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Completeness of Bethe Ansatz for Gaudin Models with $\mathfrak{gl}(1|1)$ Symmetry and Diagonal Twists

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Abstract: We studied the Gaudin models with $\mathfrak{gl}(1|1)$ symmetry that are twisted by a diagonal matrix and defined on tensor products of polynomial evaluation $\mathfrak{gl}(1|1)[t]$-modules. Namely, we gave an explicit description of the algebra of Hamiltonians (Gaudin Hamiltonians) acting on tensor products of polynomial evaluation $\mathfrak{gl}(1|1)[t]$-modules and showed that a bijection exists between common eigenvectors (up to proportionality) of the algebra of Hamiltonians and monic divisors of an explicit polynomial written in terms of the highest weights and evaluation parameters. In particular, our result implies that each common eigenspace of the algebra of Hamiltonians has dimension one. We also gave dimensions of the generalized eigenspaces.

Keywords: Gaudin models; Lie superalgebras; Bethe ansatz; pseudo-differential operators; Berezinian

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1. Introduction

In the last half of a century, Gaudin models for simple Lie algebras have been intensively studied by many mathematicians and physicists using various methods, producing numerous spectacular results. For example, the simplicity of the spectrum of Gaudin algebra (Bethe algebra) was used to solve two long-standing conjectures: the transversality conjecture of the intersection of Schubert varieties and the Shapiro–Shapiro conjecture in real algebraic geometry; see [1]. Another example is that the monodromy of the joint eigenvectors of Gaudin algebra was proved to be given by the internal cactus group action on $\mathfrak{g}$-crystals, where $\mathfrak{g}$ is the corresponding finite-dimensional simple Lie algebra; see [2].

In recent years, the Gaudin models for Lie superalgebras have steadily gained attention within the mathematical community. For instance, the algebraic Bethe ansatz for Gaudin models of $\mathfrak{osp}(1|2)$ symmetry was carried out in [3]. Higher Gaudin Hamiltonians for Gaudin models of $\mathfrak{gl}(m|n)$ symmetry were constructed in [4] via studying the MacMahon Master Theorem related to Manin matrices. The completeness of Bethe ansatz for Gaudin models of $\mathfrak{gl}(m|n)$ symmetry that are defined on tensor products of vector representations was proved for the case of generic evaluation parameters in [5]. The relation between SPL_2-superopers and the Bethe ansatz equations of $\mathfrak{osp}(1|2)$ Gaudin model was discussed in [6]. The reproduction procedure for Bethe ansatz equations of $\mathfrak{gl}(m|n)$ Gaudin models was introduced in [7]. Moreover, it was shown in [7] that the reproduction procedure gives rise to a variety that is isomorphic to the superflag variety. The duality between the quasi-periodic Gaudin model associated with Lie superalgebra $\mathfrak{gl}(m|n)$ and the quasi-periodic Gaudin model associated with Lie algebra $\mathfrak{gl}(k)$ was established in [8]. The reproduction procedure for Bethe ansatz equations of Gaudin models associated with orthosymplectic Lie superalgebras was introduced in [9]. In particular, this research developed the missing part of the reproduction procedure when the corresponding Lie algebras are of type D. Finally, in a previous work [10], we gave a complete answer of the periodic (twistless) Gaudin models of $\mathfrak{gl}(1|1)$ symmetry when the underlying Hilbert space is an arbitrary irreducible tensor product of evaluation polynomial modules. In this paper, we obtained the analogues for quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models; namely, we proved the completeness of Bethe ansatz for $\mathfrak{gl}(1|1)$ Gaudin models with diagonal twists.

The results of this paper are quite similar to those of [10,11], with suitable modifications, following the strategy of [1,12]. Surprisingly, to the best of our knowledge, most of the previous work on Gaudin models for Lie superalgebras was carried out in the periodic case, except, e.g., [8]. Therefore, we also need to establish the results on the algebraic Bethe ansatz for $\mathfrak{gl}(1|1)$ Gaudin models in the quasi-periodic case; see Section 2.4. In particular, we showed that the Bethe ansatz is complete for generic evaluation parameters; see Theorem 2. Using the completeness of the Bethe ansatz for generic parameters, we were able to describe the image of the algebra of Hamiltonians (Bethe algebra) explicitly and show that the quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models are perfectly integrable, cf. [13]. Consequently, we obtained the completeness of the Bethe ansatz for quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models with pairwise distinct evaluation parameters.

Note that the perfect integrability for the quasi-periodic $\mathfrak{gl}(m|n)$ Gaudin models defined on tensor products of symmetric powers of the vector representations was established in [8] [Corollary 5.3] by studying the duality between $\mathfrak{gl}(m|n)$ and $\mathfrak{gl}(k)$ Gaudin models and using the known results from [12]. In particular, it gives rise to the perfect integrability for the quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models defined on tensor products of polynomial modules. However, an explicit description of the image of Bethe algebra and the complete spectrum of Bethe algebra were not discussed in [8].

The paper is organized as follows. In Section 2, we fix notations and discuss basic facts of the algebraic Bethe ansatz for quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models. Then, we recall the space $\mathcal{V}^{\mathfrak{S}}$ and Weyl modules and their properties in Section 3. Section 4 contains the main theorems, where we also discuss the higher Gaudin transfer matrices and the relations between higher Gaudin transfer matrices and the first two Gaudin transfer matrices. Section 5 is dedicated to the proofs of main theorems.

2. Preliminaries

2.1. Lie Superalgebra $\mathfrak{gl}(1|1)$ and Its Representations

A *vector superspace* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space. Elements of $V_{\bar{0}}$ are called *even*; elements of $V_{\bar{1}}$ are called *odd*. We write $|v| \in \{\bar{0}, \bar{1}\}$ for the parity of a homogeneous element $v \in V$. Set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

Consider the vector superspace $\mathbb{C}^{1|1}$, where $\dim(\mathbb{C}^{1|1}_{\bar{0}}) = 1$ and $\dim(\mathbb{C}^{1|1}_{\bar{1}}) = 1$. We chose a homogeneous basis v_1, v_2 of $\mathbb{C}^{1|1}$ such that $|v_1| = \bar{0}$ and $|v_2| = \bar{1}$. For brevity, we shall write their parities as $|v_i| = |i|$. Denote by $E_{ij} \in \mathrm{End}(\mathbb{C}^{1|1})$ the linear operator of parity $|i| + |j|$ such that $E_{ij}v_r = \delta_{jr}v_i$ for $i, j, r = 1, 2$.

The Lie superalgebra $\mathfrak{gl}(1|1)$ is spanned by elements e_{ij}, $i, j = 1, 2$, with parities $e_{ij} = |i| + |j|$, and the supercommutator relations are given by

$$[e_{ij}, e_{rs}] = \delta_{jr}e_{is} - (-1)^{(|i|+|j|)(|r|+|s|)}\delta_{is}e_{rj}.$$

Let $\mathfrak{h}$ be the commutative Lie subalgebra of $\mathfrak{gl}(1|1)$ spanned by e_{11}, e_{22}. Denote the universal enveloping algebras of $\mathfrak{gl}_{1|1}$ and $\mathfrak{h}$ by $\mathrm{U}(\mathfrak{gl}_{1|1})$ and $\mathrm{U}(\mathfrak{h})$, respectively.

We call a pair $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ of complex numbers a $\mathfrak{gl}(1|1)$-*weight* . Set $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2$. A $\mathfrak{gl}(1|1)$-weight $\boldsymbol{\lambda}$ is *non-degenerate* if $\lambda_1 + \lambda_2 \neq 0$.

Let M be a $\mathfrak{gl}(1|1)$-module. A non-zero vector $v \in M$ is called *singular* if $e_{12}v = 0$. Denote the subspace of all singular vectors of M by $(M)^{\mathrm{sing}}$. A non-zero vector $v \in M$ is called *of weight* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ if $e_{11}v = \lambda_1 v$ and $e_{22}v = \lambda_2 v$. Denote by $(M)_{\boldsymbol{\lambda}}$ the subspace of M spanned by vectors of weight $\boldsymbol{\lambda}$.

Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ be a sequence of $\mathfrak{gl}(1|1)$-weights. Set $|\boldsymbol{\Lambda}| = \sum_{s=1}^{k} |\boldsymbol{\lambda}^{(s)}|$.

Denote by $L_{\boldsymbol{\lambda}}$ the irreducible $\mathfrak{gl}(1|1)$-module generated by an even singular vector $v_{\boldsymbol{\lambda}}$ of weight $\boldsymbol{\lambda}$. Then, $L_{\boldsymbol{\lambda}}$ is two-dimensional if $\boldsymbol{\lambda}$ is non-degenerate and one-dimensional otherwise. Clearly, $\mathbb{C}^{1|1} \cong L_{\omega_1}$, where $\omega_1 = (1, 0)$, if we identify the action of e_{ij} on $\mathbb{C}^{1|1}$ with the operator E_{ij}.

A $\mathfrak{gl}(1|1)$-module M is called a *polynomial module* if M is a submodule of $(\mathbb{C}^{1|1})^{\otimes n}$ for some $n \in \mathbb{Z}_{\geqslant 0}$. We say that $\boldsymbol{\lambda}$ is a *polynomial weight* if $L_{\boldsymbol{\lambda}}$ is a polynomial module. Weight

$\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is a polynomial weight if and only if $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geqslant 0}$ and either $\lambda_1 > 0$ or $\lambda_1 = \lambda_2 = 0$. We also write $L_{(\lambda_1,\lambda_2)}$ for $L_{\boldsymbol{\lambda}}$.

For non-degenerate polynomial weights $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)$, we have

$$L_{(\lambda_1,\lambda_2)} \otimes L_{(\mu_1,\mu_2)} = L_{(\lambda_1+\mu_1,\lambda_2+\mu_2)} \oplus L_{(\lambda_1+\mu_1-1,\lambda_2+\mu_2+1)}.$$

2.2. Current Superalgebra $\mathfrak{gl}(1|1)[t]$

Denote by $\mathfrak{gl}(1|1)[t]$ the Lie superalgebra $\mathfrak{gl}(1|1) \otimes \mathbb{C}[t]$ of $\mathfrak{gl}_{1|1}$-valued polynomials with the point-wise supercommutator. Call $\mathfrak{gl}(1|1)[t]$ the *current superalgebra* of $\mathfrak{gl}(1|1)$. We identify $\mathfrak{gl}(1|1)$ with the subalgebra $\mathfrak{gl}(1|1) \otimes 1$ of constant polynomials in $\mathfrak{gl}(1|1)[t]$.

We write $e_{ij}[r]$ for $e_{ij} \otimes t^r$, $r \in \mathbb{Z}_{\geqslant 0}$. A basis of $\mathfrak{gl}(1|1)[t]$ is given by $e_{ij}[r]$, $i, j = 1, 2$ and $r \in \mathbb{Z}_{\geqslant 0}$. They satisfy the supercommutator relations

$$[e_{ij}[r], e_{kl}[s]] = \delta_{jk} e_{il}[r+s] - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} e_{kj}[r+s].$$

In particular, one has

$$(e_{12}[r])^2 = (e_{21}[r])^2 = 0, \quad e_{21}[r]e_{21}[s] = -e_{21}[s]e_{21}[r] \tag{1}$$

in the universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}(1|1)[t])$. The universal enveloping superalgebra $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ is a Hopf superalgebra with the coproduct given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \text{for } X \in \mathfrak{gl}(1|1)[t].$$

Let $e_{ij}(x) = \sum_{r=0}^{\infty} e_{ij}[r] x^{-r-1}$, where x is a formal variable. Then, we have

$$(u - v)[e_{ij}(u), e_{rs}(v)] = -[e_{ij}, e_{rs}](u) + [e_{ij}, e_{rs}](v). \tag{2}$$

In particular,

$$[e_{ij}(x), e_{rs}(x)] = -\partial_x [e_{ij}, e_{rs}](x). \tag{3}$$

For each $a \in \mathbb{C}$, there exists an automorphism of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$, $\rho_a : e_{ij}(x) \to e_{ij}(x-a)$. Given a $\mathfrak{gl}(1|1)[t]$-module M, denote by $M(a)$ the pull-back of M through the automorphism ρ_a.

