symmetry

Article

An Algorithm for Solving Common Points of Convex Minimization Problems with Applications

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Abstract: In algorithm development, symmetry plays a vital part in managing optimization problems in scientific models. The aim of this work is to propose a new accelerated method for finding a common point of convex minimization problems and then use the fixed point of the forward-backward operator to explain and analyze a weak convergence result of the proposed algorithm in real Hilbert spaces under certain conditions. As applications, we demonstrate the suggested method for solving image inpainting and image restoration problems.

Keywords: Hilbert space; forward-backward algorithm; convergence theorems; convex minimization problems; fixed point

MSC: 47H10; 47J25; 65K05; 90C30

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1. Introduction

In this study, let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{R}$ be the set of all real numbers. The operator $\mathcal{I} : \mathcal{H} \to \mathcal{H}$ denotes the identity operator. Weak and strong convergence are denoted by the symbols $\rightharpoonup$ and $\to$, respectively.

In recent years, the convex minimization problem in the form of the sum of two convex functions plays and important role in solving real-world problems such as in signal and image processing, machine learning and medical image reconstruction, see [1–10], for instance. This problem can be written in the following form:

$$\underset{z \in \mathcal{H}}{\text{minimize}} \ \phi_1(z) + \phi_2(z), \tag{1}$$

where $\phi_1 : \mathcal{H} \to \mathbb{R}$ is a convex and differentiable function such that $\nabla \phi_1$ is $\mathcal{L}$-Lipschitz continuous and $\phi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a convex and proper lower semi-continuous function. Symmetry, or invariance, serves as the foundation for the solution of problem (1). The solution set for problem (1) is equivalent to the fixed point Equation (2),

$$z = \text{prox}_{\sigma\phi_2}(\mathcal{I} - \sigma\nabla\phi_1)(z), \tag{2}$$

where $\sigma > 0$, prox_{ϕ_2} is the proximity operator of ϕ_2 and $\nabla\phi_1$ stands for the gradient of ϕ_1. It is known that if the step size $\sigma \in (0, 2/\mathcal{L})$, then $\text{prox}_{\sigma\phi_2}(\mathcal{I} - \sigma\nabla\phi_1)(z)$ is nonexpansive. For the past decade, many algorithms based on fixed point method were proposed to solve the problem (1), see [4,8,11–15].

Lions and Mercier proposed the *forward-backward splitting* (FBS) algorithm [6] as the following:

$$z^{k+1} = \text{prox}_{\sigma_k \phi_2}(\mathcal{I} - \sigma_k \nabla \phi_1)(z^k), \quad \forall k \in \mathbb{N}, \tag{3}$$

where $z^1 \in \mathcal{H}$ and $0 < \sigma_k < 2/\mathcal{L}$.

Combettes and Wajs [3] studied the *relaxed forward-backward splitting* (R-FBS) method in 2005, which was defined as follows:

$$y^k = z^k - \sigma_k \nabla \phi_1(z^k), \quad z^{k+1} = z^k + \beta_k(\text{prox}_{\sigma_k \phi_2}(y^k) - z^k), \quad \forall k \in \mathbb{N}, \tag{4}$$

where $\varepsilon \in (0, \min(1, \frac{1}{\mathcal{L}}))$, $z^1 \in \mathbb{R}^N$, $\sigma_k \in [\varepsilon, \frac{2}{\mathcal{L}} - \varepsilon]$ and $\beta_k \in [\varepsilon, 1]$.

An inertial technique is often used to speed up the forward-backward splitting procedure. As a result, numerous inertial algorithms were created and explored in order to speed up the algorithms' convergence behavior, see [14,16–18] for example. Beck and Teboulle [17] recently published FISTA, a *fast iterative shrinkage-thresholding algorithm* to solve the problem (1). The following are the characteristics of FISTA:

$$\begin{aligned} t_{k+1} &= \frac{1 + \sqrt{1 + 4t_k^2}}{2}, \quad \alpha_k = \frac{t_k - 1}{t_{k+1}}, \\ y^k &= \text{prox}_{\frac{1}{\mathcal{L}}\phi_2}(\mathcal{I} - \frac{1}{\mathcal{L}} \nabla \phi_1)(z^k), \\ z^{k+1} &= y^k + \alpha_k(y^k - y^{k-1}), \quad k \in \mathbb{N}, \end{aligned} \tag{5}$$

where $z^1 = y^0 \in \mathbb{R}^N$, $t_1 = 1$. It is worth noting that α_k is an *inertial parameter* that determines the momentum $y^k - y^{k-1}$.

In this work, we are interested to construct a new accelerated algorithm for finding a common element of the convex minimization problems (6) by using inertial and fixed point techniques of forward-backward operators:

$$\min_{x \in \mathcal{H}} \phi_1(x) + \phi_2(x), \quad \text{and} \quad \min_{x \in \mathcal{H}} \omega_1(x) + \omega_2(x), \tag{6}$$

where $\phi_1 : \mathcal{H} \to \mathbb{R}, \phi_1 : \mathcal{H} \to \mathbb{R}, \omega_1 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ and $\omega_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ are convex and proper lower semi-continuous function. Then, we prove a weak convergence result of the proposed algorithm in real Hilbert spaces under certain conditions and illustrate the theoretical results via some numerical experiments in image inpainting and image restoration problems.

2. Preliminaries

Basic concepts, definitions, notations and some relevant lemmas for usage in the following parts will be discussed in this section.

Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a convex and proper lower semi-continuous function. The proximity operator can be written in the equivalent form:

$$\text{prox}_{\phi} = (\mathcal{I} + \partial \phi)^{-1} : \mathcal{H} \to \mathcal{H}, \tag{7}$$

when $\partial \phi$ is the subdifferential of ϕ given by

$$\partial \phi(z) := \{u \in \mathcal{H} : \phi(z) + \langle u, y - z \rangle \le \phi(y), \forall y \in \mathcal{H}\}, \quad \forall z \in \mathcal{H}.$$

We notice that $\text{prox}_{\delta_{\mathcal{C}}} = \mathbb{P}_{\mathcal{C}}$, where $\mathcal{C} \subseteq \mathcal{H}$ is a nonempty closed convex set, $\delta_{\mathcal{C}}$ is the indicator function and $\mathbb{P}_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is the *orthogonal projection operator* on $\mathcal{C}$. The subdifferential operator $\partial\phi$ is a maximal monotone (for additional information, see [19]), and the solution of (1) is a fixed point of the operator below:

$$z \in \operatorname{Argmin}(\phi_1 + \phi_2) \Longleftrightarrow z = \operatorname{prox}_{\sigma\phi_2}(\mathcal{I} - \sigma\nabla\phi_1)(z),$$

where $\sigma > 0$, and $\operatorname{Argmin}(\phi_1 + \phi_2)$ is solution set for problem (1).

