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Extension of a Unique Solution in Generalized Neutrosophic Cone Metric Spaces

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Abstract: In order to solve issues that arise in various branches of mathematical analysis, such as split feasibility problems, variational inequality problems, nonlinear optimization issues, equilibrium problems, complementarity issues, selection and matching problems, and issues proving the existence of solutions to integral and differential equations, fixed point theory provides vital tools. In this study, we discuss topological structure and several fixed-point theorems in the context of generalized neutrosophic cone metric spaces. In these spaces, the symmetric properties play an important role. We examine the existence and a uniqueness of a solution by utilizing new types of contraction mappings under some circumstances. We provide an example in which we show the existence and a uniqueness of a solution by utilizing our main result. These results are more generalized in the existing literature.

Keywords: cone metric space; intuitionistic fuzzy metric space; contraction mappings; fixed point; generalized cone metric space

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Citation: Ishtiaq, U.; Asif, M.; Hussain, A.; Ahmad, K.; Saleem, I.; Al Sulami, H. Extension of a Unique Solution in Generalized Neutrosophic Cone Metric Spaces. *Symmetry* **2023**, *15*, 94. https://doi.org/10.3390/sym15010094

Academic Editor: Wei-Shih Du

Received: 5 December 2022
Revised: 21 December 2022
Accepted: 24 December 2022
Published: 29 December 2022

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1. Introduction

The theory of fixed points has emerged as a very powerful and vital tool in the study of nonlinear phenomena over the past 100 years or so. Fixed point methods in particular have been used in a wide range of disciplines, including biology, chemistry, economics, engineering, game theory, computer science, physics, geometry, astronomy, fluid and elastic mechanics, physics, control theory, image processing and economics. The criteria for single or multivalued maps to admit fixed points $x = f(x)$, or inclusions of the form $x \in F(x)$ are given by fixed point theorems. The theory itself combines pure and applied analysis with topology and geometry. In 1912, the renowned Brouwer's fixed point theorem was validated. Since then, a number of fixed-point theorems have been validated under various circumstances. The Banach Contraction Principle (BCP), the first metric fixed-point theorem published by Stefan Banach a century ago, serves as an example of the unifying nature of functional analytic techniques and the practicality of fixed-point theory. The Banach contraction principle's key characteristic is that it specifies the presence, singularity, and order of successive approximations that converge to a solution to a problem. A shape is said to be symmetrical if it can be moved, rotated, or flipped without changing its appearance. An object is said to be asymmetrical if it lacks symmetry. In a metric space the symmetric property plays an important role in various applications including linear programming.

In 1965, Zadeh [1], made a great contribution to the field of mathematics by proposing the concept of fuzzy set, which deals with uncertainty or those problems that do have not any clear boundary. In the year 1986, Atanassov [2] extended the Zadeh's concept of fuzzy sets and introduced the notion of an intuitionistic fuzzy set, which have made a definite change and promoted the field of applied research. In 2002, Smarandache [3] proposed the concept of neutrosophic sets as a generalization of IFSs. These three ideas actually paved a great path that has led to several generalized metric spaces.

Huang and Zhang [4] defined the notion of cone metric spaces which generalized the notion of a metric spaces and established several fixed-point results for contraction mappings. In 2017, Mohamed and Ranjith [5] established the notion of intuitionistic fuzzy cone metric spaces (IFCMSs), which combined the notions of intuitionistic fuzzy sets and cone metric space. Recently, intuitionistic generalized fuzzy cone metric spaces (IGFCMSs) were introduced by Jeyaraman and Sowndrarajan [6] as a generalization of IFCMS and they extended the notion of a (φ, ψ)-weak contraction to IGFCMS by employing the idea of altering distance function. They also obtained common fixed-point theorems in IGCFMS.

Gregori and Sapena [7] established the notion of a fuzzy contractive mappings and used it to expand the Banach's fixed point theorem. Further, Ramachandran [8] generalized the Banach contraction theorem in the context of IGFCMS. Omeri et al. [9] introduced the notion of a neutrosophic cone metric space and derived several fixed-point results for contraction mappings. Omeri et al. [10] established a number of common fixed-point results in the sense of neutrosophic cone metric space. Additionally, the idea of changing the distance function is used to define the concept of $(\Phi,\ \Psi)$-weak contraction in the neutrosophic cone metric space (for more details see [11–19]). Recently, Riaz et al. [20] introduced the notions of generalized neutrosophic cone metric spaces (GNCMSs) and ξ-chainable neutrosophic cone metric spaces and established several common fixed-point results in both spaces. Several authors [21–27] have worked on different interesting applications including image encryption, image encryption based on a roulette-cascaded chaotic system and alienated image library and fractional and differential equations. Hamidi et al. [28] introduced the notion of KM-single valued neutrosophic metric spaces and established the several topological properties and provided its interesting applications.

In this manuscript, we aim to establish some fixed-point results in the context of GNCMSs. We examine the existence and a uniqueness of a solution by utilizing contraction mappings under some circumstances. We will provide an example in which we show the existence and a uniqueness of a solution by utilizing our main result.

2. Preliminaries

In this section, we discuss some important definitions which are helpful to understand the main results.

Definition 1 ([19])**.** *A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm (CTN) if it satisfies the following conditions:*

(i) $*$ *is associative and commutative;*
(ii) $*$ *is continuous;*
(iii) $\hbar * 1 = \hbar$ *for all* $\hbar \in [0, 1]$;
(iv) $\hbar * \ell \leq c * d$ *whenever* $\hbar \leq c$ *and* $\ell \leq d$, *for all* $\hbar, \ell, c, d \in [0, 1]$.

Definition 2 ([19])**.** *A binary operation $\circ$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-conorm (CTCN) if it meets the below assertions:*

T1 $\circ$ *is associative and commutative;*
T2 $\circ$ *is continuous;*
T3 $\hbar \circ 0 = 0,$ *for all* $\hbar \in [0, 1]$;
T4 $\hbar \circ \ell \leq c \circ d$ *whenever* $\hbar \leq c$ *and* $\ell \leq d$, *for all* $\hbar, \ell, c, d \in [0, 1]$.

Definition 3 ([4])**.** *Let E be a real Banach space and $\mathcal{C}$ be a subset of E.$\mathcal{C}$ is called a cone if only if*

(i) $\mathcal{C}$ *is closed, nonempty, and* $\mathcal{C} \neq 0$,
(ii) $\sigma, \vartheta \in \mathbb{R},\ \sigma, \vartheta \geq 0,\ c_1, c_2 \in \mathcal{C} \Rightarrow \sigma c_1 + \vartheta c_2 \in \mathcal{C}$,
(iii) $c \in \mathcal{C}$ *and* $-c \in \mathcal{C} \Rightarrow c = 0$.

The cones being evaluated here have interiors that are not empty.

Definition 4 ([6])**.** *A 5-tuple $(\Xi, \Psi, \Upsilon, *, \diamond)$ is said to be IGFCMS if $\mathcal{C}$ is a cone of E, Ξ is an arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN and Ψ, Υ are fuzzy sets in $\Xi^3 \times int(\mathcal{C})$ satisfying the following conditions: For all $\varpi, \omega, \varrho, \sigma \in \Xi$ and $\tau, \xi \in int(\mathcal{C})$,*
(IGF1) $\Psi(\varpi, \omega, \varrho, \tau) + \Upsilon(\varpi, \omega, \varrho, \tau) \leq 1$,
(IGF2) $\Psi(\varpi, \omega, \varrho, \tau) > 0$,
(IGF3) $\Psi(\varpi, \omega, \varrho, \tau) = 1 \Leftrightarrow \varpi = \omega = \varrho$,
(IGF4) $\Psi(\varpi, \omega, \varrho, \tau) = \Psi(p\{\varpi, \omega, \varrho\}, \tau)$, *where p is a permutation function,*
(IGF5) $\Psi(\varpi, \omega, \varrho, \tau + \xi) \geq \Psi(\varpi, \omega, \sigma, \tau) * \Psi(\sigma, \varrho, \varrho, \tau)$,
(IGF6) $\Psi(\varpi, \omega, \varrho, .) : int(\mathcal{C}) \to (0, 1]$ *is continuous,*
(IGF7) $\Upsilon(\varpi, \omega, \varrho, \tau) > 0$,
(IGF8) $\Upsilon(\varpi, \omega, \varrho, \tau) = 0 \Leftrightarrow \varpi = \omega = \varrho$,
(IGF9) $\Upsilon(\varpi, \omega, \varrho, \tau) = \Upsilon(p\{\varpi, \omega, \varrho\}, \tau)$, *where p is a permutation function,*
(IGF10) $\Upsilon(\varpi, \omega, \varrho, \tau + \xi) \leq \Upsilon(\varpi, \omega, \sigma, \tau) \diamond \Upsilon(\sigma, \varrho, \varrho, \tau)$,
(IGF11) $\Upsilon(\varpi, \omega, \varrho, .) : int(\mathcal{C}) \to (0, 1]$ *is continuous.*
Then (Ψ, Υ) is called an intuitionistic generalized fuzzy cone metric on Ξ.

