symmetry

MDPI

Article

Plane Partitions as Sums over Partitions

Mircea Merca [1,2,*] and Iulia-Ionelia Radu [3]

[1] Department of Mathematical Methods and Models, Fundamental Sciences Applied in Engineering Research Center, University Politehnica of Bucharest, 060042 Bucharest, Romania
[2] Academy of Romanian Scientists, 050044 Bucharest, Romania
[3] Aurel Vlaicu High School, 105400 Breaza, Romania; iulia.radu@liceulteoreticaurelvlaicu.ro
* Correspondence: mircea.merca@upb.ro

Abstract: In this paper, we consider complete homogeneous symmetric functions and provide a new formula for the number of plane partitions of n. This formula expresses the number of plane partitions of n in terms of binomial coefficients as a sum over all the partitions of n, considering the multiplicity of the parts greater than one. We obtain similar results for the number of strict plane partition of n and the number of symmetric plane partitions of n.

Keywords: partitions; plane partitions; symmetric functions; binomial coefficients

MSC: 11P81; 11P82; 05A19; 05A20

check for updates

Citation: Merca, M.; Radu, I.-I. Plane Partitions as Sums over Partitions. *Symmetry* **2023**, *15*, 1820. https://doi.org/10.3390/sym15101820

Academic Editor: Serkan Araci

Received: 14 August 2023
Revised: 15 September 2023
Accepted: 20 September 2023
Published: 25 September 2023

1. Introduction

We recall that a composition of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers whose sum is n, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k. \tag{1}$$

The positive integers in the sequence are called parts [1]. When the order of integers λ_i does not matter, Representation (1) is known as an integer partition and can be rewritten as

$$n = t_1 + 2t_2 + \cdots + nt_n,$$

where each positive integer i appears t_i times in the partition. For consistency, a partition of n is written with the summands in nonincreasing order. As usual, we denote by $p(n)$ the number of integer partitions of n. Partitions can be graphically visualized with Young diagrams. For example, the five partitions of four can be seen in Figure 1. It is clear that $p(4) = 5$. For convenience, we define $p(0) = 1$.

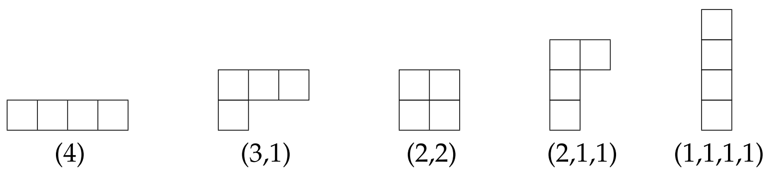

Figure 1. The partitions of 4.

Euler showed that the generating function of $p(n)$ can be expressed as an elegant infinite product

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{(q;q)_\infty}. \tag{2}$$

Here and throughout the paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we assume $|q| < 1$. All identities may be understood in the sense of formal power series in q.

A plane partition π of the positive integer n is a two-dimensional array $\pi = (\pi_{i,j})_{i,j\geqslant 1}$ of non-negative integers $\pi_{i,j}$ such that

$$n = \sum_{i,j\geqslant 1} \pi_{i,j},$$

which is weakly decreasing in rows and columns:

$$\pi_{i,j} \geqslant \pi_{i+1,j}, \qquad \pi_{i,j} \geqslant \pi_{i,j+1}, \qquad \text{for all } i,j \geqslant 1.$$

It can be considered as the filling of a Young diagram with weakly decreasing rows and columns, where the sum of all these numbers is equal to n. Different configurations are counted as different plane partitions. As usual, we denote by $PL(n)$ the number of plane partitions of n. The plane partitions of four are presented in Figure 2. We see that $PL(4) = 13$. For convenience, we define $PL(0) = 1$.

$$\begin{array}{|c|}\hline 4\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array}$$

$$\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline 1\\\hline\end{array}$$

Figure 2. The plane partitions of 4.

An equivalent definition is that a plane partition is a finite subset of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with the property that if $(r,s,t) \in \pi$ and $(1,1,1) \leqslant (h,j,k) \leqslant (r,s,t)$, then (h,j,k) must be an element of π. Here, $(h,j,k) \leqslant (r,s,t)$ means $h \leqslant r$, $j \leqslant s$ and $k \leqslant t$. There is a nice way to represent a plane partition as a three-dimensional object: this is achieved by replacing each part k of the plane partition by a stack of k unit cubes. Thus, we obtain a pile of unit cubes. The piles of cubes corresponding to the plane partitions in Figure 2 are shown in Figure 3.

