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On Nonlinear Ψ-Caputo Fractional Integro Differential Equations Involving Non-Instantaneous Conditions

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Abstract: We propose a solution to the symmetric nonlinear Ψ-Caputo fractional integro differential equations involving non-instantaneous impulsive boundary conditions. We investigate the existence and uniqueness of the solution for the proposed problem. Banach contraction theorem is employed to prove the uniqueness results, while Krasnoselkii's fixed point technique is used to prove the existence results. Additionally, an example is used to explain the results. In this manner, our results represent generalized versions of some recent interesting contributions.

Keywords: fractional differential equations; Caputo fractional derivative; fractional boundary conditions; existence and uniqueness

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Citation: Arul, R.; Karthikeyan, P.; Karthikeyan, K.; Geetha, P.; Alruwaily, Y.; Almaghamsi, L.; El-hady, E.-s. On Nonlinear Ψ-Caputo Fractional Integro Differential Equations Involving Non-Instantaneous Conditions. *Symmetry* **2023**, *15*, 5. https://doi.org/10.3390/sym15010005

Academic Editor: Alexander Zaslavski

Received: 16 November 2022
Revised: 7 December 2022
Accepted: 13 December 2022
Published: 20 December 2022

1. Introduction

Ordinary differentiation and integration principles are unified and generalized by the non-integer order integrals and derivatives known as fractional calculus (FC). For more details on fractional derivatives (FD), with geometric, physical interpretations and with a historical overview, one can refer to [1–16] and the references therein.

There has been a lot of research conducted so far on fractional differential equations (FDEs) with initial and boundary conditions (BCs). The reason for this is FDEs efficiently describe many real-world processes such as in chemistry, biology, signal processing, and many others (see, e.g., [4,7–9,13,17–21]). Additionally, FDEs have interesting applications in solving inverse problems, and in the modeling of heat flow in porous material (see, e.g., [22–24]).

Numerous models in the study of the dynamics of phenomena that experience abrupt changes in the state use differential equations with impulses. It has been observed that certain dynamics of evolution processes cannot be adequately characterized by instantaneous impulses. For instance: Pharmacotherapy, high or low levels of glucose, etc. A circumstance like that can be observed as an impulsive activity that begins suddenly at one point in time and lasts for some amount of time. Non-instantaneous (N-InI) systems are types of systems which are more suitable to study the dynamics of evolution processes. For more details, one can refer [17,25–30].

These days, one of the major topics of mathematical analysis is the study of FC domain. In [28] X. Yu discussed the existence and β-Ulam-Heyrs stabilty of fractional differential equations (FDEs) with involving of N-InI. The new class of Ulam-Heyrs stabilty of fractional integral BCs was studied in [14]. In [16] A. Zada et al. established the Ulam-stability on Caputo sense of multipoint BCs with N-InI. In [15] A. Zada et al. discussed the stability of FDEs with non instantaneous BCs of the form

$$\begin{aligned}
{}^{c}\mathcal{D}^{\mathfrak{q}}\mathfrak{y}(\mathfrak{t}) &= \mathscr{E}(\mathfrak{t},\mathfrak{y}(\mathfrak{t})), \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{j}},\mathfrak{s}_{\mathfrak{j}}], \mathfrak{q}\in(0,1],\\
\mathfrak{y}(\mathfrak{t}) &= \mathscr{G}_{\mathfrak{i}}(\mathfrak{t},\mathfrak{y}(\mathfrak{t})),\ \ \mathfrak{t}\in(\mathfrak{s}_{\mathfrak{j}-1},\mathfrak{t}_{\mathfrak{j}}],\ \ \mathfrak{i}=1,\ldots,\mathfrak{n},\\
\mathfrak{y}(0) &= \mathfrak{I}^{\mathfrak{q}}\mathfrak{y}(\mathfrak{t})|_{\mathfrak{t}=0}=0\\
\mathfrak{y}(\mathscr{T}) &= \mathfrak{I}^{\mathfrak{q}}\mathfrak{y}(\mathfrak{t})|_{\mathfrak{t}=\mathscr{T}};
\end{aligned}$$

where ${}^{c}\mathcal{D}^{\mathfrak{q}}$ and $I^{\mathfrak{q}}$ is a Caputo and Riemann-Liouville fractional integral, respectively.

Recently, R. Agarwal et al. [17] established the N-InI and BCs in Caputo FDEs. In [26] C. Long et al. studied the N-InI FDEs with integral BCs. Non instantaneous impulses with the fractional boundary value problems was referred to in [29]. In [25] V. Gupta et al. established the nonlinear fractional boundary value with N-InI using the Caputo fractional derivative. In [26] C. Long et al. discussed the following FDEs to solve the new boundary value problem for N-InI

$$\begin{aligned}
{}^{c}\mathcal{D}^{\mathfrak{p}}_{0,\mathtt{t}}\mathtt{w}(\mathtt{t}) &= \mathscr{E}(\mathtt{t},\ \mathtt{w}(\mathtt{t})), \mathtt{t}\in(\mathtt{s}_{\mathtt{i}},\mathtt{t}_{\mathtt{i}+1}]\subset[0,\mathscr{T}], \mathfrak{p}\in(0,1),\\
\mathtt{w}(\mathtt{t}) &= \mathscr{H}_{\mathtt{i}}(\mathtt{t},\mathtt{w}(\mathtt{t})),\ \ \mathtt{t}\in(\mathtt{t}_{\mathtt{i}},\mathtt{s}_{\mathtt{i}}],\ \ \mathtt{i}=1,\ldots,\mathtt{m},\\
\mathtt{w}(\mathscr{T}) &= \mathtt{w}(0)+\chi\int_0^{\mathscr{T}}\mathtt{w}(\mathtt{s})\mathtt{ds};
\end{aligned}$$

where $\mathscr{E}, \mathscr{H}_{\mathtt{i}}$ are continuous and χ is constant.