For each $a \in \mathbb{C}$, we have the evaluation map

$$\mathsf{ev}_a : \mathrm{U}(\mathfrak{gl}(1|1)[t]) \to \mathrm{U}(\mathfrak{gl}(1|1)), \quad e_{ij}(x) \mapsto e_{ij}/(x-a).$$

For a $\mathfrak{gl}(1|1)$-module L, denote by $L(a)$ the $\mathfrak{gl}(1|1)[t]$-module obtained by pulling back L through the evaluation map ev_a. We call $L(a)$ an *evaluation module* at a.

Given any series $\zeta(x) \in x^{-1}\mathbb{C}[x^{-1}]$, we have the one-dimensional $\mathfrak{gl}(1|1)[t]$-module generated by an even vector v satisfying $e_{ij}(x)v = \delta_{ij}(-1)^{|j|}\zeta(x)v$. We denote this module by $\mathbb{C}_\zeta$.

If $b_1, \ldots, b_n$ are pairwise distinct complex numbers and $L_1, \ldots, L_n$ are finite-dimensional irreducible $\mathfrak{gl}(1|1)$-modules, then the $\mathfrak{gl}(1|1)[t]$-module $\bigotimes_{s=1}^n L_s(b_s)$ is irreducible.

There is a natural $\mathbb{Z}_{\geqslant 0}$-gradation on $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ such that $\deg(e_{ij}[r]) = r$ which induces the filtration $\mathscr{F}_0\mathrm{U}(\mathfrak{gl}(1|1)[t]) \subset \mathscr{F}_1\mathrm{U}(\mathfrak{gl}(1|1)[t]) \subset \cdots \subset \mathrm{U}(\mathfrak{gl}(1|1)[t])$, where $\mathscr{F}_s\mathrm{U}(\mathfrak{gl}(1|1)[t])$ is the subspace of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ spanned by all elements of degree $\leqslant s$.

Let M be a $\mathbb{Z}_{\geqslant 0}$-graded space with finite-dimensional homogeneous components. Let $M_j \subset M$ be the homogeneous component of degree j. We call the formal power series in variable q,

$$\mathrm{ch}(M) = \sum_{j=0}^{\infty} \dim(M_j)\, q^j, \tag{4}$$

the *graded character* of M.

2.3. Gaudin Hamiltonians

In this section, we discuss the inhomogeneous Gaudin Hamiltonians. Throughout the paper, we shall fix two complex numbers $\boldsymbol{q} = (q_1, q_2)$. Moreover, we assume that $q_1 \neq q_2$; see the end of this section.

Let $\boldsymbol{b} = (b_1, \dots, b_k)$ be a sequence of distinct complex numbers and $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$ a sequence of polynomial $\mathfrak{gl}(1|1)$-weights, where $\boldsymbol{\lambda}^{(s)} = (\alpha_s, \beta_s)$.

Set $n = |\boldsymbol{\Lambda}| = \sum_{s=1}^{k}(\alpha_s + \beta_s)$ and $L_{\boldsymbol{\Lambda}} = \bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}$. The *quadratic Gaudin Hamiltonians* are the linear maps $\mathcal{H}_r \in \mathrm{End}(L_{\boldsymbol{\Lambda}})$ given by

$$\mathcal{H}_r := q_1 e_{11}^{(r)} + q_2 e_{22}^{(r)} + \sum_{s=1, s\neq r}^{k} \frac{e_{11}^{(r)} e_{11}^{(s)} - e_{12}^{(r)} e_{21}^{(s)} + e_{21}^{(r)} e_{12}^{(s)} - e_{22}^{(r)} e_{22}^{(s)}}{b_r - b_s}, \quad 1 \leqslant r \leqslant k. \tag{5}$$

where $e_{ab}^{(r)} = 1^{\otimes(r-1)} \otimes e_{ab} \otimes 1^{\otimes(k-r)}$.

Lemma 1. *The Gaudin Hamiltonians* $\mathcal{H}_r$

1. *Are mutually commuting:* $[\mathcal{H}_r, \mathcal{H}_s] = 0$ *for all* r, s;
2. *Commute with the action of* $\mathfrak{h}$: $[\mathcal{H}_r, X] = 0$ *for all* r *and* $X \in \mathfrak{h}$.

Proof. This follows immediately from [5] [Proposition 3.1] for non-twisted (i.e., $q_1 = q_2 = 0$) Gaudin Hamiltonians. □

Instead of working on Gaudin Hamiltonians $\mathcal{H}_s$, we work on the generating function of Gaudin Hamiltonians,

$$\mathscr{H}(x) := \sum_{r=1}^{\infty} \mathscr{H}_r x^{-r} = q_1 e_{11}(x) + q_2 e_{22}(x) + \frac{1}{2} \sum_{a,b=1}^{2} e_{ab}(x) e_{ba}(x) (-1)^{|b|}. \tag{6}$$

The operator $\mathscr{H}(x)$ acts on the tensor product of the evaluation $\mathfrak{gl}(1|1)[t]$-modules

$$L_{\boldsymbol{\Lambda}}(\boldsymbol{b}) := \bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s).$$

Note that $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$ and $L_{\boldsymbol{\Lambda}}$ are isomorphic as $\mathfrak{gl}(1|1)$-modules via the identity map; then, we have

$$\mathscr{H}(x) = \frac{1}{2} \sum_{s=1}^{k} \frac{\alpha_s(\alpha_s - 1) - \beta_s(\beta_s + 1)}{(x - b_s)^2} \mathrm{Id} + \sum_{s=1}^{k} \frac{1}{x - b_s} \mathcal{H}_s, \tag{7}$$

as operators in $\mathrm{End}(L_{\boldsymbol{\Lambda}}) = \mathrm{End}(L_{\boldsymbol{\Lambda}}(\boldsymbol{b}))$. We call $\mathcal{H}(x)$ the *Gaudin transfer matrix*.

We are interested in finding the eigenvalues and eigenvectors of the Gaudin transfer matrix in $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$. To be more precise, we call

$$\xi(x) = \sum_{r=1}^{\infty} \xi_r x^{-r}, \qquad \xi_r \in \mathbb{C}, \tag{8}$$

an *eigenvalue* of $\mathscr{H}(x)$ if there exists a non-zero vector $v \in L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$ such that $\mathscr{H}_r v = \xi_r v$ for all $r \in \mathbb{Z}_{\geqslant 1}$. If $\xi(x)$ is a rational function, we consider it as a power series in x^{-1} as in (8). The vector v is called an *eigenvector* of $\mathscr{H}(x)$ corresponding to eigenvalue $\xi(x)$. We also define the *eigenspace of* $\mathscr{H}(x)$ *in* $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$ *corresponding to eigenvalue* $\xi(x)$ as $\bigcap_{r=1}^{\infty} \ker(\mathscr{H}_r|_{L_{\boldsymbol{\Lambda}}(\boldsymbol{b})} - \xi_r)$.

It is sufficient to consider $L_{\boldsymbol{\Lambda}}$ with $\beta_s = 0$ for all s. Indeed, if $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$ is an arbitrary tensor product and

$$\xi(x) = \sum_{s=1}^{k} \frac{\beta_s}{x - b_s},$$

then
$$L_{\mathbf{\Lambda}}(\boldsymbol{b}) \otimes \mathbb{C}_{\xi} \cong L_{\widetilde{\mathbf{\Lambda}}}(\boldsymbol{b}), \quad \tilde{\boldsymbol{\lambda}}^{(s)} = (\alpha_s + \beta_s, 0).$$

Identify $L_{\mathbf{\Lambda}}(\boldsymbol{b}) \otimes \mathbb{C}_{\xi}$ with $L_{\mathbf{\Lambda}}(\boldsymbol{b})$ as vector spaces. Then, $\mathscr{H}(x)$ acting on $L_{\mathbf{\Lambda}}(\boldsymbol{b}) \otimes \mathbb{C}_{\xi}$ coincides with $\mathscr{H}(x) + \zeta(x)(e_{11}(x) + e_{22}(x)) + (q_1 - q_2)\xi(x)$ acting on $L_{\mathbf{\Lambda}}(\boldsymbol{b})$. Note that the coefficients of $e_{11}(x) + e_{22}(x)$ are central in $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ and hence $e_{11}(x) + e_{22}(x)$ acts on $L_{\mathbf{\Lambda}}(\boldsymbol{b})$ by the scalar series
$$\sum_{s=1}^{k} \frac{\alpha_s + \beta_s}{x - b_s};$$
therefore, the problem of the diagonalization of the Gaudin transfer matrix in $L_{\mathbf{\Lambda}}(\boldsymbol{b})$ is reduced to diagonalization of the Gaudin transfer matrix in $L_{\widetilde{\mathbf{\Lambda}}}(\boldsymbol{b})$.

Again, by the fact that the coefficients of $e_{11}(x) + e_{22}(x)$ are central, if $q_1 = q_2$, then the diagonalization problem of $\mathcal{H}(x)$ is the same as the one for the homogeneous case $q_1 = q_2 = 0$, which was discussed in [10]. Thus, for the rest of the paper, we shall assume that $q_1 \neq q_2$.

Since $L_{\boldsymbol{\lambda}}$ is one-dimensional if $\boldsymbol{\lambda}$ is degenerate, similarly, it suffices to consider the case that all participant $\mathfrak{gl}(1|1)$-weights are non-degenerate. Hence, we shall always assume throughout the paper that $\boldsymbol{\lambda}^{(s)}$ are non-degenerate for all $1 \leqslant s \leqslant k$.

2.4. Bethe Ansatz

The main method to find eigenvalues and eigenvectors of the Gaudin transfer matrix in $L_{\mathbf{\Lambda}}$ is the algebraic Bethe ansatz. We give the results for the algebraic Bethe ansatz of quasi-periodic $\mathfrak{gl}(1|1)$ Gaudin models in this section following e.g., [5] [Section VI].

Fix a non-negative integer l. Let $\boldsymbol{t} = (t_1, \ldots, t_l)$ be a sequence of complex numbers. Define the polynomial $y_{\boldsymbol{t}} = \prod_{i=1}^{l}(x - t_i)$. We say that polynomial $y_{\boldsymbol{t}}$ *represents* $\boldsymbol{t}$.

Set
$$\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x) := q_1 - q_2 + \sum_{s=1}^{k} \frac{\alpha_s + \beta_s}{x - b_s}. \tag{9}$$

A sequence of complex numbers $\boldsymbol{t}$ is called a *solution to the Bethe ansatz equation associated to* $\mathbf{\Lambda}$, $\boldsymbol{b}$, l if
$$y_{\boldsymbol{t}}(x) \quad \text{divides the polynomial} \quad \varphi_{\mathbf{\Lambda},\boldsymbol{b}}(x) := \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x) \prod_{s=1}^{k}(x - b_s). \tag{10}$$

We do not distinguish solutions that differ by a permutation of coordinates (that is represented by the same polynomial).

Let v_s be the highest weight vector of $L_{\boldsymbol{\lambda}^{(s)}}$, and set $|0\rangle = v_1 \otimes \cdots \otimes v_k$. We call $|0\rangle$ the *vacuum vector*.

Define the *off-shell Bethe vector* $\mathbb{B}_l(\boldsymbol{t}) \in (L_{\mathbf{\Lambda}})_{(n-l,l)}$ by
$$\mathbb{B}_l(\boldsymbol{t}) = e_{21}(t_1) \cdots e_{21}(t_l)\,|0\rangle. \tag{11}$$

Since $e_{21}(x)e_{21}(u) = -e_{21}(u)e_{21}(x)$, the order of t_i is not important. Moreover, the off-shell Bethe vector is zero if $t_i = t_j$ for some $1 \leqslant i \neq j \leqslant l$.