Example 1. *Let $E = \mathbb{R}^2$ and consider the cone $\mathcal{C} = \{(c_1, c_2) \in \mathbb{R}^2 : c_1 \geq 0, c_2 \geq 0\}$ in E. Let $\Xi = \mathbb{R}$ and the norms $*$ and $\diamond$ be define by $\sigma * \vartheta = \sigma\vartheta$ and $\sigma \diamond \vartheta = max\{\sigma, \vartheta\}$. Define the functions $\Psi : \Xi^3 \times int(\mathcal{C}) \to [0, 1]$ and $\Upsilon : \Xi^3 \times int(\mathcal{C}) \to [0, 1]$ by*

$$\Psi(\varpi, \omega, \varrho, \tau) = \frac{1}{e^{\frac{|\varpi - \omega| + |\omega - \varrho| + |\varrho - \varpi|}{\|\tau\|}}},$$

$$\Upsilon(\varpi, \omega, \varrho, \tau) = \frac{e^{\frac{|\varpi - \omega| + |\omega - \varrho| + |\varrho - \varpi|}{\|\tau\|}} - 1}{e^{\frac{|\varpi - \omega| + |\omega - \varrho| + |\varrho - \varpi|}{\|\tau\|}}},$$

for all $\varpi, \omega, \varrho \in \Xi$ and $\tau \in int(\mathcal{C})$. Then, $(\Xi, \Psi, \Upsilon, *, \diamond)$ is an IGFCMS.

Definition 5 ([20])**.** *A 6-tuple $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ is said to be a GNCMS if $\mathcal{C}$ is a cone of E, Ξ is an arbitrary set, $*$ is a CTC, $\diamond$ is a CTCN and Ψ, Υ, Φ are neutrosophic sets in $\Xi^3 \times int(\mathcal{C})$ fulfill the following circumstances, for all $\varpi, \omega, \varrho, \sigma \in \Xi$ and $\tau, \xi \in int(\mathcal{C})$,*
(GNC1) $\Psi(\varpi, \omega, \varrho, \tau) + \Upsilon(\varpi, \omega, \varrho, \tau) + \Phi(\varpi, \omega, \varrho, \tau) \leq 3$,
(GNC2) $\Psi(\varpi, \omega, \varrho, \tau) > 0$,
(GNC3) $\Psi(\varpi, \omega, \varrho, \tau) = 1 \Leftrightarrow \varpi = \omega = \varrho$,
(GNC4) $\Psi(\varpi, \omega, \varrho, \tau) = \Psi(p\{\varpi, \omega, \varrho\}, \tau)$, *where p is a permutation function,*
(GNC5) $\Psi(\varpi, \omega, \varrho, \tau + \xi) \geq \Psi(\varpi, \omega, \sigma, \tau) * \Psi(\sigma, \varrho, \varrho, \tau)$,
(GNC6) $\Psi(\varpi, \omega, \varrho, .) : int(\mathcal{C}) \to (0, 1]$ *is continuous,*
(GNC7) $\Upsilon(\varpi, \omega, \varrho, \tau) > 0$,
(GNC8) $\Upsilon(\varpi, \omega, \varrho, \tau) = 0 \Leftrightarrow \varpi = \omega = \varrho$,
(GNC9) $\Upsilon(\varpi, \omega, \varrho, \tau) = \Upsilon(p\{\varpi, \omega, \varrho\}, \tau)$, *where p is a permutation function,*
(GNC10) $\Upsilon(\varpi, \omega, \varrho, \tau + \xi) \leq \Upsilon(\varpi, \omega, \sigma, \tau) \diamond \Upsilon(\sigma, \varrho, \varrho, \tau)$,
(GNC11) $\Upsilon(\varpi, \omega, \varrho, .) : int(\mathcal{C}) \to (0, 1]$ *is continuous.*
(GNC12) $\Phi(\varpi, \omega, \varrho, \tau) > 0$,
(GNC13) $\Phi(\varpi, \omega, \varrho, \tau) = 0 \Leftrightarrow \varpi = \omega = \varrho$,
(GNC14) $\Phi(\varpi, \omega, \varrho, \tau) = \Phi(p\{\varpi, \omega, \varrho\}, \tau)$, *where p is a permutation function,*
(GNC15) $\Phi(\varpi, \omega, \varrho, \tau + \xi) \leq \Phi(\varpi, \omega, \sigma, \tau) \diamond \Phi(\sigma, \varrho, \varrho, \tau)$,
(GNC16) $\Phi(\varpi, \omega, \varrho, .) : int(\mathcal{C}) \to (0, 1]$ *is continuous.*
Then (Ψ, Υ, Φ) is called a generalized neutrosophic cone metric on Ξ.

3. Main Results

In this section, we prove some fixed-point result in the sense of GNCMS, and also give some non-trivial examples which support our main results.

Definition 6. *Suppose $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ is called a symmetric GNCMS if, for all $\varpi, \omega \in \Xi$ and $\tau \in int(\mathcal{C})$, Ψ, Υ and Φ satisfy the following circumstances:*

$$\left(\frac{1}{\Psi(\varpi,\omega,\omega,\tau)} - 1\right) = \left(\frac{1}{\Psi(\omega,\varpi,\varpi,\tau)} - 1\right),$$

$$\Upsilon(\varpi,\omega,\omega,\tau) = \Upsilon(\omega,\varpi,\varpi,\tau),$$

$$\Phi(\varpi,\omega,\omega,\tau) = \Phi(\omega,\varpi,\varpi,\tau).$$

Definition 7. *Let $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ be a GNCMS and $f : \Xi \to \Xi$ be a self-mapping. Then f is said to be a generalized neutrosophic cone contractive if there exists $c \in (0,1)$ such that*

$$\left(\frac{1}{\Psi(f(\varpi), f(\omega), f(\varrho), \tau)} - 1\right) \leq c\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right),$$

$$\Upsilon((f(\varpi), f(\omega), f(\varrho), \tau) \leq c\Upsilon(\varpi,\omega,\varrho,\tau),$$

$$\Phi((f(\varpi), f(\omega), f(\varrho), \tau) \leq c\Phi(\varpi,\omega,\varrho,\tau).$$

for each $\varpi, \omega, \varrho, \tau \in \Xi$ and $\tau \in int(\mathcal{C})$.

Definition 8. *Suppose $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ be a GNCMS. Ψ, Υ and Φ are said to be triangular if, for all $\varpi, \omega, \varrho, \mu \in \Xi$ and $\tau \in int(\mathcal{C})$,*

$$\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right) \leq \left(\frac{1}{\Psi(\varpi,\omega,\mu,\tau)} - 1\right) + \left(\frac{1}{\Psi(\mu,\varrho,\varrho,\tau)} - 1\right),$$

$$\Upsilon(\varpi,\omega,\varrho,\tau) \leq \Upsilon(\varpi,\omega,\mu,\tau) + \Upsilon(\mu,\varrho,\varrho,\tau),$$

$$\Phi(\varpi,\omega,\varrho,\tau) \leq \Phi(\varpi,\omega,\mu,\tau) + \Phi(\mu,\varrho,\varrho,\tau).$$

Definition 9. *Let $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ be a GNCMS, for $\varpi \in \Xi, r > 0$ and $\tau \in int(\mathcal{C})$, the open ball $\mathcal{B}_{\mathcal{C}}(\varpi, r, \tau)$ with center at ϖ and radius r is defined by*

$$\mathcal{B}_{\mathcal{C}}(\varpi,r,\tau) = \left\{ \begin{array}{c} \omega \in \Xi \ : \Psi(\varpi,\omega,\omega,\tau) > 1 - r, \\ \Upsilon(\varpi,\omega,\omega,\tau) < r \ \text{ and } \Phi(\varpi,\omega,\omega,\tau) < r \end{array} \right\}.$$

Lemma 1 ([4])**.** *For each $c_1 \in int(\mathcal{C})$ and $c_2 \in int(\mathcal{C})$, there exists $c \in int(\mathcal{C})$ such that $c_1 - c_2 \in int(\mathcal{C})$ and $c_2 - c \in int(\mathcal{C})$.*

Theorem 1. *Let $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ be a GNCMS. Then $\tau_{\mathcal{C}}$ defined below is a topology:*

$$\tau_{\mathcal{C}} = \{\mathcal{D} \subseteq \Xi : \varpi \in \mathcal{D} \; if \text{ and only } if \; \exists \, r \in (0,1), \tau \in int(\mathcal{C})$$

such that $\mathcal{B}_{\mathcal{C}}(\varpi, r, \tau) \subset \mathcal{D}\}$.

Proof.

i. It is obvious that $\varnothing \in \tau_{\mathcal{C}}$ and $\Xi \in \tau_{\mathcal{C}}$.
ii. Suppose $\mathcal{D}_1 \in \tau_{\mathcal{C}}$ and $\mathcal{D}_2 \in \tau_{\mathcal{C}}$ and $\varpi \in \mathcal{D}_1 \cap \mathcal{D}_2$. Then $\varpi \in \mathcal{D}_1$ and $\varpi \in \mathcal{D}_2$. Implies that, $\exists \, r_1, r_2 \in (0,1)$ and $\tau_1, \tau_2 \in \text{int}(\mathcal{C})$ such that $\mathcal{B}_{\mathcal{C}}(\varpi, r_1, \tau_1) \subset \mathcal{D}_1$ and $\mathcal{B}_{\mathcal{C}}(\varpi, r_2, \tau_2) \subset \mathcal{D}_2$.