The following Lipschitz continuous and nonexpansive operators are considered. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is called *Lipschitz continuous* if there exists $\mathcal{L} > 0$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \mathcal{L}\|x - y\|, \ \forall x, y \in \mathcal{H}.$$

When $\mathcal{T}$ is 1-Lipschitz continuous, it is referred to as *nonexpansive*. If $z = \mathcal{T}z$, a point $z \in \mathcal{H}$ is called *fixed point* of $\mathcal{T}$ and $\operatorname{Fix}(\mathcal{T})$ denotes the set of fixed points for $\mathcal{T}$.

The operator $\mathcal{I} - \mathcal{T}$ is called *demiclosed at zero* if any sequence $\{z^k\}$ converges weakly to z and the sequence $\{z^k - \mathcal{T}z^k\}$ converges strongly to zero, then $z \in \operatorname{Fix}(\mathcal{T})$. If $\mathcal{T}$ is a nonexpansive operator, then $\mathcal{I} - \mathcal{T}$ is known to be demiclosed at zero [20].

Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ and $\{\mathcal{T}_k : \mathcal{H} \to \mathcal{H}\}$ be such that $\emptyset \neq \operatorname{Fix}(\mathcal{T}) \subseteq \bigcap_{k=1}^{\infty} \operatorname{Fix}(\mathcal{T}_k)$. Then, $\{\mathcal{T}_k\}$ is said to satisfy *NST-condition (I)* with $\mathcal{T}$ [21] if for each bounded sequence $\{z^k\} \subset \mathcal{H}$,

$$\lim_{k\to\infty} \|z^k - \mathcal{T}_k z^k\| = 0 \ \text{ implies } \ \lim_{k\to\infty} \|z^k - \mathcal{T}z^k\| = 0.$$

The following basic property on $\mathcal{H}$ will be used in the study (see [22]): for all $x, y \in \mathcal{H}$ and $\gamma \in [0, 1]$,

$$\|\gamma x + (1-\gamma)y\|^2 = \gamma\|x\|^2 + (1-\gamma)\|y\|^2 - \gamma(1-\gamma)\|x - y\|^2, \tag{8}$$
$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2. \tag{9}$$

Lemma 1 ([18])**.** *Let $\phi_1 : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function such that $\nabla\phi_1$ is $\mathcal{L}$-Lipschitz continuous and $\phi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a convex and proper lower semi-continuous function. Let $\mathcal{T}_k := \operatorname{prox}_{\lambda_k\phi_2}(\mathcal{I} - \lambda_k\nabla\phi_1)$ and $\mathcal{T} := \operatorname{prox}_{\lambda\phi_2}(\mathcal{I} - \lambda\nabla\phi_1)$, where $\lambda_k, \lambda \in (0, 2/\mathcal{L})$ with $\lambda_k \to \lambda$. Then $\{\mathcal{T}_k\}$ satisfies NST-condition (I) with $\mathcal{T}$.*

Lemma 2 ([14])**.** *Let $\{z^k\}$ and $\{\alpha_k\}$ be two sequences of non-negative real numbers such that*

$$z^{k+1} \leq (1 + \alpha_k)z^k + \alpha_k z^{k-1}, \quad \forall k \geq 1.$$

Then $z^{k+1} \leq \mathcal{E} \cdot \prod_{j=1}^{k}(1 + 2\alpha_j)$, where $\mathcal{E} = \max\{z^1, z^2\}$. Moreover, if $\sum_{k=1}^{\infty} \alpha_k < \infty$, then $\{z^k\}$ is bounded.

Lemma 3 ([23])**.** *Let $\{z^k\}$ and $\{w^k\}$ be two sequences of non-negative real numbers such that*

$$z^{k+1} \leq z^k + w^k,$$

for all $k \geq 1$. If $\sum_{k=1}^{\infty} w^k < \infty$, then $\lim_{k\to\infty} z^k$ exists.

Lemma 4 ([24])**.** *Let $\{z^k\}$ be a sequence in $\mathcal{H}$ and $\emptyset \neq \Theta \subset \mathcal{H}$ that satisfies*

(I) For every $z^* \in \Theta$, $\lim_{k\to\infty} \|z^k - z^*\|$ exists;

(II) $\omega_w(z^k) \subset \Theta$, where $\omega_w(z^k)$ is the set of all weak-cluster points of $\{z^k\}$.

Then, $\{z^k\}$ converges weakly to a point in Θ.

3. Main Results

In this section, we suggest an inertial forward-backward splitting algorithm to solve common points of convex minimization problems and prove weak convergence of the proposed algorithm. Assumptions that will be used throughout this section are as follows:

- ► ϕ_1 and ω_1 are convex and differentiable functions from $\mathcal{H}$ to $\mathbb{R}$;

- $\nabla\phi_1$ and $\nabla\omega_1$ are Lipschitz continuous with constants $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively;
- ϕ_2 and ω_2 are convex and proper lower semi-continuous functions from $\mathcal{H}$ to $\mathbb{R}\cup\{\infty\}$;
- $\Theta := \text{Argmin}(\phi_1+\phi_2)\cap\text{Argmin}(\omega_1+\omega_2)\neq\emptyset$.

Remark 1. *Let* $\mathcal{U}_k := \text{prox}_{\sigma_k\phi_2}(\mathcal{I}-\sigma_k\nabla\phi_1)$ *and* $\mathcal{U} := \text{prox}_{\sigma\phi_2}(\mathcal{I}-\sigma\nabla\phi_1)$. *If* $0<\sigma_k,\sigma<2/\mathcal{L}_1$, *then* $\mathcal{U}_k$ *and* $\mathcal{U}$ *are nonexpansive operators with* $\text{Fix}(\mathcal{U}) = \text{Argmin}(\phi_1+\phi_2) = \bigcap_{k=1}^{\infty}\text{Fix}(\mathcal{U}_k)$. *Moreover, if* $\sigma_k\to\sigma$, *then Lemma 1 asserts that* $\{\mathcal{U}_k\}$ *satisfies NST-condition (I) with* $\mathcal{U}$.

