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On a Certain Subclass of p-Valent Analytic Functions Involving q-Difference Operator

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Abstract: This paper introduces and studies a new class of analytic p-valent functions in the open symmetric unit disc involving the Sălăgean-type q-difference operator. Furthermore, we present several interesting subordination results, coefficient inequalities, fractional q-calculus applications, and distortion theorems.

Keywords: analytic functions; q-difference operator; q-binomial theorem; Sălăgean differential operator; fractional q-calculus operators; q-Bernardi integral operator

MSC: 30C45; 30C50; 30C55; 30C80

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1. Introduction

As a result of Euler and Heine's pioneering work, Frank Hilton Jackson developed q-calculus in a systematic manner at the beginning of the previous century. In his work, Jackson systematically developed the concepts of the q-derivative (Jackson [1]), as well as the q-integral (Jackson [2]). Calculus without limits is called q-calculus. Due to its applications in mathematics, mechanics, and physics, symmetric q-calculus is experiencing rapid growth. Ismail et al. [3] were the first to apply q-calculus to geometric function theory (GFT) by generalizing the set of starlike functions into q-analogs, called q-starlike functions. Several authors have extensively investigated the q-difference operator in GFT based on the same idea. Some recent works related to this operator on analytic functions include [4–22]. Several properties of certain analytic multivalent functions are considered in this paper using the q-analog of the Sălăgean differential operator. Let $\mathcal{A}_p(j)$ denote the class of functions that have the form

$$f(z) = z^p + \sum_{l=p+j}^{\infty} a_l z^l, \qquad (p, j \in \mathbb{N} := \{1, 2, \dots\}), \tag{1}$$

that are analytic in the open unit disc $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A} = \mathcal{A}_1(1)$. In [1,2], the q-derivative operator ∂_q of a function f was defined by Jackson as follows:

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0), \\ f'(0) & (z = 0). \end{cases} \tag{2}$$

For a function $f(z) \in \mathcal{A}_p(j)$, we deduce that

$$\partial_q f(z) = [p]_q z^{p-1} + \sum_{l=2}^{\infty} [l]_q a_l z^{l-1}, \tag{3}$$

where

$$[l]_q = \frac{1-q^l}{1-q}.$$

As $q \to 1^-$, $[l]_q \to l$. Jackson [1] introduced the q-integral

$$\int_0^z f(t)d_qt = z(1-q)\sum_{l=0}^{\infty} q^l f(zq^l),$$

as long as the series converges. For a function $f(z) = z^l$, one can observe that

$$\int_0^z f(t)d_qt = \int_0^z t^l d_qt = \frac{1}{[l+1]_q} z^{l+1} \qquad (l \neq -1).$$

For a function $f(z) \in \mathcal{A}_p(j)$, El-Qadeem and Mamon [23] defined the p-valent q-Sălăgean operator by

$$\begin{aligned} D^0_{p,q}f(z) &= f(z), \\ D^1_{p,q}f(z) &= D_{p,q}f(z) = \frac{z\partial_q f(z)}{[p]_q} = z^p + \sum_{l=p+j}^{\infty} a_l \frac{[l]_q}{[p]_q} z^l, \\ D^2_{p,q}f(z) &= D_{p,q}\left(D_{p,q}f(z)\right) = z^p + \sum_{l=p+j}^{\infty} a_l \left(\frac{[l]_q}{[p]_q}\right)^2 z^l, \end{aligned}$$

therefore,

$$D^n_{p,q}f(z) = D_{p,q}\left(D^{n-1}_{p,q}f(z)\right) = z^p + \sum_{l=p+j}^{\infty} a_l \left(\frac{[l]_q}{[p]_q}\right)^n z^l. \tag{4}$$

When $p = 1$, the q-Sălăgean operator was introduced by Govindaraj and Sivasubramanian [24]. The q-shifted factorial, see [25], is defined for $a \in \mathbb{C}$ by