In [27], A. Salim et al. established the following Hilfer-type fractional derivative with N-InI involving BCs

$$\begin{aligned}
&({}^{c}\mathcal{D}^{\mathfrak{p},\mathfrak{r}}_{\omega_{\mathfrak{i}}}\wp)(\mathfrak{t}) = \mathscr{E}(\mathfrak{t},\wp(\mathfrak{t})), {}^{c}\mathcal{D}^{\mathfrak{p},\mathfrak{r}}_{\omega_{\mathfrak{i}}\wp(\mathfrak{t})},\ \ \mathfrak{t}\in\mathscr{J}_{\mathfrak{i}}, \mathfrak{i}=0,\ldots,\mathfrak{m}\\
&\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t},\wp(\mathfrak{t})),\ \ \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\ \ \mathfrak{i}=1,\ldots,\mathfrak{m},\\
&\varpi_1\left({}^{\beta}\mathscr{J}^{1-\iota}_{\mathfrak{a}^+}\wp\right)+\varpi_2\left({}^{\beta}\mathscr{J}^{1-\iota}_{\mathfrak{b}}\wp\right)=\varpi_3;
\end{aligned}$$

where ${}^{c}\mathcal{D}^{\mathfrak{p},\mathfrak{r}}_{\omega_{\mathfrak{i}}}$ and ${}^{\beta}\mathscr{J}^{1-\iota}_{\mathfrak{a}^+}$-are the generalized Hilfer derivative of order $\mathfrak{r}\in(0,1)$ and the function $\mathscr{E}$ is continuous.

In [18] M. S. Abdo et al. discussed the Ψ-Caputo FDE with fractional BCs, as follows

$$\begin{aligned}
&{}^{c}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t},\wp(\mathfrak{t})),\ \mathfrak{t}\in[\mathfrak{a},\mathfrak{b}],\\
&\wp^{[\mathfrak{k}]}_{\Psi}(\mathfrak{a}) = \wp^{\mathfrak{k}}_{\mathfrak{a}},\ \mathfrak{k}=0,1,\ldots,\mathfrak{n}-2,\\
&\wp^{[\mathfrak{n}-1]}_{\Psi}(\mathfrak{b}) = \wp_{\mathfrak{b}},\ \mathfrak{k}=0,1,\ldots,\mathfrak{n}-2;
\end{aligned}$$

where ${}^{c}\mathcal{D}^{\mathfrak{p};\Psi}$-$\Psi$ is the Caputo derivative and $\mathscr{F}$ is the continuous function.

In [31] D. B. Dhaigude et al. established the solution of the following nonlinear Ψ-Caputo fractional differential equations involving BCs

$$\begin{aligned}
&{}^{c}\mathcal{D}^{\mathfrak{p};\Psi}_{\mathfrak{t}}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t},\wp(\mathfrak{t})),\ 0<\mathfrak{t}\leq\mathscr{T},\\
&\mathscr{G}(\wp(0),\wp(\mathscr{T}))=0;
\end{aligned}$$

where ${}^{c}\mathcal{D}^{\mathfrak{p};\Psi}_{\mathfrak{t}}$-$\Psi$-Caputo derivative and $\mathscr{F}$ is continuous function.

In this paper, we examine the symmetric Ψ-Caputo fractional integro-differential equations with non instantaneous impulsive BCs of the form

$$ {}^{c}\mathcal{D}^{\mathfrak{p};\Psi}\wp(t) = \mathscr{F}(t, \wp(t), \mathscr{B}\wp(t)), t \in (\mathfrak{s}_i, t_{i+1}], 0 < \mathfrak{p} < 1, \quad (1) $$

$$ \wp(t) = \mathscr{H}_i(t, \wp(t)), \ t \in (t_i, \mathfrak{s}_i], \ i = 1, \ldots, \mathfrak{m}, \quad (2) $$

$$ \mathfrak{a}\wp(0) + \mathfrak{b}\wp(\mathscr{T}) = \mathfrak{c}; \quad (3) $$

where ${}^{c}\mathcal{D}^{\mathfrak{p};\Psi}$ is the Ψ-Caputo FD of order $\mathfrak{p}$. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are real constants with $\mathfrak{a} + \mathfrak{b} \neq 0$ and $0 = \mathfrak{s}_0 < t_1 \leq t_2 < \ldots < t_m \leq \mathfrak{s}_m \leq \mathfrak{s}_{m+1} = \mathscr{T}$,- pre-fixed, $\mathscr{F} : [0, \mathscr{T}] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathscr{H}_i : [t_i, \mathfrak{s}_i] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Moreover, $\mathscr{B}\wp(t) = \int_0^t \mathfrak{k}(t, \mathfrak{s})\wp(\mathfrak{s})\partial\mathfrak{s}$ and $\mathfrak{k} \in \mathscr{C}(D, \mathbb{R}^+)$ with domain $D = \{(t, \mathfrak{s}) \in \mathbb{R}^2 : 0 \leq \mathfrak{s} \leq t \leq \mathscr{T}\}$.