Algorithm 1: Given: $z^0, z^1\in\mathcal{H}$. Choose $\{\alpha_k\},\{\beta_k\},\{\gamma_k\},\{\sigma_k\}$ and $\{\sigma_k^*\}$.

For $k=1,2,\ldots,$ **do**

$$\begin{aligned} w^k &= z^k+\alpha_k(z^k-z^{k-1});\\ y^k &= w^k+\beta_k(\text{prox}_{\sigma_k\phi_2}(\mathcal{I}-\sigma_k\nabla\phi_1)w^k-w^k);\\ z^{k+1} &= (1-\gamma_k)\,\text{prox}_{\sigma_k\phi_2}(\mathcal{I}-\sigma_k\nabla\phi_1)w^k+\gamma_k\,\text{prox}_{\sigma_k^*\omega_2}(\mathcal{I}-\sigma_k^*\nabla\omega_1)y^k, \end{aligned}$$

end for.

Next, the convergence result of Algorithm 1 can be shown as follows:

Theorem 1. *Let* $\{z^k\}$ *be the sequence created by Algorithm 1. Suppose that* $\{\alpha_k\},\{\beta_k\},\{\gamma_k\},\{\sigma_k\}$ *and* $\{\sigma_k^*\}$ *are the sequences which satisfy the following conditions:*

(A1) $\beta_k\in[a,b]\subset(0,1), \gamma_k\in[c,d]\subset(0,1)\ \forall k\in\mathbb{N}$, *for some* $a,b,c,d\in\mathbb{R}$ *with* $a<b$ *and* $c<d$;

(A2) $\alpha_k\geq 0, \forall k\in\mathbb{N}$ *and* $\sum_{k=1}^{\infty}\alpha_k<\infty$;

(A3) $0<\sigma_k,\sigma<2/\mathcal{L}_1, 0<\sigma_k^*,\sigma^*<2/\mathcal{L}_2, \forall k\in\mathbb{N}$ *such that* $\sigma_k\to\sigma$ *and* $\sigma_k^*\to\sigma^*$ *as* $k\to\infty$.

Then, the following holds:

(i) $\|z^{k+1}-z^*\|\leq\mathcal{E}\prod_{j=1}^{k}(1+2\alpha_j)$, *where* $\mathcal{E}=\max\{\|z^1-z^*\|,\|z^2-z^*\|\}$ *and* $z^*\in\Theta$.

(ii) $\{z^k\}$ *converges weakly to common point in* $\Theta := \text{Argmin}(\phi_1+\phi_2)\cap\text{Argmin}(\omega_1+\omega_2)$.

Proof. Define operators $\mathcal{U}_k,\mathcal{T}_k,\mathcal{U},\mathcal{T}:\mathcal{H}\to\mathcal{H}$ as follows:

$$\mathcal{U}_k := \text{prox}_{\sigma_k\phi_2}(\mathcal{I}-\sigma_k\nabla\phi_1),\quad \mathcal{U} := \text{prox}_{\sigma\phi_2}(\mathcal{I}-\sigma\nabla\phi_1),$$
$$\mathcal{T}_k := \text{prox}_{\sigma_k^*\omega_2}(\mathcal{I}-\sigma_k^*\nabla\omega_1)\ \text{and}\ \mathcal{T} := \text{prox}_{\sigma^*\omega_2}(\mathcal{I}-\sigma^*\nabla\omega_1).$$

Then, Algorithm 1 can be written as follows:

$$w^k = z^k+\alpha_k(z^k-z^{k-1}); \tag{10}$$
$$y^k = w^k+\beta_k(\mathcal{U}_k w^k-w^k); \tag{11}$$
$$z^{k+1} = (1-\gamma_k)\mathcal{U}_k w^k+\gamma_k\mathcal{T}_k y^k. \tag{12}$$

Let $z^*\in\Theta$. By (10), we have

$$\|w^k-z^*\|\leq\|z^k-z^*\|+\alpha_k\|z^k-z^{k-1}\|. \tag{13}$$

By (11)–(13) and the nonexpansiveness of $\mathcal{U}_k$ and $\mathcal{T}_k$, we have

$$\begin{aligned}
\|z^{k+1}-z^*\| &\le (1-\gamma_k)\|\mathcal{U}_k w^k - z^*\| + \gamma_k\|\mathcal{T}_k y^k - z^*\| \\
&\le (1-\gamma_k)\|w^k - z^*\| + \gamma_k\|y^k - z^*\| \\
&\le (1-\gamma_k)\|w^k - z^*\| + \gamma_k\Big[(1-\beta_k)\|w^k - z^*\| + \beta_k\|\mathcal{U}_k w^k - z^*\|\Big] \\
&\le \|w^k - z^*\| \\
&\le \|z^k - z^*\| + \alpha_k\|z^k - z^{k-1}\|. \qquad (14)
\end{aligned}$$

This implies

$$\|z^{k+1}-z^*\| \le (1+\alpha_k)\|z^k - z^*\| + \alpha_k\|z^{k-1} - z^*\|. \qquad (15)$$

When we apply Lemma 2 to the Equation (15), we obtain $\|z^{k+1}-z^*\| \le \mathcal{E}\cdot\prod_{j=1}^{k}(1+2\alpha_j)$, where $\mathcal{E} = \max\{\|z^1 - z^*\|, \|z^2 - z^*\|\}$. Hence, the proof of (i) is now complete.

