \\
\mathbb{Z}(\wp,\rho,0) &= \exp(-\wp+\rho).
\end{aligned} \tag{42}$$

Applying the NT, we have

$$\begin{aligned}
\mathbf{N}\{D_\delta^\kappa \mathbb{X}(\wp,\rho,\delta)\} &= -\mathbf{N}\{\mathbb{Y}_\wp \mathbb{Z}_\rho - \mathbb{Y}_\rho \mathbb{Z}_\wp - \mathbb{X}\}, \\
\mathbf{N}\{D_\delta^\kappa \mathbb{Y}(\wp,\rho,\delta)\} &= -\mathbf{N}\{\mathbb{Z}_\wp \mathbb{X}_\rho - \mathbb{X}_\rho \mathbb{Z}_\wp - \mathbb{Y}\}, \\
\mathbf{N}\{D_\delta^\kappa \mathbb{Z}(\wp,\rho,\delta)\} &= -\mathbf{N}\{\mathbb{X}_\wp \mathbb{Y}_\rho - \mathbb{X}_\rho \mathbb{Y}_\wp - \mathbb{Z}\}.
\end{aligned} \tag{43}$$

By using the transform property, we have

$$\begin{aligned}
\frac{1}{\xi^\kappa}\mathbf{N}\{\mathbb{X}(\wp,\rho,\delta)\} - \xi^{2-\kappa}\mathbb{X}(\wp,0) &= -\mathbf{N}\{\mathbb{Y}_\wp \mathbb{Z}_\rho - \mathbb{Y}_\rho \mathbb{Z}_\wp - \mathbb{X}\}, \\
\frac{1}{\xi^\kappa}\mathbf{N}\{\mathbb{Y}(\wp,\rho,\delta)\} - \xi^{2-\kappa}\mathbb{Y}(\wp,0) &= -\mathbf{N}\{\mathbb{Z}_\wp \mathbb{X}_\rho - \mathbb{X}_\rho \mathbb{Z}_\wp - \mathbb{Y}\}, \\
\frac{1}{\xi^\kappa}\mathbf{N}\{\mathbb{Z}(\wp,\rho,\delta)\} - \xi^{2-\kappa}\mathbb{Z}(\wp,0) &= -\mathbf{N}\{\mathbb{X}_\wp \mathbb{Y}_\rho - \mathbb{X}_\rho \mathbb{Y}_\wp - \mathbb{Z}\}.
\end{aligned} \tag{44}$$

The above algorithm's simplified form is

$$\begin{aligned}
\mathbf{N}\{\mathbb{X}(\wp,\rho,\delta)\} &= \xi^2\exp(\wp+\rho) - \frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Y}_\wp \mathbb{Z}_\rho - \mathbb{Y}_\rho \mathbb{Z}_\wp - \mathbb{X}\}, \\
\mathbf{N}\{\mathbb{Y}(\wp,\rho,\delta)\} &= \xi^2\exp(\wp-\rho) - \frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Z}_\wp \mathbb{X}_\rho - \mathbb{X}_\rho \mathbb{Z}_\wp - \mathbb{Y}\}, \\
\mathbf{N}\{\mathbb{Z}(\wp,\rho,\delta)\} &= \xi^2\exp(-\wp+\rho) - \frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{X}_\wp \mathbb{Y}_\rho - \mathbb{X}_\rho \mathbb{Y}_\wp - \mathbb{Z}\}.
\end{aligned} \tag{45}$$

Using the inverse NT, we obtain

$$\begin{aligned}
\mathbb{X}(\wp,\rho,\delta) &= \exp(\wp+\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Y}_\wp \mathbb{Z}_\rho - \mathbb{Y}_\rho \mathbb{Z}_\wp - \mathbb{X}\}\right\}, \\
\mathbb{Y}(\wp,\rho,\delta) &= \exp(\wp-\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{Z}_\wp \mathbb{X}_\rho - \mathbb{X}_\rho \mathbb{Z}_\wp - \mathbb{Y}\}\right\}, \\
\mathbb{Z}(\wp,\rho,\delta) &= \exp(-\wp+\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\{\mathbb{X}_\wp \mathbb{Y}_\rho - \mathbb{X}_\rho \mathbb{Y}_\wp - \mathbb{Z}\}\right\}.
\end{aligned} \tag{46}$$

Solution by Means of NDM_{CF}

Assume that the unknown functions $\mathbb{X}(\wp,\rho,\delta)$, $\mathbb{Y}(\wp,\rho,\delta)$ *and* $\mathbb{Z}(\wp,\rho,\delta)$ *have the following solution in the infinite series form:*

$$\mathbb{X}(\wp,\rho,\delta) = \sum_{l=0}^{\infty}\mathbb{X}_l(\wp,\rho,\delta), \mathbb{Y}(\wp,\rho,\delta) = \sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\rho,\delta) \ \ and \ \ \mathbb{Z}(\wp,\rho,\delta) = \sum_{l=0}^{\infty}\mathbb{Z}_l(\wp,\rho,\delta).$$

Remember that $\mathbb{Y}_{\wp}\mathbb{Z}_{\rho} = \sum_{m=0}^{\infty}\mathcal{A}_m$, $\mathbb{Y}_{\rho}\mathbb{Z}_{\wp} = \sum_{m=0}^{\infty}\mathcal{B}_m$, $\mathbb{Z}_{\wp}\mathbb{X}_{\rho} = \sum_{m=0}^{\infty}\mathcal{C}_m$, $\mathbb{X}_{\rho}\mathbb{Z}_{\wp} = \sum_{m=0}^{\infty}\mathcal{D}_m$, $\mathbb{X}_{\wp}\mathbb{Y}_{\rho} = \sum_{m=0}^{\infty}\mathcal{E}_m$ *and* $\mathbb{X}_{\rho}\mathbb{Y}_{\wp} = \sum_{m=0}^{\infty}\mathcal{F}_m$ *represent the nonlinear terms. Thus, (46) can be rewritten using certain terms as*

$$\begin{aligned}
\sum_{l=0}^{\infty}\mathbb{X}_{l+1}(\wp,\rho,\delta) &= \exp(\wp+\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l - \sum_{l=0}^{\infty}\mathcal{B}_l - \mathbb{X}\right\}\right\},\\
\sum_{l=0}^{\infty}\mathbb{Y}_{l+1}(\wp,\rho,\delta) &= \exp(\wp-\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{C}_l - \sum_{l=0}^{\infty}\mathcal{D}_l - \mathbb{Y}\right\}\right\}, \qquad (47)\\
\sum_{l=0}^{\infty}\mathbb{Z}_{l+1}(\wp,\rho,\delta) &= \exp(-\wp+\rho) - \mathbf{N}^{-1}\left\{\frac{\kappa(\xi-\kappa(\xi-\kappa))}{\xi^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{E}_l - \sum_{l=0}^{\infty}\mathcal{F}_l - \mathbb{Z}\right\}\right\}.
\end{aligned}$$