Main Contributions:

1. The main motivation for this work is to use the Ψ-Caputo fractional derivative to present a new class of N-InI Ψ-CFIDE with BCs;
2. Moreover, we investigate the existence and uniqueness of the solutions of Equations (1)–(3) using Krasnoselkii's and Banach's FPT;
3. We extend the results studied in [18,32] by including Ψ-Caputo FD, nonlinear integral terms and N-InI conditions.

The remainder of the article is organized as follows: In Section 2, the basic definitions and lemmas will be used in the main results. In Section 3, we used the suitable conditions for the existence and uniqueness of solution for the system (1)–(3). The application is also presented in Section 4.

2. Supporting Notes

Let the space $\mathscr{PC}([0, \mathscr{T}], \mathbb{R}) = \{\wp : [0, \mathscr{T}] \to \mathbb{R} : \wp \in \mathscr{C}(t_{\mathfrak{k}}, t_{\mathfrak{k}+1}], \mathbb{R}\}$ be continuous and there exists $\wp(t_{\mathfrak{k}}^-)$ and $\wp(t_{\mathfrak{k}}^+)$ with $\wp(t_{\mathfrak{k}}^-) = \wp(t_{\mathfrak{k}}^+)$ satisfying the norm $\|\wp\|_{\mathscr{PC}} = \sup\{|\wp(t)| : 0 \leq t \leq \mathscr{T}\}$.

Set $\mathscr{PC}([0, \mathscr{T}], \mathbb{R}) : \{\wp \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R}) : \wp' \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R})\}$ with norm $\|\wp\|_{\mathscr{PC}} := \max\{\|\wp\|_{\mathscr{PC}}, \|\wp'\|_{\mathscr{PC}}\}$. Clearly, $\mathscr{PC}([0, \mathscr{T}], \mathbb{R})$ ended with norm $\|.\|_{\mathscr{PC}}$.

Definition 1 ([33]). *For a continuous function $\mathscr{F}$, the Riemann-Liouville fractional derivative of order $\mathfrak{q} > 0$ is given by*

$$ \mathcal{D}^{\mathfrak{p}}_{0^+}\mathscr{F}(t) = \frac{1}{\Gamma(\mathfrak{n} - \mathfrak{p})}\left(\frac{\partial}{\partial t}\right)^{\mathfrak{n}} \int_0^t (t - \mathfrak{s})^{\mathfrak{n}-\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\partial\mathfrak{s}, \ \mathfrak{n} - 1 < \mathfrak{p} < \mathfrak{n}. $$

Definition 2 ([33]). *For a continuous function $\mathscr{F}$, the Riemann-Liouville fractional integral of order $\mathfrak{p} > 0$ is given by*

$$ \mathcal{J}^{\mathfrak{p}}\mathscr{F}(t) = \frac{1}{\Gamma(\mathfrak{p})}\int_0^t (t - \mathfrak{s})^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\partial\mathfrak{s}, $$

where Γ is defined by $\Gamma(\mathfrak{p}) = \int_0^\infty \mathfrak{e}^{-\mathfrak{s}}\mathfrak{s}^{\mathfrak{p}-1}\partial\mathfrak{s}$.

Definition 3 ([33]). *For the function $\mathscr{F} : [0, \infty) \to \mathbb{R}$, the Caputo derivative of order $\mathfrak{p}$ is defined as*

$$ {}^{c}\mathcal{D}^{\mathfrak{p}}\mathscr{F}(t) = \frac{1}{\Gamma(\mathfrak{n} - \mathfrak{p})}\int_0^t \frac{\mathscr{F}^{(n)}(s)}{(t\text{-}s)^{\mathfrak{p}+1-n}}ds = \mathfrak{I}^{\mathfrak{n}-\mathfrak{p}}\mathscr{F}^{(n)}(t), \ t > 0, \mathfrak{n} - 1 < \mathfrak{p} < \mathfrak{n}. $$

Definition 4 ([34]). *A function $\mathscr{F}$ is fractional integrals and FDs with regard to another function Ψ are defined as follows:*

$$\mathfrak{I}^{\mathfrak{p};\Psi}\mathscr{F}(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\partial\mathfrak{s},$$

and

$$\mathfrak{D}^{\mathfrak{p};\Psi}\mathscr{F}(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{n}-\mathfrak{p})}(\frac{1}{\Psi'(\mathfrak{t})}\frac{\partial}{\partial\mathfrak{t}})^{\mathfrak{n}}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{n}-\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\partial\mathfrak{s},$$

respectively.

Definition 5 ([17]). *For noninstantaneous impulsive fractional differential differential Equations (1)–(3) the intervals $(t_i, s_i], \mathfrak{i} = 1, \ldots, \mathfrak{m}$ are called intervals of N-InI, and $\mathscr{H}_{\mathfrak{i}}(\mathfrak{t}, \wp(\mathfrak{t}))$, $\mathfrak{i} = 1, \ldots, \mathfrak{m}$ are called N-InI functions.*

Definition 6 ([32]). *A function $\mathfrak{w} \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R})$ is said to be a solution of (1)–(3) if u satisfied the equation ${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t}, \wp(\mathfrak{t}), \Psi\wp(\mathfrak{t}))$ on $\mathfrak{J}$, and the conditions $\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}, \wp(\mathfrak{t})), \mathfrak{a}\wp(0) + \mathfrak{b}\wp(\mathscr{T}) = \mathfrak{c}$.*