By Lemma 3.1, $\exists\ \tau \in \mathrm{int}(\mathcal{C})$ such that $\tau_1 - \tau \in \mathrm{int}(\mathcal{C}),\ \tau_2 - \tau \in \mathrm{int}(\mathcal{C})$. Let $r = \min\{r_1, r_2\}$. Then

$$\mathcal{B}_{\mathcal{C}}(\varpi, r, \tau) \subset \mathcal{B}_{\mathcal{C}}(\varpi, r_1, \tau_1) \cap \mathcal{B}_{\mathcal{C}}(\varpi, r_2, \tau_2) \subset \mathcal{D}_1 \cap \mathcal{D}_2.$$

Hence, $\mathcal{D}_1 \cap \mathcal{D}_2 \in \tau_{\mathcal{C}}$.

iii. Let $\mathcal{D}_j \in \tau_{\mathcal{C}}$ for each $j \in J$, an index set and let $\varpi \in U_{j\in J}\mathcal{D}_j$. then $\varpi \in \mathcal{D}_{j_0}$ for some $j_0 \in J$, implies that $\exists\ r \in (0,1)$ and $\tau \in \mathrm{int}(\mathcal{C})$ such that $\mathcal{B}_{\mathcal{C}}(\varpi, r, \tau) \subset \mathcal{D}_{j_0}$, as $\mathcal{D}_{j_0} \subset U_{j\in J}\mathcal{D}_j$, we have that $\mathcal{B}_{\mathcal{C}}(\varpi, r, \tau) \subset U_{j\in J}\mathcal{D}_j$. Thus $U_{j\in J}\mathcal{D}_j \in \tau_c$. From (i), (ii), and (iii), τ_c is a topology. □

Remark 1. *For any $r_1 > r_2$, there exists r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 there exists $r_5 \in (0,1)$ such that $r_5 * r_5 \geq r_4$, where $r_1, r_2, r_3, r_4, r_5 \in (0,1)$.*

Theorem 2. *Suppose $(\Xi, \Psi, Y, \Phi, *, \diamond)$ be a GNCMS. Then $(\Xi, \tau_{\mathcal{C}})$ is Hausdorff.*

Proof. Let $\varpi, \omega \in \Xi$ and $\varpi \neq \omega$. Then

$$0 < \Psi(\varpi, \omega, \omega, \tau) < 1,$$

$$0 < Y(\varpi, \omega, \omega, \tau) < 1,$$

$$0 < \Phi(\varpi, \omega, \omega, \tau) < 1.$$

Let

$$\Psi(\varpi, \omega, \omega, \tau) = r_1,$$
$$Y(\varpi, \omega, \omega, \tau) = r_2,$$
$$\Phi(\varpi, \omega, \omega, \tau) = r_3.$$

Take $r = \max\{r_1, r_2,\ r_3\}$. now, for each $r_0 \in (r, 1)$, there exists $r_4, r_5 \in (0,1)$ such that $r_4 * r_4 \geq r_0$ and $(1 - r_5) \diamond (1 - r_5) \leq 1 - r_0$. suppose $r_6 = \max\{r_4, r_5\}$, we obtain

$$\mathcal{B}_{\mathcal{C}}\Big(\varpi, 1 - r_1, \frac{\tau}{2}\Big) \cap \mathcal{B}_{\mathcal{C}}\Big(\omega, 1 - r_2, \frac{\tau}{2}\Big) \neq \varnothing.$$

Then, there exists

$$\varrho \in \mathcal{B}_{\mathcal{C}}\Big(\varpi, 1 - r_1, \frac{\tau}{2}\Big) \cap \mathcal{B}_{\mathcal{C}}\Big(\omega, 1 - r_2, \frac{\tau}{2}\Big),$$

and, we have that

$$r_1 = \Psi(\varpi, \omega, \omega, \tau) \geq \Psi\Big(\varpi, \omega, \varrho, \frac{\tau}{2}\Big) * \Psi\Big(\varrho, \omega, \omega, \frac{\tau}{2}\Big) \geq r_6 * r_6 \geq r_4 * r_4 \geq r_0 > r_1,$$

$$r_2 = Y(\varpi, \omega, \omega, \tau) \leq Y\Big(\varpi, \omega, \varrho, \frac{\tau}{2}\Big) \diamond Y\Big(\varrho, \omega, \omega, \frac{\tau}{2}\Big),$$

$$\leq (1 - r_6) \diamond (1 - r_6) \leq (1 - r_5) \diamond (1 - r_5) \leq (1 - r_0) < r_2,$$

$$r_3 = \Phi(\varpi, \omega, \omega, \tau) \leq \Phi\Big(\varpi, \omega, \varrho, \frac{\tau}{2}\Big) \diamond \Phi\Big(\varrho, \omega, \omega, \frac{\tau}{2}\Big),$$

$$\leq (1 - r_6) \diamond (1 - r_6) \leq (1 - r_5) \diamond (1 - r_5) \leq (1 - r_0) < r_2.$$

This is a contradiction. Hence

$$\mathcal{B}_{\mathcal{C}}\Big(\varpi, 1 - r_1, \frac{\tau}{2}\Big) \cap \mathcal{B}_{\mathcal{C}}\Big(\omega, 1 - r_2, \frac{\tau}{2}\Big) \neq \varnothing.$$

Therefore $(\Xi, \tau_{\mathcal{C}})$ is Hausdorff. □

Definition 10. *Let $(\Xi, \Psi, Y, \Phi, *, \diamond)$ be a GNCMS, $\varpi \in \Xi$ and $\{\varpi_n\}$ be a sequence in Ξ.*

i. $\{\varpi_n\}$ *is said to converge to ϖ if for all $\tau \in int(\mathcal{C})$,*

$$\lim_{n\to+\infty}\left(\frac{1}{\Psi(\varpi_n, \varpi, \varpi, \tau)} - 1\right) = 0,$$

$$\lim_{n\to+\infty} Y(\varpi_n, \varpi, \varpi, \tau) = 0,$$

$$\lim_{n\to+\infty} \Phi(\varpi_n, \varpi, \varpi, \tau) = 0.$$

$\lim_{n\to+\infty}\left(\frac{1}{\Psi(\varpi_n,\varpi,\varpi,\tau)} - 1\right) = 0,$ *It is denoted by* $\lim_{n\to+\infty} \varpi_n = \varpi$ *or by* $\varpi_n \to \varpi$ *as* $n \to +\infty$.

ii. $\{\varpi_n\}$ *is said to Cauchy sequence if for all $\tau \in int(\mathcal{C})$ and $m \in \mathbb{N}$, we have that*

$$\lim_{n\to+\infty}\left(\frac{1}{\Psi(\varpi_{n+m},\varpi_n,\varpi_n,\tau)} - 1\right) = 0,$$

$$\lim_{n\to+\infty} Y(\varpi_{n+m}, \varpi_n, \varpi_n, \tau) = 0,$$

$$\lim_{n\to+\infty} \Phi(\varpi_{n+m}, \varpi_n, \varpi_n, \tau) = 0.$$

iii. $(\Xi, \Psi, Y, \Phi, *, \diamond)$ *is called complete GNCMS, if every Cauchy sequence in Ξ converges.*

Remark 2. *The convergence of sequences in a GNCMS is considered in the sense of the topology defined here. Therefore, each converging sequence in a GNCMS has a unique limit and this makes the definition of convergence meaningful.*

Definition 11. *Let $(\Xi, \Psi, Y, \Phi, *, \diamond)$ be a GNCMS. A sequence $\{\varpi_n\}$ in Ξ is cone contractive if there exists $c \in (0,1)$ such that*

$$\Omega\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)} - 1\right) \le \Omega\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)} - 1\right),$$

$$Y(\xi_n, \xi_{n+1}, \xi_{n+1}\ , \tau) \le cY(\xi_{n-1}, \xi_n, \xi_n, \tau),$$

$$\Phi(\xi_n, \xi_{n+1}, \xi_{n+1}\ , \tau) \le c\Phi(\xi_{n-1}, \xi_n, \xi_n, \tau).$$

For all $\tau \in int(\mathcal{C})$.