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), & \text{if } n \in \mathbb{N} = \{1,2,\dots\}, \end{cases}$$

let $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$. Recalling the q-analog definitions given by Gasper and Rahman [26], the q-Gamma function is given by

$$\Gamma_q(z) = \frac{(q,q)_\infty}{(q^z,q)_\infty}(1-q)^{1-z} \quad (0 < q < 1),$$

and the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu\left(\frac{y}{x},q\right)_\nu = x^\nu \prod_{n=0}^{\infty} \frac{1-\left(\frac{y}{x}\right)q^n}{1-\left(\frac{y}{x}\right)q^{n+\nu}}.$$

For functions f and g analytic in $\mathbb{E}$, one can say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{E})$, if there exists a Schwarz function ω, that is analytic in $\mathbb{E}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ $(z \in \mathbb{E})$. In addition, if the function g is univalent in $\mathbb{E}$, then the following equivalence will occur

$$f(z) \prec g(z) \Leftrightarrow f(0) \prec g(0),$$

and

$$f(\mathbb{E}) \subset g(\mathbb{E}).$$

For functions $f_j(z) = \sum_{l=0}^{\infty} a_{l,j} z^l$ $(j = 1, 2)$ analytic in $\mathbb{E}$, the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = \sum_{l=0}^{\infty} a_{l,1} a_{l,2} z^l = (f_2 * f_1)(z) \quad (z \in \mathbb{E}).$$

A function $f \in \mathcal{A}$ is convex, if and only if $f(\mathbb{E})$ is a convex domain. We denote this subclass of $\mathcal{A}$ by K. Analytically, a function $f \in \mathcal{A}$ belongs to the class K if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{E}).$$

The proof can be found in [27]. A similar characterization can be made for the class S^* of functions starlike in $\mathbb{E}$. A function $f \in \mathcal{A}$ belongs to the class S^* if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{E}).$$

More details on the classes of starlike and convex functions can be found in [28,29]. A univalent function $f : \mathbb{E} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is said to be concave if the complement $\hat{\mathbb{C}} \backslash f$ is convex (functions mapping on the exterior of a convex curve). An analytic, univalent function $f \in \mathcal{A}$ is said to be in the class $C_o(\alpha)$, if it is concave, satisfies $f(1) = \infty$ with an opening angle of $f(\mathbb{E})$ at ∞ less than or equal to $\alpha\pi$ with $\alpha \in (1, 2]$. Due to the similarity with convex functions, sometimes the inequality

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < 0 \quad (z \in \mathbb{E}),$$

is also used as a definition for concave functions (see e.g., [30]) see also, Avkhadiev et al. [31], Cruz and Pommerenke [32], and the references within. Recently, Nishiwaki and Owa [33] defined and studied the subclasses $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ of $\mathcal{A}$ as follows: for some $\beta(\beta > 1)$, let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$, which satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \quad (z \in \mathbb{E}),$$

let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$, which satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{E}),$$

(see [33–36]). With the use of the differential operator $D^n_{p,q}$, we introduce class $\mathcal{A}_{p,q}(n, j, \beta)$, which generalizes the above-mentioned classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$.

Definition 1. *We say that a function $f(z) \in \mathcal{A}_p(j)$ belongs to the class $\mathcal{A}_{p,q}(n, j, \beta)$, if it satisfies the condition*

$$\Re\left\{\frac{z\partial_q(D^n_{p,q} f(z))}{D^n_{p,q} f(z)}\right\} < \beta \quad (z \in \mathbb{E}), \tag{5}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, $\beta > [p]_q$, and $0 < q < 1$.

As $f(z) = z^p$ belongs to the class $\mathcal{A}_{p,q}(n,j,\beta)$, it is not empty. $\mathcal{A}_{p,q}(n,j,\beta)$ generalizes the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ as follows

Remark 1. *1.* $\lim_{q\to 1}\mathcal{A}_{1,q}(0,1,\beta) = \mathcal{M}(\beta)$;

2. $\lim_{q\to 1}\mathcal{A}_{1,q}(1,1,\beta) = \mathcal{N}(\beta)$.

In this paper, we derive some interesting subordination results, coefficient inequalities, and various distortion theorems involving fractional q-calculus operators for functions in the class $\mathcal{A}_{p,q}(n,j,\beta)$. Moreover, some special cases are also indicated.