Thus, by comparing both sides of (47), we obtain

$$\begin{aligned}
\mathbb{X}_0(\wp,\rho,\delta) &= \exp(\wp+\rho),\\
\mathbb{Y}_0(\wp,\rho,\delta) &= \exp(\wp-\rho),\\
\mathbb{Z}_0(\wp,\rho,\delta) &= \exp(-\wp+\rho),\\
\mathbb{X}_1(\wp,\rho,\delta) &= -\exp(\wp+\rho)(\kappa(\delta-1)+1),\\
\mathbb{Y}_1(\wp,\rho,\delta) &= \exp(\wp-\rho)(\kappa(\delta-1)+1),\\
\mathbb{Z}_1(\wp,\rho,\delta) &= \exp(-\wp+\rho)(\kappa(\delta-1)+1),\\
\mathbb{X}_2(\wp,\rho,\delta) &= \exp(\wp+\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right),\\
\mathbb{Y}_2(\wp,\rho,\delta) &= \exp(\wp-\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right),\\
\mathbb{Z}_2(\wp,\rho,\delta) &= \exp(-\wp+\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right).
\end{aligned}$$

In the same manner, the remaining $\mathbb{X}_l, \mathbb{Y}_l$ *and* $\mathbb{Z}_l$ $(l \geq 3)$ *elements are easy to obtain. So, we describe the alternative sequence as*

$$\begin{aligned}
\mathbb{X}(\wp,\rho,\delta) &= \sum_{l=0}^{\infty}\mathbb{X}_l(\wp,\rho,\delta) = \mathbb{X}_0(\wp,\rho,\delta)+\mathbb{X}_1(\wp,\rho,\delta)+\mathbb{X}_2(\wp,\rho,\delta)+\cdots\\
&= \exp(\wp+\rho) - \exp(\wp+\rho)(\kappa(\delta-1)+1)\\
&\quad + \exp(\wp+\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right)+\cdots.\\
\mathbb{Y}(\wp,\rho,\delta) &= \sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\rho,\delta) = \mathbb{Y}_0(\wp,\rho,\delta)+\mathbb{Y}_1(\wp,\rho,\delta)+\mathbb{Y}_2(\wp,\rho,\delta)+\cdots\\
&= \exp(\wp-\rho) + \exp(\wp-\rho)(\kappa(\delta-1)+1)\\
&\quad + \exp(\wp-\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right)+\cdots.\\
\mathbb{Z}(\wp,\rho,\delta) &= \sum_{l=0}^{\infty}\mathbb{Z}_l(\wp,\rho,\delta) = \mathbb{Z}_0(\wp,\rho,\delta)+\mathbb{Z}_1(\wp,\rho,\delta)+\mathbb{Z}_2(\wp,\rho,\delta)+\cdots\\
&= \exp(-\wp+\rho) + \exp(-\wp+\rho)(\kappa(\delta-1)+1)\\
&\quad + \exp(-\wp+\rho)\left((1-\kappa)^2+2\kappa(1-\kappa)\delta+\frac{\kappa^2\delta^2}{2}\right)+\cdots.
\end{aligned}$$

Solution by Means of NDM_{ABC}

Assume that the unknown functions $\mathbb{X}(\wp,\rho,\delta)$, $\mathbb{Y}(\wp,\rho,\delta)$ *and* $\mathbb{Z}(\wp,\rho,\delta)$ *have the following solution in the infinite series form:*

$$\mathbb{X}(\wp,\rho,\delta)=\sum_{l=0}^{\infty}\mathbb{X}_l(\wp,\rho,\delta),\mathbb{Y}(\wp,\rho,\delta)=\sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\rho,\delta)\ \text{ and }\ \mathbb{Z}(\wp,\rho,\delta)=\sum_{l=0}^{\infty}\mathbb{Z}_l(\wp,\rho,\delta).$$

Remember that $\mathbb{Y}_\wp\mathbb{Z}_\rho=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\mathbb{Y}_\rho\mathbb{Z}_\wp=\sum_{m=0}^{\infty}\mathcal{B}_m$, $\mathbb{Z}_\wp\mathbb{X}_\rho=\sum_{m=0}^{\infty}\mathcal{C}_m$, $\mathbb{X}_\rho\mathbb{Z}_\wp=\sum_{m=0}^{\infty}\mathcal{D}_m$, $\mathbb{X}_\wp\mathbb{Y}_\rho=\sum_{m=0}^{\infty}\mathcal{E}_m$ *and* $\mathbb{X}_\rho\mathbb{Y}_\wp=\sum_{m=0}^{\infty}\mathcal{F}_m$ *represent the nonlinear terms. Thus,* (46) *can be rewritten using certain terms as*

$$\begin{aligned}
\sum_{l=0}^{\infty}\mathbb{X}_{l+1}(\wp,\rho,\delta)&=\exp(\wp+\rho)+\mathbf{N}^{-1}\left\{\frac{\vartheta^\kappa(\xi^\kappa+\kappa(\vartheta^\kappa-\xi^\kappa))}{\xi^{2\kappa}}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l-\sum_{l=0}^{\infty}\mathcal{B}_l-\mathbb{X}\right\}\right\},\\
\sum_{l=0}^{\infty}\mathbb{Y}_{l+1}(\wp,\rho,\delta)&=\exp(\wp-\rho)+\mathbf{N}^{-1}\left\{\frac{\vartheta^\kappa(\xi^\kappa+\kappa(\vartheta^\kappa-\xi^\kappa))}{\xi^{2\kappa}}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{C}_l-\sum_{l=0}^{\infty}\mathcal{D}_l-\mathbb{Y}\right\}\right\},\\
\sum_{l=0}^{\infty}\mathbb{Z}_{l+1}(\wp,\rho,\delta)&=\exp(-\wp+\rho)+\mathbf{N}^{-1}\left\{\frac{\vartheta^\kappa(\xi^\kappa+\kappa(\vartheta^\kappa-\xi^\kappa))}{\xi^{2\kappa}}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{E}_l-\sum_{l=0}^{\infty}\mathcal{F}_l-\mathbb{Z}\right\}\right\}.
\end{aligned}\tag{48}$$