By (15) and condition (A2), we have that $\{z^k\}$ is bounded. This implies $\sum_{k=1}^{\infty}\alpha_k\|z^k - z^{k-1}\| < \infty$. By (14) and Lemma 3, we obtain that $\lim_{k\to\infty}\|z^k - z^*\|$ exists. By (9) and (10), we obtain

$$\|w^k - z^*\|^2 \le \|z^k - z^*\|^2 + \alpha_k^2\|z^k - z^{k-1}\|^2 + 2\alpha_k\|z^k - z^*\|\|z^k - z^{k-1}\|. \qquad (16)$$

By (8), (11) and the nonexpansiveness of $\mathcal{U}_k$, we obtain

$$\begin{aligned}
\|y^k - z^*\|^2 &= (1-\beta_k)\|w^k - z^*\|^2 + \beta_k\|\mathcal{U}_k w^k - z^*\|^2 - \beta_k(1-\beta_k)\|w^k - \mathcal{U}_k w^k\|^2 \\
&\le \|w^k - z^*\|^2 - \beta_k(1-\beta_k)\|w^k - \mathcal{U}_k w^k\|^2. \qquad (17)
\end{aligned}$$

By (8), (12), (16), (17) and the nonexpansiveness of $\mathcal{U}_k$ and $\mathcal{T}_k$, we have

$$\begin{aligned}
\|z^{k+1}-z^*\|^2 &\le (1-\gamma_k)\|\mathcal{U}_k w^k - z^*\|^2 + \gamma_k\|\mathcal{T}_k y^k - z^*\|^2 - \gamma_k(1-\gamma_k)\|\mathcal{T}_k y^k - \mathcal{U}_k w^k\|^2 \\
&\le (1-\gamma_k)\|w^k - z^*\|^2 + \gamma_k\|y^k - z^*\|^2 - \gamma_k(1-\gamma_k)\|\mathcal{T}_k y^k - \mathcal{U}_k w^k\|^2 \\
&\le \|w^k - z^*\|^2 - \gamma_k\beta_k(1-\beta_k)\|w^k - \mathcal{U}_k w^k\|^2 - \gamma_k(1-\gamma_k)\|\mathcal{T}_k y^k - \mathcal{U}_k w^k\|^2 \\
&\le \|z^k - z^*\|^2 + \alpha_k^2\|z^k - z^{k-1}\|^2 + 2\alpha_k\|z^k - z^*\|\|z^k - z^{k-1}\| \\
&\quad - \gamma_k\beta_k(1-\beta_k)\|w^k - \mathcal{U}_k w^k\|^2 - \gamma_k(1-\gamma_k)\|\mathcal{T}_k y^k - \mathcal{U}_k w^k\|^2. \qquad (18)
\end{aligned}$$

From (18) and by condition (A1), (A2), $\sum_{k=1}^{\infty}\alpha_k\|z^k - z^{k-1}\| < \infty$ and $\lim_{k\to\infty}\|z^k - z^*\|$ exists, we obtain

$$\lim_{k\to\infty}\|\mathcal{T}_k y^k - \mathcal{U}_k w^k\| = \lim_{k\to\infty}\|w^k - \mathcal{U}_k w^k\| = 0 \text{ and } \lim_{k\to\infty}\|y^k - w^k\| = 0. \qquad (19)$$

From (19), we obtain

$$\|\mathcal{T}_k y^k - y^k\| \le \|\mathcal{T}_k y^k - \mathcal{U}_k w^k\| + \|\mathcal{U}_k w^k - w^k\| + \|w^k - y^k\| \to 0 \quad \text{as } k\to\infty. \qquad (20)$$

From (10) and $\sum_{k=1}^{\infty}\alpha_k\|z^k - z^{k-1}\| < \infty$, we have

$$\|w^k - z^k\| = \alpha_k\|z^k - z^{k-1}\| \to 0 \quad \text{as } k\to\infty. \qquad (21)$$

Since $\{z^k\}$ is bounded, we have $\omega_w(z^k) \neq \emptyset$. By (19) and (21), we obtain $\omega_w(z^k) \subseteq \omega_w(w^k) \subseteq \omega_w(y^k)$. By Condition (A3) and Remark 1, we know that $\{\mathcal{U}_k\}$ and $\{\mathcal{T}_k\}$ satisfies NST-condition (I) with $\mathcal{U}$ and $\mathcal{T}$, respectively. From (19), (20) and by using the demiclosedness of $\mathcal{I}-\mathcal{U}$ and $\mathcal{I}-\mathcal{T}$, we obtain $\omega_w(z^k) \subset \mathrm{Fix}(\mathcal{U})\cap\mathrm{Fix}(\mathcal{T}) = \Theta$. From Lemma 4, we conclude that $\{z^k\}$ converges weakly to a point in Θ. This completes the proof. $\square$

Open Problem: Can we choose the step size σ_k and σ_k^* that does not depend on the Lipschitz constant of the gradient of the function $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and the obtained convergence result of the proposed algorithm?

If we set $\phi_1 = \omega_1, \phi_2 = \omega_2$ and $\sigma_k = \sigma_k^*$ for all $k \geq 1$, then Algorithm 1 is reduced to Algorithm 2.

Algorithm 2: Given: $z^0, z^1 \in \mathcal{H}$. Choose $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\sigma_k\}$.

For $k = 1, 2, \ldots,$ **do**

$$\begin{aligned} w^k &= z^k + \alpha_k(z^k - z^{k-1}); \\ y^k &= w^k + \beta_k(\mathrm{prox}_{\sigma_k\phi_2}(\mathcal{I} - \sigma_k\nabla\phi_1)w^k - w^k); \\ z^{k+1} &= (1-\gamma_k)\,\mathrm{prox}_{\sigma_k\phi_2}(\mathcal{I} - \sigma_k\nabla\phi_1)w^k + \gamma_k\,\mathrm{prox}_{\sigma_k\phi_2}(\mathcal{I} - \sigma_k\nabla\phi_1)y^k, \end{aligned}$$

end for.

The following result is immediately obtained by Theorem 1.

Corollary 1. *Let $\{z^k\}$ be the sequence created by Algorithm 2. Suppose that $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\sigma_k\}$ are the sequences which satisfy the following conditions:*

(A1) $\beta_k \in [a, b] \subset (0, 1), \gamma_k \in [c, d] \subset (0, 1)\ \forall k \in \mathbb{N}$, *for some $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$;*

(A2) $\alpha_k \geq 0, \forall k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} \alpha_k < \infty$;

(A3) $0 < \sigma_k, \sigma < 2/\mathcal{L}_1, \forall k \in \mathbb{N}$ *such that $\sigma_k \to \sigma$ as $k \to \infty$.*

Then the following hold:

(i) $\|z^{k+1} - z^*\| \leq \mathcal{E} \prod_{j=1}^{k}(1 + 2\alpha_j)$, *where* $\mathcal{E} = \max\{\|z^1 - z^*\|, \|z^2 - z^*\|\}$ *and* $z^* \in \mathrm{Argmin}(\phi_1 + \phi_1)$.