Lemma 2. *Suppose $(\Xi, \Psi, Y, \Phi, *, \diamond)$ be a GNCMS is symmetric.*

Proof. Let $\varpi, \omega \in \Xi$ and $\tau \in \mathrm{int}(\mathcal{C})$. Then

$$\lim_{r\to 0}\Psi(\varpi, \varpi, \omega, \tau + r) \ge \lim_{r\to 0}(\Psi(\varpi, \varpi, \varpi, r) * \Psi(\varpi, \omega, \omega, \tau)),$$

$$\lim_{r\to 0}\Psi(\omega, \omega, \varpi, \tau + r) \ge \lim_{r\to 0}(\Psi(\omega, \omega, \omega, r) * \Psi(\varpi, \omega, \omega, \tau)),$$

implies

$$M(\varpi, \varpi, \omega, \tau) \ge M(\varpi, \omega, \omega, \tau) \text{ and } M(\omega, \omega, \varpi, \tau) \ge M(\omega, \varpi, \varpi, \tau)$$

$$\lim_{r\to 0}Y(\varpi, \varpi, \omega, \tau + r) \le \lim_{r\to 0}(Y(\varpi, \varpi, \varpi, r) \diamond Y(\varpi, \omega, \omega, \tau)),$$

$$\lim_{r\to 0}Y(\omega, \omega, \varpi, \tau + r) \le \lim_{r\to 0}(Y(\omega, \omega, \omega, r) \diamond Y(\omega, \varpi, \varpi, \tau)),$$

implies
$$N(\varpi,\varpi,\omega,\tau) \leq N(\varpi,\omega,\omega,\tau) \text{and } N(\omega,\omega,\varpi,\tau) \leq N(\omega,\varpi,\varpi,\tau),$$
$$\lim_{r\to 0}\Phi(\varpi,\varpi,\omega,\tau+r) \leq \lim_{r\to 0}(\Phi(\varpi,\varpi,\varpi,r) \diamond \Phi(\varpi,\omega,\omega,\tau)),$$
$$\lim_{r\to 0}\Phi(\omega,\omega,\varpi,\tau+r) \leq \lim_{r\to 0}(\Phi(\omega,\omega,\omega,r) \diamond \Phi(\omega,\varpi,\varpi,\tau)),$$
implies
$$\Phi(\varpi,\varpi,\omega,\tau) \leq \Phi(\varpi,\omega,\omega,\tau) \text{ and } \Phi(\omega,\omega,\varpi,\tau) \leq \Phi(\omega,\varpi,\varpi,\tau).$$

Hence
$$M(\varpi,\omega,\omega,\tau) = M(\omega,\varpi,\varpi,\tau),$$
$$N(\varpi,\omega,\omega,\tau) = N(\omega,\varpi,\varpi,\tau),$$
$$\Phi(\varpi,\omega,\omega,\tau) = \Phi(\omega,\varpi,\varpi,\tau).$$

□

Lemma 3. *Let* $(\Xi,\Psi,\Upsilon,\Phi,*,\diamond)$ *be a GNCMS where* Ψ , Υ *and* Φ *are triangular. Then any cone contractive sequence in* Ξ *is a Cauchy sequence.*

Proof. *Let the sequence* $\{\xi_n\}$ *be cone contractive* Ξ*. Then there exists* $c \in (0,1)$ *such that*

$$\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)} - 1\right) \leq c\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)} - 1\right)$$
$$\Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \leq c\Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau)$$
$$\Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \leq c\Phi(\xi_{n-1},\xi_n,\xi_n,\tau).$$

Now, Ψ, Υ and Φ are triangular. By Lemma 3, for $m > n > n_0, n_0 \in N$,

$$\left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_m,\tau)} - 1\right) \leq \left(\left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_{n+1},\tau)} - 1\right) + \left(\frac{1}{\Psi(\xi_{n+1},\xi_{n+1},\xi_m,\tau)} - 1\right)\right)$$
$$\leq \begin{pmatrix} \left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_{n+1},\tau)} - 1\right) + \left(\frac{1}{\Psi(\xi_{n+1},\xi_{n+1},\xi_{n+2},\tau)} - 1\right) \\ +\left(\frac{1}{\Psi(\xi_{n+2},\xi_{n+2},\xi_m,\tau)} - 1\right) \end{pmatrix},$$

$$\Upsilon(\xi_n,\xi_n,\xi_m,\tau) \leq \Upsilon(\xi_n,\xi_n,\xi_{n+1},\tau) + \Upsilon(\xi_{n+1},\xi_{n+1},\xi_m,\tau)$$
$$\leq \begin{pmatrix} \Upsilon(\xi_n,\xi_n,\xi_{n+1},\tau) + \Upsilon(\xi_{n+1},\xi_{n+1},\xi_{n+2},\tau) \\ +\Upsilon(\xi_{n+2},\xi_{n+2},\xi_m,\tau) \end{pmatrix},$$

and

$$\Phi(\xi_n,\xi_n,\xi_m,\tau) \leq \Phi(\xi_n,\xi_n,\xi_{n+1},\tau) + \Phi(\xi_{n+1},\xi_{n+1},\xi_m,\tau)$$
$$\leq \begin{pmatrix} \Phi(\xi_n,\xi_n,\xi_{n+1},\tau) + \Phi(\xi_{n+1},\xi_{n+1},\xi_{n+2},\tau) \\ +\Phi(\xi_{n+2},\xi_{n+2},\xi_m,\tau) \end{pmatrix}.$$

Continuing the process, and, using (1), (2) and (3), we finally arrive at

$$\left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_m,\tau)}-1\right) \leq \begin{pmatrix} \left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_{n+1},\tau)}-1\right)+\left(\frac{1}{\Psi(\xi_{n+1},\xi_{n+1},\xi_{n+2},\tau)}-1\right) \\ +\cdots+\left(\frac{1}{\Psi(\xi_{m-1},\xi_{m-1},\xi_m,\tau)}-1\right) \end{pmatrix}$$

$$\leq c^n\left(\frac{1}{\Psi(\xi_0,\xi_0,\xi_1,\tau)}-1\right)+\cdots+c^{m-1}\left(\frac{1}{\Psi(\xi_0,\xi_0,\xi_1,\tau)}-1\right)$$

$$=\left(c^n+\cdots+c^{m-1}\right)\left(\frac{1}{\Psi(\xi_0,\xi_0,\xi_1,\tau)}-1\right)$$

$$\leq \frac{c^n}{1-c}\left(\frac{1}{\Psi(\xi_0,\xi_0,\xi_1,\tau)}-1\right),$$

$$\begin{aligned} \Upsilon(\xi_n,\xi_n,\xi_m,\tau) &\leq \Upsilon(\xi_n,\xi_n,\xi_{n+1},\tau)+\Upsilon(\xi_{n+1},\xi_{n+1},\xi_m,\tau)+\cdots+\Upsilon(\xi_{m-1},\xi_{m-1},\xi_m,\tau) \\ &\leq c^n\Upsilon(\xi_0,\xi_0,\xi_1,\tau)+\cdots+c^{m-1}\Upsilon(\xi_0,\xi_0,\xi_1,\tau), \\ &=\left(c^n+\cdots+c^{m-1}\right)\Upsilon(s_0,\xi_0,\xi_1,\tau), \\ &\leq \frac{c^n}{1-c}\Upsilon(s_0,\xi_0,\xi_1,\tau), \end{aligned}$$

and

$$\begin{aligned} \Phi(\xi_n,\xi_n,\xi_m,\tau) &\leq \Phi(\xi_n,\xi_n,\xi_{n+1},\tau)+\Phi(\xi_{n+1},\xi_{n+1},\xi_m,\tau)+\cdots+\Phi(\xi_{m-1},\xi_{m-1},\xi_m,\tau) \\ &\leq c^n\Phi(\xi_0,\xi_0,\xi_1,\tau)+\cdots+c^{m-1}\Phi(\xi_0,\xi_0,\xi_1,\tau), \\ &=\left(c^n+\cdots+c^{m-1}\right)\Phi(s_0,\xi_0,\xi_1,\tau), \\ &\leq \frac{c^n}{1-c}\Phi(s_0,\xi_0,\xi_1,\tau). \end{aligned}$$

We have that

$$\begin{aligned} \left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_m,\tau)}-1\right) &\to 0, \\ \Upsilon(\xi_n,\xi_n,\xi_m,\tau) &\to 0, \\ \Phi(\xi_n,\xi_n,\xi_m,\tau) &\to 0, \end{aligned}$$

as $n \to +\infty$. Therefore $\{\xi_n\}$ is a Cauchy sequence. □

Theorem 3. *Let $(\Xi,\Psi,\Upsilon,\Phi,*,\diamond)$ be a complete GNCMS, where Ψ, Υ and Φ are triangular. If $\Gamma:\Xi\to\Xi$ is such that for all $\varpi,\omega,\varrho\in X$ and $\tau\in int(\mathcal{C})$,*

$$\left(\frac{1}{\Psi(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau)}-1\right) \leq \left\{\begin{array}{c} c_1\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)}-1\right)+c_2\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)}-1\right)+c_3\left(\frac{1}{\Psi(\omega,\Gamma\varrho,\Gamma\varrho,\tau)}-1\right) \\ +c_4\left(\frac{1}{\Psi(\omega,\Gamma\omega,\Gamma\omega,\tau)}-1\right)+c_5\left(\frac{1}{\Psi(\varrho,\Gamma\varrho,\Gamma\varrho,\tau)}-1\right) \\ +c_6\left(\frac{1}{\Psi(\varrho,\Gamma\omega,\Gamma\omega,\tau)}-1\right)+c_7\left(\frac{1}{\Psi(\omega,\Gamma\varpi,\varrho,\tau)}-1\right) \end{array}\right\} \quad (1)$$