Thus, by comparing both sides of (48), *we obtain*

$$\begin{aligned}
\mathbb{X}_0(\wp,\rho,\delta)&=\exp(\wp+\rho),\\
\mathbb{Y}_0(\wp,\rho,\delta)&=\exp(\wp-\rho),\\
\mathbb{Z}_0(\wp,\rho,\delta)&=\exp(-\wp+\rho),\\
\mathbb{X}_1(\wp,\rho,\delta)&=\exp(\wp+\rho)\left(1-\kappa+\frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right),\\
\mathbb{Y}_1(\wp,\rho,\delta)&=\exp(\wp-\rho)\left(1-\kappa+\frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right),\\
\mathbb{Z}_1(\wp,\rho,\delta)&=\exp(-\wp+\rho)\left(1-\kappa+\frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right),\\
\mathbb{X}_2(\wp,\rho,\delta)&=\exp(\wp+\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)}+(1-\kappa)^2\right),\\
\mathbb{Y}_2(\wp,\rho,\delta)&=\exp(\wp-\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)}+(1-\kappa)^2\right),\\
\mathbb{Z}_2(\wp,\rho,\delta)&=\exp(-\wp+\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)}+(1-\kappa)^2\right).
\end{aligned}$$

In the same manner, the remaining $\mathbb{X}_l$,$\mathbb{Y}_l$ *and* $\mathbb{Z}_l$ $(l\geq 3)$ *elements are easy to obtain. So, we describe the alternative sequence as*

$$\begin{aligned}
\mathbb{X}(\wp,\rho,\delta)=\sum_{l=0}^{\infty}\mathbb{X}_l(\wp,\rho,\delta)&=\mathbb{X}_0(\wp,\rho,\delta)+\mathbb{X}_1(\wp,\rho,\delta)+\mathbb{X}_2(\wp,\rho,\delta)+\cdots\\
&=\exp(\wp+\rho)-\exp(\wp+\rho)\left(1-\kappa+\frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right)\\
&\quad+\exp(\wp+\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^\kappa}{\Gamma(\kappa+1)}+(1-\kappa)^2\right)+\cdots,\\
\mathbb{Y}(\wp,\rho,\delta)=\sum_{l=0}^{\infty}\mathbb{Y}_l(\wp,\rho,\delta)&=\mathbb{Y}_0(\wp,\rho,\delta)+\mathbb{Y}_1(\wp,\rho,\delta)+\mathbb{Y}_2(\wp,\rho,\delta)+\cdots\\
&=\exp(\wp-\rho)+\exp(\wp-\rho)\left(1-\kappa+\frac{\kappa\delta^\kappa}{\Gamma(\kappa+1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\exp(\wp-\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^{\kappa}}{\Gamma(\kappa+1)}+(1-\kappa)^2\right)+\cdots,\\
\mathbb{Z}(\wp,\rho,\delta)&=\sum_{l=0}^{\infty}\mathbb{Z}_l(\wp,\rho,\delta)=\mathbb{Z}_0(\wp,\rho,\delta)+\mathbb{Z}_1(\wp,\rho,\delta)+\mathbb{Z}_2(\wp,\rho,\delta)+\cdots\\
&=\exp(-\wp+\rho)+\exp(-\wp+\rho)\left(1-\kappa+\frac{\kappa\delta^{\kappa}}{\Gamma(\kappa+1)}\right)\\
&+\exp(-\wp+\rho)\left(\frac{\kappa^2\delta^{2\kappa}}{\Gamma(2\kappa+1)}+2\kappa(1-\kappa)\frac{\delta^{\kappa}}{\Gamma(\kappa+1)}+(1-\kappa)^2\right)+\cdots.
\end{aligned}
$$

At $\kappa=1$, the exact solution to (41) *is*

$$
\begin{aligned}
\mathbb{X}(\wp,\rho,\delta)&=\exp(\wp+\rho-\delta),\\
\mathbb{Y}(\wp,\rho,\delta)&=\exp(\wp-\rho+\delta),\\
\mathbb{Y}(\wp,\rho,\delta)&=\exp(-\wp+\rho+\delta).
\end{aligned}
$$

In Figure 4, the exact approximate solutions, respectively, to $\mathbb{X}(\wp,\delta)$ for system (41). In Figure 5, the approximate solution when $\kappa=0.8,0.6$ for system (41) of $\mathbb{X}(\wp,\delta)$. In Figure 6, the approximate solution to system (41) at various values of κ for $\mathbb{X}(\wp,\delta)$. In Figure 7, the exact approximate solutions, respectively, to $\mathbb{Y}(\wp,\delta)$ for system (41). In Figure 8, approximate solution when $\kappa=0.8,0.6$ for system (41) of $\mathbb{Y}(\wp,\delta)$. In Figure 9, The approximate solution to system (41) at various values of κ for $\mathbb{Y}(\wp,\delta)$. In Figure 10, The exact approximate solutions, respectively, to $\mathbb{Z}(\wp,\delta)$ for system (41). In Figure 11, approximate solution when $\kappa=0.8,0.6$ for system (41) of $\mathbb{Z}(\wp,\delta)$. In Figure 12, the approximate solution to system (41) at various values of κ for $\mathbb{Z}(\wp,\delta)$. In Tables 3, 4 and 5 show that the absolute error obtained for different values of δ for system (41).

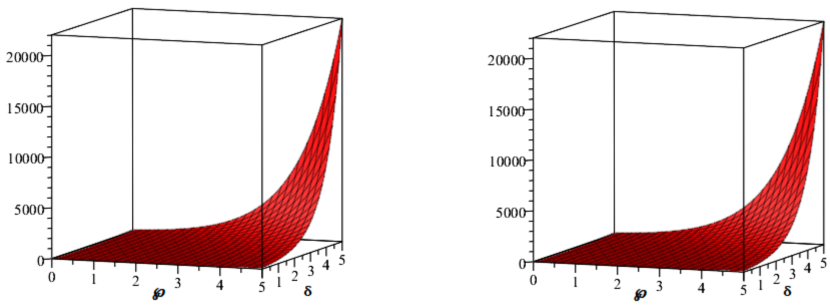

Figure 4. The exact approximate solutions, respectively, to $\mathbb{X}(\wp,\delta)$ for system (41).

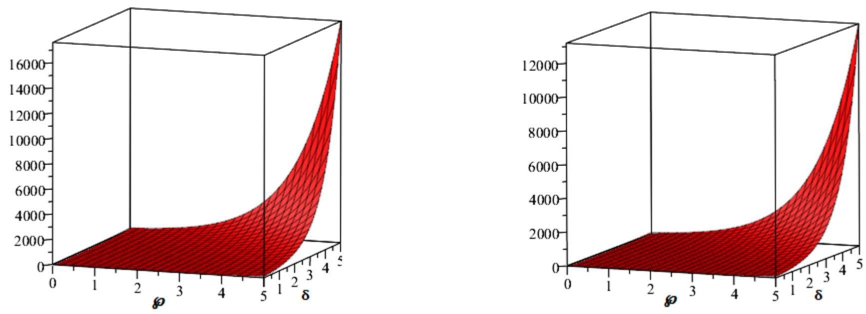

Figure 5. The approximate solution when $\kappa=0.8,0.6$ for system (41) of $\mathbb{X}(\wp,\delta)$.