(ii) $\{z^k\}$ *converges weakly to a point in* $\mathrm{Argmin}(\phi_1 + \phi_1)$.

4. Applications

For this part, we apply the Algorithm 1 to solving constrained image inpainting problems (22) and apply the Algorithm 2 to solving image restoration problems (24). As image quality metrics, we utilize the peak signal-to-noise ratio (PSNR) in decibel (dB) [25], which is formulated as follows:

$$PSNR := 10 \log_{10}\left(\frac{255^2}{\frac{1}{M}\|z^k - z\|_2^2}\right),$$

where z and M are the original image and the number of image samples, respectively. All experimental simulations are performed in MATLAB\R2022a on a PC with an Intel Core-i5 processor and 4.00 GB of RAM running Windows 8 64-bit.

4.1. Image Inpainting Problems

In this experiment, we apply the Algorithm 1 to solving the following constrained image inpainting problems [13]:

$$\min_{z \in \mathcal{C}} \frac{1}{2}\|\mathcal{P}_\Lambda(z^0) - \mathcal{P}_\Lambda(z)\|_F^2 + \tau\|z\|_*, \tag{22}$$

where $z^0 \in \mathbb{R}^{m \times n}$ is a given image, $\{z_{ij}^0\}_{(i,j)\in\Lambda}$ are observed, Λ is a subset of the index set $\{1, 2, 3 \ldots, m\} \times \{1, 2, 3 \ldots, n\}$, which indicates where data are available in the image domain and the rest are missed, $\mathcal{C} = \{z \in \mathbb{R}^{m \times n} \mid z_{ij} \geq 0\}$ and define $\mathcal{P}_\Lambda$ by

$$\mathcal{P}_\Lambda(z^0) = \begin{cases} z_{ij}^0, & (i,j) \in \Lambda, \\ 0, & \text{otherwise}. \end{cases}$$

In Algorithm 1, we set

$$\phi_1(z) = \frac{1}{2}\|\mathcal{P}_\Lambda(z^0) - \mathcal{P}_\Lambda(z)\|_F^2, \ \phi_2(z) = \tau\|z\|_*, \ \omega_1(z) = 0 \text{ and } \omega_2(z) = \delta_{\mathcal{C}}(z),$$

where $\tau > 0$ is regularization parameter, $\|\cdot\|_F$ is the Frobenius matrix norm and $\|\cdot\|_*$ is the nuclear matrix norm. Then, $\phi_1(z)$ is convex differentiable and $\nabla\phi_1(z) = \mathcal{P}_\Lambda(z^0) - \mathcal{P}_\Lambda(z)$ with 1-Lipschitz continuous. We note that the proximity operator of $\phi_2(z)$ can be computed by the singular value decomposition (SVD), see [26], and the proximity operator of $\omega_2(z)$ is the orthogonal projection onto the closed convex set $\mathcal{C}$. Therefore, Algorithm 1 is reduced to Algorithm 3 which can be used for solving constrained image inpainting problems (22), we have the following algorithm:

Algorithm 3: Given: $z^0, z^1 \in \mathcal{H}$. Choose $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\sigma_k\}$.

For $k = 1, 2, \ldots,$ **do**

$$w^k = z^k + \alpha_k(z^k - z^{k-1});$$
$$y^k = w^k + \beta_k(\text{prox}_{\sigma_k\phi_2}(\mathcal{I} - \sigma_k\nabla\phi_1)w^k - w^k);$$
$$z^{k+1} = (1-\gamma_k)\,\text{prox}_{\sigma_k\phi_2}(\mathcal{I} - \sigma_k\nabla\phi_1)w^k + \gamma_k\mathbb{P}_{\mathcal{C}}y^k,$$

end for.

In the standard Gallery, we marked and fixed the damaged portion of the image, and we compared Algorithm 3 with different inertial parameters settings. The following are the details of the parameters for Algorithm 3:

$$\beta_k = \frac{0.9k}{k+1}, \ \gamma_k = \frac{0.01k}{k+1}, \ \alpha_k = \begin{cases} \rho_k & \text{if } \ 1 \le k \le \mathcal{M} \\ \frac{1}{2^k} & \text{otherwise,} \end{cases}$$

where $\mathcal{M}$ is a positive integer depending on the number of iterations of Algorithm 3.

The regularization parameter was set to $\tau = 0.01$ and the stopping criterion is as follows:

$$\frac{\|z^{k+1} - z^k\|_F}{\|z^k\|_F} \le \varepsilon,$$

where ε is a given small constant. The number of iterations is indicated by Iter., and CPU time is indicated by CPU (second). We use the parameters selection cases I–V in Table 1 to evaluate the performance of Algorithm 3. Table 2 displays the results that were achieved. We observe from Table 2 that when the stopping criterion $\varepsilon = 10^{-5}$ or at the 2000th iteration, Algorithm 3 with inertial parameter (Case V) outperforms the other cases in terms of PSNR performance. We may infer from Table 2 that Algorithm 3 is more effective at recovering images when inertial parameters are added. The test image and the restored images are shown in Figures 1 and 2.

Figure 1. Test image.

Table 1. The different inertial parameters settings.

Cases	Inertial Parameters
I	$\rho_k = 0$
II	$\rho_k = 0.5$
III	$\rho_k = 0.9$
IV	$\rho_k = \frac{t_k - 1}{t_{k+1}}, t_1 = 1, t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$
V	$\rho_k = \frac{k}{k+1}$

Table 2. Results of comparing the selection of inertial parameters in terms of number of iterations, CPU time, PSNR, and the stopping criteria for Algorithm 3.