If $\boldsymbol{t}$ is a solution of the Bethe ansatz Equation (10), we call $\mathbb{B}_l(\boldsymbol{t})$ an *on-shell Bethe vector*.

Let $\boldsymbol{t}$ be a solution of the Bethe ansatz equation associated to $\mathbf{\Lambda}$, $\boldsymbol{b}$, l.

Theorem 1. *If the on-shell Bethe vector $\mathbb{B}_l(\boldsymbol{t})$ is non-zero, then $\mathbb{B}_l(\boldsymbol{t})$ is an eigenvector of the Gaudin transfer matrix $\mathscr{H}(x)$ with the corresponding eigenvalue*
$$\mathcal{E}_{y_{\boldsymbol{t}},\mathbf{\Lambda},\boldsymbol{b}}(x) = \frac{1}{2}\zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(x) - \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)\frac{y'_{\boldsymbol{t}}(x)}{y_{\boldsymbol{t}}(x)} + \sum_{r,s=1}^{k} \frac{\alpha_r\alpha_s - \beta_r\beta_s}{2(x - b_r)(x - b_s)} + \sum_{s=1}^{k} \frac{q_1\alpha_s + q_2\beta_s}{x - b_s}. \tag{12}$$
where $\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)$ is given by (9).

Proof. By (2) and the fact that coefficients of $e_{11}(x)+e_{22}(x)$ are central in $\mathrm{U}(\mathfrak{gl}(1|1)[t])$, we have

$$[\mathcal{H}(x), e_{21}(t)] = -\frac{1}{u-t}\big(\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)e_{21}(t) - \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(t)e_{21}(x)\big),$$

as operators on $L_{\mathbf{\Lambda}}(\boldsymbol{b})$. Note that if t is a coordinate of a solution of the Bethe ansatz equation, then $\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(t)=0$. Therefore, we have

$$[\mathcal{H}(x), e_{21}(t_i)] = -\frac{1}{u-t}\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)e_{21}(t_i)$$

for $1 \leqslant i \leqslant l$. Hence, we conclude that

$$\mathcal{H}(x)\mathbb{B}_l(\boldsymbol{t}) = -\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)\sum_{j=1}^{l}\frac{1}{x-t_j}\mathbb{B}_l(\boldsymbol{t}) + e_{21}(t_1)\cdots e_{21}(t_l)\mathcal{H}(x)|0\rangle.$$

The theorem now follows from the straightforward computation of the eigenvalue of $\mathcal{H}(x)$ corresponding to the vector $|0\rangle$. □

Consider another Gaudin transfer matrix

$$\mathcal{T}(x) = \frac{1}{2}\big(\dot{e}_{11}(x)+\dot{e}_{22}(x)\big) + \frac{1}{2}\big(e_{11}(x)+e_{22}(x)\big)^2 + q_1\big(e_{11}(x)+e_{22}(x)\big) - \mathscr{H}(x), \quad (13)$$

where $\dot{e}_{ii}(x) = \partial_x(e_{ii}(x))$, $i=1,2$. Then, the eigenvalue of $\mathcal{T}(x)$ acting on the on-shell Bethe vector $\mathbb{B}_l(\boldsymbol{t})$ is

$$\begin{aligned}\mathscr{E}_{y_t,\mathbf{\Lambda},\boldsymbol{b}}(x) &= \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)\frac{y_{\boldsymbol{t}}'(x)}{y_{\boldsymbol{t}}(x)} + \sum_{r,s=1}^{k}\frac{\alpha_r\beta_s+\alpha_s\beta_r+2\beta_r\beta_s}{2(x-b_r)(x-b_s)} + \sum_{s=1}^{k}\frac{(q_1-q_2)\beta_s}{x-b_s} \\ &= \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(x)\Big(\frac{y_{\boldsymbol{t}}'(x)}{y_{\boldsymbol{t}}(x)} + \sum_{s=1}^{k}\frac{\beta_s}{x-b_s}\Big).\end{aligned} \quad (14)$$

It is important to know if the on-shell Bethe vectors are non-zero.

Proposition 1. *Suppose that the polynomial $\varphi_{\mathbf{\Lambda},\boldsymbol{b}}(x)$ only has simple roots; then, the on-shell Bethe vector $\mathbb{B}_l(\boldsymbol{t})$ is nonzero.*

Proof. Since $\varphi_{\mathbf{\Lambda},\boldsymbol{b}}(x)$ only has simple roots, we have $t_i \neq t_j$ for $i \neq j$. Note that b_s are distinct and $\alpha_s+\beta_s>0$ (since the weights are nondegenerate by our assumption); then, we have $b_s \neq t_i$. Hence, $\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(t_i)=0$. Moreover, we have

$$0 \neq \varphi'_{\mathbf{\Lambda},\boldsymbol{b}}(t_i) = \zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(t_i)\prod_{s=1}^{k}(t_i-b_s) + \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(t_i)\Big(\prod_{s=1}^{k}(x-b_s)\Big)'\Big|_{x=t_i} = \zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(t_i)\prod_{s=1}^{k}(t_i-b_s).$$

Therefore, $\zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(t_i) \neq 0$.

By (2) and the fact that coefficients of $e_{11}(x)+e_{22}(x)$ are central in $\mathrm{U}(\mathfrak{gl}(1|1)[t])$, we have

$$[e_{12}(t), e_{21}(\tilde{t})] = -\frac{1}{t-\tilde{t}}\big(\zeta_{\mathbf{\Lambda},\boldsymbol{b}}(t) - \zeta_{\mathbf{\Lambda},\boldsymbol{b}}(\tilde{t})\big)$$

as operators on $L_{\mathbf{\Lambda}}(\boldsymbol{b})$. Therefore, we have $[e_{12}(t), e_{21}(\tilde{t})]=0$ if t and $\tilde{t}$ are distinct coordinates of $\boldsymbol{t}$ while $[e_{12}(t), e_{21}(t)] = -\zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(t)$. One finds that

$$e_{12}(t_l)\cdots e_{12}(t_1)e_{21}(t_1)\cdots e_{21}(t_l)|0\rangle = (-1)^l\prod_{i=1}^{l}\zeta'_{\mathbf{\Lambda},\boldsymbol{b}}(t_i)|0\rangle \neq 0,$$

completing the proof. □

The conjecture of the completeness of the Bethe ansatz for Gaudin models associated with $\mathfrak{gl}(1|1)$ was formulated as follows, cf. [7] [Conjecture 8.3].

Conjecture 1. *Suppose all weights $\boldsymbol{\lambda}^{(s)}$, $1 \leqslant s \leqslant k$ are polynomial $\mathfrak{gl}(1|1)$-weights. Then, the Gaudin transfer matrix $\mathscr{H}(x)$ has a simple spectrum in $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$. There exists a bijective correspondence between the monic divisors y of the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ and the eigenvectors v of the Gaudin transfer matrices (up to multiplication by a non-zero constant). Moreover, this bijection is such that $\mathscr{H}(x)v = \mathcal{E}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}(x)v$, where $\mathcal{E}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}(x)$ is given by (12).*

By simple spectrum, we mean that if v_1, v_2 are eigenvectors of $\mathscr{H}(x)$ and $v_1 \neq cv_2$, $c \in \mathbb{C}^{\times}$, then the eigenvalues of $\mathscr{H}(x)$ on v_1 and v_2 are different.

The conjecture follows from Theorem 4 proved in Section 5.3.

The conjecture is clear for the case when $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ only has simple roots. Note that $\dim L_{\boldsymbol{\Lambda}}(\boldsymbol{b}) = 2^k$. If the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ has no multiple roots, then $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ has the desired number of distinct monic divisors. Therefore, we have the desired number of on-shell Bethe vectors, which are also nonzero by Proposition 1. By Theorem 1, it implies that we do have an eigenbasis of the Gaudin transfer matrix consisting of on-shell Bethe vectors in $L_{\boldsymbol{\Lambda}}(\boldsymbol{b})$ with different eigenvalues. Thus, the algebraic Bethe ansatz works well for this situation.

Theorem 2. *Suppose that all weights $\boldsymbol{\lambda}^{(s)}$, $1 \leqslant s \leqslant k$ are polynomial $\mathfrak{gl}(1|1)$-weights. If the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ has no multiple roots, then the Gaudin transfer matrix $\mathscr{H}(x)$ is diagonalizable and the Bethe ansatz is complete. In particular, for any given $\boldsymbol{\Lambda}$ and generic $\boldsymbol{b}$, the Gaudin transfer matrix $\mathscr{H}(x)$ is diagonalizable and the Bethe ansatz is complete.*

3. Space $\mathcal{V}^{\mathfrak{S}}$ and Weyl Modules

In this section, we discuss the super-analog of $\mathcal{V}^{S}$ in [1] [Section 2.5], cf. [11] [Section 3].

The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathbb{C}[z_1,\dots,z_n]$ by permuting variables. Denote by $\sigma_i(\boldsymbol{z})$ the i-th elementary symmetric polynomial in $z_1,\dots,z_n$. The algebra of symmetric polynomials $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}}$ is freely generated by $\sigma_1(\boldsymbol{z}),\dots,\sigma_n(\boldsymbol{z})$.

Fix $\ell \in \{0,1,\dots,n\}$. We have a subgroup $\mathfrak{S}_\ell \times \mathfrak{S}_{n-\ell} \subset \mathfrak{S}_n$. Then, $\mathfrak{S}_\ell$ permutes the first ℓ variables, whereas $\mathfrak{S}_{n-\ell}$ permutes the last $n-\ell$ variables. Denote by

$$\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}_\ell\times\mathfrak{S}_{n-\ell}}$$

the subalgebra of $\mathbb{C}[z_1,\dots,z_n]$ consisting of $\mathfrak{S}_\ell \times \mathfrak{S}_{n-\ell}$-invariant polynomials. It is known that $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}_\ell\times\mathfrak{S}_{n-\ell}}$ is a free $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}}$-module of rank $\binom{n}{\ell}$.

3.1. Definition of $\mathcal{V}^{\mathfrak{S}}$

Let $V = (\mathbb{C}^{1|1})^{\otimes n}$ be the tensor power of the vector representation of $\mathfrak{gl}(1|1)$. The $\mathfrak{gl}(1|1)$-module V has weight decomposition

$$V = \bigoplus_{\ell=0}^{n} (V)_{(n-\ell,\ell)}.$$

Let $\mathcal{V}$ be the space of polynomials in variables $\boldsymbol{z} = (z_1, z_2, \dots, z_n)$ with coefficients in V,

$$\mathcal{V} = V \otimes \mathbb{C}[z_1, z_2, \dots, z_n].$$

The space V is identified with the subspace $V \otimes 1$ of constant polynomials in $\mathcal{V}$. The space $\mathcal{V}$ has a natural grading induced from the grading on $\mathbb{C}[z_1,\dots,z_n]$ with $\deg(z_i) = 1$. Namely, the degree of an element $v \otimes p$ in $\mathcal{V}$ is given by the degree of the polynomial p, $\deg(v \otimes p) = \deg\, p$. Clearly, the space $\mathrm{End}(\mathcal{V})$ has a gradation structure induced from that on $\mathcal{V}$.

Let $P^{(i,j)}$ be the graded flip operator that acts on the i-th and j-th factors of V. Let s_1, $s_2, \dots, s_{n-1}$ be the simple permutations of the symmetric group $\mathfrak{S}_n$. Define the $\mathfrak{S}_n$-action on $\mathcal{V}$ by the rule:

$$s_i : f(z_1, \dots, z_n) \mapsto P^{(i,i+1)} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n),$$

for $f(z_1, \dots, z_n) \in \mathcal{V}$. Note that the $\mathfrak{S}_n$-action respects the gradation on $\mathcal{V}$. Denote the subspace of all vectors in $\mathcal{V}$ invariant with respect to the $\mathfrak{S}_n$-action by $\mathcal{V}^{\mathfrak{S}}$.