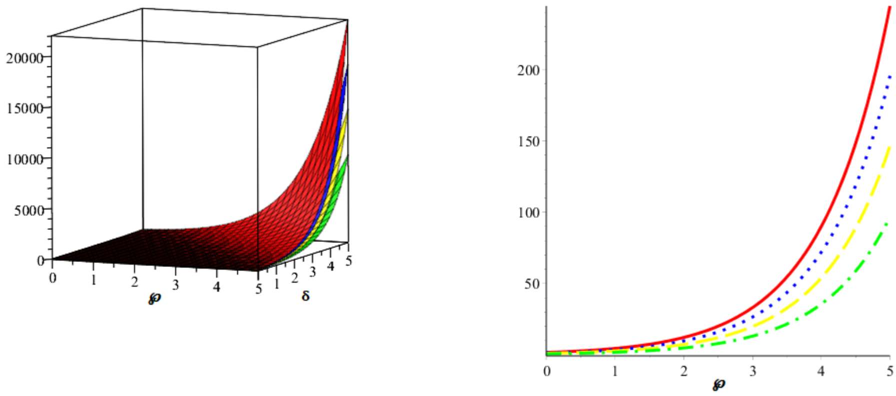

Figure 6. The approximate solution to system (41) at various values of κ for $\mathbb{X}(\wp, \delta)$.

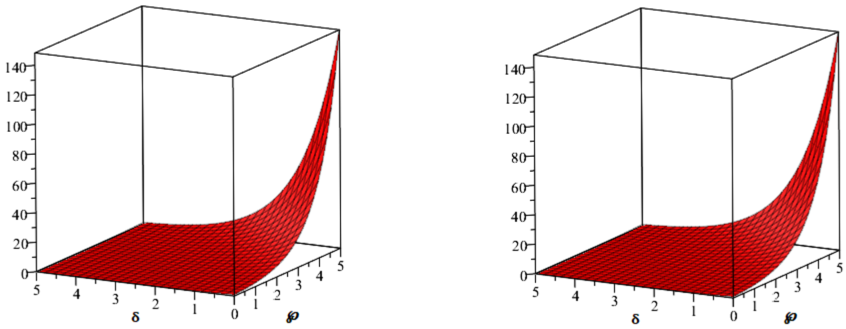

Figure 7. The exact approximate solutions, respectively, to $\mathbb{Y}(\wp, \delta)$ for system (41).

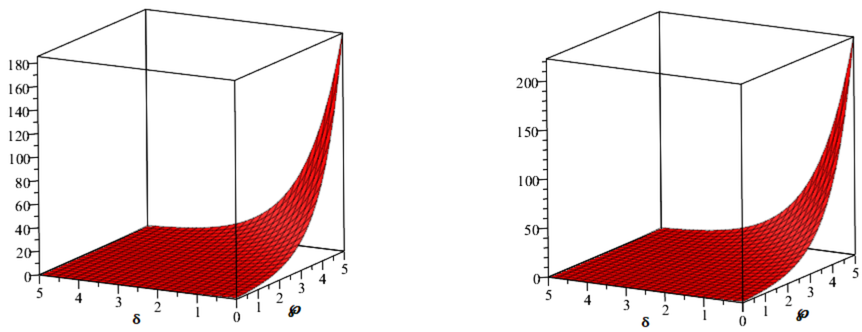

Figure 8. The approximate solution when $\kappa = 0.8, 0.6$ for system (41) of $\mathbb{Y}(\wp, \delta)$.

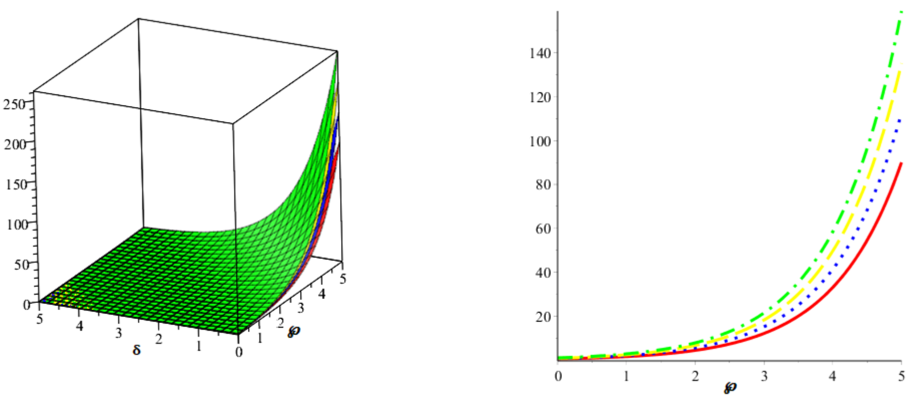

Figure 9. The approximate solution to system (41) at various values of κ for $\mathbb{Y}(\wp, \delta)$.

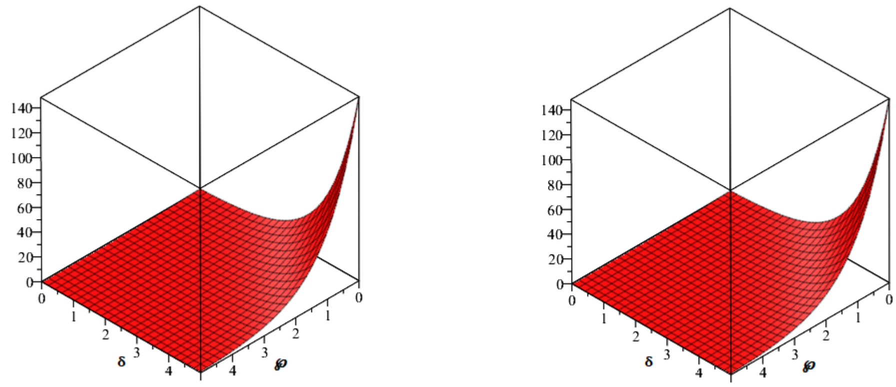

Figure 10. The exact approximate solutions, respectively, to $\mathbb{Z}(\wp, \delta)$ for system (41).