Lemma 1. *Let $0 < \alpha < 1$ and let $\mathscr{F} : \mathfrak{J} \longrightarrow \mathbb{R}$ be continuous. A function $\wp$ is a solution of the Ψ-fractional integral equation*

$$\begin{cases} \mathscr{H}_{\mathfrak{m}}(\mathfrak{s}_{\mathfrak{m}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{t})\partial\mathfrak{s} + \wp_0, \ \ \mathfrak{t} \in [0, \mathfrak{t}_1], \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \qquad \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}, \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{t})\partial\mathfrak{s} \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_{\mathfrak{i}}} \Psi'(\mathfrak{s})(\Psi\mathfrak{s}_{\mathfrak{i}} - \Psi\mathfrak{s})^{\mathfrak{p}-1}\omega(\mathfrak{t})\partial\mathfrak{s}, \ \ \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}. \end{cases} \tag{4}$$

if $\wp$ is a solution of the initial value problem of the system,

$$ {}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \omega(\mathfrak{t}) \ \ \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}] \subset [0, \mathscr{T}], 0 < \mathfrak{p} < 1, \tag{5}$$

$$\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \ \ \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \ \ \mathfrak{i} = 1, \ldots, \mathfrak{m}, \tag{6}$$

$$\wp(0) = \wp_0. \tag{7}$$

We obtain the following lemma as a result of Lemma 1.

Lemma 2. *A function $\wp \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R})$ is given by, $\wp(\mathfrak{t}) =$*

$$\begin{cases} \mathscr{H}_{\mathfrak{m}}(\mathfrak{s}_{\mathfrak{m}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s} \\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^{\mathscr{T}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s} - \mathfrak{c}\right], \ \ \mathfrak{t} \in [0, \mathfrak{t}_1], \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \qquad \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}, \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s} \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_{\mathfrak{i}}} \Psi'(\mathfrak{s})(\Psi\mathfrak{s}_{\mathfrak{i}} - \Psi\mathfrak{s})^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s}, \ \ \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}. \end{cases} \tag{8}$$

is a solution of the system given by

$$\begin{aligned} &{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \omega(\mathfrak{t}) \ \ \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}], 0 < \mathfrak{p} < 1, \\ &\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \ \ \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \ \ \mathfrak{i} = 1, \ldots, \mathfrak{m}, \\ &\mathfrak{a}\wp(0) + \mathfrak{b}\wp(\mathscr{T}) = \mathfrak{c}. \end{aligned} \tag{9}$$

Proof. Assume that $\wp(\mathfrak{t})$ is satisfied for Equation (9). Integrating the first equation of (9) for $\mathfrak{t} \in [0, \mathfrak{t}_1]$, we have

$$\wp(\mathfrak{t}) = \wp(\mathscr{T}) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s}. \tag{10}$$

On the other hand, if $\mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}], i = 1, 2, \ldots, \mathfrak{m}$ and again integrating the first equation of (9), we have

$$\wp(\mathfrak{t}) = \wp(\mathfrak{s}_i) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_i}^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s}. \tag{11}$$

Now, by applying impulsive condition, $\wp(\mathfrak{t}) = \mathscr{H}_i(\mathfrak{t}),\ \mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i]$, we obtain,

$$\wp(\mathfrak{s}_i) = \mathscr{H}_i(\mathfrak{s}_i). \tag{12}$$

Consequently, from (11) and (12), we obtain,

$$\wp(\mathfrak{t}) = \mathscr{H}_i(\mathfrak{s}_i) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s}, \tag{13}$$

and

$$\begin{aligned} \wp(\mathfrak{t}) = {} & \mathscr{H}_i(\mathfrak{s}_i) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s} \\ & - \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{s}_i} (\Psi'(\mathfrak{s}) \Psi \mathfrak{s}_i - \Psi \mathfrak{s})^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s},\ \mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}]. \end{aligned} \tag{14}$$

Now, using the BCs $\mathfrak{a}\wp(0) + \mathfrak{b}\wp(\mathscr{T}) = \mathfrak{c}$, we obtain

$$\wp(\mathscr{T}) = \mathscr{H}_{\mathfrak{m}}(\mathfrak{s}_{\mathfrak{m}}) - \frac{1}{\mathfrak{a} + \mathfrak{b}} \Big[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})} \int_0^{\mathscr{T}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \partial \mathfrak{s} - \mathfrak{c} \Big],\ \mathfrak{t} \in [0, \mathfrak{t}_1]. \tag{15}$$

Hence, with the direct applications of the FDs, integral definitions and lemmas, it is clear that (10), (14) and (15) $\Rightarrow$ (8). Hence the proof: □

FPT play a key role in many interesting recent outputs see, e.g., [20,21,35].

Theorem 1 ([36])**.** *(Banach FPT)*
If $\mathscr{Q}$ is a closed nonempty subset of a Banach space (BSp.) $\mathbb{B}$. Let $\mathscr{N} : \mathscr{Q} \to \mathscr{Q}$, be a contraction mapping, then $\mathscr{N}$ has a unique FP.

Theorem 2 ([37])**.** *(Krasnoselkii's FPT)*
Suppose a Banach space $\mathbb{Y}$, select a closed, bounded, and convex set $\emptyset \neq B \subset \mathbb{Y}$. Let A_1 and A_2 be two operators: (1) $A_1 x + A_2 y \in B$ whenever $x, y \in B$; (2) A_1 is compact and continuous; (3) A_2 is a contraction mapping. Therefore, $\exists\, z \in B$: $z = A_1 z + A_2 z$.