Clearly, the $\mathfrak{gl}(1|1)$-action on $\mathcal{V}$ commutes with the $\mathfrak{S}_n$-action on $\mathcal{V}$ and preserves the grading. Therefore, $\mathcal{V}^{\mathfrak{S}}$ is a graded $\mathfrak{gl}(1|1)$-module. Hence, we have the weight decomposition for both $\mathcal{V}^{\mathfrak{S}}$ and $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}$:

$$\mathcal{V}^{\mathfrak{S}} = \bigoplus_{\ell=0}^{n} (\mathcal{V}^{\mathfrak{S}})_{(n-\ell,\ell)}, \qquad (\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}} = \bigoplus_{\ell=0}^{n} (\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}_{(n-\ell,\ell)}.$$

Note that $(\mathcal{V}^{\mathfrak{S}})_{(n-\ell,\ell)}$ and $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}_{(n-\ell,\ell)}$ are also graded $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-modules.

The space $\mathcal{V}$ is a $\mathfrak{gl}(1|1)[t]$-module where $e_{ij}[r]$ acts by

$$\begin{aligned} &e_{ij}[r](p(z_1, \dots, z_n) w_1 \otimes \cdots \otimes w_n) \\ &\quad = p(z_1, \dots, z_n) \sum_{s=1}^{n} (-1)^{(|w_1|+\cdots+|w_{s-1}|)(|i|+|j|)} z_s^r \, w_1 \otimes \cdots \otimes e_{ij} w_s \otimes \cdots \otimes w_n, \end{aligned} \tag{15}$$

for $p(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$ and $w_s \in \mathbb{C}^{1|1}$.

Lemma 2. *The $\mathfrak{gl}(1|1)[t]$-action on $\mathcal{V}$ commutes with the $\mathfrak{S}_n$-action on $\mathcal{V}$. Both $\mathcal{V}$ and $\mathcal{V}^{\mathfrak{S}}$ are graded $\mathfrak{gl}(1|1)[t]$-modules.*

3.2. Properties of $\mathcal{V}^{\mathfrak{S}}$ and $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}$

In this section, we recall properties of $\mathcal{V}^{\mathfrak{S}}$ and $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}$ from [11] [Section 3].

Lemma 3. *The space $(\mathcal{V}^{\mathfrak{S}})_{(n-\ell,\ell)}$ is a free $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-module of rank $\binom{n}{\ell}$. In particular, the space $\mathcal{V}^{\mathfrak{S}}$ is a free $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-module of rank 2^n.*

Set $v^+ = v_1^{\otimes n} = v_1 \otimes \cdots \otimes v_1$.

Lemma 4. *The $\mathfrak{gl}(1|1)[t]$-module $\mathcal{V}^{\mathfrak{S}}$ is a cyclic module generated by v^+.*

Lemma 5. *The set*

$$\{e_{21}[r_1] e_{21}[r_2] \cdots e_{21}[r_\ell] v^+ \mid 0 \leqslant r_1 < r_2 < \cdots < r_\ell \leqslant n-1\} \tag{16}$$

is a free generating set of $(\mathcal{V}^{\mathfrak{S}})_{(n-\ell,\ell)}$ over $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$.

Lemma 6. *The space $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}_{(n-\ell,\ell)}$ is a free $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-module of rank $\binom{n-1}{\ell}$ with a free generating set given by*

$$\{e_{12}[0] e_{21}[0] e_{21}[r_1] \cdots e_{21}[r_\ell] v^+, \quad 1 \leqslant r_1 < r_2 < \cdots < r_\ell \leqslant n-1\}. \tag{17}$$

In particular, the space $(\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}$ is a free $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-module of rank 2^{n-1}.

Set $(q)_r = \prod_{i=1}^{r} (1 - q^i)$.

Proposition 2. *We have*

$$\mathrm{ch}\big((\mathcal{V}^{\mathfrak{S}})_{(n-\ell,\ell)}\big) = \frac{q^{\ell(\ell-1)/2}}{(q)_\ell (q)_{n-\ell}}, \qquad \mathrm{ch}\big((\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}_{(n-\ell,\ell)}\big) = \frac{q^{\ell(\ell+1)/2}}{(q)_\ell (q)_{n-1-\ell}(1-q^n)}.$$

Given $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, let $I_{\boldsymbol{a}}$ be the ideal of $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$ generated by $\sigma_i(\boldsymbol{z}) - \boldsymbol{a}$, $i = 1, \dots, n$. Then, for any $\boldsymbol{a}$, by Lemmas 2 and 3, the quotient space $\mathcal{V}^{\mathfrak{S}} / I_{\boldsymbol{a}} \mathcal{V}^{\mathfrak{S}}$ is a $\mathfrak{gl}(1|1)[t]$-module of dimension 2^n over $\mathbb{C}$. Denote by $\bar{v}^+$ the image of v^+ under this quotient.

3.3. Weyl Modules

In this section, we recall a special family of Weyl modules for $\mathfrak{gl}(1|1)[t]$ and their properties from [10] [Section 3.3].

Let $\eta(x)$ be a monic polynomial of degree m with complex coefficients, where $m \in \mathbb{Z}_{\geqslant 0}$,

$$\eta(x) = \sum_{i=0}^{m} \gamma_i x^i, \qquad \gamma_m = 1.$$

Denote by W_η the $\mathfrak{gl}(1|1)[t]$-module generated by an even vector w subject to the relations:

$$e_{11}(x)w = \eta'(x)/\eta(x)w, \qquad e_{22}(x)w = e_{12}(x)w = 0, \tag{18}$$

$$\sum_{i=0}^{m} \gamma_i e_{21}[i]w = 0. \tag{19}$$

It is convenient to write (19) as $(e_{21} \otimes \eta(t))w = 0$.

Clearly, we have $\dim W_\eta \leqslant 2^m$ by the PBW theorem and (1), (19). The module W_η is the universal $\mathfrak{gl}(1|1)[t]$-module satisfying (18), (19), which we call a *Weyl module*.

If $\eta(x) = (x-b)^m$, we write W_η as $W_m(b)$.

Lemma 7. *Let $\boldsymbol{a} = (0, \dots, 0) \in \mathbb{C}^n$. Then, $\mathcal{V}^{\mathfrak{S}} / I_{\boldsymbol{a}} \mathcal{V}^{\mathfrak{S}}$ is isomorphic to $W_n(0)$ as $\mathfrak{gl}(1|1)[t]$-modules.*

In particular, we have $\dim W_m(b) = 2^m$.

Lemma 8. *Let $\eta(x) = \prod_{s=1}^{k} (x - b_s)^{n_s}$, where $b_s \neq b_r$ for $1 \leqslant s \neq r \leqslant k$. Then, W_η is isomorphic to $\bigotimes_{s=1}^{k} W_{n_s}(b_s)$ as $\mathfrak{gl}(1|1)[t]$-modules.*

Given sequences $\boldsymbol{n} = (n_1, \dots, n_k)$ of non-negative integers and $\boldsymbol{b} = (b_1, \dots, b_s)$ of distinct complex numbers, by Lemma 8, we call $\bigotimes_{s=1}^{k} W_{n_s}(b_s)$ the *Weyl module associated with $\boldsymbol{n}$ and $\boldsymbol{b}$.*

Given $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, define $k \in \mathbb{Z}_{>0}$, $b_s \in \mathbb{C}$ and $n_s \in \mathbb{Z}_{>0}$ for $1 \leqslant s \leqslant k$ by

$$x^n + \sum_{i=1}^{n} (-1)^i a_i x^{n-i} = \prod_{s=1}^{k} (x - b_s)^{n_s}, \tag{20}$$

where $b_1, \dots, b_k$ are distinct. Note that $n = \sum_{s=1}^{k} n_s$.

Lemma 9. *The $\mathfrak{gl}(1|1)[t]$-module $\mathcal{V}^{\mathfrak{S}} / I_{\boldsymbol{a}} \mathcal{V}^{\mathfrak{S}}$ is isomorphic to $\bigotimes_{s=1}^{k} W_{n_s}(b_s)$.*

We also need the following statements.

Lemma 10. *Let $b \in \mathbb{C}$. We have the following properties for $W_m(b)$.*

1. *As a $\mathfrak{gl}(1|1)$-module, $W_m(b)$ is isomorphic to $(\mathbb{C}^{1|1})^{\otimes m}$.*
2. *A $\mathfrak{gl}(1|1)[t]$-module M is an irreducible subquotient of $W_m(b)$ if and only if M has the form $L_{\boldsymbol{\lambda}}(b)$, where $\boldsymbol{\lambda}$ is a polynomial weight such that $|\boldsymbol{\lambda}| = m$.*

Corollary 1. *A $\mathfrak{gl}(1|1)[t]$-module M is an irreducible subquotient of $\bigotimes_{s=1}^{k} W_{n_s}(b_s)$ if and only if M has the form $\bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)$, where $\boldsymbol{\lambda}^{(s)}$ is a polynomial weight such that $|\boldsymbol{\lambda}^{(s)}| = n_s$ for each $1 \leqslant s \leqslant k$.*

4. Main Theorems

4.1. The Algebra $\mathcal{O}_l$

Let Ω_l be the n-dimensional affine space with coordinates $f_1, \dots, f_l, g_1, \dots, g_{n-l}$. Introduce two polynomials

$$f(x) = x^l + \sum_{i=1}^{l} f_i x^{l-i}, \quad g(x) = x^{n-l} + \sum_{i=1}^{n-l} g_i x^{n-l-i}. \tag{21}$$

Denote by $\mathcal{O}_l$ the algebra of regular functions on Ω_l, namely

$$\mathcal{O}_l = \mathbb{C}[f_1, \dots, f_l, g_1, \dots, g_{n-l-1}, g_{n-l}].$$

Define the degree function by

$$\deg f_i = i, \qquad \deg g_j = j,$$

for all $i = 1, \dots, l$ and $j = 1, \dots, n-l$. The algebra $\mathcal{O}_l$ is graded with the graded character given by

$$\mathrm{ch}(\mathcal{O}_l) = \frac{1}{(q)_l (q)_{n-l}}. \tag{22}$$

Let $\mathscr{F}_0\mathcal{O}_l \subset \mathscr{F}_1\mathcal{O}_l \subset \cdots \subset \mathcal{O}_l$ be the increasing filtration corresponding to this grading, where $\mathscr{F}_s\mathcal{O}_l$ consists of elements of a degree of at most s.

Let $\Sigma_1, \dots, \Sigma_n$ be the elements of $\mathcal{O}_l$ such that

$$(q_1 - q_2) f(x) g(x) = (q_1 - q_2) x^n + \sum_{i=1}^{n} (-1)^i \big((q_1 - q_2)\Sigma_i - (n+1-i)\Sigma_{i-1} \big) x^{n-i}, \tag{23}$$

where $\Sigma_0 = 1$. The homomorphism

$$\pi_l : \mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}} \to \mathcal{O}_l, \qquad \sigma_i(z) \mapsto \Sigma_i, \qquad i = 1, \dots, n, \tag{24}$$

is injective and induces a $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}}$-module structure on $\mathcal{O}_l$.

Express $f'(x) g(x)$ as follows:

$$(q_1 - q_2) f'(x) g(x) = (q_1 - q_2) l x^{n-1} + \sum_{i=1}^{n-1} G_i x^{n-1-i}, \tag{25}$$

where $G_i \in \mathcal{O}_l$.