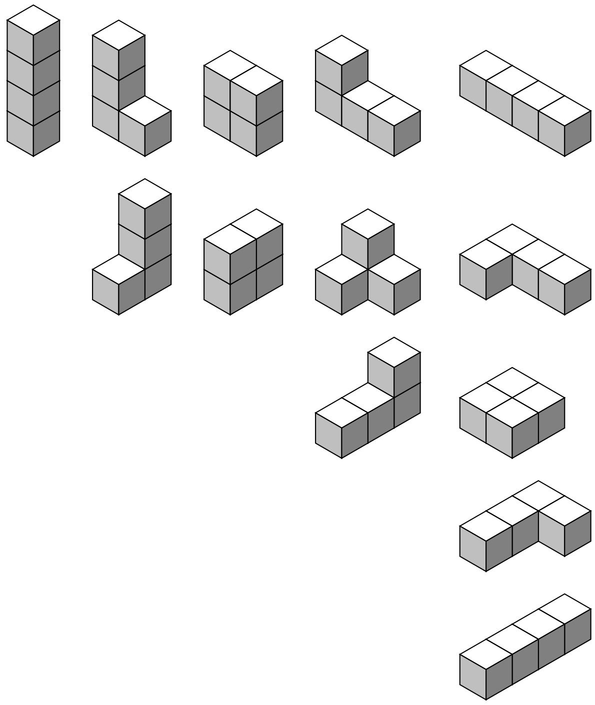

Figure 3. The plane partitions of 4 as piles of cubes.

Plane partitions were introduced to mathematics by Major Percy Alexander MacMahon [2] as generalizations of partitions of integers. MacMahon [3] (Section 429) offered a surprisingly simple formula for the generating function for all plane partitions π contained in an $a \times b \times c$ box. According to Macdonald [4] (Equation (2) on p. 81), this formula can be written as

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

Letting $a, b, c \to \infty$, we obtain an elegant product formula for the generating function for all plane partitions, i.e.,

$$\sum_{n=0}^{\infty} PL(n)\, q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n},$$

and the expansion starts as

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^n} = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + 86\,q^7 + 160\,q^8 + \cdots. \quad (3)$$

The arrangement of the plane partitions of four in Figure 2 or Figure 3 is not random. According to M. K. Azarian [5] (Theorem 1.1), $p(n)$ can be interpreted as the number of different ways to run up a staircase with n steps, taking steps of possibly different sizes, where the order is not important and there is no restriction on the number or the size of each step taken. Any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be considered a staircase with n steps where λ_k $(1 \leqslant k \leqslant n)$ is just a label associated with the kth step. On the other hand, any partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be converted into plane partitions by insertion of line feeds at some or all places of the commas. The plane partition obtained in this way can be interpreted as a way to run up the staircase labeled by λ, i.e., the number of parts on the kth line of the plane partition represents the size of the kth step. We remark that there are plane partitions that cannot be obtained in this way. For example, the plane partition in Figure 4 cannot be obtained from the partition $(2, 2, 1)$ by insertion of line feeds at some or all places of the commas.

2	1
2	

Figure 4. A plane partitions of 5

Since the length of rows in plane partitions must be nonincreasing, there are only $p(n)$ ways to comply with this rule. Thus, we easily deduce the following inequality for $PL(n)$:

Theorem 1. *For $n \geqslant 0$,*

$$PL(n) \geqslant \sum_{t_1+2t_2+\cdots+nt_n=n} p(t_1+t_2+\cdots+t_n),$$

with strict inequality if and only if $n > 4$.

Rewriting the partitions of four as

$$\begin{array}{c} 1\cdot 0+2\cdot 0+3\cdot 0+4\cdot 1 \\ 1\cdot 1+2\cdot 0+3\cdot 1+4\cdot 0 \\ 1\cdot 0+2\cdot 2+3\cdot 0+4\cdot 0 \\ 1\cdot 2+2\cdot 1+3\cdot 0+4\cdot 0 \\ 1\cdot 4+2\cdot 0+3\cdot 0+4\cdot 0, \end{array} \tag{4}$$

the case $n = 4$ of Theorem 1 reads as follows:

$$\begin{aligned} PL(4) &\geqslant p(0+0+0+1)+p(1+0+1+0) \\ &\qquad +p(0+2+0+0)+p(2+1+0+0)+p(4+0+0+0) \\ &= 1+2+2+3+5 = 13. \end{aligned}$$

In this paper, motivated by Theorem 1, we want to show that $PL(n)$ can be expressed as a sum over all the partitions of n in terms of binomial coefficients. This new formula considers the multiplicity of the parts greater than one.

Theorem 2. *For $n \geqslant 0$,*

$$PL(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{1+t_2}{t_2}\binom{2+t_3}{t_3}\cdots\binom{n-1+t_n}{t_n}.$$

Considering (4), the case $n = 4$ of Theorem 2 reads as follows:

$$\begin{aligned} PL(4) &= \binom{1+0}{0}\binom{2+0}{0}\binom{3+1}{1}+\binom{1+0}{0}\binom{2+1}{1}\binom{3+0}{0} \\ &\qquad +\binom{1+2}{2}\binom{2+0}{0}\binom{3+0}{0}+\binom{1+1}{1}\binom{2+0}{0}\binom{3+0}{0} \\ &\qquad +\binom{1+0}{0}\binom{2+0}{0}\binom{3+0}{0} \\ &= 4+3+3+2+1 = 13. \end{aligned}$$

In this context, we remark that the first differences of $PL(n)$ can be be expressed as a sum over the partitions of n into parts greater than one in terms of binomial coefficients.