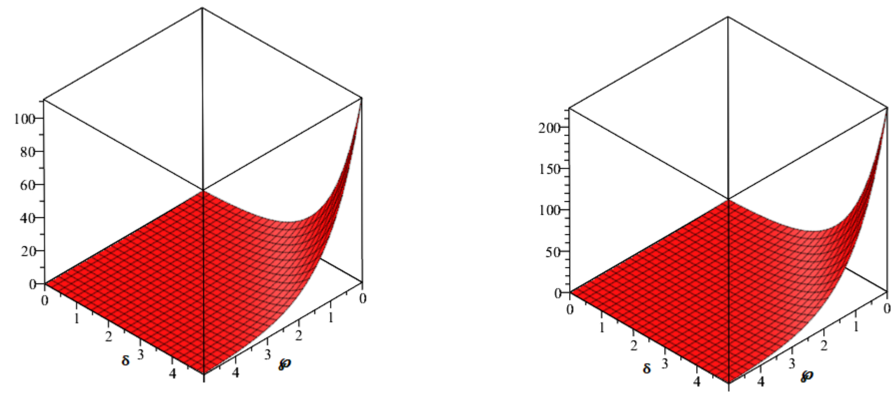

Figure 11. The approximate solution when $\kappa = 0.8, 0.6$ for system (41) of $\mathbb{Z}(\wp, \delta)$.

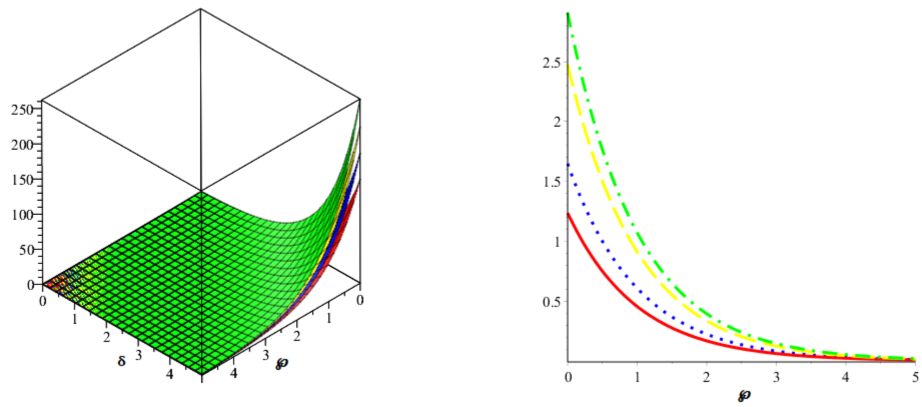

Figure 12. The approximate solution to system (41) at various values of κ for $\mathbb{Z}(\wp, \delta)$.

Table 3. The absolute error obtained at different values of δ for system (41).

$(\wp, \delta)$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.4$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.6$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.8$	$(NTDM_{CF})$ at $\kappa = 1$	$(NTDM_{ABC})$ at $\kappa = 1$
(0.3,0.01)	$4.4552686000 \times 10^{-2}$	$2.2275469000 \times 10^{-2}$	$2.2274830000 \times 10^{-3}$	$1.2000000000 \times 10^{-8}$	$1.2000000000 \times 10^{-8}$
(0.5,0.01)	$5.4416773000 \times 10^{-2}$	$2.7207319000 \times 10^{-2}$	$2.7206530000 \times 10^{-3}$	$1.4000000000 \times 10^{-8}$	$1.4000000000 \times 10^{-8}$
(0.7,0.01)	$6.6464797000 \times 10^{-2}$	$3.3231094000 \times 10^{-2}$	$3.3230120000 \times 10^{-3}$	$1.7000000000 \times 10^{-8}$	$1.7000000000 \times 10^{-8}$
(0.3,0.02)	$4.4587295000 \times 10^{-2}$	$2.2292198000 \times 10^{-2}$	$2.2291340000 \times 10^{-3}$	$4.5000000000 \times 10^{-8}$	$4.5000000000 \times 10^{-8}$
(0.5,0.02)	$5.4459045000 \times 10^{-2}$	$2.7227751000 \times 10^{-2}$	$2.7226690000 \times 10^{-3}$	$5.4000000000 \times 10^{-8}$	$5.4000000000 \times 10^{-8}$
(0.7,0.02)	$6.6516428000 \times 10^{-2}$	$3.3256051000 \times 10^{-2}$	$2.7226690000 \times 10^{-3}$	$6.6000000000 \times 10^{-8}$	$6.6000000000 \times 10^{-8}$
(0.3,0.03)	$4.4619247000 \times 10^{-2}$	$2.2307698000 \times 10^{-2}$	$2.2306910000 \times 10^{-3}$	$1.0000000000 \times 10^{-7}$	$1.0000000000 \times 10^{-7}$
(0.5,0.03)	$5.4498071000 \times 10^{-2}$	$2.7246685000 \times 10^{-2}$	$2.7245720000 \times 10^{-3}$	$1.2300000000 \times 10^{-7}$	$1.2300000000 \times 10^{-7}$
(0.7,0.03)	$6.6564094000 \times 10^{-2}$	$3.3279175000 \times 10^{-2}$	$3.3277990000 \times 10^{-3}$	$1.4900000000 \times 10^{-7}$	$1.4900000000 \times 10^{-7}$
(0.3,0.04)	$4.4649499000 \times 10^{-2}$	$2.2322416000 \times 10^{-2}$	$2.2321930000 \times 10^{-3}$	$1.7800000000 \times 10^{-7}$	$1.7800000000 \times 10^{-7}$
(0.5,0.04)	$5.4535021000 \times 10^{-2}$	$2.7264661000 \times 10^{-2}$	$2.7264070000 \times 10^{-3}$	$2.1800000000 \times 10^{-7}$	$2.1800000000 \times 10^{-7}$
(0.7,0.04)	$6.6609225000 \times 10^{-2}$	$3.3301131000 \times 10^{-2}$	$3.3300410000 \times 10^{-3}$	$2.6600000000 \times 10^{-7}$	$2.6600000000 \times 10^{-7}$
(0.3,0.05)	$4.4678498000 \times 10^{-2}$	$2.2336557000 \times 10^{-2}$	$2.2336620000 \times 10^{-3}$	$2.7800000000 \times 10^{-7}$	$2.7800000000 \times 10^{-7}$
(0.5,0.05)	$5.4570440000 \times 10^{-2}$	$2.7281932000 \times 10^{-2}$	$2.7282010000 \times 10^{-3}$	$3.4000000000 \times 10^{-7}$	$3.4000000000 \times 10^{-7}$
(0.7,0.05)	$6.6652485000 \times 10^{-2}$	$3.3322227000 \times 10^{-2}$	$3.3322310000 \times 10^{-3}$	$4.1400000000 \times 10^{-7}$	$4.1400000000 \times 10^{-7}$

Table 4. The absolute error obtained at various values of δ for system (41).