2. Coefficient Estimates

Theorem 1. *If $f(z) \in \mathcal{A}_p(j)$ satisfies the condition*

$$\sum_{l=p+j}^{\infty}\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q - [p]_q + \left|[l]_q + [p]_q - 2\beta\right|\right)|a_l| \le 2\left(\beta - [p]_q\right), \tag{6}$$

for some $\beta\left(\beta > [p]_q\right)$; $n \in \mathbb{N}_0$, then $f(z) \in \mathcal{A}_{p,q}(n,j,\beta)$.

Proof. Let condition (6) be true. Then, we have

$$\begin{aligned}
&\left|\frac{\frac{z\partial_q(D^n_{p,q}f(z))}{D^n_{p,q}f(z)} - [p]_q}{\frac{z\partial_q(D^n_{p,q}f(z))}{D^n_{p,q}f(z)} - \left(2\beta - [p]_q\right)}\right| \\
= &\left|\frac{\sum_{l=p+j}^{\infty} a_l\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q - [p]_q\right)z^l}{-2\left(\beta - [p]_q\right)z^p + \sum_{l=p+j}^{\infty} a_l\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q + [p]_q - 2\beta\right)z^l}\right| \\
\le &\frac{|z|^{p+j}\sum_{l=p+j}^{\infty}|a_l|\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q - [p]_q\right)}{2\left(\beta - [p]_q\right) - |z|^{p+j}\sum_{l=p+j}^{\infty}|a_l|\left(\frac{[l]_q}{[p]_q}\right)^n\left|[l]_q + [p]_q - 2\beta\right|} \\
\le &\frac{\sum_{l=p+j}^{\infty}|a_l|\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q - [p]_q\right)}{2\left(\beta - [p]_q\right) - \sum_{l=p+j}^{\infty}|a_l|\left(\frac{[l]_q}{[p]_q}\right)^n\left|[l]_q + [p]_q - 2\beta\right|}
\end{aligned}$$

the last expression is bounded above by 1 if

$$\sum_{l=p+j}^{\infty}\left(\frac{[l]_q}{[p]_q}\right)^n\left([l]_q - [p]_q + \left|[l]_q + [p]_q - 2\beta\right|\right)|a_l| \le 2\left(\beta - [p]_q\right).$$

This completes the proof of Theorem 1. □

Remark 2. *Letting $p = 1, q \to 1$, and $n = 1$ in Theorem 1, we obtain the result obtained by Nishwaki and Owa [33].*

Corollary 1. *If $f(z) \in \mathcal{A}_p(j)$ satisfies the condition*

$$\sum_{l=p+j}^{\infty} \left(\frac{[l]_q}{[p]_q} \right)^n \left([l]_q - \beta \right) |a_l| \leq \left(\beta - [p]_q \right),$$

for some $\beta\left([p]_q \leq \beta < \frac{[p+j]_q + [p]_q}{2} \right); n \in \mathbb{N}_0$, then $f(z) \in \mathcal{A}_{p,q}(n, j, \beta)$.

Proof. Since $([l]_q + [p]_q - 2\beta)$ is an increasing function of $l(l \geq p + j)$, we have

$$[l]_q + [p]_q - 2\beta \geq 0 \; if \; [p+j]_q + [p]_q - 2\beta \geq 0,$$

or

$$\beta \leq \frac{[p+j]_q + [p]_q}{2}.$$

□

3. Subordination Results

Definition 2 ([37]). *A sequence $\{b_l\}_{l=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if, whenever $f(z) = z + \sum_{l=2}^{\infty} a_l z^l$, $a_1 = 1$ is analytic, univalent, and convex in $\mathbb{E}$, we have*

$$\sum_{l=1}^{\infty} b_l a_l z^l \prec f(z) \quad (z \in \mathbb{E}).$$

Lemma 1 ([37]). *The sequence $\{b_l\}_{l=1}^{\infty}$ is subordinating factor sequence if and only if*

$$\Re\left(1 + 2 \sum_{l=1}^{\infty} b_l z^l \right) > 0 \quad (z \in \mathbb{E}).$$

Let $\mathcal{A}^*_{p,q}(n, j, \beta)$ denoted the class of functions $f(z) \in \mathcal{A}_p(j)$ whose coefficients satisfy the condition (6).