$$\Upsilon(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau) \leq \left\{\begin{array}{c} c_1\Upsilon(\varpi,\omega,\varrho,\tau)+c_2\Upsilon(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)+c_3\Upsilon(\omega,\Gamma\varrho,\Gamma\varrho,\tau) \\ +c_4\Upsilon(\omega,\Gamma\omega,\Gamma\omega,\tau)+c_5\Upsilon(\varrho,\Gamma\varrho,\Gamma\varrho,\tau) \\ +c_6\Upsilon(\varrho,\Gamma\omega,\Gamma\omega,\tau)+c_7\Upsilon(\omega,\Gamma\varpi,\varrho,\tau) \end{array}\right\} \quad (2)$$

$$\Phi(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau) \leq \left\{ \begin{array}{c} c_1\Phi(\varpi,\omega,\varrho,\tau)+c_2\Phi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)+c_3\Phi(\omega,\Gamma\varrho,\Gamma\varrho,\tau) \\ +c_4\Phi(\omega,\Gamma\omega,\Gamma\omega,\tau)+c_5\Phi(\varrho,\Gamma\varrho,\Gamma\varrho,\tau) \\ +c_6\Phi(\varrho,\Gamma\omega,\Gamma\omega,\tau)+c_7\Phi(\omega,\Gamma\varpi,\varrho,\tau) \end{array} \right\} \quad (3)$$

where $c_i \in [0,+\infty]$, $i = 1,\cdots,6$ *and* $\sum_{i=1}^{6} c_i < 1$. *Then* Γ *has a fixed point and such a point is unique if* $c_1 + c_7 < 1$.

Proof. Let $\xi_0 \in X$ be arbitrary. Generate a sequence $\{\xi_n\}$ with $\xi_n = \Gamma\xi_{n-1}$ for $n \in \mathbb{N}$. If there exists a nonnegative integer m such that $\xi_{m+1} = \xi_m$, then $\Gamma\xi_m = \xi_m$ and ξ_m becomes a fixed point of Γ.

Suppose $S_n \neq S_{n-1}$ for any $n \in \mathbb{N}$. From (1), we have

$$\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \leq \left(\frac{1}{\Psi(\Gamma\xi_{n-1},\Gamma\xi_n,\Gamma\xi_n,\tau)}-1\right)$$

$$\leq \left\{ \begin{array}{c} c_1\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right)+c_2\left(\frac{1}{\Psi(\xi_{n-1},\Gamma\xi_{n-1},\Gamma\xi_{n-1},\tau)}-1\right)+c_3\left(\frac{1}{\Psi(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)}-1\right) \\ +c_4\left(\frac{1}{\Psi(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)}-1\right)+c_5\left(\frac{1}{\Psi(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)}-1\right) \\ +c_6\left(\frac{1}{\Psi(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)}-1\right)+c_7\left(\frac{1}{\Psi(\xi_n,\Gamma\xi_{n-1},\xi_n,\tau)}-1\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} c_1\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right)+c_2\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right)+c_3\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \\ +c_4\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right)+c_5\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \\ +c_6\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right)+c_7\left(\frac{1}{\Psi(\xi_n,\xi_n,\xi_n,\tau)}-1\right) \end{array} \right\},$$

$$= \left\{(c_1+c_2)\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right)+(c_3+c_4+c_5+c_6)\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right)\right\}.$$

Hence, we have that

$$\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \leq \frac{c_1+c_2}{1-(c_3+c_4+c_5+c_6)}\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right). \quad (4)$$

Assume that

$$c = \frac{c_1+c_2}{1-(c_3+c_4+c_5+c_6)},$$

then $c < 1$ and (4) becomes

$$\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \leq c\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right). \quad (5)$$

From (2), we have

$$\begin{aligned} \Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau) &\leq \Upsilon(\Gamma\xi_{n-1},\Gamma\xi_n,\Gamma\xi_n,\tau) \\ &\leq \left\{ \begin{array}{c} c_1\Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau)+c_2\Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau)+c_3\Upsilon(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau) \\ +c_4\Upsilon(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)+c_5\Upsilon(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau)+c_6\Upsilon(\xi_n,\Gamma\xi_n,\Gamma\xi_n,\tau) \\ +c_7\Upsilon(\xi_n,\xi_n,\xi_n,\tau) \end{array} \right\}, \end{aligned}$$

$$= \left\{ \begin{array}{c} c_1 Y(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_2 Y(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_3 Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \\ +c_4 Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) + c_5 Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) + c_6 Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \\ +c_7 Y(\xi_n, \xi_n, \xi_n, \tau) \end{array} \right\},$$

$$= \{(c_1 + c_2) Y(\xi_{n-1}, \xi_n, \xi_n, \tau) + (c_3 + c_4 + c_5 + c_6) Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau)\}.$$

Hence, we have that

$$Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le \frac{(c_1 + c_2)}{(c_3 + c_4 + c_5 + c_6)} Y(\xi_{n-1}, \xi_n, \xi_n, \tau),$$

implies

$$Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le c Y(\xi_{n-1}, \xi_n, \xi_n, \tau). \tag{6}$$

From (3), we have

$$\Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le \Phi(\Gamma\xi_{n-1}, \Gamma\xi_n, \Gamma\xi_n, \tau)$$

$$\le \left\{ \begin{array}{c} c_1 \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_2 \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_3 \Phi(\xi_n, \Gamma\xi_n, \Gamma\xi_n, \tau) \\ +c_4 \Phi(\xi_n, \Gamma\xi_n, \Gamma\xi_n, \tau) + c_5 \Phi(\xi_n, \Gamma\xi_n, \Gamma\xi_n, \tau) + c_6 \Phi(\xi_n, \Gamma\xi_n, \Gamma\xi_n, \tau) \\ +c_7 \Phi(\xi_n, \xi_n, \xi_n, \tau) \end{array} \right\},$$

$$= \left\{ \begin{array}{c} c_1 \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_2 \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau) + c_3 \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \\ +c_4 \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) + c_5 \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) + c_6 \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \\ +c_7 \Phi(\xi_n, \xi_n, \xi_n, \tau) \end{array} \right\},$$

$$= \{(c_1 + c_2) \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau) + (c_3 + c_4 + c_5 + c_6) \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau)\}.$$

Hence, we have that,

$$\Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le \frac{(c_1 + c_2)}{(c_3 + c_4 + c_5 + c_6)} \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau),$$

implies

$$\Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le c \Phi(\xi_{n-1}, \xi_n, \xi_n, \tau). \tag{7}$$

By utilizing inequalities (5), (6) and (7) make the sequence $\{\xi_n\}$ is cone contractive. Hence by Lemma 2, $\{\xi_n\}$ is Cauchy in Ξ. As Ξ is complete, there exists $\dot{s} \in X$ such that

$$\left. \begin{array}{c} \lim_{n \to +\infty} \left(\frac{1}{\Psi(\xi_n, \dot{s}, \dot{s}, \tau)} - 1 \right) = 0 \\ \lim_{n \to +\infty} (Y(\xi_n, \dot{s}, \dot{s}, \tau)) = 0 \\ \lim_{n \to +\infty} (\Phi(\xi_n, \dot{s}, \dot{s}, \tau)) = 0 \end{array} \right\}. \tag{8}$$

By repeated application of (5), (6) and (7), we obtain that

$$\left(\frac{1}{\Psi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau)} - 1 \right) \le c^n \left(\frac{1}{\Psi(\xi_0, \xi_1, \xi_1, \tau)} - 1 \right),$$
$$Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le c^n Y(\xi_0, \xi_1, \xi_1, \tau),$$
$$\Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) \le c^n \Phi(\xi_0, \xi_1, \xi_1, \tau),$$

implies that

$$\lim_{n \to +\infty} \left(\frac{1}{\Psi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau)} - 1 \right) = 0,$$
$$\lim_{n \to +\infty} Y(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) = 0,$$
$$\lim_{n \to +\infty} \Phi(\xi_n, \xi_{n+1}, \xi_{n+1}, \tau) = 0.$$

Now,

$$\left(\tfrac{1}{\Psi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)=\left(\tfrac{1}{\Psi(\Gamma\xi_{n},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right),$$
$$\leq\left\{\begin{array}{c} c_1\left(\tfrac{1}{\Psi(\xi_{n},\dot{s},\dot{s},\tau)}-1\right)+c_2\left(\tfrac{1}{\Psi(\xi_{n},\Gamma\xi_{n},\Gamma\xi_{n},\tau)}-1\right)+c_3\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\\ +c_4\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)+c_5\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\\ +c_6\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)+c_7\left(\tfrac{1}{\Psi(\dot{s},\Gamma\xi_{n},\dot{s},\tau)}-1\right)\end{array}\right\},$$
$$=\left\{\begin{array}{c} c_1\left(\tfrac{1}{\Psi(\xi_{n},\dot{s},\dot{s},\tau)}-1\right)+c_2\left(\tfrac{1}{\Psi(\xi_{n},\xi_{n+1},\xi_{n+1},\tau)}-1\right)+c_3\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\\ +c_4\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)+c_5\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\\ +c_6\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)+c_7\left(\tfrac{1}{\Psi(\dot{s},\xi_{n+1},\dot{s},\tau)}-1\right)\end{array}\right\},$$
$$\to d\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\text{ as }n\to+\infty,$$

where $d=c_3+c_4+c_5+c_6$. Since by (6) and (7). Hence

$$\limsup_{n\to+\infty}\left(\frac{1}{\Psi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\leq d\left(\frac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right).$$