Lemma 11. *The elements G_i and Σ_j, $i = 1, \dots, n-1$, $j = 1, \dots, n$ generate the algebra $\mathcal{O}_l$.*

Lemma 12. *We have $G_i \in \mathscr{F}_i\mathcal{O}_l \setminus \mathscr{F}_{i-1}\mathcal{O}_l$ and $\Sigma_j \in \mathscr{F}_j\mathcal{O}_l \setminus \mathscr{F}_{j-1}\mathcal{O}_l$, $i = 1, \dots, n-1$, $j = 1, \dots, n$.*

4.2. Bethe Algebra

We call the unital subalgebra of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ generated by the coefficients of

$$e_{11}(x) + e_{22}(x), \quad \mathscr{H}(x) = q_1 e_{11}(x) + q_2 e_{22}(x) + \frac{1}{2}\sum_{i,j=1}^{2} e_{ij}(x) e_{ji}(x)(-1)^{|j|}$$

the *Bethe algebra*. We denote the Bethe algebra by $\mathcal{B}$. Note that the coefficients of $e_{11}(x) + e_{22}(x)$ generate the center of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$.

Lemma 13 ([4]). *The Bethe algebra $\mathcal{B}$ is commutative. The Bethe algebra $\mathcal{B}$ commutes with the subalgebra $\mathrm{U}(\mathfrak{h}) \subset \mathrm{U}(\mathfrak{gl}(1|1)[t])$.*

Being a subalgebra of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$, the Bethe algebra $\mathcal{B}$ acts on any $\mathfrak{gl}(1|1)[t]$-module M. Since $\mathcal{B}$ commutes with $\mathrm{U}(\mathfrak{h})$, the Bethe algebra preserves the subspace $(M)_{\boldsymbol{\lambda}}$ for any weight $\boldsymbol{\lambda}$. If $K \subset M$ is a $\mathcal{B}$-invariant subspace, then we call the image of $\mathcal{B}$ in $\mathrm{End}(K)$ the *Bethe algebra associated with* K.

Let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$. Define $k \in \mathbb{Z}_{>0}$, a sequence of positive integers $\boldsymbol{n} = (n_1, \dots, n_k)$ and a sequence of distinct complex numbers $\boldsymbol{b} = (b_1, \dots, b_k)$ by (20). Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$ be a sequence of polynomial $\mathfrak{gl}(1|1)$-weights such that $|\boldsymbol{\lambda}^{(s)}| = n_s$.

We study the action of the Bethe algebra $\mathcal{B}$ on the following $\mathcal{B}$-modules:

$$\mathcal{M}_l = (\mathcal{V}^{\mathfrak{S}})_{(n-l,l)}, \quad \mathcal{M}_{l,\boldsymbol{a}} = \Big(\bigotimes_{s=1}^{k} W_{n_s}(b_s)\Big)_{(n-l,l)}, \quad \mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \Big(\bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_s)\Big)_{(n-l,l)}.$$

Denote the Bethe algebras associated with $\mathcal{M}_l$, $\mathcal{M}_{l,\boldsymbol{a}}$, $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ by $\mathcal{B}_l$, $\mathcal{B}_{l,\boldsymbol{a}}$, $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$, respectively. For any element $X \in \mathcal{B}$, we denote by $X(\boldsymbol{z})$, $X(\boldsymbol{a})$, $X(\boldsymbol{\Lambda}, \boldsymbol{b})$ the respective linear operators.

Since, by Lemma 4, the $\mathfrak{gl}(1|1)[t]$-module $\mathcal{V}^{\mathfrak{S}}$ is generated by $v_1^{\otimes n} = v_1 \otimes \cdots \otimes v_1$, the series $e_{11}(x) + e_{22}(x)$ acts on $\mathcal{V}^{\mathfrak{S}}$ by multiplication by the series

$$\sum_{i=1}^{n} \frac{1}{x - z_i} = \sum_{i=1}^{n} \sum_{j=0}^{\infty} z_i^j x^{-j-1}.$$

Therefore, there exist unique central elements $C_1, \dots, C_n$ of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ of minimal degrees such that each C_i acts on $\mathcal{V}^{\mathfrak{S}}$ by multiplication by $\sigma_i(\boldsymbol{z})$.

Define $B_i \in \mathcal{B}$ by

$$\Big(x^n + \sum_{i=1}^{n} (-1)^i C_i x^{n-i}\Big) \mathcal{T}(x) = x^n \sum_{i=1}^{\infty} B_i x^{-i}, \tag{26}$$

where $\mathcal{T}(x)$ is defined in (13).

Lemma 14. *We have $B_i(\boldsymbol{z}) = 0$ for $i > n$ and $B_1(\boldsymbol{z}) = (q_1 - q_2)l$.*

Proof. Let $V(\boldsymbol{c}) = \bigotimes_{i=1}^{n} \mathbb{C}^{1|1}(c_i)$, where $c_i \in \mathbb{C}$. Note that $B_i(\boldsymbol{z})$ is a polynomial in $\boldsymbol{z}$ with values in $\mathrm{End}((V)_{(n-l,l)})$. For any sequence of complex numbers $\boldsymbol{c} = (c_1, \dots, c_n)$, we can evaluate $B_i(\boldsymbol{z})$ at $\boldsymbol{z} = \boldsymbol{c}$ to an operator on $(V(\boldsymbol{c}))_{(n-l,l)}$. By Theorem 2, the Gaudin transfer matrix $\mathscr{H}(x)$ is diagonalizable and the Bethe ansatz is complete for $(V(\boldsymbol{c}))_{(n-l,l)}$ when $\boldsymbol{c} \in \mathbb{C}^n$ is generic. Hence, by (14) and (26), $\big(x^n + \sum_{i=1}^{n}(-1)^i C_i x^{n-i}\big)\mathcal{T}(x)$ acts on $(V(\boldsymbol{c}))_{(n-l,l)}$ as a polynomial in x for generic $\boldsymbol{c}$. In particular, it implies that B_i, $i > n$ acts on $(V(\boldsymbol{c}))_{(n-l,l)}$ by zero for generic $\boldsymbol{c}$. Therefore, $B_i(\boldsymbol{z})$, $i > n$ is identically zero.

By the same reasoning, one shows that $B_1(\boldsymbol{z}) = (q_1 - q_2)l$. Alternatively, it also follows from $B_1 = (q_1 - q_2)e_{22}$. □

Lemma 15. *The elements $B_i(z)$ and $C_j(z)$, for $1 < i \leqslant n$ and $1 \leqslant j \leqslant n$, generate the algebra $\mathcal{B}_l$.*

Proof. It follows from the definition of $\mathcal{B}$, (26) and Lemma 14. □

One can restrict the filtration on $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ to the Bethe algebra, $\mathscr{F}_0\mathcal{B} \subset \mathscr{F}_1\mathcal{B} \subset \cdots \subset \mathcal{B}$.

Lemma 16. *We have $B_i \in \mathscr{F}_{i-1}\mathcal{B}/\mathscr{F}_{i-2}\mathcal{B}$ and $C_j \in \mathscr{F}_j\mathcal{B}/\mathscr{F}_{j-1}\mathcal{B}$ for $1 < i \leqslant n$ and $1 \leqslant j \leqslant n$.*

4.3. Main Theorems

Recall from Proposition 2 that there exists a unique vector (up to proportionality) of degree $l(l-1)/2$ in $\mathcal{M}_l$ explicitly given by

$$\mathfrak{u}_l := e_{21}[0]e_{21}[1]\cdots e_{21}[l-1]v^+;$$

see Lemma 5.

Any commutative algebra $\mathcal{A}$ is a module over itself induced by left multiplication. We call it the *regular representation of* $\mathcal{A}$. The dual space $\mathcal{A}^*$ is naturally an $\mathcal{A}$-module, which is called the *coregular representation*. A bilinear form $(\cdot|\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ is called *invariant* if $(ab|c) = (a|bc)$ for all $a, b, c \in \mathcal{A}$. A finite-dimensional commutative algebra $\mathcal{A}$ admitting an invariant non-degenerate symmetric bilinear form $(\cdot|\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ is called a *Frobenius algebra*. The regular and coregular representations of a Frobenius algebra are isomorphic.

Let M be an $\mathcal{A}$-module and $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ a character; then, the *$\mathcal{A}$-eigenspace associated to* $\mathcal{E}$ in M is defined by $\bigcap_{a\in\mathcal{A}} \ker(a|_M - \mathcal{E}(a))$. The *generalized $\mathcal{A}$-eigenspace associated to* $\mathcal{E}$ in M is defined by $\bigcap_{a\in\mathcal{A}} \big(\bigcup_{m=1}^{\infty} \ker(a|_M - \mathcal{E}(a))^m\big)$.

Theorem 3. *The action of the Bethe algebra $\mathcal{B}_l$ on $\mathcal{M}_l$ has the following properties.*

1. *The map $\eta_l : G_i \mapsto B_{i+1}(z)$, $\Sigma_j \mapsto C_j(z)$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$ extends uniquely to an isomorphism $\eta_l : \mathcal{O}_l \to \mathcal{B}_l$ of filtered algebras. Moreover, the isomorphism η_l is an isomorphism of $\mathbb{C}[z_1, \ldots, z_n]^{\mathfrak{S}}$-modules.*
2. *The map $\rho_l : \mathcal{O}_l \mapsto \mathcal{M}_l$, $F \mapsto \eta_l(F)\mathfrak{u}_l$ is an isomorphism of filtered vector spaces identifying the $\mathcal{B}_l$-module $\mathcal{M}_l$ with the regular representation of $\mathcal{O}_l$.*

Theorem 3 is proved in Section 5.

Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Define $k \in \mathbb{Z}_{>0}$, a sequence of positive integers $\boldsymbol{n} = (n_1, \ldots, n_k)$ and a sequence of distinct complex numbers $\boldsymbol{b} = (b_1, \ldots, b_k)$ by (20). Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)})$ be a sequence of non-degenerate polynomial weights such that $|\boldsymbol{\lambda}^{(s)}| = n_s$ for each $1 \leqslant s \leqslant k$.

Theorem 4. *The action of the Bethe algebra $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ on $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ has the following properties.*

1. *The Bethe algebra $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is isomorphic to*

$$\mathbb{C}[w_1, \ldots, w_k]^{\mathfrak{S}_l \times \mathfrak{S}_{k-l}} / \langle \sigma_i(\boldsymbol{w}) - \varepsilon_i \rangle_{i=1,\ldots,k},$$

where ε_i is given by

$$\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x) := \prod_{s=1}^{k}(x - b_s)\Big(q_1 - q_2 + \sum_{s=1}^{k} \frac{n_s}{x - b_s}\Big) = (q_1 - q_2)\Big(x^k + \sum_{i=1}^{k}(-1)^i \varepsilon_i x^{k-i}\Big)$$

and $\sigma_i(\boldsymbol{w})$ are elementary symmetric functions in $w_1, \ldots, w_k$.

2. *The Bethe algebra $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is a Frobenius algebra. Moreover, the $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$-module $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is isomorphic to the regular representation of $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$.*
3. *The Bethe algebra $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is a maximal commutative subalgebra in $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ of dimension $\binom{k}{l}$.*

4. *Every $\mathcal{B}$-eigenspace in $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ has dimension one.*
5. *The $\mathcal{B}$-eigenspaces in $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ bijectively correspond to the monic degree l divisors $y(x)$ of the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x)$. Moreover, the eigenvalue of $\mathscr{H}(x)$ corresponding to the monic divisor y is described by $\mathcal{E}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}(x)$; see (12).*
6. *Every generalized $\mathcal{B}$-eigenspace in $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is a cyclic $\mathcal{B}$-module.*
7. *The dimension of the generalized $\mathcal{B}$-eigenspace associated to $\mathcal{E}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}(x)$ is*

$$\prod_{a\in\mathbb{C}} \binom{\mathrm{Mult}_a(\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}})}{\mathrm{Mult}_a(y)},$$

where $\mathrm{Mult}_a(p)$ *is the multiplicity of a as a root of the polynomial p.*

Theorem 4 is proved in Section 5.

Note that its results are quite parallel to that of XXX spin chains; see [11] [Theorem 4.11].