$(\wp, \delta)$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.4$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.6$	$\mathbb{X}(\wp, \delta)$ at $\kappa = 0.8$	$(NTDM_{CF})$ at $\kappa = 1$	$(NTDM_{ABC})$ at $\kappa = 1$
(0.3,0.01)	$1.639000890 \times 10^{-2}$	$8.194678600 \times 10^{-3}$	$8.194366000 \times 10^{-4}$	$4.000000000 \times 10^{-9}$	$4.000000000 \times 10^{-9}$
(0.5,0.01)	$2.001880200 \times 10^{-2}$	$1.000900300 \times 10^{-2}$	$1.000862000 \times 10^{-3}$	$5.000000000 \times 10^{-9}$	$5.000000000 \times 10^{-9}$
(0.7,0.01)	$2.445101900 \times 10^{-2}$	$1.222502300 \times 10^{-2}$	$1.222455000 \times 10^{-3}$	$7.000000000 \times 10^{-9}$	$7.000000000 \times 10^{-9}$
(0.3,0.02)	$1.640271640 \times 10^{-2}$	$8.200808400 \times 10^{-3}$	$8.200195000 \times 10^{-4}$	$1.630000000 \times 10^{-8}$	$1.630000000 \times 10^{-8}$
(0.5,0.02)	$2.003432300 \times 10^{-2}$	$1.001649000 \times 10^{-2}$	$1.001574000 \times 10^{-3}$	$2.000000000 \times 10^{-8}$	$2.000000000 \times 10^{-8}$
(0.7,0.02)	$2.446997700 \times 10^{-2}$	$1.223416800 \times 10^{-2}$	$1.223325000 \times 10^{-3}$	$2.400000000 \times 10^{-8}$	$2.400000000 \times 10^{-8}$
(0.3,0.03)	$1.641442990 \times 10^{-2}$	$8.206469900 \times 10^{-3}$	$8.205516000 \times 10^{-4}$	$3.690000000 \times 10^{-8}$	$3.690000000 \times 10^{-8}$
(0.5,0.03)	$2.004863000 \times 10^{-2}$	$1.002340500 \times 10^{-2}$	$1.002224000 \times 10^{-3}$	$4.500000000 \times 10^{-8}$	$4.500000000 \times 10^{-8}$
(0.7,0.03)	$2.448745200 \times 10^{-2}$	$1.224261400 \times 10^{-2}$	$1.224119000 \times 10^{-3}$	$5.500000000 \times 10^{-8}$	$5.500000000 \times 10^{-8}$
(0.3,0.04)	$1.642550160 \times 10^{-2}$	$8.211826900 \times 10^{-3}$	$8.210469000 \times 10^{-4}$	$6.550000000 \times 10^{-8}$	$6.550000000 \times 10^{-8}$
(0.5,0.04)	$2.006215300 \times 10^{-2}$	$1.002994800 \times 10^{-2}$	$1.002829000 \times 10^{-3}$	$8.000000000 \times 10^{-8}$	$8.000000000 \times 10^{-8}$
(0.7,0.04)	$2.450396900 \times 10^{-2}$	$1.225060600 \times 10^{-2}$	$1.224858000 \times 10^{-3}$	$9.800000000 \times 10^{-8}$	$9.800000000 \times 10^{-8}$
(0.3,0.05)	$1.643609600 \times 10^{-2}$	$8.216955400 \times 10^{-3}$	$8.215136000 \times 10^{-4}$	$1.023000000 \times 10^{-7}$	$1.023000000 \times 10^{-7}$
(0.5,0.05)	$2.007509300 \times 10^{-2}$	$1.003621200 \times 10^{-2}$	$1.003399000 \times 10^{-3}$	$1.250000000 \times 10^{-7}$	$1.250000000 \times 10^{-7}$
(0.7,0.05)	$2.451977400 \times 10^{-2}$	$1.225825700 \times 10^{-2}$	$1.225554000 \times 10^{-3}$	$1.530000000 \times 10^{-7}$	$1.530000000 \times 10^{-7}$

Table 5. The absolute error obtained for different values of δ for system (41).

$(\wp,\delta)$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.4$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.6$	$\mathbb{X}(\wp,\delta)$ at $\kappa=0.8$	$(NTDM_{CF})$ at $\kappa=1$	$(NTDM_{ABC})$ at $\kappa=1$
(0.3,0.01)	2.445101900×10^{-2}	1.222502300×10^{-2}	1.222455000×10^{-3}	7.000000000×10^{-9}	7.000000000×10^{-9}
(0.5,0.01)	2.001880200×10^{-2}	1.000900300×10^{-2}	1.000862000×10^{-3}	5.000000000×10^{-9}	5.000000000×10^{-9}
(0.7,0.01)	1.639000890×10^{-2}	8.194678600×10^{-3}	8.194366000×10^{-4}	4.000000000×10^{-9}	4.000000000×10^{-9}
(0.3,0.02)	2.446997700×10^{-2}	1.223416800×10^{-2}	1.223325000×10^{-3}	2.400000000×10^{-8}	2.400000000×10^{-8}
(0.5,0.02)	2.003432300×10^{-2}	1.001649000×10^{-2}	1.001574000×10^{-3}	2.000000000×10^{-8}	2.000000000×10^{-8}
(0.7,0.02)	1.640271640×10^{-2}	8.200808400×10^{-3}	8.200195000×10^{-4}	1.630000000×10^{-8}	1.630000000×10^{-8}
(0.3,0.03)	2.448745200×10^{-2}	1.224261400×10^{-2}	1.224119000×10^{-3}	5.500000000×10^{-8}	5.500000000×10^{-8}
(0.5,0.03)	2.004863000×10^{-2}	1.002340500×10^{-2}	1.002224000×10^{-3}	4.500000000×10^{-8}	4.500000000×10^{-8}
(0.7,0.03)	1.641442990×10^{-2}	8.206469900×10^{-3}	8.205516000×10^{-4}	3.690000000×10^{-8}	3.690000000×10^{-8}
(0.3,0.04)	2.450396900×10^{-2}	1.225060600×10^{-2}	1.224858000×10^{-3}	9.800000000×10^{-8}	9.800000000×10^{-8}
(0.5,0.04)	2.006215300×10^{-2}	1.002994800×10^{-2}	1.002829000×10^{-3}	8.000000000×10^{-8}	8.000000000×10^{-8}
(0.7,0.04)	1.642550160×10^{-2}	8.211826900×10^{-3}	8.210469000×10^{-4}	6.550000000×10^{-8}	6.550000000×10^{-8}
(0.3,0.05)	2.451977400×10^{-2}	1.225825700×10^{-2}	1.225554000×10^{-3}	1.530000000×10^{-7}	1.530000000×10^{-7}
(0.5,0.05)	2.007509300×10^{-2}	1.003621200×10^{-2}	1.003399000×10^{-3}	1.250000000×10^{-7}	1.250000000×10^{-7}
(0.7,0.05)	1.643609600×10^{-2}	8.216955400×10^{-3}	8.215136000×10^{-4}	1.023000000×10^{-7}	1.023000000×10^{-7}

6. Conclusions

This study uses the natural transform decomposition method to solve various fractional-order partial differential equations. In terms of the CF and ABC, the fractional derivatives are expressed. The proposed method is used to obtain the solution to a number of numerical problems. The solutions to the presented problems are determined in various fractional and integer orders. The approximate solutions to the problems are observed to agree with the exact solutions to the problems. Furthermore, it has been revealed that the fractional-order solutions converge to an integer-order solution to the problems. The suggested technique is shown to be simple and effective and may be implemented to solve various differential equation systems with fractional order.