3. Main Results

Theorem 3. *Suppose that the following assumption holds.*
($\mathtt{Al}_1$)*: There exists a positive constant $\mathscr{L}, \mathscr{G}, \mathscr{M}, \mathscr{L}_{\mathfrak{h}_i}$ such that*

$$|\mathscr{F}(\mathfrak{t}, \wp_1, \omega_1) - \mathscr{F}(\mathfrak{t}, \wp_2, \omega_2)| \leq \mathscr{L}|\wp_1 - \wp_2| + \mathscr{G}|\omega_1 - \omega_2|,\ for\ \ \mathfrak{t} \in [0, \mathscr{T}],\ \wp_1, \wp_2, \omega_1, \omega_2 \in \mathbb{R}.$$

$$|\mathfrak{k}(\mathfrak{t}, \mathfrak{s}, \vartheta) - \mathfrak{k}(\mathfrak{t}, \mathfrak{s}, \nu)| \leq \mathscr{M}|\vartheta - \nu|, for\ \mathfrak{t} \in [\mathfrak{t}_i, \mathfrak{s}_i]\ \vartheta, \nu \in \mathbb{R}.$$

$$|\mathscr{H}_i(\mathfrak{t}, \mathfrak{v}_1) - \mathscr{H}_i(\mathfrak{t}, \mathfrak{v}_2)| \leq \mathscr{L}_{\mathfrak{h}_i}|\mathfrak{v}_1 - \mathfrak{v}_2|, for\ \mathfrak{v}_1, \mathfrak{v}_2 \in \mathbb{R}.$$

If

$$\mathscr{Z}:\max\{\max_{\mathfrak{i}=1,2,\ldots,\mathfrak{m}}\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{t}_{\mathfrak{i}+1}^{\mathfrak{p}}+\mathfrak{s}_\mathfrak{i}^{\mathfrak{p}}),$$
$$\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})(\Psi(\mathscr{T}))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1+\frac{|\mathfrak{b}|(\mathscr{L}+\mathscr{G}\mathscr{M})}{|\mathfrak{a}+\mathfrak{b}|}\right]\}<1, \tag{16}$$

then the problems (1)–(3) have a unique solution on $[0,\mathscr{T}]$.

Proof. We define an operator $\mathscr{N}:\mathscr{PC}([0,\mathscr{T}],\mathbb{R})\longrightarrow\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ by

$$(\mathscr{N}\wp)(\mathfrak{t})=\begin{cases}\mathscr{H}_\mathfrak{m}(\mathfrak{s}_\mathfrak{m},\wp(\mathfrak{s}_\mathfrak{m}))+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\mathfrak{d}\mathfrak{s}\\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^{\mathscr{T}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\mathfrak{d}\mathfrak{s}\right],\ \ \mathfrak{t}\in[0,\mathfrak{t}_1],\\ \mathscr{H}_\mathfrak{i}(\mathfrak{t}),\quad \mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}],\mathfrak{i}=1,2,\ldots,\mathfrak{m},\\ \mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i})+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\mathfrak{d}\mathfrak{s}\\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_\mathfrak{i}}\Psi'(\mathfrak{s})(\Psi\mathfrak{s}_\mathfrak{i}-\Psi\mathfrak{s})^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\mathfrak{d}\mathfrak{s},\ \ \mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}],\mathfrak{i}=1,2,\ldots,\mathfrak{m}.\end{cases}$$

It is obvious that $\mathscr{N}$ is well defined and $\mathscr{N}\wp\in\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$. We now prove that $\mathscr{N}$ is a contraction mapping.
Case 1: For $\wp,\overline{\wp}\in\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ and $\mathfrak{t}\in[0,\mathfrak{t}_1]$, we obtain

$$\begin{aligned}&|(\mathscr{N}\wp)(\mathfrak{t})-(\mathscr{N}\overline{\wp})(\mathfrak{t})|\\ &\leq\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}|\wp(\mathfrak{s}_\mathfrak{m})-\overline{\wp}(\mathfrak{s}_\mathfrak{m})|\mathfrak{d}\mathfrak{s}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}\int_0^{\mathfrak{t}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}|\wp-\overline{\wp}|\mathfrak{d}\mathfrak{s}\\ &+\frac{|\mathfrak{b}|(\mathscr{L}+\mathscr{G}\mathscr{M})}{|\mathfrak{a}+\mathfrak{b}|\Gamma(\mathfrak{p})}\int_0^{\mathscr{T}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}|\wp-\overline{\wp}|\mathfrak{d}\mathfrak{s}\\ &\leq\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})(\Psi(\mathscr{T}))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right]\|\wp-\overline{\wp}\|_{\mathscr{PC}}.\end{aligned}$$

Case 2: For $\mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}]$, we find that

$$\begin{aligned}|(\mathscr{N}\wp)(\mathfrak{t})-(\mathscr{N}\overline{\wp})(\mathfrak{t})|&\leq|\mathscr{H}_\mathfrak{i}(\mathfrak{t},\wp(\mathfrak{t}))-\mathscr{H}_\mathfrak{i}(\mathfrak{t},\overline{\wp}(\mathfrak{t}))|\\ &\leq\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}\|\wp-\overline{\wp}\|_{\mathscr{PC}}.\end{aligned}$$