Similarly,

$$\limsup_{n\to+\infty}\Upsilon(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$
$$\limsup_{n\to+\infty}\Phi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau).$$

As Ψ, Υ and Φ are triangular

$$\left(\tfrac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)}-1\right)\leq\left(\tfrac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)}-1\right)+\left(\tfrac{1}{\Psi(\xi_{n+1},\dot{s},\dot{s},\tau)}-1\right),$$
$$\Upsilon(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)\leq\Upsilon(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)+\Upsilon(\xi_{n+1},\dot{s},\dot{s},\tau),$$
$$\Phi(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)\leq\Phi(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)+\Phi(\xi_{n+1},\dot{s},\dot{s},\tau).$$

From (6) to (8), we can bring that

$$\left(\tfrac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)}-1\right)\leq d\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right),$$
$$\Upsilon(\dot{s},\dot{s},\Gamma\dot{s},\tau)\leq d\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$
$$\Phi(\dot{s},\dot{s},\Gamma\dot{s},\tau)\leq d\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$

implies

$$\left(\tfrac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)}-1\right)=0,$$
$$\Upsilon(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)=0,$$
$$\Phi(\Gamma\dot{s},\Gamma\dot{s},\dot{s},\tau)=0.$$

Since $d<1$, it implies that

$$\Gamma\dot{s}=\dot{s}.$$

Thus, we can conclude that $\dot{s}$ is a fixed point of Γ. Suppose $\Gamma\ddot{\xi} = \ddot{\xi}$. The form (1),

$$\left(\frac{1}{\Psi\left(\Gamma\dot{s},\Gamma\ddot{\xi},\Gamma\ddot{\xi},\tau\right)} - 1\right)$$

$$\leq \left\{\begin{array}{c} c_1\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)} - 1\right) + c_2\left(\frac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)} - 1\right) + c_3\left(\frac{1}{\Psi\left(\ddot{\xi},\Gamma\ddot{\xi},\Gamma\ddot{\xi},\tau\right)} - 1\right) \\ +c_4\left(\frac{1}{\Psi\left(\ddot{\xi},\Gamma\ddot{\xi},\Gamma\ddot{\xi},\tau\right)} - 1\right) + c_5\left(\frac{1}{\Psi\left(\ddot{\xi},\Gamma\ddot{\xi},\Gamma\ddot{\xi},\tau\right)} - 1\right) \\ +c_6\left(\frac{1}{\Psi\left(\ddot{\xi},\Gamma\ddot{\xi},\Gamma\ddot{\xi},\tau\right)} - 1\right) + c_7\left(\frac{1}{\Psi\left(\ddot{\xi},\Gamma\dot{s},\ddot{\xi},\tau\right)} - 1\right) \end{array}\right\}$$

implies

$$\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)} - 1\right) \leq (c_1 + c_7)\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)} - 1\right).$$

$$\left(\frac{1}{M\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)} - 1\right) = 0 \text{ if } c_1 + c_7 < 1.$$

Similarly,

$$Y\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right) = 0,$$

$$\Phi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right) = 0.$$

Hence, we can conclude that Γ has a unique fixed-point if $c_1 + c_7 < 1$. □

Example 2. *Let $\Xi = [0,+\infty)$ with metric $d(\varpi,\omega) = |\varpi - \omega|$ for all $\varpi,\omega \in \Xi$ and let $\mathcal{C} = \mathbb{R}^+$. Define the t-norm $*$ and the t-conorm $\diamond$ by $\sigma * \vartheta = min\{\sigma,\vartheta\}$ and $\sigma \diamond \vartheta = max\{\sigma,\vartheta\}$. Define the Ψ, Y and Φ by*

$$\Psi(\varpi,\omega,\varrho,\tau) = \frac{\tau}{\tau+l(|\varpi-\omega|+|\omega-\varrho|+|\varrho-\varpi|)},$$

$$Y\,(\varpi,\omega,\varrho,\tau) = \frac{l(|\varpi-\omega|+|\omega-\varrho|+|\varrho-\varpi|)}{\tau+l(|\varpi-\omega|+|\omega-\varrho|+|\varrho-\varpi|)},$$

$$\Phi\,(\varpi,\omega,\varrho,\tau) = \frac{l(|\varpi-\omega|+|\omega-\varrho|+|\varrho-\varpi|)}{\tau}$$

for all $\varpi,\omega,\varrho \in \Xi$ and $\tau \in int(\mathcal{C})$ where $l = 10$. Then it is clear that $(\Xi,\Psi,Y,\Phi,*,\diamond)$ be a complete GNCMS and that Ψ and Y are triangular. Consider the self-map $\Gamma : \Xi \to \Xi$ given by

$$\Gamma\varpi = \begin{cases} \frac{5}{4}\varpi + 3, & \varpi \in [0,1] \\ \frac{3}{4}\varpi + \frac{7}{2}, & \varpi \in [1,+\infty). \end{cases}$$

Then

$$\left(\frac{1}{\Psi(\,\Gamma\varpi,\ \Gamma\omega,\ \Gamma\varrho,\tau)} - 1\right) = \frac{5}{4}\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right)$$

and

$$Y(\,\Gamma\varpi,\ \Gamma\omega,\ \Gamma\varrho,\tau) \geq \frac{5}{4}\,Y(\varpi,\omega,\varrho,\tau),$$

$$\Phi(\,\Gamma\varpi,\ \Gamma\omega,\ \Gamma\varrho,\tau) \geq \frac{5}{4}\,\Phi(\varpi,\omega,\varrho,\tau),$$

when $\varpi, \omega, \varrho \in [0,1]$. *Hence* Γ *is not fuzzy cone contractive. Therefore, we cannot use the contraction theorem to assure the existence of fixed points. However, here* Γ *satisfied the conditions* (1) *and* (2) *with*

$$c_1 = \frac{3}{80},\ c_2 = \frac{17}{80},\ c_3 = c_4 = c_5 = \frac{1}{20},\ c_6 = 0,\ c_7 = \frac{1}{20}.$$

Therefore, Γ *has a unique fixed point and this point is* $\varpi = 14$.

Corollary 1. *Let* $(\Xi, \Psi, Y, \Phi, *, \diamond)$ *be a complete GNCMS where* Ψ, Y *and* Φ *are triangular. If* $\Gamma : \Xi \to \Xi$ *is such that for all* $\varpi, \omega, \varrho \in \Xi$, $\tau \in int(\mathcal{C})$,

$$\left(\frac{1}{\Psi(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau)} - 1\right) \geq \begin{Bmatrix} c_1\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right) + c_2\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)} - 1\right) + \\ c_3\left(\frac{1}{\Psi(\omega, \Gamma\varrho,\Gamma\varrho,\tau)} - 1\right) + c_4\left(\frac{1}{\Psi(\omega,\Gamma\varpi,\varrho,\tau)} - 1\right) \end{Bmatrix},$$

$$Y(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1 Y(\varpi, \omega, \varrho, \tau) + c_2 Y(\Gamma\varpi, \Gamma\varpi, \Gamma\varpi, \tau) + \\ c_3 Y(\omega, \Gamma\varrho, \Gamma\varrho, \tau) + c_4 Y(\omega, \Gamma\varpi, \varrho, \tau) \end{Bmatrix},$$

$$\Phi(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1 \Phi(\varpi, \omega, \varrho, \tau) + c_2 \Phi(\Gamma\varpi, \Gamma\varpi, \Gamma\varpi, \tau) + \\ c_3 \Phi(\omega, \Gamma\varrho, \Gamma\varrho, \tau) + c_4 \Phi(\omega, \Gamma\varpi, \varrho, \tau) \end{Bmatrix},$$

where $c_i \in [0, +\infty)$, $i = 1, \cdots, 4$ *and*

$$c_1 + c_2 + c_3 < 1.$$

Then Γ *has a fixed point and such a point is unique if* $c_1 + c_4 < 1$.