σ_k	Inertial Parameters	Iter.	CPU	PSNR (dB)	ε
0.5	Case I	2000	148.6537	23.1486	4.6305×10^{-5}
	Case II	2000	148.7307	27.1841	5.4313×10^{-5}
	Case III	1225	91.3319	33.1603	9.9616×10^{-6}
	Case IV	2000	148.1541	33.3112	1.8945×10^{-5}
	Case V	878	65.1786	33.3264	9.9611×10^{-6}
1	Case I	2000	147.9165	27.1766	5.4335×10^{-5}
	Case II	2000	148.2205	32.1462	2.5990×10^{-5}
	Case III	682	50.4207	33.2415	9.9935×10^{-6}
	Case IV	1692	125.4178	33.3025	9.9841×10^{-6}
	Case V	852	62.9013	33.3276	9.9929×10^{-6}
1.3	Case I	2000	150.4054	29.2670	4.8888×10^{-5}
	Case II	2000	147.7252	32.9289	1.2150×10^{-5}
	Case III	542	40.1375	33.2605	9.9835×10^{-6}
	Case IV	1485	109.8176	33.3038	9.9924×10^{-6}
	Case V	835	61.5336	33.3123	9.9484×10^{-6}

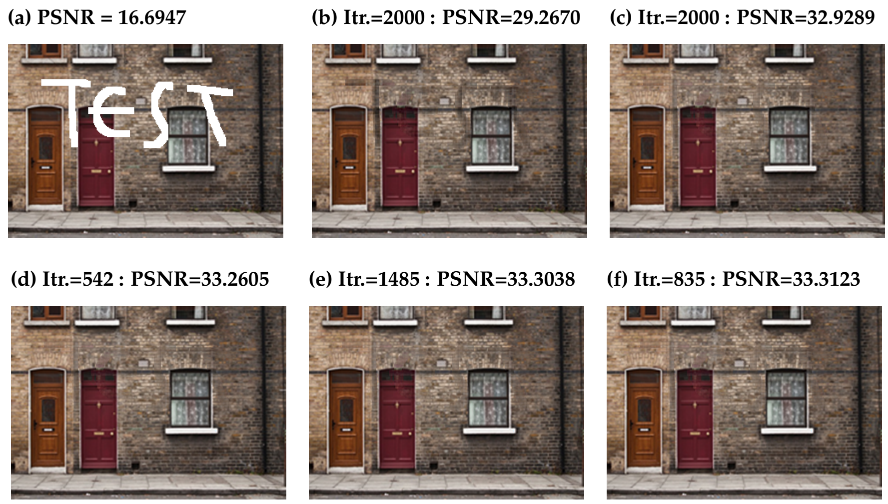


Figure 2. The painted image and restored images. (**a**) The painted image; (**b**–**f**) Images that have been recovered for cases I through V with $\sigma_k = 1.3$, respectively.

To solve a general convex optimization problem, model the sum of three convex functions in the form:

$$\min_{x \in \mathcal{H}} \phi_1(x) + \phi_2(x) + \phi_3(x), \tag{23}$$

where $\phi_1 : \mathcal{H} \to \mathbb{R}$, $\phi_2 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ and $\phi_3 : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ are convex and proper lower semi-continuous function and ϕ_1 is a differentiable function with a $\mathcal{L}$-Lipschitz continuous gradient. Cui et al. introduced an *inertial three-operator splitting (iTOS) algorithm* [13] which can be applied to solving constrained image inpainting problems (22).

Next experiment, we set $\phi_1(z) = \frac{1}{2}\|\mathcal{P}_\Lambda(z^0) - \mathcal{P}_\Lambda(z)\|_F^2$, $\phi_2(z) = \tau\|z\|_*$, and $\phi_3(z) = \delta_C(z)$, for Algorithm 4 (iTOS algorithm) and use the parameters selection as in Table 3 to evaluate the performance. Table 3 displays the results that were achieved. We observe from Tables 2 and 3 that when the stopping criterion $\varepsilon = 10^{-5}$ or at the 2000th iteration, the Algorithm 3 with inertial parameter (Case V) outperforms all cases of the iTOS algorithm in terms of PSNR performance.

Algorithm 4: An inertial three-operator splitting (iTOS) algorithm [13].

Let $z^0, z^1 \in \mathcal{H}$ and $\lambda \in (0, \frac{2}{\mathcal{L}}\bar{\varepsilon})$, where $\bar{\varepsilon} \in (0,1)$. For $k \geq 1$, let

$$\begin{aligned} w^k &= z^k + \alpha_k(z^k - z^{k-1}); \\ y^k_{\phi_3} &= \text{prox}_{\lambda\phi_3} w^k; \\ y^k_{\phi_2} &= \text{prox}_{\lambda\phi_2}(2y^k_{\phi_3} - y^k - \lambda\nabla\phi_1(y^k_{\phi_3})); \\ z^{k+1} &= w^k + \beta_k(y^k_{\phi_2} - y^k_{\phi_3}), \end{aligned}$$

where $\{\alpha_k\}$ is nondecreasing with $k \geq 1$, $0 \leq \alpha_k \leq \alpha < 1$ and for all $k \geq 1$, and $\beta, a, b > 0$ such that

$$b > \frac{\alpha^2(1+\alpha) + \alpha a}{1-\alpha^2} \text{ and } 0 < \beta \leq \beta_k \leq \frac{b - \alpha[\alpha(1+\alpha) + \alpha b + a]}{\bar{\alpha} b[1 + \alpha(1+\alpha) + \alpha b + a]}, \text{ where } \bar{\alpha} = \frac{1}{2-\bar{\varepsilon}}.$$

Table 3. Results of comparing the selection of parameters in terms of number of iterations, CPU time, PSNR, and the stopping criteria for iTOS algorithm.

σ_k	Parameters	Iter.	CPU	PSNR (dB)	ε
0.5	$\alpha_k = 0.1, \beta_k = 1.4$	2000	150.3057	25.3434	5.5297×10^{-5}
	$\alpha_k = 0.2, \beta_k = 0.8$	2000	152.3218	23.0876	4.6797×10^{-5}
	$\alpha_k = 0.5, \beta_k = 0.3$	2000	151.0935	21.4506	3.5078×10^{-5}
	$\alpha_k = 0.8, \beta_k = 0.4$	2000	161.5143	27.0492	5.5804×10^{-5}
	$\alpha_k = 0.9, \beta_k = 0.5$	2000	163.1106	30.4406	3.9901×10^{-5}
1	$\alpha_k = 0.1, \beta_k = 1.4$	2000	150.6947	30.2252	4.7538×10^{-5}
	$\alpha_k = 0.2, \beta_k = 0.8$	2000	150.9510	27.0585	5.5603×10^{-5}
	$\alpha_k = 0.5, \beta_k = 0.3$	2000	164.3304	23.9033	5.0955×10^{-5}
	$\alpha_k = 0.8, \beta_k = 0.4$	2000	156.7255	30.9755	4.0485×10^{-5}
	$\alpha_k = 0.9, \beta_k = 0.5$	2000	158.6223	25.3198	7.2758×10^{-5}
1.3	$\alpha_k = 0.1, \beta_k = 1.4$	2000	149.7497	30.9921	4.0100×10^{-5}
	$\alpha_k = 0.2, \beta_k = 0.8$	2000	151.2015	29.0476	5.2936×10^{-5}
	$\alpha_k = 0.5, \beta_k = 0.3$	2000	153.6181	25.3584	5.5326×10^{-6}
	$\alpha_k = 0.8, \beta_k = 0.4$	2000	155.3317	30.3421	3.9970×10^{-5}
	$\alpha_k = 0.9, \beta_k = 0.5$	2000	155.7551	22.9716	9.3178×10^{-5}