Theorem 2. *Let $f(z) \in \mathcal{A}^*_{p,q}(n, j, \beta)$, $g(z) \in K$, and*

$$\varepsilon = \frac{\left(\frac{[p+j]_q}{[p]_q} \right)^n \left(q^p [j]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right)}{2\left\{ \left(\frac{[p+j]_q}{[p]_q} \right)^n \left(q^p [j]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right) + \left(\beta - [p]_q \right) \right\}},$$

then

$$\left(\varepsilon z^{1-p} f(z) \right) * g(z) \prec g(z) \;\; (z \in \mathbb{E}), \tag{7}$$

and

$$\Re\left(\frac{f(z)}{z^{p-1}} \right) > \frac{-1}{2\varepsilon}. \tag{8}$$

The constant

$$\frac{\left(\frac{[p+j]_q}{[p]_q} \right)^n \left(q^p [j]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right)}{2\left\{ \left(\frac{[p+j]_q}{[p]_q} \right)^n \left(q^p [j]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right) + \left(\beta - [p]_q \right) \right\}},$$

is the best estimate.

Proof. Let $f(z) \in \mathcal{A}^{*}_{p,q}(n,j,\beta)$, and let $g(z) = z + \sum_{l=2}^{\infty} c_l z^l$ belong to the subclass K. Then

$$\left(\varepsilon z^{1-p} f(z)\right) * g(z) = \sum_{l=1}^{\infty} b_l c_l z^l \quad (z \in \mathbb{E}),$$

where

$$b_l = \begin{cases} \varepsilon & (l = 1), \\ 0 & (2 \le l \le j), \\ \varepsilon a_{p+l-1} & (l \ge j+1). \end{cases}$$

Hence, by using Definition 2, the subordination result (7) will be true, if $\{b_l\}_{l=1}^{\infty}$ is the subordinating factor sequence. Since

$$\Psi(l) = \left(\frac{[l]_q}{[p]_q}\right)^n \left([l]_q - [p]_q + \left|[l]_q + [p]_q - 2\beta\right|\right), \tag{9}$$

is an increasing function of $l (l \ge j+1)$, we have

$$\begin{aligned}
&\Re\left\{1 + 2\sum_{l=1}^{\infty} b_l z^l\right\} = \Re\left\{1 + 2\varepsilon z + 2\sum_{l=j+1}^{\infty} b_l z^l\right\} \\
=\ & \Re\left\{1 + \frac{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right)}{\left\{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)\right\}} z \right.\\
&+ \frac{1}{\left\{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)\right\}} \\
&\left.\times \sum_{l=j+p}^{\infty} \left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) a_l z^{l+1-p}\right\} \\
\ge\ & \Re\left\{1 - \frac{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right)}{\left\{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)\right\}} r \right.\\
&\left. - \sum_{l=j+p}^{\infty} \frac{\left(\frac{[l]_q}{[p]_q}\right)^n \left([l]_q - [p]_q + \left|[l]_q + [p]_q - 2\beta\right|\right)}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)} |a_l| r^{j+1}\right\}
\end{aligned}$$

Thus, by using Theorem 1, and Lemma 1 we deduce that

$$\begin{aligned}
&\Re\left\{1 + 2\sum_{l=1}^{\infty} b_l z^k\right\} \\
\ge\ & 1 - \frac{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) r}{\left\{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)\right\}} \\
&- \frac{\left(\beta - [p]_q\right)}{\left\{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left(q^p[j]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right) + \left(\beta - [p]_q\right)\right\}} r \\
>\ & 0.
\end{aligned}$$

This proves the subordination result (7). Letting $g(z) = \frac{z}{1-z} = \sum_{l=1}^{\infty} z^l \ \ (z \in \mathbb{E})$ in (7), we easily get the result (8). □