Corollary 2. *Let* $(\Xi, \Psi, Y, \Phi, *, \diamond)$ *be a complete GNCMS, where* Ψ *and* Y *are triangular. If* $\Gamma : \Xi \to \Xi$ *is such that for all* $\varpi, \omega, \varrho \in \Xi$, $\tau \in int(\mathcal{C})$,

$$\begin{aligned} &\left(\frac{1}{\Psi(\Gamma\varpi, \Gamma\omega,\Gamma\varrho,\tau)} - 1\right) \\ &\leq \begin{Bmatrix} c_1\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right) + c_2\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)} - 1\right) + c_3\left(\frac{1}{\Psi(\omega, \Gamma\varrho,\Gamma\varrho,\tau)} - 1\right) + \\ c_4\left(\frac{1}{\Psi(\omega, \Gamma\omega,\Gamma\omega,\tau)} - 1\right) + c_5\left(\frac{1}{\Psi(\varrho,\Gamma\varrho,\Gamma\varrho,\tau)} - 1\right) + c_6\left(\frac{1}{\Psi(\varrho, \Gamma\omega,\Gamma\omega,\tau)} - 1\right) \end{Bmatrix} \\ &Y(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1 Y(\varpi, \omega, \varrho, \tau) + c_2 Y(\Gamma\varpi, \Gamma\varpi, \Gamma\varpi, \tau) + c_3 Y(\omega, \Gamma\varrho, \Gamma\varrho, \tau) + \\ c_4 Y(\omega, \Gamma\omega, \Gamma\omega, \tau) + c_5 Y(\varrho, \Gamma\varrho, \Gamma\varrho, \tau) + c_6 Y(\varrho, \Gamma\omega, \Gamma\omega, \tau) \end{Bmatrix} \\ &Y(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1 \Phi(\varpi, \omega, \varrho, \tau) + c_2 \Phi(\Gamma\varpi, \Gamma\varpi, \Gamma\varpi, \tau) + c_3 \Phi(\omega, \Gamma\varrho, \Gamma\varrho, \tau) + \\ c_4 \Phi(\omega, \Gamma\omega, \Gamma\omega, \tau) + c_5 \Phi(\varrho, \Gamma\varrho, \Gamma\varrho, \tau) + c_6 \Phi(\varrho, \Gamma\omega, \Gamma\omega, \tau) \end{Bmatrix} \end{aligned}$$

where $c_i \in [0, +\infty]$, $i = 1, \cdots, 6$ *and* $\sum_{i=1}^{6} c_i < 1$. *Then* Γ *has a fixed point.*

Corollary 3. *Let* $(\Xi, \Psi, Y, \Phi *, \diamond)$ *be a complete GNCMS, where* Ψ *and* Y *are triangular. If* $\Gamma : \Xi \to \Xi$ *satisfied* (1) *and* (2) *with* $\sum_{i=1}^{7} c_i < 1$, *then* Γ *has a unique fixed point.*

Theorem 4. *Let* $(\Xi, \Psi, Y, \Phi, *, \diamond)$ *be a complete GNCMS, where* Ψ *and* Y *are triangular. If* $\Gamma : \Xi \to \Xi$ *is such that for all* $\varpi, \omega, \varrho \in \Xi$, $\tau \in int(\mathcal{C})$,

$$\left(\frac{1}{\Psi(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau)}-1\right) \leq \left\{\begin{array}{c} c_1\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)}-1\right)+c_2\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\varrho,\tau)}-1\right)+c_3\left(\frac{1}{\Psi(\varpi,\varpi,\Gamma\varpi,\tau)}-1\right)+ \\ c_4\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)}-1\right)+c_5\left(\frac{1}{\Psi(\omega,\Gamma\omega,\varrho,\tau)}-1\right)+c_6\left(\frac{1}{\Psi(\omega,\Gamma\varrho,\varrho,\tau)}-1\right)+ \\ c_7\left(\frac{1}{\Psi(\Gamma\varpi,\Gamma\omega,\varrho,\tau)}-1\right)+c_8\left(\frac{1}{\Psi(\Gamma\varpi,\Gamma\varrho,\omega,\tau)}-1\right)+c_9\left(\frac{1}{\Psi(\omega,\omega,\Gamma\omega,\tau)}-1\right)+ \\ c_{10}\left(\frac{1}{\Psi(\varrho,\varrho,\Gamma\varrho,\tau)}-1\right)+c_{11}\left(\frac{1}{\Psi(\omega,\Gamma\omega,\Gamma\omega,\tau)}-1\right)+c_{12}\left(\frac{1}{\Psi(\varrho,\Gamma\varrho,\varrho,\tau)}-1\right)+ \\ c_{13}\left(\frac{1}{\Psi(\varrho,\Gamma\omega,\Gamma\omega,\tau)}-1\right)+c_{14}\left(\frac{1}{\Psi(\varpi,\Gamma\varrho,\Gamma\varrho,\tau)}-1\right) \\ c_{15}\left(\frac{1}{\Psi(\Gamma\omega,\Gamma\varrho,\varpi,\tau)}-1\right)+c_{16}\left(\frac{1}{\Psi(\varpi,\Gamma\varrho,\Gamma\varrho,\tau)}-1\right) \end{array}\right\} \tag{9}$$

$$Y(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau) \leq \left\{\begin{array}{c} c_1Y(\varpi,\omega,\varrho,\tau)+c_2Y(\varpi,\Gamma\varpi,\varrho,\tau)+c_3Y(\varpi,\varpi,\Gamma\varpi,\tau)+ \\ c_4Y(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)+c_5Y(\varrho,\Gamma\varpi,\varrho,\tau)+c_6Y(\omega,\Gamma\varrho,\varrho,\tau)+ \\ c_7Y(\Gamma\varpi,\Gamma\omega,\varrho,\tau)+c_8Y(\Gamma\varpi,\Gamma\varrho,\omega,\tau)+c_9Y(\omega,\omega,\Gamma\omega,\tau)+ \\ c_{10}Y(\varrho,\varrho,\Gamma\varrho,\tau)+c_{11}Y(\omega,\Gamma\omega,\Gamma\omega,\tau)+c_{12}Y(\varrho,\Gamma\varrho,\Gamma\varrho,\tau)+ \\ c_{13}Y(\varrho,\Gamma\omega,\Gamma\omega,\tau)+c_{14}Y(\omega,\Gamma\varrho,\Gamma\varrho,\tau) \\ c_{15}Y(\Gamma\omega,\Gamma\varrho,\varpi,\tau)+c_{16}Y(\varpi,\Gamma\varrho,\Gamma\varrho,\tau) \end{array}\right\} \tag{10}$$

$$\Phi(\Gamma\varpi,\Gamma\omega,\Gamma\varrho,\tau) \leq \left\{\begin{array}{c} c_1\Phi(\varpi,\omega,\varrho,\tau)+c_2\Phi(\varpi,\Gamma\varpi,\varrho,\tau)+c_3\Phi(\varpi,\varpi,\Gamma\varpi,\tau)+ \\ c_4\Phi(\varpi,\Gamma\varpi,\Gamma\varpi,\tau)+c_5\Phi(\varrho,\Gamma\omega,\varrho,\tau)+c_6\Phi(\omega,\Gamma\varrho,\varrho,\tau)+ \\ c_7\Phi(\Gamma\varpi,\Gamma\omega,\varrho,\tau)+c_8\Phi(\Gamma\varpi,\Gamma\varrho,\omega,\tau)+c_9\Phi(\omega,\omega,\Gamma\omega,\tau)+ \\ c_{10}\Phi(\varrho,\varrho,\Gamma\varrho,\tau)+c_{11}\Phi(\omega,\Gamma\omega,\Gamma\omega,\tau)+c_{12}\Phi(\varrho,\Gamma\varrho,\Gamma\varrho,\tau)+ \\ c_{13}\Phi(\varrho,\Gamma\omega,\Gamma\omega,\tau)+c_{14}\Phi(\omega,\Gamma\varrho,\Gamma\varrho,\tau) \\ c_{15}\Phi(\Gamma\omega,\Gamma\varrho,\varpi,\tau)+c_{16}\Phi(\varpi,\Gamma\varrho,\Gamma\varrho,\tau) \end{array}\right\} \tag{11}$$

where $c_i \in [0,+\infty]$, $i = 1, \cdots, 16$ *and* $c_1 + \cdots + c_{14} + 2(c_{15} + c_{16}) < 1$. *Then* Γ *has a fixed point.*

Proof: Let $\xi_0 \in \Xi$ be an arbitrary point. Generate a sequence $\{\xi_n\}$ with $\xi_n = \Gamma\xi_{n-1}$ for $n \in \mathbb{N}$. If there exists a non-negative integer m such that $\xi_{m+1} = \xi_m$. Then $\Gamma\xi_m = \xi_m$ and ξ_m becomes a fixed point of Γ.

Suppose $\xi_n \neq \xi_{n-1}$ for any $n \in \mathbb{N}$. As Ψ, Y and Φ are triangular and Lemma 3, we have

$$\begin{aligned} \left(\frac{1}{\Psi(\xi_{n+1},\xi_{n+1},\xi_{n-1},\tau)}-1\right) &= \left(\frac{1}{\Psi(\xi_{n-1},\xi_{n-1},\xi_{n+1},\tau)}-1\right) \\ &\leq \left(\frac{1}{\Psi(\xi_{n-1},\xi_{n-1},\xi_n,\tau)}-1\right)+\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right), \end{aligned} \tag{12}$$

$$\begin{aligned} Y(\xi_{n+1},\xi_{n+1},\xi_{n-1},\tau) &\leq Y(\xi_{n-1},\xi_{n-1},\xi_{n+1},\tau) \\ \leq Y(\xi_{n-1},\xi_{n-1},\xi_n,\tau) &+ Y(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \\ \Phi(\xi_{n+1},\xi_{n+1},\xi_{n-1},\tau) &\leq \Phi(\xi_{n-1},\xi_{n-1},\xi_{n+1},\tau) \\ \leq \Phi(\xi_{n-1},\xi_{n-1},\xi_n,\tau) &+ \Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \end{aligned} \tag{13}$$

$$\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_{n+1},\tau)}-1\right) \leq \left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right)+\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \tag{14}$$

$$\Upsilon(\xi_{n-1},\xi_n,\xi_{n+1},\tau) \le \Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau) + \Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau). \tag{15}$$

$$\Phi(\xi_{n-1},\xi_n,\xi_{n+1},\tau) \le \Phi(\xi_{n-1},\xi_n,\xi_n,\tau) + \Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau). \tag{16}$$