Case 3: For $\mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}]$, we obtain

$$\begin{aligned}&|(\mathscr{N}\wp)(\mathfrak{t})-(\mathscr{N}\overline{\wp})(\mathfrak{t})|\\ &\leq|\mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i},\wp(\mathfrak{s}_\mathfrak{i})-\mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i},\overline{\wp}(\mathfrak{s}_\mathfrak{i})|+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))-\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}\\ &+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_\mathfrak{i}}(\mathfrak{s}_\mathfrak{i}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))-\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\mathfrak{d}\mathfrak{s},\\ &\leq\left[\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{t}_{\mathfrak{i}+1}^{\mathfrak{p}}+\mathfrak{s}_\mathfrak{i}^{\mathfrak{p}})\right]\|\wp-\overline{\wp}\|_{\mathscr{PC}}.\end{aligned}$$

Therefore, $\mathscr{N}$ is a contraction, as in the above inequality

$$\mathscr{Z}=\left[\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{t}_{\mathfrak{i}+1}^{\mathfrak{p}}+\mathfrak{s}_\mathfrak{i}^{\mathfrak{p}})\right]<1.$$

Thus, the problem (1)–(3) has a unique solution for each $\wp\in\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$. □

Theorem 4. *Suppose that the condition* $(\mathtt{Al}_1)$ *is satisfied and the following assumption holds well:* $(\mathtt{Al}_2)$*: There exists a constant* $\mathscr{L}_{\mathtt{g}_\mathtt{i}} > 0$*, such that*

$$|\mathscr{F}(\mathtt{t},\mathtt{W}_1,\omega_1)| \leq \mathscr{L}_{\mathtt{g}_\mathtt{i}}(1+|\mathtt{W}_1|+|\omega_1|),\ \ \mathtt{t}\in[\mathtt{s}_\mathtt{i},\mathtt{t}_{\mathtt{i}+1}], \forall\, \mathtt{W}_1,\omega_1\in\mathbb{R}.$$

$(\mathtt{Al}_3)$*: There exists a function* $\kappa_i(\mathtt{t}), \mathtt{i}=1,2,\ldots,\mathtt{m}$*, such that*

$$|\mathscr{H}_i(\mathtt{t},\mathtt{W}_1,\omega_1)| \leq \kappa_\mathtt{i}(\mathtt{t}),\ \ \mathtt{t}\in[\mathtt{t}_\mathtt{i},\mathtt{s}_\mathtt{i}], \forall\ \mathtt{W}_1,\omega_1\in\mathbb{R}.$$

Assume that $\mathscr{M}_\mathtt{i} : \sup_{\mathtt{t}\in[\mathtt{t}_\mathtt{i},\mathtt{s}_i]} \kappa_\mathtt{i}(\mathtt{t}) < \infty$*, and* $\mathscr{K} := \max \mathscr{L}_{\mathtt{h}_i} < 1$*, for all* $\mathtt{i} = 1,2,..,\mathtt{m}$*. Then, the problem (1)–(3) has at least one solution on* $[0,\mathscr{T}]$*.*

Proof. Let us consider $\mathscr{B}_{\mathfrak{p},\mathfrak{r}} = \{\wp \in \mathscr{PC}([0,\mathscr{T}],\mathbb{R}) : \|\wp\|_{\mathscr{PC}} \leq \mathfrak{r}\}$. Let $\mathscr{Q}$ and $\mathscr{R}$ be two operators on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ defined as follows:

$$\mathscr{Q}\wp(\mathfrak{t}) = \begin{cases} \mathscr{H}_\mathfrak{m}(\mathfrak{s}_\mathfrak{m},\wp(\mathfrak{s}_\mathfrak{m})), & \mathfrak{t}\in[0,\mathfrak{t}_1],\\ \mathscr{H}_\mathfrak{i}(\mathfrak{t},\wp(\mathfrak{t})), & \mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}], \mathfrak{i}=1,2,\ldots,\mathfrak{m},\\ \mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i},\wp(\mathfrak{s}_\mathfrak{i})), & \mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i}=1,2,\ldots,\mathfrak{m}. \end{cases}$$

and

$$\mathscr{R}\wp(\mathfrak{t}) = \begin{cases} \frac{1}{\Gamma(\mathfrak{p})}\int_0^\mathfrak{t} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s} \\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^\mathscr{T} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s}\right],\ \ \mathfrak{t}\in[0,\mathfrak{t}_1],\\ 0, \mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}], \mathfrak{i}=1,2,\ldots,\mathfrak{m},\\ \frac{1}{\Gamma(\mathfrak{p})}\int_0^\mathfrak{t} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s} \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_\mathfrak{i}} \Psi'(\mathfrak{s})(\Psi\mathfrak{s}_{(i)}-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s},\ \ \mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i}=1,2,\ldots,\mathfrak{m}. \end{cases}$$

step 1: For $\wp\in\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ then $\mathscr{Q}\wp+\mathscr{R}\wp\in\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
case 1: For $\mathfrak{t}\in[0,\mathfrak{t}_1]$,

$$\begin{aligned} |\mathscr{Q}\wp+\mathscr{R}\overline{\wp}| &\leq |\mathscr{H}_\mathfrak{m}(\mathfrak{s}_\mathfrak{m},\wp(\mathfrak{s}_\mathfrak{m}))| + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\mathfrak{t}(\mathfrak{t}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\partial\mathfrak{s} \\ &+ \frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^\mathscr{T}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s}\right], \\ &\leq \left[\mathscr{L}_{\mathfrak{g}_\mathfrak{i}} + \frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}(\Psi(\mathscr{T}))^\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right]\right](1+\mathfrak{r}) \leq \mathfrak{r}. \end{aligned}$$