Theorem 3. *Let $f(z)$ be in the class $\mathcal{A}^*_{p,q}(n,j,\beta)$, defined by (1). Then for $|z| = r < 1$, we have*

$$\left(\frac{[p]_q!}{[p-m]_q!} - \frac{2\left(\beta - [p]_q\right) \frac{[p+j]_q!}{[p+j-m]_q!} r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left\{ [p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right\}} \right) r^{p-m}$$
$$\leq \quad \left| \partial_q^m (f(z)) \right| \leq$$
$$\left(\frac{[p]_q!}{[p-m]_q!} + \frac{2\left(\beta - [p]_q\right) \frac{[p+j]_q!}{[p+j-m]_q!} r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left\{ [p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right\}} \right) r^{p-m}.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{2\left(\beta - [p]_q\right) \frac{[p+j]_q!}{[p+j-m]_q!}}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left\{ [p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right\}} z^{p+j}. \tag{10}$$

Proof. Since $\Psi(l)$ given by (9) is an increasing function of $l (l \geq j+1)$, Theorem 1 gives

$$\left(\frac{[p+j]_q}{[p]_q}\right)^n \left([p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right) \sum_{l=p+j}^{\infty} |a_l|$$
$$\leq \sum_{l=p+j}^{\infty} \left(\frac{[l]_q}{[p]_q}\right)^n \left([l]_q - [p]_q + \left| [l]_q + [p]_q - 2\beta \right| \right) |a_l| \leq 2\left(\beta - [p]_q\right).$$

That is

$$\sum_{l=p+j}^{\infty} |a_l| \leq \frac{2\left(\beta - [p]_q\right)}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left([p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right)}. \tag{11}$$

The m^{th} q-derivative of the functions $f(z) \in \mathcal{A}_p(j)$ is given by

$$\partial_q^m (f(z)) = \frac{[p]_q!}{[p-m]_q!} z^{p-m} + \sum_{l=p+j}^{\infty} \frac{[l]_q!}{[l-m]_q!} a_l z^{l-m},$$

then we have

$$\begin{aligned} \left| \partial_q^m (f(z)) \right| &\geq \frac{[p]_q!}{[p-m]_q!} |z|^{p-m} - \sum_{l=p+j}^{\infty} \frac{[l]_q!}{[l-m]_q!} |a_l| |z|^{l-m} \\ &\geq \frac{[p]_q!}{[p-m]_q!} r^{p-m} - r^{p+j-m} \frac{[p+j]_q!}{[p+j-m]_q!} \sum_{l=p+j}^{\infty} |a_l| \end{aligned}$$

$$\geq \left(\frac{[p]_q!}{[p-m]_q!} - \frac{2\left(\beta - [p]_q\right) \frac{[p+j]_q!}{[p+j-m]_q!} r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n \left([p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right)} \right) r^{p-m},$$

and

$$\begin{aligned}\left|\partial_q^m(f(z))\right| &\leq \frac{[p]_q!}{[p-m]_q!}|z|^{p-m}+\sum_{l=p+j}^{\infty}\frac{[l]_q!}{[l-m]_q!}|a_l||z|^{l-m}\\ &\leq \frac{[p]_q!}{[p-m]_q!}r^{p-m}+\frac{[p+j]_q!}{[p+j-m]_q!}r^{p+j-m}\sum_{l=p+j}^{\infty}|a_l|\end{aligned}$$

$$\leq \left(\frac{[p]_q!}{[p-m]_q!}+\frac{2\left(\beta-[p]_q\right)\frac{[p+j]_q!}{[p+j-m]_q!}r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n\left([p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right)}\right)r^{p-m}.$$

□

Putting $m=0$ in Theorem 3 we have the following corollary

Corollary 2. *Let $f(z)$ defined by (1) be in the class $\mathcal{A}^*_{p,q}(n,j,\beta)$.Then for $|z|=r<1$, we have*

$$|f(z)|\geq\left(1-\frac{2\left(\beta-[p]_q\right)r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}}\right)r^p,$$

and

$$|f(z)|\leq\left(1+\frac{2\left(\beta-[p]_q\right)r^j}{\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}}\right)r^p.$$

This result is sharp.

4. Application of q-Fractional Calculus Operators

Let the function $f(z)$ be defined by (1). Then the q-Bernardi integral operator $\mathcal{J}^q_{c,p}$ is given by

$$\mathcal{J}^q_{c,p}f(z)=\frac{[c+p]_q}{z^c}\int_0^z t^{c-1}f(z)d_qt=z^p+\sum_{l=p+j}^{\infty}\frac{[c+p]_q}{[c+l]_q}a_lz^l\quad(c>-p),\tag{12}$$

this operator introduced by El-Qadeem and Mamon [23] (see also [17,38]). For $f(z)\in\mathcal{A}_p(j)$, we define the following q-fractional calculus operators given by Purohit and Raina [39,40].