4.4. Higher Gaudin Transfer Matrices

To define higher Gaudin transfer matrices, we first recall basics about pseudo-differential operators. Let $\mathscr{A}$ be a differential superalgebra with an even derivation $\partial : \mathscr{A} \to \mathscr{A}$. For $r \in \mathbb{Z}_{>0}$, denote the r-th derivative of $a \in \mathscr{A}$ by $a_{[r]}$. Define the *superalgebra of pseudo-differential operators* $\mathscr{A}((\partial^{-1}))$ as follows. Elements of $\mathscr{A}((\partial^{-1}))$ are Laurent series in ∂^{-1} with coefficients in $\mathscr{A}$, and the product is given by

$$\partial\partial^{-1} = \partial^{-1}\partial = 1, \quad \partial^r a = \sum_{s=0}^{\infty} \binom{r}{s} a_{[s]}\partial^{r-s}, \quad r \in \mathbb{Z}, \quad a \in \mathscr{A},$$

where

$$\binom{r}{s} = \frac{r(r-1)\cdots(r-s+1)}{s!}.$$

Let

$$\mathscr{A}_x^{m|n} = \mathrm{U}(\mathfrak{gl}(1|1)[t])((x^{-1})) = \Big\{ \sum_{r=-\infty}^{s} g_r x^r,\ r \in \mathbb{Z},\ g_r \in \mathrm{U}(\mathfrak{gl}(1|1)[t]) \Big\}.$$

Consider the operator in $\mathrm{End}(\mathbb{C}^{1|1}) \otimes \mathscr{A}_x^{m|n}((\partial_x^{-1}))$,

$$\mathfrak{Z}(x,\partial_x) := \sum_{a,b=1}^{2} E_{ab} \otimes \Big(\delta_{ab}(\partial_x - q_a) - e_{ab}(x)(-1)^{|a|}\Big),$$

which is a Manin matrix; see [4] [Lemma 3.1] and [8] [Lemma 4.2]. Define the *Berezinian*—see [14]—of $\mathfrak{Z}(x,\partial_x)$ by

$$\begin{aligned}&\mathrm{Ber}\big(\mathfrak{Z}(x,\partial_x)\big)\\ &= \big(\partial_x - q_1 - e_{11}(x)\big)\Big(\partial_x - q_2 + e_{22}(x) + e_{21}(x)\big(\partial_x - q_1 - e_{11}(x)\big)^{-1} e_{12}(x)\Big)^{-1}.\end{aligned} \tag{27}$$

Denote the Berezinian by $\mathfrak{D}(x,\partial_x)$ and expand it as an element in $\mathscr{A}_x^{m|n}((\partial_x^{-1}))$,

$$\mathfrak{D}(x,\partial_x) = \sum_{r=0}^{\infty} (-1)^r \mathcal{G}_r(x)\partial_x^{-r}. \tag{28}$$

We call the series $\mathcal{G}_r(x) \in \mathscr{A}_x^{m|n}$, $r \in \mathbb{Z}_{\geqslant 0}$ the *higher Gaudin transfer matrices*. In particular, we call $\mathcal{G}_1(x)$ and $\mathcal{G}_2(x)$ the *first and second Gaudin transfer matrices*, respectively.

Example 1. We have $\mathcal{G}_0(x) = 1$,

$$\mathcal{G}_1(x) = q_1 - q_2 + e_{11}(x) + e_{22}(x),$$

$$\mathcal{G}_2(x) = \big(q_1 - q_2 + e_{11}(x) + e_{22}(x)\big)\big(-q_2 + e_{22}(x)\big) - e_{21}(x)e_{12}(x).$$

Remark 1. *In principle, the Bethe algebra should be the unital subalgebra of* $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ *generated by coefficients* $\mathcal{G}_r(x)$, $r \in \mathbb{Z}_{>0}$, *cf.* [15]*. However, it turns out that the first two transfer matrices already give (almost) complete information about the Bethe algebra; see the discussion below.*

Now, we describe the eigenvalues of higher Gaudin transfer matrices acting on the on-shell Bethe vector.

Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$ be a sequence of $\mathfrak{gl}(1|1)$-weights and $\boldsymbol{b} = (b_1, \dots, b_k)$ a sequence of distinct complex numbers, where $\boldsymbol{\lambda}^{(s)} = (\alpha_s, \beta_s)$. Let $\boldsymbol{t} = (t_1, \dots, t_l)$, where $0 \leqslant l < k$. Suppose that $y_{\boldsymbol{t}}$ divides the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}$ (namely $\boldsymbol{t}$ satisfies the Bethe ansatz equation); see (10).

Theorem 5 ([16] [Theorem 5.2])**.** *If* $t_i \neq t_j$ *for* $1 \leqslant i < j \leqslant l$*, then*

$$\mathfrak{D}(x, \partial_x)\mathbb{B}_l(\boldsymbol{t}) = \mathbb{B}_l(\boldsymbol{t})\Big(\partial_x - q_1 - \sum_{s=1}^{k} \frac{\alpha_s}{x - b_s} + \frac{y'_{\boldsymbol{t}}}{y_{\boldsymbol{t}}}\Big)\Big(\partial_x - q_2 + \sum_{s=1}^{k} \frac{\beta_s}{x - b_s} + \frac{y'_{\boldsymbol{t}}}{y_{\boldsymbol{t}}}\Big)^{-1}. \tag{29}$$

The theorem is a differential analog of [11] [Theorem 6.4]. Note that the pseudo-differential operator in the right-hand side of (29), denoted by $\mathfrak{D}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}$, was introduced [7] [Section 5.3]. This theorem is generalized to the $\mathfrak{gl}(m|n)$ case in [16] [Theorem 5.2] where, on the right-hand side, the pseudo-differential operator describing the eigenvalues of higher Gaudin transfer matrices should be replaced by the pseudo-differential operator in [7] [Equation (6.5)]. This generalization is a classical limit of [17] [Conjecture 5.15] and [16] [Corollary 3.6] that connects the rational difference operator introduced in [18] [Equation (5.6)] with the eigenvalues of higher transfer matrices on the on-shell Bethe vector for XXX spin chains associated with $\mathfrak{gl}(m|n)$. The method used in the proof of [16] [Theorem 5.2] is motivated by [19,20] via the nested algebraic Bethe ansatz introduced in [21].

Remark 2. *As shown in [7] [Lemma 5.7], the odd reflection of* $\mathfrak{D}_{y,\boldsymbol{\Lambda},\boldsymbol{b}}$*, cf. [7] [Equation (3.1)], which comes from the study of the fermionic reproduction procedure of the Bethe ansatz equation, is compatible with the odd reflection of Lie superalgebras. The difference analog of this fact was used in [22] to investigate the relations between the odd reflections of the super Yangian of type A and the fermionic reproduction procedure of the Bethe ansatz equation for XXX spin chains.*

We conclude this section by discussing the connections between $\mathcal{G}_i(x)$, $i \geqslant 3$ and $\mathcal{G}_1(x)$, $\mathcal{G}_2(x)$.

Let

$$\mu(x) = q_1 + \sum_{s=1}^{k} \frac{\alpha_s}{x - b_s} - \frac{y'_{\boldsymbol{t}}}{y_{\boldsymbol{t}}}, \quad \nu(x) = -q_2 + \sum_{s=1}^{k} \frac{\beta_s}{x - b_s} + \frac{y'_{\boldsymbol{t}}}{y_{\boldsymbol{t}}}.$$

For simplicity, we do not write the dependence of $\mu(x)$ and $\nu(x)$ on $\boldsymbol{\Lambda}, \boldsymbol{b}, \boldsymbol{t}$ explicitly. Then, the eigenvalue of $\mathfrak{D}(x, \partial_x)$ acting on $\mathbb{B}_l(\boldsymbol{t})$ is given by

$$(\partial_x - \mu(x))(\partial_x + \nu(x))^{-1} = 1 - (\mu(x) + \nu(x))(\partial_x + \nu(x))^{-1}. \tag{30}$$

Hence, the eigenvalues of $\mathcal{G}_i(x)$ are essentially only determined by $\mu(x) + \nu(x)$ and $\nu(x)$. Comparing (28) and the expansion of (30), we have

$$\mathcal{G}_1(x)\mathbb{B}_l(\boldsymbol{t}) = (\mu(x) + \nu(x))\mathbb{B}_l(\boldsymbol{t}), \quad \mathcal{G}_2(x)\mathbb{B}_l(\boldsymbol{t}) = (\mu(x) + \nu(x))\nu(x)\mathbb{B}_l(\boldsymbol{t}); \tag{31}$$

see also (14). Therefore, the spectrum of all higher transfer matrices are simply determined by that of the first two transfer matrices, which justifies our definition of Bethe algebra.

Lemma 17. *Let the complex parameters $c_1, \dots, c_m$ and the positive integer m vary. Then, the kernels of the representations $\bigotimes_{i=1}^m \mathbb{C}^{1|1}(c_i)$ of $\mathrm{U}(\mathfrak{gl}(1|1)[t])$ have a zero intersection.*

Proof. The proof is contained in the proof of [23] [Proposition 1.7]. □

Corollary 2. *We have*

$$\mathfrak{D}(x, \partial_x) = \Big(\partial_x - \mathcal{G}_1(x) + \frac{\mathcal{G}_2(x)}{\mathcal{G}_1(x)}\Big)\Big(\partial_x + \frac{\mathcal{G}_2(x)}{\mathcal{G}_1(x)}\Big)^{-1}. \tag{32}$$

Proof. By Lemma 17, it suffices to check that the left-hand side and the right-hand side of (32) act identically on a basis of $\bigotimes_{i=1}^m \mathbb{C}^{1|1}(c_i)$ for all $m \in \mathbb{Z}_{>0}$ and generic $\boldsymbol{c} = (c_1, \dots, c_m)$.

By Theorem 2, there is a basis of $\bigotimes_{i=1}^m \mathbb{C}^{1|1}(c_i)$ consisting of on-shell Bethe vectors for generic $\boldsymbol{c}$. Therefore, the statement follows from Theorem 5, (30) and (31). □

5. Proof of Main Theorems

In this section, we prove the main theorems. For completeness, we provide all details, even if they are parallel to those in [10] [Section 5].

5.1. The First Isomorphism

Proof of Theorem 3. We first show that the homomorphism defined by η_l is well-defined.

Consider the tensor product $V(\boldsymbol{c}) = \bigotimes_{i=1}^n \mathbb{C}^{1|1}(c_i)$, where $c_i \in \mathbb{C}$, and the corresponding Bethe ansatz equation associated to weight $(n-l, l)$. Let $\boldsymbol{t}$ be a solution with distinct coordinates and $\mathbb{B}_l(\boldsymbol{t})$ be the corresponding on-shell Bethe vector. Denote $\mathcal{E}_{i,\boldsymbol{t}}$ the eigenvalues of B_i acting on $\mathbb{B}_l(\boldsymbol{t})$; see Theorem 1 and Equation (14).

Define a character $\pi : \mathcal{O}_l \to \mathbb{C}$ by sending

$$f(x) \mapsto y_{\boldsymbol{t}}(x), \qquad \Sigma_n \mapsto \prod_{i=1}^n c_i,$$

$$g(x) \mapsto \frac{1}{(q_1 - q_2) y_{\boldsymbol{t}}(x)} \prod_{i=1}^n (x - c_i)\Big(q_1 - q_2 + \sum_{i=1}^n \frac{1}{x - c_i}\Big).$$

Then,

$$\pi(\Sigma_i) = \sigma_i(\boldsymbol{c}), \qquad \pi(G_i) = \mathcal{E}_{i,\boldsymbol{t}}, \tag{33}$$

by (23) and by (12), (14), (25), respectively.

Now, let $P(G_i, \Sigma_j)$ be a polynomial in G_i, Σ_j such that $P(G_i, \Sigma_j)$ is equal to zero in $\mathcal{O}_l$. It suffices to show that $P(B_i(\boldsymbol{z}), C_j(\boldsymbol{z}))$ is equal to zero in $\mathcal{B}_l$.