case 2: For each $\mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}]$,

$$|\mathscr{Q}\wp+\mathscr{R}\overline{\wp}| \leq |\mathscr{H}_\mathfrak{i}(\mathfrak{t},\mathscr{W}_1(\mathfrak{t}))| \leq \mathscr{M}_\mathfrak{i}.$$

case 3: For each $\mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}]$,

$$\begin{aligned} |\mathscr{Q}\wp+\mathscr{R}\overline{\wp}(\mathfrak{t})| &\leq |\mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i},\wp(\mathfrak{s}_\mathfrak{i}))| + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\mathfrak{t}(\mathfrak{t}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\partial\mathfrak{s} \\ &+ \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_\mathfrak{i}}(\mathfrak{s}_\mathfrak{i}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\partial\mathfrak{s}, \\ &\leq \mathscr{M}_\mathfrak{i} + \left[\frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}}(\mathfrak{s}_\mathfrak{i}^\mathfrak{p}+\mathfrak{t}_{\mathfrak{i}+1}^\mathfrak{p})}{\Gamma(\mathfrak{p}+1)}\right](1+\mathfrak{r}) \leq \mathfrak{r}. \end{aligned}$$

Thus

$$\mathscr{Q}\wp+\mathscr{R}\wp\in\mathscr{B}_{\mathfrak{p},\mathfrak{r}}.$$

step 2: $\mathscr{Q}$ is contraction on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
case 1: $\wp_1, \wp_2 \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ then $\mathfrak{t} \in [0, \mathfrak{t}_1]$,

$$|\mathscr{Q}\wp_1(\mathfrak{t}) - \mathscr{Q}\wp_2(\mathfrak{t})| \leq \mathscr{L}_{\mathfrak{g}_\mathfrak{m}} |\wp_1(\mathfrak{s}_\mathfrak{m}) - \wp_2(\mathfrak{s}_\mathfrak{m})| \leq \mathscr{L}_{\mathfrak{g}_\mathfrak{m}} \|\wp_1 - \wp_2\|_{\mathscr{PC}}.$$

case 2: For each $\mathfrak{t} \in (\mathfrak{t}_\mathfrak{i}, \mathfrak{s}_\mathfrak{i}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}$,

$$|\mathscr{Q}\wp_1(\mathfrak{t}) - \mathscr{Q}\wp_2(\mathfrak{t})| \leq \mathscr{L}_{\mathfrak{g}_\mathfrak{i}} \|\wp_1 - \wp_2\|_{\mathscr{PC}}.$$

case 3: For $\mathfrak{t} \in (\mathfrak{s}_\mathfrak{i}, \mathfrak{t}_{\mathfrak{i}+1}]$,

$$|\mathscr{Q}\wp_1(\mathfrak{t}) - \mathscr{Q}\wp_2(\mathfrak{t})| \leq \mathscr{L}_{\mathfrak{g}_\mathfrak{i}} \|\wp_1 - \wp_2\|_{\mathscr{PC}}.$$

We can deduce the following from above inequalities:

$$|\mathscr{Q}\wp_1(\mathfrak{t}) - \mathscr{Q}\wp_2(\mathfrak{t})| \leq \mathscr{K} \|\wp_1 - \wp_2\|_{\mathscr{PC}}.$$

Hence, $\mathscr{Q}$ is a contraction.
step 3: We prove that $\mathscr{R}$ is continuous.
Let $\wp_\mathfrak{n}$ be a sequence $\ni \wp_\mathfrak{n} \to \overline{\wp}$ in $\mathscr{PC}([0, \mathscr{T}], \mathbb{R})$.
case 1: For each $\mathfrak{t} \in [0, \mathfrak{t}_1]$,

$$|\mathscr{Q}\wp_\mathfrak{n}(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})| \leq \left[\frac{(\Psi(\mathscr{T}))^\mathfrak{p}}{\Gamma(\mathfrak{p}+1)}\left[1 + \frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right]\right] \|\mathscr{F}(., \wp_\mathfrak{n}(.), .,) - \mathscr{F}(., \wp(.), .,)\|_{\mathscr{PC}}.$$

case 2: For each $\mathfrak{t} \in (\mathfrak{t}_\mathfrak{i}, \mathfrak{s}_\mathfrak{i}]$, we obtain

$$|\mathscr{Q}\wp_\mathfrak{n}(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})| = 0.$$

case 3: For each $\mathfrak{t} \in (\mathfrak{s}_\mathfrak{i}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}$,

$$|\mathscr{Q}\wp_\mathfrak{n}(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})| \leq \frac{(\mathfrak{t}_{\mathfrak{i}+1} - \mathfrak{s}_\mathfrak{i})}{\Gamma(\mathfrak{p}+1)} \|\mathscr{F}(., \wp_\mathfrak{n}(.), .,) - \mathscr{F}(., \wp(.), .,)\|_{\mathscr{PC}}.$$