Using inequalities (9), (10) and (11) and above inequalities, we obtain

$$\begin{gathered}\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \le \frac{c_1+\cdots c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right),\\ \Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le \frac{c_1+\cdots c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}\Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau),\\ \Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le \frac{c_1+\cdots c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})}\Phi(\xi_{n-1},\xi_n,\xi_n,\tau).\end{gathered}$$

Putting

$$c = \frac{c_1+\cdots c_4+c_{15}+c_{16}}{1-(c_5+\cdots+c_{16})},$$

the above inequality becomes

$$\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \le c\left(\frac{1}{\Psi(\xi_{n-1},\xi_n,\xi_n,\tau)}-1\right), \tag{17}$$

$$\Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le c\Upsilon(\xi_{n-1},\xi_n,\xi_n,\tau), \tag{18}$$

$$\Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le c\Phi(\xi_{n-1},\xi_n,\xi_n,\tau). \tag{19}$$

By utilizing inequalities (13), (14) and (15) made the sequence $\{\xi_n\}$ fuzzy cone contractive. Hence by Lemma 2 $\{\xi_n\}$ is Cauchy in Ξ. As Ξ is complete, there exists $\dot{s} \in \Xi$ such that

$$\left.\begin{gathered}\lim_{n\to+\infty}\left(\frac{1}{\Psi(\xi_n,\dot{s},\dot{s},\tau)}-1\right)=0\\ \lim_{n\to+\infty}\Upsilon(\xi_n,\dot{s},\dot{s},\tau)=0\\ \lim_{n\to+\infty}\Phi(\xi_n,\dot{s},\dot{s},\tau)=0\end{gathered}\right\}. \tag{20}$$

By repeated application of (18), (19) and (20), we obtain that

$$\begin{gathered}\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right) \le c^n\left(\frac{1}{\Psi(\xi_0,\xi_1,\xi_1,\tau)}-1\right),\\ \Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le c^n\Upsilon(\xi_0,\xi_1,\xi_1,\tau),\\ \Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau) \le c^n\Phi(\xi_0,\xi_1,\xi_1,\tau).\end{gathered}$$

Implies that,

$$\left.\begin{gathered}\lim_{n\to+\infty}\left(\frac{1}{\Psi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)}-1\right)=0\\ \lim_{n\to+\infty}\Upsilon(\xi_n,\xi_{n+1},\xi_{n+1},\tau)=0\\ \lim_{n\to+\infty}\Phi(\xi_n,\xi_{n+1},\xi_{n+1},\tau)=0\end{gathered}\right\}. \tag{21}$$

From (9), we have

$$\begin{aligned}\left(\frac{1}{\Psi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right) &= \left(\frac{1}{\Psi(\Gamma\xi_n,\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\\ &\le d\left(\frac{1}{\Psi(\Gamma\dot{s}\Gamma\dot{s},\dot{s},\tau)}-1\right),\end{aligned}$$

where $d = c_5 + \cdots + c_{16}$, hence

$$\text{limsup}_{n\to+\infty}\left(\frac{1}{\Psi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\leq d\left(\frac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right).$$

Similarly,

$$\text{limsup}_{n\to+\infty}\Upsilon(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$

$$\text{limsup}_{n\to+\infty}\Phi(\xi_{n+1},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau).$$

As Ψ, Υ and Φ are triangular

$$\left(\frac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\leq d\left(\frac{1}{\Psi(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)}-1\right)+\left(\frac{1}{\Psi(\xi_{n+1},\dot{s},\dot{s},\tau)}-1\right)\tag{22}$$

$$\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq\Upsilon(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)+\Upsilon(\xi_{n+1},\dot{s},\dot{s},\tau)\tag{23}$$

$$\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq\Phi(\Gamma\dot{s},\Gamma\dot{s},\xi_{n+1},\tau)+\Phi(\xi_{n+1},\dot{s},\dot{s},\tau).\tag{24}$$

From (22) to (24), we can bring that

$$\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)\leq d\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right),$$
$$\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$
$$\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)\leq d\Phi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau),$$

implies

$$\left(\tfrac{1}{\Psi(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)}-1\right)=0,$$
$$\Upsilon(\dot{s},\Gamma\dot{s},\Gamma\dot{s},\tau)=0,$$
$$\Phi(\dot{s},\Gamma\dot{s}\Gamma\dot{s},\tau)=0,\ \text{as } d<1.$$
$$\Gamma\dot{s}=\dot{s}.$$

Thus, we can conclude that $\dot{s}$ is a fixed point of Γ. Suppose $\Gamma\ddot{\xi}=\ddot{\xi}$. Then from (9), (10), (11) and by Lemma 1, we have

$$\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)}-1\right)\leq d\prime\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)}-1\right),$$
$$\Upsilon\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)\leq d'\Upsilon\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right),$$
$$\Phi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)\leq d'\Phi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right),$$

where, $d' = c_1 + c_2 + c_7 + c_8 + c_{15} + c_{16}$.

These inequalities imply that

$$\left(\frac{1}{\Psi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)}-1\right)=0,\Upsilon\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)=0\text{ and }\Phi\left(\dot{s},\ddot{\xi},\ddot{\xi},\tau\right)=0.$$

Since, $d' < 1$. Thus, we can conclude that Γ is a fixed point. □

Corollary 4. *Suppose* $(\Xi, \Psi, \Upsilon, \Phi, *, \diamond)$ *be a complete GNCMS, where* Ψ, Υ *and* Φ *are triangular. If* $\Gamma : \Xi \to \Xi$ *is a self-mapping such that for all* $\omega, \varrho \in \Xi, \tau \in int(\mathcal{C})$,

$$\left(\frac{1}{\Psi(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau)} - 1\right) \leq \begin{Bmatrix} c_1\left(\frac{1}{\Psi(\varpi,\omega,\varrho,\tau)} - 1\right) + c_2\left(\frac{1}{\Psi(\varpi,\Gamma\varpi,\varrho,\tau)} - 1\right) + c_3\left(\frac{1}{\Psi(\Gamma\varpi,\Gamma\omega,\varrho,\tau)} - 1\right) + \\ c_4\left(\frac{1}{\Psi(\omega,\Gamma\varrho,\Gamma\varrho,\tau)} - 1\right) + c_5\left(\frac{1}{\Psi(\Gamma\omega,\Gamma\varrho,\varpi,\tau)} - 1\right) + c_6\left(\frac{1}{\Psi(\varpi,\Gamma\omega,\varrho,\tau)} - 1\right) \end{Bmatrix},$$

$$\Upsilon(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1\Upsilon(\varpi,\omega,\varrho,\tau) + c_2\Upsilon(\varpi,\Gamma\varpi,\varrho,\tau) + c_3\Upsilon(\Gamma\varpi,\Gamma\omega,\varrho,\tau) + \\ c_4\Upsilon(\omega,\Gamma\varrho,\Gamma\varrho,\tau) + c_5\Upsilon(\Gamma\omega,\Gamma\varrho,\varpi,\tau) + c_6\Upsilon(\varpi,\Gamma\omega,\varrho,\tau) \end{Bmatrix},$$

$$\Phi(\Gamma\varpi, \Gamma\omega, \Gamma\varrho, \tau) \leq \begin{Bmatrix} c_1\Phi(\varpi,\omega,\varrho,\tau) + c_2\Phi(\varpi,\Gamma\varpi,\varrho,\tau) + c_3\Phi(\Gamma\varpi,\Gamma\omega,\varrho,\tau) + \\ c_4\Phi(\omega,\Gamma\varrho,\Gamma\varrho,\tau) + c_5\Phi(\Gamma\omega,\Gamma\varrho,\varpi,\tau) + c_6\Phi(\varpi,\Gamma\omega,\varrho,\tau) \end{Bmatrix},$$

where

$$c_i \in [0, +\infty],\ i = 1, \cdots, 6 \text{ and } c_1 + c_2 + c_3 + c_4 + 2(c_5 + c_6) < 1.$$

Then Γ has a unique fixed point.

4. Conclusions

In this paper, we established several fixed-point results for new types of contraction mappings in the context of GNCMSs and derived an example to show the validity of our main result. If the triangular condition does not hold then these results cannot be fulfilled under the given conditions. This work could be extended to increase the number of self-mappings, i.e., two self-mappings, three-self-mappings, etc., and in different structures such as generalized neutrosophic cone b-metric spaces, generalized neutrosophic cone controlled-metric spaces, etc.

Author Contributions: Conceptualization, U.I. and K.A.; methodology, U.I. and I.S.; software, K.A.; validation, A.H., M.A. and H.A.S.; formal analysis, A.H.; investigation, U.I.; resources, M.A.; data curation, I.S.; writing—original draft preparation, K.A.; writing—review and editing, U.I. and M.A.; visualization, A.H.; supervision, U.I.; project administration, H.A.S.; funding acquisition, A.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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