Theorem 3. *For $n \geqslant 0$,*

$$PL(n)-PL(n-1) = \sum_{2t_2+3t_3+\cdots+nt_n=n} \binom{1+t_2}{t_2}\binom{2+t_3}{t_3}\cdots\binom{n-1+t_n}{t_n}.$$

According to (3), we have

$$PL(4) - PL(3) = 13 - 6 = 7.$$

The partitions of four into parts greater than pne are (4) and $(2,2)$. Therefore, case $n = 4$ of Theorem 2 reads as follows:

$$\begin{aligned} PL(4) - PL(3) &= \binom{1+0}{0}\binom{2+0}{0}\binom{3+1}{1} + \binom{1+2}{2}\binom{2+0}{0}\binom{3+0}{0} \\ &= 4 + 3 = 7. \end{aligned}$$

The following result shows that the partial sums of $PL(n)$ can be be expressed as a sum over all the partitions of n in terms of binomial coefficients.

Theorem 4. *For $n \geqslant 0$,*

$$\sum_{k=0}^{n} PL(k) = \sum_{t_1+2t_2+3t_3+\cdots+nt_n=n} \binom{1+t_1}{t_1}\binom{1+t_2}{t_2}\binom{2+t_3}{t_3}\cdots\binom{n-1+t_n}{t_n}.$$

According to (3), we have

$$PL(0) + PL(1) + PL(2) + PL(3) + PL(4) = 1 + 1 + 3 + 6 + 13 = 24.$$

Considering (4), case $n = 4$ of Theorem 4 reads as follows:

$$\begin{aligned} &PL(0) + PL(1) + PL(2) + PL(3) + PL(4) \\ &\quad = \binom{1+0}{0}\binom{1+0}{0}\binom{2+0}{0}\binom{3+1}{1} + \binom{1+1}{1}\binom{1+0}{0}\binom{2+1}{1}\binom{3+0}{0} \\ &\qquad + \binom{1+0}{0}\binom{1+2}{2}\binom{2+0}{0}\binom{3+0}{0} + \binom{1+2}{2}\binom{1+1}{1}\binom{2+0}{0}\binom{3+0}{0} \\ &\qquad + \binom{1+4}{4}\binom{1+0}{0}\binom{2+0}{0}\binom{3+0}{0} \\ &\quad = 4 + 6 + 3 + 6 + 5 = 24. \end{aligned}$$

Upon reflection, one expects that there might be more general results where our Theorems 2–4 are the first entries. For any positive integer m, we denote by $PL^{(m)}(n)$ the number of m-tuples of plane partitions of non-negative integers $n_1, n_2, \ldots, n_m$ where $n_1 + n_2 + \cdots + n_m = n$. It is clear that $PL(n) = PL^{(1)}(n)$ and

$$PL^{(m)}(n) = \sum_{n_1+n_2+\cdots+n_m=n} PL(n_1)\, PL(n_2)\, \cdots\, PL(n_m).$$

For any positive integer m, $PL^{(m)}(n)$, the partial sums of $PL^{(m)}(n)$ and the first differences of $PL^{(m)}(n)$ can be expressed as a sum over all the partitions of n in terms of binomial coefficients.

Theorem 5. *For $m \geqslant 1$ and $n \geqslant 0$,*

1. $$PL^{(m)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+t_1}{t_1}\prod_{j=2}^{n}\binom{jm-1+t_j}{t_j};$$
2. $$\sum_{k=0}^{n} PL^{(m)}(k) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m+t_1}{t_1}\prod_{j=2}^{n}\binom{jm-1+t_j}{t_j};$$
3. $$PL^{(m)}(n) - PL^{(m)}(n-1) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-2+t_1}{t_1}\prod_{j=2}^{n}\binom{jm-1+t_j}{t_j}.$$

The remainder of the paper is organized as follows. In Section 2, we consider the complete homogeneous symmetric functions and introduce Theorem 6. This general result allows us provision of an analytic proof of Theorem 5 by considering specializations of complete homogeneous symmetric functions. In Section 3, we provide other applications of Theorem 6 by considering the strict plane partitions and the symmetric plane partitions. In the last section, we consider a sum over all the partitions of n in order to provide a new expression for the generating function of $PL(n)$. Finding a combinatorial interpretation in terms of plane partitions for this sum over all partitions of n remains an open problem.