Definition 3. *The fractional q-integral operator of order $m(m>0)$ is defined, for a function f, by*

$$\Omega^{-m}_{q,z}f(z)=\frac{1}{\Gamma_q(m)}\int_0^z(z-qt)_{m-1}f(t)d_qt,$$

where f is analytic in a simply-connected region of the z-plane containing the origin and the function $(z-qt)_{-m}$ is single-valued when $\left|\arg(-\frac{-tq^m}{z})\right|<\pi$, $\left|\frac{tq^m}{z})\right|<1$ and $|\arg z|<\pi$.

Definition 4. *The fractional q-derivative operator of order m is defined, for a function f, by*

$$\Omega^m_{q,z}(f(z))=\frac{1}{\Gamma_q(1-m)}\partial_q\int_0^z(z-qt)_{-m}f(t)d_qt\quad(1>m\geq0),$$

where f suitably constrained and removing the multiplicity of $(z-qt)_{-m}$ as in Definition 3 above.

Remark 3. *From Definitions 3 and 4, we see that*

$$\begin{aligned} \Omega_{q,z}^{m} z^{\gamma} &= \frac{\Gamma_q(\gamma+1)}{\Gamma_q(\gamma-m+1)} z^{\gamma-m} \quad (m \geq 0, \gamma > -1), \\ \Omega_{q,z}^{-m} z^{\gamma} &= \frac{\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+m+1)} z^{\gamma+m} \quad (m > 0, \gamma > -1). \end{aligned}$$

This gives that, for $f(z) \in \mathcal{A}_p(j)$,

$$\Omega_{q,z}^{-m} f(z) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} z^{p+m} + \sum_{l=p+j}^{\infty} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1+m)} a_l z^{l+m}, \tag{13}$$

and

$$\Omega_{q,z}^{m} f(z)) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)} z^{p-m} + \sum_{l=p+j}^{\infty} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1-m)} a_l z^{l-m}. \tag{14}$$

Using the formulas (13), (14), and (12), we have

$$\Omega_{q,z}^{-m} (\mathcal{J}_{c,p}^{q} f(z)) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} z^{p+m} + \sum_{l=p+j}^{\infty} \frac{[c+p]_q}{[c+l]_q} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1+m)} a_l z^{l+m}, \tag{15}$$

$$\Omega_{q,z}^{m} (\mathcal{J}_{c,p}^{q} f(z)) = \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)} z^{p-m} + \sum_{l=p+j}^{\infty} \frac{[c+p]_q}{[c+l]_q} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1-m)} a_l z^{l-m}, \tag{16}$$

$$\begin{aligned} \mathcal{J}_{c,p}^{q} (\Omega_{q,z}^{-m} f(z)) &= \frac{[c+p]_q}{[c+p+m]_q} \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} z^{p+m} \\ &+ \sum_{l=p+j}^{\infty} \frac{[c+p]_q}{[c+l+m]_q} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1+m)} a_l z^{l+m}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mathcal{J}_{c,p}^{q} (\Omega_{q,z}^{m} f(z)) &= \frac{[c+p]_q}{[c+p-m]_q} \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)} z^{p-m} \\ &+ \sum_{l=p+j}^{\infty} \frac{[c+p]_q}{[c+l-m]_q} \frac{\Gamma_q(l+1)}{\Gamma_q(l+1-m)} a_l z^{l-m}. \end{aligned} \tag{18}$$

Here, we investigate the distortion properties of functions in the class $\mathcal{A}_{p,q}^{*}(n,j,\beta)$ involving the operators $\mathcal{J}_{q,z}^{q}$, $\Omega_{q,z}^{-m}$, and $\Omega_{q,z}^{m}$.