Note that $P(B_i(\boldsymbol{z}), C_j(\boldsymbol{z}))$ is a polynomial in $z_1, \dots, z_n$ with values in $\mathrm{End}((V)_{(n-l,l)})$. For any sequence $\boldsymbol{c}$ of complex numbers, we can evaluate $P(B_i(\boldsymbol{z}), C_j(\boldsymbol{z}))$ at $\boldsymbol{z} = \boldsymbol{c}$ to an operator on $(V(\boldsymbol{c}))_{(n-l,l)}$. By Theorem 2, the transfer matrix $\mathcal{T}(x)$ is diagonalizable and the Bethe ansatz is complete for $(V(\boldsymbol{c}))_{(n-l,l)}$ when $\boldsymbol{c} \in \mathbb{C}^n$ is generic. Hence, by (33), the value of $P(B_i(\boldsymbol{z}), C_j(\boldsymbol{z}))$ at $\boldsymbol{z} = \boldsymbol{c}$ is also equal to zero for generic $\boldsymbol{c}$. Therefore, $P(B_i(\boldsymbol{z}), C_j(\boldsymbol{z}))$ is identically zero and the map η_l is well-defined.

Let us now show that the map η_l is injective. Let $P(G_i, \Sigma_j)$ be a polynomial in G_i, Σ_j such that $P(G_i, \Sigma_j)$ is non-zero in $\mathcal{O}_l$. Then, the value at a generic point of Ω_l (e.g., the non-vanishing points of $P(G_i, \Sigma_j)$ such that f and g are relatively prime and have only simple zeros) is not equal to zero. Moreover, at those points, the transfer matrix $\mathcal{T}(x)$ is diagonalizable and the Bethe ansatz is complete again by Theorem 2. Therefore, again by (33), the polynomial $P(B_i(z), C_j(z))$ is a non-zero element in $\mathcal{B}_l$. Thus, the map η_l is injective.

The surjectivity of η_l follows from Lemma 15. Hence, η_l is an isomorphism of algebras.

The fact that η_l is an isomorphism of graded algebra respecting the gradation follows from Lemmas 12 and 16. This completes the proof of part (i).

The kernel of ρ_l is an ideal of $\mathcal{O}_l$. If we identify $\sigma_i(z)$ with Σ_i, then the algebra $\mathcal{O}_l$ contains the algebra $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}}$; see (24). The kernel of ρ_l intersects $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}}$ trivially. Therefore, the kernel of ρ_l is trivial as well. Hence, ρ_l is an injective map. Comparing Equation (22) and Proposition 2, we have $\mathrm{ch}(\mathcal{M}_l) = q^{l(l-1)/2}\mathrm{ch}(\mathcal{O}_l)$. Thus, ρ_l is an isomorphism of graded vector spaces, which shifts the degree by $l(l-1)/2$, completing the proof of part (ii). □

5.2. The Second Isomorphism

Let $\boldsymbol{a} = (a_1,\dots,a_n)$ be a sequence of complex numbers. Define $k \in \mathbb{Z}_{>0}$, a sequence of positive integers $\boldsymbol{n} = (n_1,\dots,n_k)$, and a sequence of distinct complex numbers $\boldsymbol{b} = (b_1,\dots,b_k)$ by (20). Let $I^{\mathcal{O}}_{l,\boldsymbol{a}}$ be the ideal of $\mathcal{O}_l$ generated by the elements $\Sigma_i - a_i$, $i = 1,\dots,n$, where $\Sigma_1,\dots,\Sigma_{n-1}$ are defined in (23). Let $\mathcal{O}_{l,\boldsymbol{a}}$ be the quotient algebra

$$\mathcal{O}_{l,\boldsymbol{a}} = \mathcal{O}_l / I^{\mathcal{O}}_{l,\boldsymbol{a}}.$$

Let $I^{\mathcal{B}}_{l,\boldsymbol{a}}$ be the ideal of $\mathcal{B}_l$ generated by $C_i(z) - a_i$, $i = 1,\dots,n$. Consider the subspace

$$I^{\mathcal{M}}_{l,\boldsymbol{a}} = I^{\mathcal{B}}_{l,\boldsymbol{a}}\mathcal{M}_l = (I_{\boldsymbol{a}}\mathcal{V}^{\mathfrak{S}})_{(n-l,l)},$$

where $I_{\boldsymbol{a}}$, as before, is the ideal of $\mathbb{C}[z_1,\dots,z_n]^{\mathfrak{S}}$ generated by $\sigma_i(z) - a_i$, $i = 1,\dots,n$.

Lemma 18. *We have*

$$\eta_l(I^{\mathcal{O}}_{l,\boldsymbol{a}}) = I^{\mathcal{B}}_{l,\boldsymbol{a}}, \quad \rho_l(I^{\mathcal{O}}_{l,\boldsymbol{a}}) = I^{\mathcal{M}}_{l,\boldsymbol{a}}, \quad \mathcal{B}_{l,\boldsymbol{a}} = \mathcal{B}_l / I^{\mathcal{B}}_{l,\boldsymbol{a}}, \quad \mathcal{M}_{l,\boldsymbol{a}} = (\mathcal{V}^{\mathfrak{S}})^{\mathrm{sing}}_{(n-l,l)} / I^{\mathcal{M}}_{l,\boldsymbol{a}}.$$

Proof. The lemma follows from Theorem 3 and Lemma 9. □

By Lemma 18, the maps η_l and ρ_l induce the maps

$$\eta_{l,\boldsymbol{a}} : \mathcal{O}_{l,\boldsymbol{a}} \to \mathcal{B}_{l,\boldsymbol{a}}, \qquad \rho_{l,\boldsymbol{a}} : \mathcal{O}_{l,\boldsymbol{a}} \to \mathcal{M}_{l,\boldsymbol{a}}.$$

The map $\eta_{l,\boldsymbol{a}}$ is an isomorphism of algebras. Since $\mathcal{B}_{l,\boldsymbol{a}}$ is finite-dimensional, by, e.g., [1] [Lemma 3.9], $\mathcal{O}_{l,\boldsymbol{a}}$ is a Frobenius algebra, and so is $\mathcal{B}_{l,\boldsymbol{a}}$. The map $\rho_{l,\boldsymbol{a}}$ is an isomorphism of vector spaces. Moreover, it follows from Theorem 3 and Lemma 18 that $\rho_{l,\boldsymbol{a}}$ identifies the regular representation of $\mathcal{O}_{l,\boldsymbol{a}}$ with the $\mathcal{B}_{l,\boldsymbol{a}}$-module $\mathcal{M}_{l,\boldsymbol{a}}$.

The statement of this section implies, by, e.g., [13] [Lemma 1.3], the following. Set

$$\zeta_{\boldsymbol{n},\boldsymbol{b}}(x) = q_1 - q_2 + \sum_{s=1}^{k} \frac{n_s}{x - b_s}, \qquad \psi_{\boldsymbol{n},\boldsymbol{b}}(x) := \zeta_{\boldsymbol{n},\boldsymbol{b}}(x) \prod_{r=1}^{k} (x - b_r)^{n_s}.$$

Theorem 6. *Suppose that $\boldsymbol{b} = (b_1,\dots,b_k)$ is a sequence of distinct complex numbers. Then, the Gaudin transfer matrix $\mathscr{H}(x)$ has a simple spectrum in $(\bigotimes_{s=1}^{k} W_{n_s}(b_s))^{\mathrm{sing}}$. There exists a bijective correspondence between the monic divisors y of the polynomial $\psi_{\boldsymbol{n},\boldsymbol{b}}$ and the eigenvectors v_y of the Gaudin transfer matrix $\mathscr{H}(x)$ (up to multiplication by a non-zero constant). Moreover, this bijection is such that*

$$\mathscr{H}(x)v_y = \Big(\frac{1}{2}\zeta'_{\boldsymbol{n},\boldsymbol{b}}(x) - \zeta_{\boldsymbol{n},\boldsymbol{b}}(x)\frac{y'(x)}{y(x)} + \frac{1}{2}\Big(\sum_{s=1}^{k}\frac{n_s}{x-b_s}\Big)\Big(\sum_{s=1}^{k}\frac{n_s}{x-b_s} + 2q_1\Big)\Big)v_y.$$

Remark 3. *Fix $l \in \mathbb{Z}_{\geqslant 0}$ and set $\boldsymbol{t} = (t_1, \dots, t_l)$. Let $y_{\boldsymbol{t}}$ represent $\boldsymbol{t}$. Then, the Bethe ansatz equation for $(\bigotimes_{s=1}^{k} W_{n_s}(b_s))$ is*

$$y_{\boldsymbol{t}}(x) \quad \text{divides the polynomial} \quad \psi_{\boldsymbol{n},\boldsymbol{b}}(x).$$

Note that, in this case, $y_{\boldsymbol{t}}$ may have multiple roots. If there are multiple roots in $y_{\boldsymbol{t}}$, then the corresponding on-shell Bethe vector is zero. Therefore, an actual eigenvector should be obtained via an appropriate derivative as pointed out in [7] [Section 8.2].

5.3. The Third Isomorphism

Recall from Section 2.3 that, without a loss of generality, we can assume that $\beta_s = 0$, $1 \leqslant s \leqslant k$. In this case, $\alpha_s = n_s$, $1 \leqslant s \leqslant k$.

Lemma 19. *There exists a surjective $\mathfrak{gl}(1|1)[t]$-module homomorphism from $\bigotimes_{s=1}^{k} W_{n_s}(b_k)$ to $\bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_k)$ that maps a vacuum vector to a vacuum vector.*

Proof. It follows from Lemma 10 and our assumption that $\beta_s = 0$ for all $1 \leqslant s \leqslant k$. □

By Lemma 9, the surjective $\mathfrak{gl}(1|1)[t]$-module homomorphism

$$\bigotimes_{s=1}^{k} W_{n_s}(b_k) \twoheadrightarrow \bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_k)$$

induces a surjective $\mathfrak{gl}(1|1)[t]$-module homomorphism

$$\mathcal{V}^{\mathfrak{S}} \twoheadrightarrow \bigotimes_{s=1}^{k} L_{\boldsymbol{\lambda}^{(s)}}(b_k).$$

The second map then induces a projection of the Bethe algebras $\mathcal{B}_l \twoheadrightarrow \mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$. We describe the kernel of this projection. We consider the corresponding ideal in the algebra $\mathcal{O}_l$.

Suppose that $l \leqslant k$. Define the polynomial $h(x)$ by

$$h(x) = \prod_{s=1}^{k} (x - b_s)^{n_s - 1}.$$

Divide the polynomial $g(x)$ in (21) by $h(x)$ and let

$$p(x) = x^{k-l} + p_1 x^{k-l-1} + \dots + p_{k-l-1} x + p_{k-l}, \tag{34}$$

$$r(x) = r_1 x^{n-k-1} + r_2 x^{n-k-2} + \dots + r_{n-k-1} x + r_{n-k} \tag{35}$$

be the quotient and the remainder, respectively. Clearly, $p_i, r_j \in \mathcal{O}_l$.

Denote by $I^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ the ideal of $\mathcal{O}_l$ generated by $r_1, \dots, r_{n-k}$ and the coefficients of polynomial

$$\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x) - (q_1 - q_2) p(x) f(x) = \prod_{s=1}^{k} (x - b_s) \Big(q_1 - q_2 + \sum_{s=1}^{k} \frac{n_s}{x - b_s} \Big) - (q_1 - q_2) p(x) f(x).$$

Let $\mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ be the quotient algebra

$$\mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \mathcal{O}_l / I^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}.$$

Clearly, if $\mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is finite-dimensional, then it is a Frobenius algebra.

Let $I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ be the image of $I^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ under the isomorphism η_l.