2. Proof of Theorem 5

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is an integer partition with $k \leqslant n$, then the monomial symmetric function

$$m_\lambda(x_1, x_2, \dots, x_n) = m_{(\lambda_1,\lambda_2,\dots,\lambda_k)}(x_1, x_2, \dots, x_n)$$

is the sum of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ and all distinct monomials obtained from this by a permutation of variables. For instance, with $\lambda = (2,1,1)$ and $n = 4$, we have

$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3, x_4) &= x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_4 + x_1 x_2^2 x_4 + x_1 x_2 x_4^2 \\ &\quad + x_1^2 x_3 x_4 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2. \end{aligned}$$

If every monomial in a symmetric function has total degree k, then we say that this symmetric function is homogeneous of degree k. Proofs and more details about monomial symmetric functions can be found in Macdonald's book [4].

The kth complete homogeneous symmetric function h_k is the sum of all monomials of total degree k in these variables, i.e.,

$$h_k(x_1, x_2, \dots, x_n) = \sum_{\lambda \vdash k} m_\lambda(x_1, x_2, \dots, x_n) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

where $\lambda \vdash n$ indicates that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition of n. It is well known that the complete homogeneous symmetric functions are characterized by the following formal power series identity in t:

$$\sum_{k=0}^{\infty} h_k(x_1, x_2, \dots, x_n)\, t^k = \prod_{i=1}^{n} \frac{1}{1 - x_i t}. \tag{5}$$

Considering the complete homogeneous symmetric functions, we introduce the following result:

Theorem 6. *Let m and n be positive integers. Then,*

$$\prod_{k=1}^{n} \frac{1}{(1 - x_k z)^{a_k(m)}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\dots+t_n=k} \prod_{j=1}^{n} \binom{a_j(m) - 1 + t_j}{t_j} x_j^{t_j} \right) z^k,$$

where $x_1, x_2, \dots, x_n$ are independent variables and $\big(a_n(m)\big)_{n\geqslant 1}$ is a sequence of non-negative integers.

Proof. We are to prove the theorem by induction on n. For $n = 1$, we have

$$\begin{aligned} \frac{1}{(1 - x_1 z)^{a_1(m)}} &= \prod_{k=1}^{a_1(m)} \frac{1}{1 - x_1 z} \\ &= \sum_{k=0}^{\infty} h_k(\underbrace{x_1, \dots, x_1}_{a_1(m)\text{ times}})\, z^k \end{aligned}$$

$$= \sum_{k=0}^{\infty} x_1^k \, h_k(\underbrace{1,\ldots,1}_{a_1(m)\text{ times}}) \, z^k$$
$$= \sum_{k=0}^{\infty} \binom{a_1(m)-1+k}{k} x_1^k \, z^k,$$

and the base case of induction is finished. We suppose that relation

$$\prod_{k=1}^{N} \frac{1}{(1-x_k z)^{a_k(m)}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_N=k} \prod_{j=1}^{N} \binom{a_j(m)-1+t_j}{t_j} x_j^{t_j} \right) z^k$$

is true for any integer N, $1 \leqslant N < n$. We can write

$$\begin{aligned}
&\prod_{k=1}^{n} \frac{1}{(1-x_k z)^{a_k(m)}} \\
&= \frac{1}{(1-x_n z)^{a_n(m)}} \prod_{k=1}^{n-1} \frac{1}{(1-x_k z)^{a_k(m)}} \\
&= \left(\prod_{k=1}^{a_n(m)} \frac{1}{1-x_n z} \right) \left(\prod_{k=1}^{n-1} \frac{1}{(1-x_k z)^{a_k(m)}} \right) \\
&= \left(\sum_{k=0}^{\infty} h_k(\underbrace{x_n,\ldots,x_n}_{a_n(m)\text{ times}}) \, z^k \right) \left(\prod_{k=1}^{n-1} \frac{1}{(1-x_k z)^{a_k(m)}} \right) \\
&= \left(\sum_{k=0}^{\infty} x_n^k \, h_k(\underbrace{1,\ldots,1}_{a_n(m)\text{ times}}) \, z^k \right) \left(\prod_{k=1}^{n-1} \frac{1}{(1-x_k z)^{a_k(m)}} \right) \\
&= \left(\sum_{k=0}^{\infty} x_n^k \binom{a_n(m)-1+k}{k} z^k \right) \left(\sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_{n-1}=k} \prod_{j=1}^{n-1} \binom{a_j(m)-1+t_j}{t_j} x_j^{t_j} \right) z^k \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{j=1}^{n} \binom{a_j(m)-1+t_j}{t_j} x_j^{t_j} \right) z^k,
\end{aligned}$$

where we invoke the well-known Cauchy multiplications of two power series. □

We are now in the position to prove Theorem 5. By Theorem 6, with x_k replaced by q^{k-1} and z replaced by q, we obtain

$$\begin{aligned}
\prod_{k=1}^{n} \frac{1}{(1-q^k)^{a_k(m)}} &= \sum_{k=0}^{\infty} q^k \sum_{t_1+t_2+\cdots+t_n=k} q^{t_2+2t_3+\cdots+(n-1)t_n} \prod_{j=1}^{n} \binom{a_j(m)-1+t_j}{t_j} \\
&= \sum_{k=0}^{\infty} \sum_{t_1+t_2+\cdots+t_n=k} q^{t_1+2t_2+3t_3+\cdots+nt_n} \prod_{j=1}^{n} \binom{a_j(m)-1+t_j}{t_j}.
\end{aligned}$$

The limiting case $n \to \infty$ of this relation reads as

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{a_k(m)}} = \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \prod_{j=1}^{k} \binom{a_j(m)-1+t_j}{t_j}. \tag{6}$$

By this identity, with $a_j(m)$ replaced by jm, we obtain

$$\sum_{k=0}^{\infty} PL^{(m)}(k) \, q^k = \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \prod_{j=1}^{k} \binom{jm-1+t_j}{t_j},$$

and the first identity of Theorem 5 follows easily by equating the coefficients of q^k in this relation.