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References

1. Debnath, L.; Feynman, R.P. Recent applications of fractional calculus to science and engineering. *Int. J. Math. Math. Sci.* **2003**, *2003*, 3413–3442. [CrossRef]
2. Ortigueira, M.D. An introduction to the fractional continuous-time linear systems: The 21st century systems. *IEEE Circuits Syst. Mag.* **2008**, *8*, 19–26. [CrossRef]
3. Kaczorek, T. Positive linear systems consisting of n subsystems with different fractional orders. *IEEE Trans. Circuits Syst. I Reg. Pap.* **2011**, *58*, 1203–1210. [CrossRef]
4. Syta, A.; Litak, G.; Lenci, S.; Scheffler, M. Chaotic vibrations of the duffing system with fractional damping. *Chaos* **2014**, *24*, 013107. [CrossRef]
5. Lv, Z.; Yu, Z.; Xie, S.; Alamri, A. Deep Learning-Based Smart Predictive Evaluation for Interactive Multimedia-Enabled Smart Healthcare. *ACM Trans. Multimed. Comput. Commun. Appl.* **2022**, *18*, 1–20. [CrossRef]
6. Lv, Z.; Chen, D.; Lou, R.; Song, H. Industrial security solution for virtual reality. *IEEE Internet Things J.* **2020**, *8*, 6273–6281. [CrossRef]
7. Lv, Z.; Qiao, L.; Hossain, M.S.; Choi, B.J. Analysis of using blockchain to protect the privacy of drone big data. *IEEE Netw.* **2021**, *35*, 44–49. [CrossRef]
8. Kothari, K.; Mehta, U.; Vanualailai, J. A novel approach of fractionalorder time delay system modeling based on Haar wavelet. *ISA Trans.* **2018**, *80*, 371–380. [CrossRef]
9. Singh, J.; Kumar, D.; Baleanu, D.; Rathore, S. An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation. *Appl. Math. Comput.* **2018**, *335*, 12–24. [CrossRef]
10. Nonlaopon, K.; Alsharif, A.M.; Zidan, A.M.; Khan, A.; Hamed, Y.S.; Shah, R. Numerical investigation of fractional-order Swift-Hohenberg equations via a Novel transform. *Symmetry* **2021**, *13*, 1263. [CrossRef]
11. Areshi, M.; Khan, A.; Shah, R.; Nonlaopon, K. Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform. *AIMS Math.* **2022**, *7*, 6936–6958. [CrossRef]
12. Jain, S. Numerical analysis for the fractional diffusion and fractional Buckmaster equation by the two-step Laplace Adam-Bashforth method. *Eur. Phys. J. Plus* **2018**, *133*, 19. [CrossRef]
13. Kumar, D.; Tchier, F.; Singh, J.; Baleanu, D. An efficient computational technique for fractal vehicular traffic flow. *Entropy* **2018**, *20*, 259. [CrossRef] [PubMed]
14. Alaoui, M.K.; Fayyaz, R.; Khan, A.; Shah, R.; Abdo, M.S. Analytical investigation of Noyes-Field model for time-fractional Belousov-Zhabotinsky reaction. *Complexity* **2021**, *2021*, 3248376. [CrossRef]
15. Vinagre, B.M.; Feliu, V. Modeling and control of dynamic system using fractional calculus: Application to electrochemical processes and flexible structures. In Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, NV, USA, 10–13 December 2002; pp. 214–239.
16. Qin, Y.; Khan, A.; Ali, I.; Al Qurashi, M.; Khan, H.; Shah, R.; Baleanu, D. An efficient analytical approach for the solution of certain fractional-order dynamical systems. *Energies* **2020**, *13*, 2725. [CrossRef]
17. Ullah, S.; Khan, M.A.; Farooq, M. A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative. *Eur. Phys. J. Plus* **2018**, *133*, 237. [CrossRef]
18. Khan, M.A.; Ullah, S.; Farooq, M. A new fractional model for tuberculosis with relapse via Atangana- Baleanu derivative. *Chaos Solitons Fractals* **2018**, *116*, 227–238. [CrossRef]
19. Bisquert, J.; Garcia-Belmonte, G.; Fabregat-Santiago, F.; Compte, A. Anomalous transport effects in the impedance of porous film electrodes. *Electrochem. Commun.* **1999**, *1*, 429–435. [CrossRef]
20. Singh, J.; Kumar, D.; Kiliçman, A. Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations. *Abstr. Appl. Anal.* **2014**, *2014*, 535793. [CrossRef]
21. Yin, C.; Chen, Y.Q.; Zhong, S.-M. Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems. *Automatica* **2014**, *50*, 3173–3181. [CrossRef]
22. Stevik, T.K.; Aa, K.; Ausland, G.; Hanssen, J.F. Retention and removal of pathogenic bacteria in wastewater percolating through porous media:A review. *Water Res.* **2004**, *38*, 1355–1367. [CrossRef]
23. Shah, N.A.; El-Zahar, E.R.; Akgül, A.; Khan, A.; Kafle, J. Analysis of Fractional-Order Regularized Long-Wave Models via a Novel Transform. *J. Funct. Spaces* **2022**, *2022*, 2754507. [CrossRef]
24. Khan, M.A.; Ullah, S.; Okosun, K.O.; Shah, K. A fractional order pine wilt disease model with Caputo-Fabrizio derivative. *Adv. Differ. Equ.* **2018**, *2018*, 410. [CrossRef]
25. Hosseini, K.; Bekir, A.; Kaplan, M.; Güner, Ö. On a new technique for solving the nonlinear conformable time-fractional differential equations. *Opt. Quantum Electron.* **2017**, *49*, 343. [CrossRef]
26. Sunthrayuth, P.; Alyousef, H.A.; El-Tantawy, S.A.; Khan, A.; Wyal, N. Solving Fractional-Order Diffusion Equations in a Plasma and Fluids via a Novel Transform. *J. Funct. Spaces* **2022**, *2022*, 1899130. [CrossRef]
27. Akram, T.; Abbas, M.; Riaz, M.B.; Ismail, A.I.; Ali, N.M. An efficient numerical technique for solving time fractional Burgers equation. *Alex. Eng. J.* **2020**, *59*, 2201–2220. [CrossRef]
28. Naber, M. Time fractional Schrödinger equation. *J. Math. Phys.* **2004**, *45*, 3339–3352. [CrossRef]
29. Yang, D.; Zhu, T.; Wang, S.; Wang, S.; Xiong, Z. LFRSNet: A Robust Light Field Semantic Segmentation Network Combining Contextual and Geometric Features. *Front. Environ. Sci.* **2022**, *10*, 996513. [CrossRef]