Theorem 4. *Let $f(z)$ be in the class $\mathcal{A}_{p,q}(n,j,\beta)$, defined by (1). Then for $|z| = r < 1$, we have*

$$\left| \Omega_{q,z}^{-m} (\mathcal{J}_{c,p}^{q} f(z)) \right| \geq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} - \Upsilon_{p,q}^{c,j,m}(n,\beta) |z|^{j} \right\} |z|^{p+m}, \tag{19}$$

$$\left| \Omega_{q,z}^{-m} (\mathcal{J}_{c,p}^{q} f(z)) \right| \leq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} + \Upsilon_{p,q}^{c,j,m}(n,\beta) |z|^{j} \right\} |z|^{p+m}, \tag{20}$$

where

$$\Upsilon_{p,q}^{c,j,m}(n,\beta) = \frac{[c+p]_q}{[c+p+j]_q} \frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1+m) \left(\frac{[p+j]_q}{[p]_q} \right)^{n} \left\{ [p+j]_q - [p]_q + \left| [p+j]_q + [p]_q - 2\beta \right| \right\}},$$

$(0 \leq m < 1, c > -p, p \in \mathbb{N})$. *Each of the assertions are sharp for f given by (10).*

Proof. By using (11) and (15), we have

$$\begin{aligned}\left|\Omega_{q,z}^{-m}(\mathcal{J}_{c,p}^{q} f(z))\right| &\geq \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)}|z|^{p+m} - \sum_{l=p+j}^{\infty} \frac{[c+p]_q}{[c+l]_q} \frac{\Gamma_q(k+1)}{\Gamma_q(l+1+m)} |a_l||z|^{l+m} \\ &\geq \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)}|z|^{p+m} - \frac{[c+p]_q}{[c+p+j]_q} \frac{\Gamma_q(p+j+1)}{\Gamma_q(p+j+1+m)} |z|^{p+j+m} \sum_{l=p+j}^{\infty} |a_l| \\ &\geq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} - \Upsilon_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p+m},\end{aligned}$$

where

$$\Upsilon_{p,q}^{c,j,m}(n,\beta) = \frac{[c+p]_q}{[c+p+j]_q} \frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1+m)\left(\frac{[p+j]_q}{[p]_q}\right)^n \left\{[p+j]_q - [p]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right\}}.$$

Similarly, using (15) and (11) we have

$$\left|\Omega_{q,z}^{-m}(\mathcal{J}_{c,p}^{q} f(z))\right| \leq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} + \Upsilon_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p+m}.$$

Thus, the proof of the theorem is completed. □

Theorem 5. *Let $f(z)$ be in the class $\mathcal{A}_{p,q}(n,j,\beta)$, defined by (1). Then for $|z| = r < 1$, we have*

$$\left|\Omega_{q,z}^{m}(\mathcal{J}_{c,p}^{q} f(z))\right| \geq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)} - \Theta_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p-m}, \tag{21}$$

$$\left|\Omega_{q,z}^{m}(\mathcal{J}_{c,p}^{q} f(z))\right| \leq \left\{ \frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)} + \Theta_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p-m}, \tag{22}$$

where

$$\begin{aligned}&\Theta_{p,q}^{c,j,m}(n,\beta) \\ &= \frac{[c+p]_q}{[c+p+j]_q} \frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1-m)\left(\frac{[p+j]_q}{[p]_q}\right)^n \left\{[p+j]_q - [p]_q + \left|[p+j]_q + [p]_q - 2\beta\right|\right\}}.\end{aligned}$$

According to (10), each assertion is sharp.

Proof. Using (11) and (16), the assertions (21) and (22) of Theorem 5 can now be proved similarly to Theorem 4. □

Theorem 6. *Let $f(z)$ be in the class $\mathcal{A}_{p,q}(n,j,\beta)$, defined by(1). Then for $|z| = r < 1$, we have*

$$\left|\mathcal{J}_{c,p}^{q}(\Omega_{q,z}^{-m} f(z))\right| \geq \left\{ \frac{[c+p]_q}{[c+p+m]_q} \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} - \Lambda_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p+m}$$

$$\left|\mathcal{J}_{c,p}^{q}(\Omega_{q,z}^{-m} f(z))\right| \leq \left\{ \frac{[c+p]_q}{[c+p+m]_q} \frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)} + \Lambda_{p,q}^{c,j,m}(n,\beta)|z|^{j} \right\} |z|^{p+m},$$

where

$$\Lambda_{p,q}^{c,j,m}(n,\beta)=\frac{[c+p]_q}{[c+p+j]_q}\frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1+m)\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}},$$