The proof of the second identity of Theorem 5 is quite similar to the proof of the first identity. We take into account the fact that the generating function for the partial sums of $PL^{(m)}(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} PL^{(m)}(k) &= \frac{1}{1-q} \sum_{n=0}^{\infty} PL^{(m)}(n)\ q^n \\ &= \frac{1}{(1-q)^{m+1}} \prod_{n=2}^{\infty} \frac{1}{(1-q^n)^{nm}}.\end{aligned}$$

We let $\delta_{i,j}$ to be the Kronecker delta function. By (6), with $a_j(m)$ replaced by $jm+\delta_{1,j}$, we obtain

$$\begin{aligned}&\frac{1}{(1-q)^{m+1}} \prod_{k=2}^{n} \frac{1}{(1-q^k)^{km}} \\ &\quad = \sum_{k=0}^{\infty} q^k \sum_{t_1+t_2+\cdots+t_n=k} q^{t_2+2t_3+\cdots+(n-1)t_n} \binom{m+t_1}{t_1} \prod_{j=2}^{n} \binom{jm-1+t_j}{t_j} \\ &\quad = \sum_{k=0}^{\infty} \sum_{t_1+t_2+\cdots+t_n=k} q^{t_1+2t_2+3t_3+\cdots+nt_n} \binom{m+t_1}{t_1} \prod_{j=2}^{n} \binom{jm-1+t_j}{t_j}.\end{aligned}$$

The limiting case $n \to \infty$ of this relation reads as

$$\sum_{k=0}^{\infty} q^k \sum_{j=0}^{k} PL^{(m)}(j) = \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{m+t_1}{t_1} \prod_{j=2}^{k} \binom{jm-1+t_j}{t_j},$$

and the second identity of Theorem 5 follows easily by equating the coefficients of q^k in this relation.

In order to prove the last identity of Theorem 5, we consider the fact that the generating function for the first differences of $PL(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} \left(PL^{(m)}(n) - PL^{(m)}(n-1)\right) q^n &= (1-q) \sum_{n=0}^{\infty} PL^{(m)}(n)\, q^n \\ &= \frac{1}{(1-q)^{m-1}} \prod_{n=2}^{\infty} \frac{1}{(1-q^n)^{nm}}.\end{aligned}$$

By (6), with $a_j(m)$ replaced by $jm-\delta_{1,j}$, we obtain

$$\begin{aligned}&\frac{1}{(1-q)^{m-1}} \prod_{k=2}^{n} \frac{1}{(1-q^k)^{km}} \\ &\quad = \sum_{k=0}^{\infty} q^k \sum_{t_1+t_2+\cdots+t_n=k} q^{t_2+2t_3+\cdots+(n-1)t_n} \binom{m-2+t_1}{t_1} \prod_{j=2}^{n} \binom{jm-1+t_j}{t_j} \\ &\quad = \sum_{k=0}^{\infty} \sum_{t_1+t_2+\cdots+t_n=k} q^{t_1+2t_2+3t_3+\cdots+nt_n} \binom{m-2+t_1}{t_1} \prod_{j=2}^{n} \binom{jm-1+t_j}{t_j}.\end{aligned}$$

The limiting case $n \to \infty$ of this relation reads as

$$\sum_{k=0}^{\infty} \left(PL^{(m)}(k) - PL^{(m)}(k-1)\right) q^k = \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{m-2+t_1}{t_1} \prod_{j=2}^{k} \binom{jm-1+t_j}{t_j},$$

and the last identity of Theorem 5 follows easily by equating the coefficients of q^k in this relation.

3. Further Applications of Theorem 6

In this section, we introduce two applications of Theorem 6 related to plane partitions with restrictions.

3.1. Strict Plane Partitions

A strict plane partition π of the positive integer n is a plane partition $\pi = (\pi_{i,j})_{i,j\geqslant 1}$ of n which is decreasing in rows, i.e., $\pi_{i,j} > \pi_{i+1,j}$, for all $i, j \geqslant 1$. We denote by $SPL(n)$ the number of strict plane partitions of n. The strict plane partitions of four are presented in Figure 5. We see that $SPL(4) = 7$. For convenience, we define $SPL(0) = 1$.

$$\begin{array}{|c|}\hline 4\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 3&1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array}$$

$$\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array}$$

Figure 5. The strict plane partitions of 4.