4.2. Image Restoration Problems

In this experiment, we apply the Algorithm 2 to solving the image restoration problems by using the LASSO model [25]:

$$\min_{z \in \mathbb{R}^N} \left\{ \frac{1}{2}\|Bz - \epsilon\|_2^2 + \tau\|z\|_1 \right\}, \tag{24}$$

where $\tau > 0$, $\|\cdot\|_1$ is the l_1-norm and $\|\cdot\|_2$ is the Euclidean norm.

In Algorithm 2, we set $\phi_1(z) = \frac{1}{2}\|\epsilon - Bz\|_2^2$ and $\phi_2(z) = \tau\|z\|_1$, where ϵ is the observed image and $B = RW$, when R and W are the kernel matrix and 2-D fast Fourier transform, respectively.

We will use two test photos (Pepper and Bird, with sizes of 512×512 and 288×288, respectively) to exhibit two scenarios of blurring processes in Table 4 and add a random Gaussian white noise 10^{-5}, with the original and blurred images shown in Figure 3.

Table 4. Processes of blurring in Detail.

Scenarios	Kernel Matrix
I	Gaussian blur of filter size 9×9 with standard deviation $\hat{\sigma} = 17$
II	Motion blur specifying with motion length of 21 pixels and motion orientation 15^o

We examine and compare the efficiency of our algorithms (Algorithm 2 := ALG 2) to that of FBS, R-FBS and FISTA algorithms. The image restoration performance of the examined methods is next tested by setting as described in (25) and using blurred images as starting points. For all algorithms, the maximum number of iterations is set at 300. The regularization parameter in the LASSO model (24) is set to $\tau = 10^{-5}$. The following are the parameters for the studied algorithms under consideration:

$$\sigma_k = \frac{1}{\mathcal{L}}, \quad \beta_k = \gamma_k = \frac{0.99k}{k+1}, \quad \alpha_k = \begin{cases} \frac{k}{k+1} & \text{if} \quad 1 \leq k \leq \mathcal{M} \\ \frac{1}{2^k} & \text{otherwise,} \end{cases} \tag{25}$$

where $\mathcal{M}$ is a positive integer depending on the number of iterations of Algorithm 2.

Figures 4–7 present the deblurring test images by the studied algorithms. In Figure 8, we see that the graph of PSNR of Algorithm 2 is higher than the others, which means that the efficiency of restored images by Algorithm 2 is better than the other methods. The number of iterations is indicated by Iter., and CPU time is indicated by CPU (second).

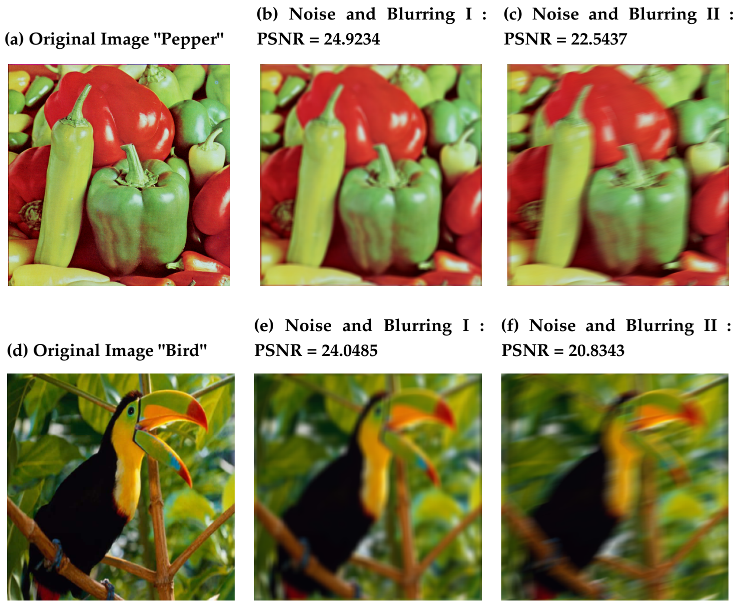


Figure 3. The deblurring images of Pepper and Bird.

(**a**) FBS : PSNR = 31.0502
Iter. = 300 : CPU = 60.6599

(**b**) R-FBS : PSNR = 31.0161
Iter. = 300 : CPU = 63.7107

(**c**) FISTA : PSNR = 35.9609
Iter. = 300 : CPU = 61.4995

(**d**) ALG 2 : PSNR = 37.4027
Iter. = 300 : CPU = 98.4046

Figure 4. The PSNR, Iter. and CPU of the FBS, R-FBS, FISTA and ALG 2 for scenario I of the Pepper.

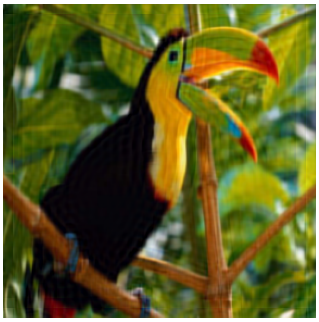
(**a**) FBS : PSNR = 30.0248
Iter. = 300 : CPU = 58.2310

(**b**) R-FBS : PSNR = 29.9726
Iter. = 300 : CPU = 64.7440

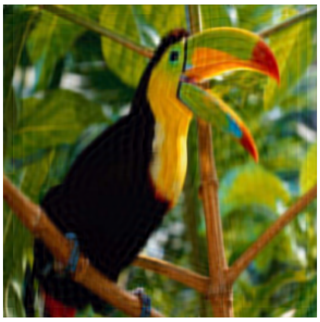
(**c**) FISTA : PSNR = 38.5825
Iter. = 300 : CPU = 62.6794

(**d**) ALG 2 : PSNR = 40.0226
Iter. = 300 : CPU = 90.0035

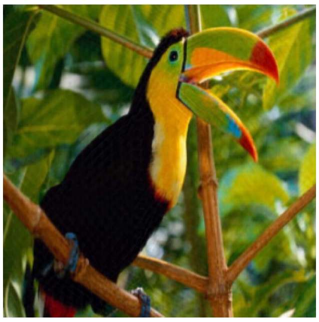
Figure 5. The PSNR, Iter. and CPU of the FBS, R-FBS, FISTA and ALG 2 for scenario II of the Pepper.