$(m>0, c>-p, p\in\mathbb{N})$. *The result is sharp for the function f given by (10).*

Proof. We only prove the first inequality. The argument for the second inequality is similar and hence omitted. Using (17) and (11), we have

$$\begin{aligned}
&\left|\mathcal{J}_{c,p}^{q}(\Omega_{q,z}^{-m}f(z))\right| \\
\geq\ & \frac{[c+p]_q}{[c+p+m]_q}\frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)}|z|^{p+m}-\frac{[c+p]_q}{[c+p+j+m]_q}\frac{\Gamma_q(p+j+1)}{\Gamma_q(p+j+1+m)}|z|^{p+j+m}\sum_{l=p+j}^{\infty}|a_l| \\
\geq\ & \frac{[c+p]_q}{[c+p+m]_q}\frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)}|z|^{p+m} \\
&-\frac{[c+p]_q}{[c+p+j+m]_q}\frac{2\Gamma_q(p+j+1)|z|^{p+j+m}(\beta-[p]_q)}{\Gamma_q(p+j+1+m)\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}} \\
=\ & \left\{\frac{[c+p]_q}{[c+p+m]_q}\frac{\Gamma_q(p+1)}{\Gamma_q(p+1+m)}-\Lambda_{p,q}^{c,j,m}(n,\beta)|z|^j\right\}|z|^{p+m},
\end{aligned}$$

where

$$\Lambda_{p,q}^{c,j,m}(n,\beta)=\frac{[c+p]_q}{[c+p+j]_q}\frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1+m)\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}}.$$

□

Theorem 7. *Let $f(z)$ be in the class $\mathcal{A}_{p,q}(n,j,\beta)$, defined by (1). Then for $|z|=r<1$, we have*

$$\left|\mathcal{J}_{c,p}^{q}(\Omega_{q,z}^{m}f(z))\right|\geq\left\{\frac{[c+p]_q}{[c+p-m]_q}\frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)}-\Phi_{p,q}^{c,j,m}(n,\beta)|z|^j\right\}|z|^{p-m},$$

$$\left|\mathcal{J}_{c,p}^{q}(\Omega_{q,z}^{m}f(z))\right|\leq\left\{\frac{[c+p]_q}{[c+p-m]_q}\frac{\Gamma_q(p+1)}{\Gamma_q(p+1-m)}+\Phi_{p,q}^{c,j,m}(n,\beta)|z|^j\right\}|z|^{p-m},$$

where

$$\begin{aligned}
&\Phi_{p,q}^{c,j,m}(n,\beta)\\
=\ &\frac{[c+p]_q}{[c+p+j]_q}\frac{2\Gamma_q(p+j+1)(\beta-[p]_q)}{\Gamma_q(p+j+1+m)\left(\frac{[p+j]_q}{[p]_q}\right)^n\left\{[p+j]_q-[p]_q+\left|[p+j]_q+[p]_q-2\beta\right|\right\}},
\end{aligned}$$

$(0\leq m<1, c>-p, p\in\mathbb{N})$. *Each of the assertions are sharp for f given by (10).*

Proof. As the same manner in proving Theorem 6, we can easily deduce the proof of this theorem. □

5. Conclusions

Quantum calculus is classical calculus without limits. The field of q-calculus has recently attracted researchers' attention. Its application in various branches of mathematics and physics is responsible for this extraordinary interest. Jackson [1,2] was one of the first to define the q-analog to derivative the integral operators and provide some applications for them. Numerous subclasses of normalized analytic functions in the open symmetric unit disc associated with q-derivatives have already been investigated in geometric function theory. Using the Sălăgean q-difference operator, we introduce a new class of analytic p-valent functions in the open symmetric unit disc. Several subordination results, coefficient inequalities, fractional q-calculus applications, and distortion theorems are also presented. The paper also generalizes some known results. For future work, we can study some new classes of analytic p-valent functions in the open symmetric unit disc in the same way.

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