According to Gordon and Houten [6], the generating function for the number of strict plane partition of n is given by

$$\sum_{n=0}^{\infty} SPL(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\lceil n/2\rceil}},$$

and the expansion starts as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\lceil n/2\rceil}} = 1 + q + 2\,q^2 + 4\,q^3 + 7\,q^4 + 12\,q^5 + 21\,q^6 + 34\,q^7 + 56\,q^8 + \cdots. \tag{7}$$

For any positive integer m, we denote by $SPL^{(m)}(n)$ the number of m-tuples of strict plane partitions of non-negative integers $n_1, n_2, \ldots, n_m$ where $n_1 + n_2 + \cdots + n_m = n$. It is clear that $SPL(n) = SPL^{(1)}(n)$ and

$$SPL^{(m)}(n) = \sum_{n_1+n_2+\cdots+n_m=n} SPL(n_1)\, SPL(n_2) \,\cdots\, SPL(n_m).$$

For any positive integer m, $SPL^{(m)}(n)$, the partial sums of $SPL^{(m)}(n)$ and the first differences of $SPL^{(m)}(n)$ can be expressed as a sum over all the partitions of n in terms of binomial coefficients.

Theorem 7. *For $m \geqslant 1$ and $n \geqslant 0$,*

1. $SPL^{(m)}(n) = \displaystyle\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+t_1}{t_1} \prod_{j=2}^{n} \binom{\lceil j/2\rceil m - 1 + t_j}{t_j};$
2. $\displaystyle\sum_{k=0}^{n} SPL^{(m)}(k) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m+t_1}{t_1} \prod_{j=2}^{n} \binom{\lceil j/2\rceil m - 1 + t_j}{t_j};$
3. $SPL^{(m)}(n) - SPL^{(m)}(n-1) = \displaystyle\sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-2+t_1}{t_1} \prod_{j=2}^{n} \binom{\lceil j/2\rceil m - 1 + t_j}{t_j}.$

Proof. The proof of this theorem is quite similar to the proof of Theorem 5. Therefore, we omit the details. □

If $m = 1$, then we have

$$\binom{m-2+t_1}{t_1} = 0.$$

Thus, the sum in the right hand side of the last identity of Theorem 7 runs over all the partitions of n into parts greater than one. The case $m = 1$ of Theorem 7 reads as follows:

Corollary 1. *For $m \geqslant 1$ and $n \geqslant 0$,*

1. $SPL(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \prod_{j=3}^{n} \binom{\lceil j/2 \rceil - 1 + t_j}{t_j};$
2. $\sum_{k=0}^{n} SPL(k) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{1+t_1}{t_1} \prod_{j=3}^{n} \binom{\lceil j/2 \rceil - 1 + t_j}{t_j};$
3. $SPL(n) - SPL(n-1) = \sum_{2t_2+\cdots+nt_n=n} \prod_{j=3}^{n} \binom{\lceil j/2 \rceil - 1 + t_j}{t_j}.$

According to (7), we have

$$\begin{aligned}
&SPL(4) = 7, \\
&SPL(0) + SPL(1) + SPL(2) + SPL(3) + SPL(4) = 1 + 1 + 2 + 4 + 7 = 15, \\
&SPL(4) - SPL(3) = 7 - 4 = 3.
\end{aligned}$$

Considering (4), the case $n = 4$ of Corollary 1 can be written as

$$\begin{aligned}
SPL(4) &= \binom{1+0}{0}\binom{1+1}{1} + \binom{1+1}{1}\binom{1+0}{0} \\
&\quad + \binom{1+0}{0}\binom{1+0}{0} + \binom{1+0}{0}\binom{1+0}{0} + \binom{1+0}{0}\binom{1+0}{0} \\
&= 2 + 2 + 1 + 1 + 1 = 7, \\
\sum_{k=0}^{4} SPL(k) &= \binom{1+0}{0}\binom{1+0}{0}\binom{1+1}{1} + \binom{1+1}{1}\binom{1+1}{1}\binom{1+0}{0} \\
&\quad + \binom{1+0}{0}\binom{1+0}{0}\binom{1+0}{0} + \binom{1+2}{2}\binom{1+0}{0}\binom{1+0}{0} \\
&\quad + \binom{1+4}{4}\binom{1+0}{0}\binom{1+0}{0} \\
&= 2 + 4 + 1 + 3 + 5 = 15.
\end{aligned}$$

The partitions of four into parts greater than one are

$$2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 \qquad \text{and} \qquad 2 \cdot 2 + 3 \cdot 0 + 4 \cdot 0.$$

When we take into account Corollary 1, we can write

$$\begin{aligned}
SPL(4) - SPL(3) &= \binom{1+0}{0}\binom{1+1}{1} + \binom{1+0}{0}\binom{1+0}{0} \\
&= 2 + 1 = 3.
\end{aligned}$$

3.2. Symmetric Plane Partitions

A symmetric plane partition π of the positive integer n is a plane partition $\pi = (\pi_{i,j})_{i,j \geqslant 1}$ of n such that $\pi_{i,j} = \pi_{j,i}$, for all $i, j \geqslant 1$. We denote by $sPL(n)$ the number of symmetric plane partitions of n. The symmetric plane partitions of six are presented in Figure 6. We see that $sPL(6) = 6$. For convenience, we define $sPL(0) = 1$.

$$\begin{array}{|c|}\hline 6\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 4&1\\\hline 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3&1\\\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&2\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 1\\\cline{1-1} 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

Figure 6. The symmetric plane partitions of 6.