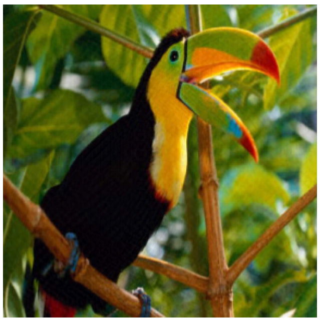
(**a**) FBS : PSNR = 31.6556
Iter. = 300 : CPU = 14.9200



(**b**) R-FBS : PSNR = 31.6111
Iter. = 300 : CPU = 16.1730

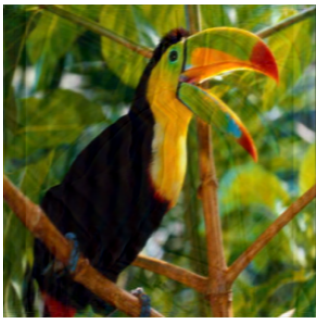


(**c**) FISTA : PSNR = 37.9241
Iter. = 300 : CPU = 15.8337

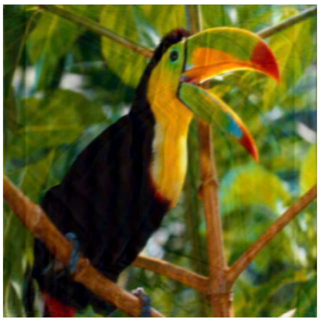


(**d**) ALG 2 : PSNR = 39.6178
Iter. = 300 : CPU = 25.3616

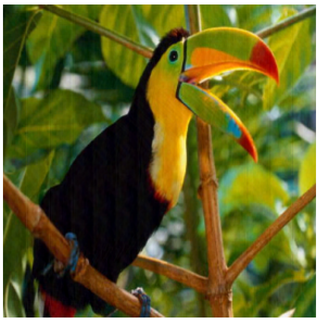
Figure 6. The PSNR, Iter. and CPU of the FBS, R-FBS, FISTA and ALG 2 for scenario I of the Bird.



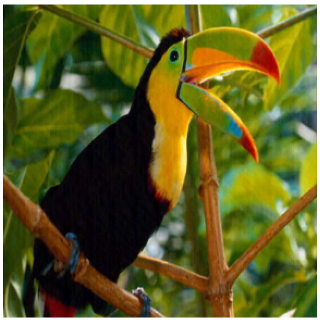
(**a**) FBS : PSNR = 29.8541
Iter. = 300 : CPU = 14.8847



(**b**) R-FBS : PSNR = 29.7941
Iter. = 300 : CPU = 16.1327

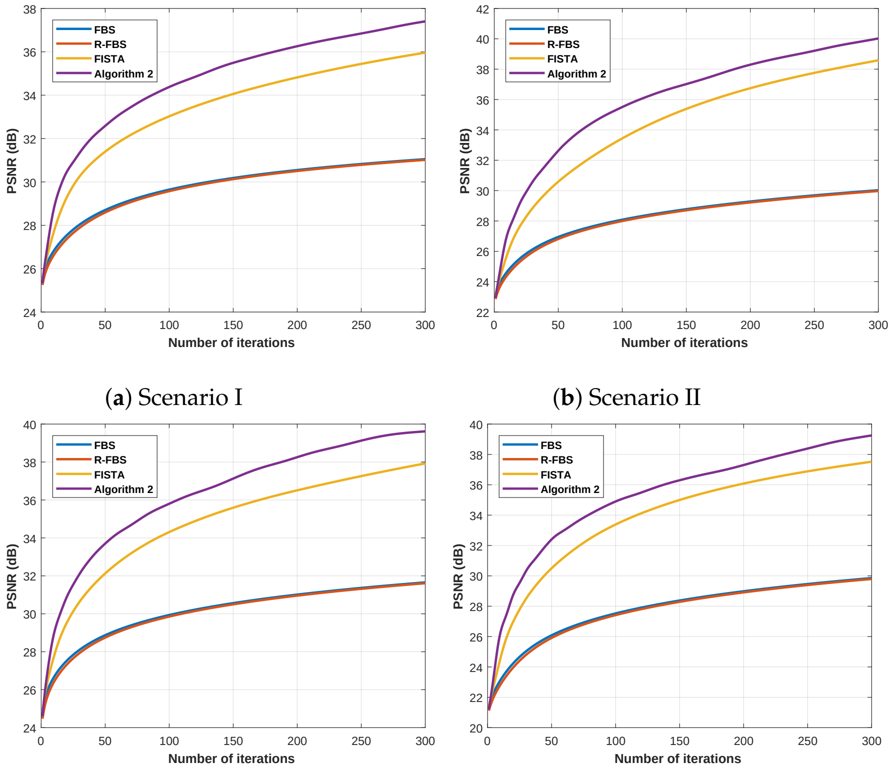


(**c**) FISTA : PSNR = 37.5210
Iter. = 300 : CPU = 15.6691



(**d**) ALG 2 : PSNR = 39.2515
Iter. = 300 : CPU = 25.1109

Figure 7. The PSNR, Iter. and CPU of the FBS, R-FBS, FISTA and ALG 2 for scenario II of the Bird.



(**a**) Scenario I (**b**) Scenario II

(**c**) Scenario I (**d**) Scenario II

Figure 8. The PSNR graphs of the studied algorithms: (**a**,**b**) for Pepper; (**c**,**d**) for Bird.

5. Conclusions

In this research, an inertial forward-backward splitting algorithm for solving a common point of convex minimization problems is developed. We investigated the weak convergence of the suggested algorithm based on the fixed point equation of the forward-backward operator under some suitable control conditions. Finally, we use numerical simulations to show the benefits of the inertial terms in the studied algorithms for the constrained image inpainting problems (22) and the image restoration problems (24).

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Conflicts of Interest: The authors declare that they have no competing interest.

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