Lemma 20. *The ideal $I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is contained in the kernel of the projection $\mathcal{B}_l \twoheadrightarrow \mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$.*

Proof. We treat $\boldsymbol{b} = (b_1, \dots, b_k)$ as variables. Note that the elements of $I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ act on $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ as polynomials in $\boldsymbol{b}$ with values in $\mathrm{End}((L_{\boldsymbol{\Lambda}})_{(n-l,l)})$. Therefore it suffices to show it for generic $\boldsymbol{b}$. Let $\mathfrak{f}(x)$ be the image of $f(x)$ under η_l. The condition that $I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ vanishes is equivalent to the condition that $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x)$ is divisible by $\mathfrak{f}(x)$.

By Theorem 2, there exists an eigenbasis of the operator $\mathcal{T}(x)$ in $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ for generic $\boldsymbol{b}$. Clearly, a solution of the Bethe ansatz equation associated to $\boldsymbol{\Lambda}, \boldsymbol{b}, l$ is also a solution to the Bethe ansatz equation for $\mathcal{M}_{l,\boldsymbol{a}}$; see Theorem 6 and Remark 3. Moreover, the expressions of corresponding on-shell Bethe vectors coincide (with different vacuum vectors). By Lemma 19 and Theorems 1 and 6, $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x)$ is divisible by $\mathfrak{f}(x)$ for generic $\boldsymbol{b}$ since the eigenvalue of $\mathfrak{f}(x)$ corresponds to $y_{\boldsymbol{t}}(x)$ in (12). Therefore, $I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ vanishes for generic $\boldsymbol{b}$, thus completing the proof. □

Therefore, we have the epimorphism

$$\mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \cong \mathcal{B}_l / I^{\mathcal{B}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \twoheadrightarrow \mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}. \tag{36}$$

We claim that the surjection in (36) is an isomorphism by checking $\dim \mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \dim \mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$.

Lemma 21. *We have* $\dim \mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \binom{k}{l}$.

Proof. Note that $\mathbb{C}[p_1, \dots, p_{k-l}, r_1, \dots, r_{n-k}] \cong \mathbb{C}[g_1, \dots, g_{n-l}]$, where p_i and r_j are defined in (34) and (35). It is not hard to check that

$$\mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \cong \mathbb{C}[f_1, \dots, f_l, p_1, \dots, p_{k-l}] / \tilde{I}^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}, \tag{37}$$

where $\tilde{I}^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is the ideal of $\mathbb{C}[f_1, \dots, f_l, p_1, \dots, p_{k-l}]$ generated by the coefficients of the polynomial $\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x) - (q_1 - q_2)p(x)f(x)$.

Introduce new variables $\boldsymbol{w} = (w_1, \dots, w_k)$ such that

$$f(x) = \prod_{i=1}^{l}(x - w_i), \quad p(x) = \prod_{i=1}^{k-l}(x - w_{l+i}).$$

Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$ be complex numbers such that

$$\varphi_{\boldsymbol{\Lambda},\boldsymbol{b}}(x) = \prod_{s=1}^{k}(x - b_s)\Big(q_1 - q_2 + \sum_{s=1}^{k} \frac{n_s}{x - b_s}\Big) = (q_1 - q_2)\Big(x^k + \sum_{i=1}^{k}(-1)^i \varepsilon_i x^{k-i}\Big).$$

Then,

$$\mathbb{C}[f_1, \dots, f_l, p_1, \dots, p_{k-l}] / \tilde{I}^{\mathcal{O}}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \cong \mathbb{C}[w_1, \dots, w_{k-1}]^{\mathfrak{S}_l \times \mathfrak{S}_{k-l}} / \langle \sigma_i(\boldsymbol{w}) - \varepsilon_i \rangle_{i=1,\dots,k}. \tag{38}$$

The lemma now follows from the fact that $\mathbb{C}[w_1, \dots, w_{k-1}]^{\mathfrak{S}_l \times \mathfrak{S}_{k-l}}$ is a free $\mathbb{C}[w_1, \dots, w_{k-1}]^{\mathfrak{S}}$-module of rank $\binom{k}{l}$. □

Note that we have the projection $(\mathcal{V}^{\mathfrak{S}})_{(n-l,l)} \twoheadrightarrow \mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$. Since, by Theorem 3, the Bethe algebra $\mathcal{B}_l$ acts on $(\mathcal{V}^{\mathfrak{S}})_{(n-l,l)}$ cyclically, the Bethe algebra $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ acts on $\mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ cyclically as well. Therefore, we have

$$\dim \mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \dim \mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} = \binom{k}{l}. \tag{39}$$

Proof of Theorem 4. Part (i) follows from Lemma 21 and (36)–(39). Clearly, we have that $\mathcal{B}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \cong \mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$ is a Frobenius algebra. Moreover, the map ρ_l from Theorem 3 induces a map

$$\rho_{l,\boldsymbol{\Lambda},\boldsymbol{b}} : \mathcal{O}_{l,\boldsymbol{\Lambda},\boldsymbol{b}} \to \mathcal{M}_{l,\boldsymbol{\Lambda},\boldsymbol{b}}$$

that identifies the regular representation of $\mathcal{O}_{l,\mathbf{\Lambda},\boldsymbol{b}}$ with the $\mathcal{B}_{l,\mathbf{\Lambda},\boldsymbol{b}}$-module $\mathcal{M}_{l,\mathbf{\Lambda},\boldsymbol{b}}$. Therefore, part (ii) is proved.

Since $\mathcal{B}_{l,\mathbf{\Lambda},\boldsymbol{b}}$ is a Frobenius algebra, the regular and coregular representations of the algebra $\mathcal{B}_{l,\mathbf{\Lambda},\boldsymbol{b}}$ are isomorphic to each other. Parts (iii)–(vi) follow from the general facts about the coregular representations; see, e.g., [1] [Section 3.3] or [13] [Lemma 1.3].

Due to part (iv), it suffices to consider the algebraic multiplicity of every eigenvalue. It is well known that roots of a polynomial continuously depend on its coefficients. Hence, the eigenvalues of $\mathcal{T}(x)$ continuously depend on $\boldsymbol{b}$. Part (vii) follows from the deformation argument and Theorem 2. □

6. Conclusions

In this paper, we investigated the $\mathfrak{gl}(1|1)$ Gaudin models that are twisted by a diagonal matrix G and defined on tensor products of polynomial evaluation $\mathfrak{gl}(1|1)[t]$-modules. Our results generalize all of the results of [10] to the twisted case. Meanwhile, we gave an explicit description of the algebra of Hamiltonians acting on tensor products of polynomial evaluation $\mathfrak{gl}(1|1)[t]$-modules by generators and relations. Moreover, we showed that there exists a bijection between common eigenvectors (up to proportionality) of the algebra of Hamiltonians and monic divisors of an explicit polynomial written in terms of the highest weights and evaluation parameters. In particular, our result implies that each common eigenspace of the algebra of Hamiltonians has dimension one. We also gave dimensions of the generalized eigenspaces. Our results give a confirmed answer to the completeness of the Bethe ansatz in the case of $\mathfrak{gl}(1|1)$ Gaudin models. We expect our results tp be an essential step towards understanding the more general $\mathfrak{gl}(m|n)$ Gaudin models.

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References

1. Mukhin, E.; Tarasov, V.; Varchenko, A. Schubert calculus and representations of general linear group. *J. Amer. Math. Soc.* **2009**, *22*, 909–940. [CrossRef]
2. Halacheva, I.; Kamnitzer, J.; Rybnikov, L.; Weekes, A. Crystals and monodromy of Bethe vectors. *Duke Math. J.* **2020**, *169*, 2337–2419. [CrossRef]
3. Kulish, P.; Manojlovic, N. Bethe vectors of the $\mathfrak{osp}(1|2)$ Gaudin model. *Lett. Math. Phys.* **2001**, *55*, 77–95. [CrossRef]
4. Molev, A.; Ragoucy, E. The MacMahon Master Theorem for right quantum superalgebras and higher Sugawara operators for $\widehat{\mathfrak{gl}}(m|n)$. *Moscow Math. J.* **2014**, *14*, 83–119. [CrossRef]
5. Mukhin, E.; Vicedo, B.; Young, C. Gaudin models for $\mathfrak{gl}(m|n)$. *J. Math. Phys.* **2015**, *56*, 051704. [CrossRef]
6. Zeitlin, A. Superopers on Supercurves. *Lett. Math. Phys.* **2015**, *105*, 149–167. [CrossRef]
7. Huang, C.-L.; Mukhin, E.; Vicedo, B.; Young, C. The solutions of $\mathfrak{gl}(m|n)$ Bethe ansatz equation and rational pseudodifferential operators. *Sel. Math. New Ser.* **2019**, *25*, 52. [CrossRef]
8. Huang, C.-L.; Mukhin, E. The duality of $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ Gaudin models. *J. Algebra* **2020**, *548*, 1–24. [CrossRef]
9. Lu, K.; Mukhin, E. Bethe ansatz equations for orthosymplectic Lie superalgebra and self-dual superspaces. *Ann. Henri Poincaré* **2021**, *22*, 4087–4130. [CrossRef]
10. Lu, K. Completeness of Bethe ansatz for Gaudin models associated with $\mathfrak{gl}(1|1)$. *Nuclear Phys. B* **2022**, *980*, 115790. [CrossRef]
11. Lu, K.; Mukhin, E. On the supersymmetric XXX spin chains associated with $\mathfrak{gl}_{1|1}$. *Commun. Math. Phys.* **2021**, *386*, 711–747. [CrossRef]
12. Mukhin, E.; Tarasov, V.; Varchenko, A. Spaces of quasi-exponentials and representations of $\mathfrak{gl}_N$. *J. Phys A: Math. Theor.* **2008**, *41*, 194017. [CrossRef]
13. Lu, K. Perfect integrability and Gaudin models. *SIGMA Symmetry Integrability Geom. Methods Appl.* **2020**, *16*, 132. [CrossRef]
14. Nazarov, M. Quantum Berezinian and the classical capelli identity. *Lett. Math. Phys.* **1991**, *21*, 123–131. [CrossRef]
15. Molev, A.; Mukhin, E. Invariants of the vacuum module associated with the Lie superalgebra $\mathfrak{gl}(1|1)$. *J. Phys A: Math. Theor.* **2015**, *48*, 314001. [CrossRef]
16. Lu, K. On Bethe eigenvectors and higher transfer matrices for supersymmetric spin chains. *arXiv* **2022**, arXiv:2209.14416.

17. Lu, K.; Mukhin, E. Jacobi-Trudi identity and Drinfeld functor for super Yangian. *Int. Math. Res. Not. IMRN* **2021**, *2021*, 16749–16808. [CrossRef]
18. Huang, C.-L.; Lu, K.; Mukhin, E. Solutions of $\mathfrak{gl}(m|n)$ XXX Bethe ansatz equation and rational difference operators. *J. Phys. A* **2019**, *52*, 375204. [CrossRef]
19. Belliard, S.; Ragoucy, E. The nested Bethe ansatz for 'all' closed spin chains. *J. Phys A Math. Theor.* **2008**, *41*, 295202. [CrossRef]
20. Mukhin, E.; Tarasov, V.; Varchenko, A. Bethe eigenvectors of higher transfer matrices. *J. Stat. Mech. Theor. Exp.* **2006**, *2006*, P08002. [CrossRef]
21. Kulish, P.; Reshetikhin, N.Y. Diagonalization of $\mathrm{GL}(N)$ invariant transfer-matrices and quantum N-wave system (Lee model). *J. Phys A: Math. Theor.* **1983**, *15*, L591–L596. [CrossRef]
22. Lu, K. A note on odd reflections of super Yangian and Bethe ansatz. *Lett. Math. Phys.* **2022**, *112*, 1–26. [CrossRef]
23. Nazarov, M. Yangian of the General Linear Lie Superalgebra. *SIGMA Symmetry Integrability Geom. Methods Appl.* **2020**, *16*, 112. [CrossRef]