According to Gordon [7], the generating function for the number of symmetric plane partition of n is given by

$$\sum_{n=0}^{\infty} sPL(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}},$$

where

$$a_n = \begin{cases} 1, & n \text{ odd}, \\ \lfloor n/4 \rfloor, & n \text{ even}. \end{cases}$$

The expansion starts as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}} = 1 + q + q^2 + 2\,q^3 + 3\,q^4 + 4\,q^5 + 6\,q^6 + 8\,q^7 + 12\,q^8 + \cdots. \tag{8}$$

We recall that the number of symmetric plane partition of n is equal to the number of strict plane partitions of n into odd parts [7]. The strict plane partitions of six into odd parts are presented in Figure 7.

$$\begin{array}{|c|c|}\hline 5&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 5\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 3&3\\\hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 3&1&1&1\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline\end{array}$$

Figure 7. The strict plane partitions of 6 into odd parts.

For any positive integer m, we denote by $sPL^{(m)}(n)$ the number of m-tuples of symmetric plane partitions of non-negative integers $n_1, n_2, \ldots, n_m$ where $n_1 + n_2 + \cdots + n_m = n$. It is clear that $sPL(n) = sPL^{(1)}(n)$ and

$$sPL^{(m)}(n) = \sum_{n_1+n_2+\cdots+n_m=n} sPL(n_1)\, sPL(n_2) \,\cdots\, sPL(n_m).$$

For any positive integer m, $sPL^{(m)}(n)$, the partial sums of $sPL^{(m)}(n)$ and the first differences of $sPL^{(m)}(n)$ can be expressed as a sum over all the partitions of n in terms of binomial coefficients.

Theorem 8. *For $m \geqslant 1$ and $n \geqslant 0$,*

1. $\displaystyle sPL^{(m)}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-1+t_1}{t_1} \prod_{j=2}^{n} \binom{a_j m - 1 + t_j}{t_j};$
2. $\displaystyle \sum_{k=0}^{n} sPL^{(m)}(k) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m+t_1}{t_1} \prod_{j=2}^{n} \binom{a_j m - 1 + t_j}{t_j};$
3. $\displaystyle sPL^{(m)}(n) - sPL^{(m)}(n-1) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{m-2+t_1}{t_1} \prod_{j=2}^{n} \binom{a_j m - 1 + t_j}{t_j}.$

Proof. The proof of this theorem is quite similar to the proof of Theorem 5. Therefore, we omit the details. □

If $j = 1$ and $m = 1$, then

$$\binom{\lfloor j/2 \rfloor m - 1 + t_{2j}}{t_{2j}} = 0.$$

Thus, the sum in the right hand side of identities of Theorem 8 runs over all the partitions of n into parts $\neq 2$. If $m = 1$, then we have

$$\binom{m-2+t_1}{t_1} = 0.$$

Thus, the sum in the right hand side of the last identity of Theorem 7 runs over all the partitions of n into parts greater than two. The case $m = 1$ of Theorem 8 reads as follows:

Corollary 2. *For $m \geqslant 1$ and $n \geqslant 0$,*

1. $$sPL(n) = \sum_{t_1+3t_3+4t_4+\cdots+nt_n=n} \prod_{j=3}^{\lfloor n/2 \rfloor} \binom{\lfloor j/2 \rfloor - 1 + t_{2j}}{t_{2j}};$$
2. $$\sum_{k=0}^{n} sPL(k) = \sum_{t_1+3t_3+4t_4+\cdots+nt_n=n} \binom{1+t_1}{t_1} \prod_{j=3}^{\lfloor n/2 \rfloor} \binom{\lfloor j/2 \rfloor - 1 + t_{2j}}{t_{2j}};$$
3. $$sPL(n) - sPL(n-1) = \sum_{3t_3+4t_4+\cdots+nt_n=n} \prod_{j=3}^{\lfloor n/2 \rfloor} \binom{\lfloor j/2 \rfloor - 1 + t_{2j}}{t_{2j}}.$$

According to (8), we have

$$\begin{aligned} &sPL(6) = 6, \\ &\sum_{k=0}^{6} sPL(k) = 1+1+1+2+3+4+6 = 18, \\ &sPL(6) - sPL(5) = 6 - 4 = 2. \end{aligned}$$