Thus, we conclude from the above cases that $\|\mathscr{Q}\wp_\mathfrak{n}(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})\|_{\mathscr{PC}} \longrightarrow 0$ as $\mathfrak{n} \to \infty$.
step 4: We prove that $\mathscr{Q}$ is compact.
First $\mathscr{Q}$ is uniformly bounded on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
Since $\|\mathscr{Q}\wp\| \leq \frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}}(\mathscr{T})}{\Gamma(1+\mathfrak{p})} < \mathfrak{r}$,
First $\mathscr{Q}$ is uniformly bounded on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
Since $\|\mathscr{Q}\wp\| \leq \frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}}(\mathscr{T})}{\Gamma(1+\mathfrak{p})} < \mathfrak{r}$, we prove that $\mathscr{Q}$ maps a bounded set to a $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ equicontinuous set.
case 1: For interval $\mathfrak{t} \in [0, \mathfrak{t}_1], 0 \leq \mathscr{E}_1 \leq \mathscr{E}_2 \leq \mathfrak{t}_1, \wp \in \mathscr{B}_\mathfrak{r}$, we obtain

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \leq \frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)}(\mathscr{E}_2 - \mathscr{E}_1).$$

case 2: For each $\mathfrak{t} \in (\mathfrak{t}_\mathfrak{i}, \mathfrak{s}_\mathfrak{i}], \mathfrak{t}_\mathfrak{i} < \mathscr{E}_1 < \mathscr{E}_2 \leq \mathfrak{s}_\mathfrak{i}, \wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$, we obtain

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| = 0.$$

case 3: For each $\mathfrak{t} \in (\mathfrak{s}_\mathfrak{i}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{s}_\mathfrak{i} < \mathscr{E}_1 < \mathscr{E}_2 \leq \mathfrak{t}_{\mathfrak{i}+1}, \wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$, we establish

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \leq \frac{\mathscr{L}_{\mathfrak{g}_\mathfrak{i}}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)}(\mathscr{E}_2 - \mathscr{E}_1).$$

From the above cases, we obtain $|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \longrightarrow 0$ as $\mathscr{E}_2 \longrightarrow \mathscr{E}_1$ and $\mathscr{Q}$ is equicontinuous. Thus $\mathscr{Q}(\mathscr{B}_{\mathfrak{p},\mathfrak{r}})$ is relatively compact, so by using the Ascoli-Arzela theorem, $\mathscr{Q}$

is compact. Hence, the problem (1)–(3) have at least one fixed point on $[0, \mathscr{T}]$. Hence the proof. □

4. Example

Let as consider the Ψ-Caputo fractional boundary value problem

$$D^{\mathfrak{p}}\wp(\mathfrak{t}) = \frac{\mathfrak{e}^{-\mathfrak{t}}|\mathfrak{w}|}{9+\mathfrak{e}^{\mathfrak{t}}(1+|\wp|} + \frac{1}{3}\int_0^{\mathfrak{t}} \mathfrak{e}^{-(\mathfrak{s}-\mathfrak{t})}\wp(\mathfrak{s})\mathfrak{d}\mathfrak{s}, \ \ \mathfrak{t} \in (0, \frac{1}{2}], \tag{17}$$

$$\wp(\mathfrak{t}) = \frac{|\wp(\mathfrak{t})|}{2(1+|\wp(\mathfrak{t})|)}, \ \mathfrak{t} \in (\frac{1}{2}, 1], \tag{18}$$

$$\wp(0) + \wp(1) = 0. \tag{19}$$

and $\mathscr{L} = \mathscr{G} = \frac{1}{10}$, $\mathscr{M} = \frac{1}{3}$, $\mathfrak{p} = \frac{5}{7}$ $\mathscr{L}_{\mathfrak{h}_1} = \frac{1}{3}$, We shall check that condition (3) is satisfied for appropriate values of $\mathfrak{p} \in (0, 1]$ with $\mathfrak{a} = \mathfrak{b} = \mathscr{T} = 1$. Indeed, by using Theorem 4, we determine that

$$\mathscr{L}_{\mathfrak{h}_i} + \frac{(\mathscr{L} + \mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{t}_{i+1}^{\mathfrak{p}} + \mathfrak{s}_i^{\mathfrak{p}}) \approx 0.41 < 1,$$

and

$$\left\{\mathscr{L}_{\mathfrak{h}_i} + \frac{(\mathscr{L} + \mathscr{G}\mathscr{M})(\Psi(\mathscr{T}))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1 + \frac{|\mathfrak{b}|(\mathscr{L} + \mathscr{G}\mathscr{M})}{|\mathfrak{a}+\mathfrak{b}|}\right]\right\} \approx 0.485 < 1.$$

Thus, all assumptions of Theorem 4 are satisfied, so the problem (17)–(19) has a unique solution $[0, \mathscr{T}]$.

5. Conclusions

In this work, we discuss the existence results for N-InI Ψ-CFIDE with BCs. Our results guarantee the existence of integral solution via FC theory and Krasnoselkii's FPT. The example is used to illustrate the results. Potential future works could be to extend the problem with more advanced delays. Moreover, we plan to investigate other kinds of fractional derivatives such as, e.g., Katugampola derivative, conformable derivative, and many others.

Author Contributions: Conceptualization, R.A., P.K., K.K., P.G., Y.A., L.A. and E.-s.E.-h.; methodology, R.A., P.K., K.K. and P.G.; software, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h.; validation, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h.; formal analysis, R.A., P.K., K.K. and P.G.; investigation, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h.; data curation, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h.; writing—original draft preparation, R.A., P.K., K.K. and P.G.; writing—review and editing, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h.; visualization, R.A., P.K., K.K., P.G., Y.A., L.A. and E.-s.E.-h.; supervision, K.K.; project administration, R.A., P.K., K.K., Y.A., L.A. and E.-s.E.-h. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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