30. Kovalnogov, V.N.; Fedorov, R.V.; Chukalin, A.V.; Simos, T.E.; Tsitouras, C. Evolutionary Derivation of Runge-Kutta Pairs of Orders 5(4) SpeciallyTuned for Problems with Periodic Solutions. *Mathematics* **2021**, *9*, 2306. [CrossRef]
31. Wang, L.; Liu, G.; Xue, J.; Wong, K. Channel Prediction Using Ordinary Differential Equations for MIMO systems. *IEEE Trans. Veh. Technol.* **2022**, 1–9. [CrossRef]
32. Kovalnogov, V.N.; Fedorov, R.V.; Karpukhina, T.V.; Simos, T.E.; Tsitouras, C. Runge-Kutta Pairs of Orders 5(4) Trained to Best Address KeplerianType Orbits. *Mathematics* **2021**, *9*, 2400. [CrossRef]
33. Batiha, B.; Noorani, M.S.M.; Hashim, I.; Batiha, K. Numerical simulations of systems of PDEs by variational iteration method. *Phys. Lett. A* **2008**, *372*, 822–829. [CrossRef]
34. Wazwaz, A.-M. The variational iteration method for solving linear and nonlinear systems of PDEs. *Comput. Math. Appl.* **2007**, *54*, 895–902. [CrossRef]
35. Botmart, T.; Agarwal, R.P.; Naeem, M.; Khan, A.; Shah, R. On the solution of fractional modified Boussinesq and approximate long wave equations with non-singular kernel operators. *AIMS Math.* **2022**, *7*, 12483–12513. [CrossRef]
36. Dang, W.; Guo, J.; Liu, M.; Liu, S.; Yang, B.; Yin, L.; Zheng, W. A Semi-Supervised Extreme Learning Machine Algorithm Based on the New Weighted Kernel for Machine Smell. *Appl. Sci.* **2022**, *12*, 9213. [CrossRef]
37. Wazwaz, A.M. *Partial Differential Equations: Methods and Applications*; Balkema Publishers: Leiden, The Netherlands, 2002.
38. Linge, S.; Sundnes, J.; Hanslien, M.; Lines, G.T.; Tveito, A. Numerical solution of the bidomain equations. *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.* **2009**, *367*, 1931–1950. [CrossRef] [PubMed]
39. Sundnes, J.; Lines, G.T.; Mardal, K.A.; Tveito, A. Multigrid block preconditioning for a coupled system of partial differential equations modeling the electrical activity in the heart. *Comput. Methods Biomech. Biomed. Eng.* **2002**, *5*, 397–409. [CrossRef]
40. Sundnes, J.; Lines, G.T.; Tveito, A. An operator splitting method for solving the bidomain equations coupled to a volume conductor model for the torso. *Math. Biosci.* **2005**, *194*, 233–248 [CrossRef]
41. Aksikas, I.; Fuxman, A.; Forbes, J.F.; Winkin, J.J. LQ control design of a class of hyperbolic PDE systems: Application to fixed-bed reactor. *Automatica* **2009**, *45*, 1542–1548. [CrossRef]
42. Moghadam, A.A.; Aksikas, I.; Dubljevic, S.; Forbes, J.F. LQ control of coupled hyperbolic PDEs and ODEs: Application to a CSTR-PFR system. *IFAC Proc. Vol.* **2010**, *43*, 721–726. [CrossRef]
43. Fackeldey, K.; Krause, R. Multiscale coupling in function space-weak coupling between molecular dynamics and continuum mechanics. *Int. J. Numer. Methods Eng.* **2009**, *79*, 1517–1535. [CrossRef]
44. Lin, L.L.; Li, Z.Y.; Lin, B. Engineering waveguide-cavity resonant side coupling in a dynamically tunable ultracompact photonic crystal filter. *Phys. Rev. B* **2005**, *72*, 165330. [CrossRef]
45. Lu, S.; Guo, J.; Liu, S.; Yang, B.; Liu, M.; Yin, L.; Zheng, W. An Improved Algorithm of Drift Compensation for Olfactory Sensors. *Applied Sciences*. **2022**, *12*, 9529. [CrossRef]
46. Adomian, G. A new approach to nonlinear partial differential equations. *J. Math. Anal. Appl.* **1984**, *102*, 420–434. [CrossRef]
47. Rawashdeh, M.S.; Al-Jammal, H. New approximate solutions to fractional nonlinear systems of partial differential equations using the FNDM. *Adv. Differ. Equ.* **2016**, *235*, 1–19. [CrossRef]
48. Rawashdeh, M.S.; Al-Jammal, H. Numerical solutions for system of nonlinear fractional ordinary differential equations using the FNDM. *Mediterr. J. Math.* **2016**, *13*, 4661–4677. [CrossRef]
49. Rawashdeh, M.S.The fractional natural decomposition method: Theories and applications. *Math. Meth. Appl. Sci.* **2017**, *40*, 2362–2376. [CrossRef]
50. Sun, L.; Hou, J.; Xing, C.; Fang, Z. A Robust Hammerstein-Wiener Model Identification Method for Highly Nonlinear Systems. *Processes* **2022**, *10*, 2664. [CrossRef]
51. Shah, N.A.; Hamed, Y.S.; Abualnaja, K.M.; Chung, J.D.; Shah, R.; Khan, A. A comparative analysis of fractional-order kaup-kupershmidt equation within different operators. *Symmetry* **2022**, *14*, 986. [CrossRef]
52. Zhou, M.; Kanth, A.R.; Aruna, K.; Raghavendar, K.; Rezazadeh, H.; Inc, M.; Aly, A.A. Numerical solutions of time fractional Zakharov-Kuznetsov equation via natural transform decomposition method with nonsingular kernel derivatives. *J. Funct. Space* **2021**, *2021*, 9884027. [CrossRef]
53. Al-Sawalha, M.M.; Khan, A.; Ababneh, O.Y.; Botmart, T. Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives. *AIMS Math.* **2022**, *7*, 18334–18359. [CrossRef]