The partitions of six into parts $\neq 2$ are

$$\begin{aligned} &1\cdot 0 + 3\cdot 0 + 4\cdot 0 + 5\cdot 0 + 6\cdot 1, \\ &1\cdot 1 + 3\cdot 0 + 4\cdot 0 + 5\cdot 1 + 6\cdot 0, \\ &1\cdot 2 + 3\cdot 0 + 4\cdot 1 + 5\cdot 0 + 6\cdot 0, \\ &1\cdot 0 + 3\cdot 2 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0, \\ &1\cdot 3 + 3\cdot 1 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0, \\ &1\cdot 6 + 3\cdot 2 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0. \end{aligned}$$

Considering Corollary 2, we can write

$$\begin{aligned} sPL(6) &= \binom{0+1}{1} + \binom{0+0}{0} + \binom{0+0}{0} + \binom{0+0}{0} + \binom{0+0}{0} + \binom{0+0}{0} \\ &= 1+1+1+1+1+1 = 6, \\ \sum_{k=0}^{6} sPL(k) &= \binom{1+0}{0}\binom{0+1}{1} + \binom{1+1}{1}\binom{0+0}{0} + \binom{1+2}{2}\binom{0+0}{0} \\ &\quad + \binom{1+0}{0}\binom{0+0}{0} + \binom{1+3}{3}\binom{0+0}{0} + \binom{1+6}{6}\binom{0+0}{0} \\ &= 1+2+3+1+4+7 = 18. \end{aligned}$$

The partitions of six into parts > 2 are

$$\begin{aligned} &3\cdot 0 + 4\cdot 0 + 5\cdot 0 + 6\cdot 1, \\ &3\cdot 2 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0, \end{aligned}$$

Considering Corollary 2, we can write

$$sPL(6) - sPL(5) = \binom{0+1}{1} + \binom{0+0}{0} = 2.$$

4. Concluding Remarks

From (5), with x_j replaced by q^{j-1} for each $j \in \{1, 2, \ldots, n\}$, we obtain a well-known identity, which was proven by Cauchy [8] (Theorem 26).

Theorem 9 (Cauchy)**.** *If n is any non-negative integer and $|q|$ and $|t|$ are both less than one, then*

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k = \frac{1}{(t;q)_{n+1}}.$$

The limiting case $n \to \infty$ of Theorem 9 is given by the following theorem of Euler [8] (Theorem 25):

Theorem 10 (Euler)**.** *If $|q| < 1$ and $|t| < 1$, then*

$$\sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} = \frac{1}{(t;q)_\infty}.$$

By this theorem, with t replaced by q, we obtain a well-known expression for the generating function of $p(n)$, i.e.,

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n}.$$

We remark an analogous result for the generating function of $PL(n)$.

Theorem 11. *For $|q| < 1$,*

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \sum_{n=0}^{\infty} \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{q^n}{(q;q)_{t_1}(q;q)_{t_2}\cdots(q;q)_{t_n}}.$$

Proof. Considering the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & n, k \text{ integers}, 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise}, \end{cases}$$

as specializations of complete homogeneous symmetric functions, namely

$$h_k(1, q, \ldots, q^n) = \begin{bmatrix} n+k \\ k \end{bmatrix}, \tag{9}$$

we can write

$$\begin{aligned} \prod_{i=1}^{n} \frac{1}{(1-q^{i-1}z)^i} &= \prod_{i=1}^{n} \left(\prod_{j=i}^{n} \frac{1}{1-q^{j-1}z} \right) \\ &= \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} h_k(q^{i-1}, q^i, \ldots, q^{n-1})\, z^k \right) \\ &= \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} (q^{i-1})^k h_k(1, q, \ldots, q^{n-i})\, z^k \right) \end{aligned}$$

$$= \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} q^{(i-1)k} \begin{bmatrix} n-i+k \\ k \end{bmatrix} z^k \right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} q^{(i-1)t_i} \begin{bmatrix} n-i+t_i \\ t_i \end{bmatrix} \right) z^k.$$

Replacing z by q, we obtain

$$\prod_{i=1}^{n} \frac{1}{(1-q^i)^i}$$
$$= \sum_{k=0}^{\infty} \sum_{t_1+t_2+\cdots+t_n=k} q^{t_1+2t_2+\cdots+nt_n} \begin{bmatrix} n-1+t_1 \\ t_1 \end{bmatrix} \begin{bmatrix} n-2+t_2 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} n-n+t_n \\ t_n \end{bmatrix}$$
$$= \sum_{N=0}^{\infty} \sum_{t_1+2t_2+\cdots+nt_n=N} \begin{bmatrix} n-1+t_1 \\ t_1 \end{bmatrix} \begin{bmatrix} n-2+t_2 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} n-n+t_n \\ t_n \end{bmatrix} q^N.$$

The limiting case $n \to \infty$ of this equation can be written as

$$\prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i} = \sum_{N=0}^{\infty} \sum_{t_1+2t_2+\cdots+Nt_N=N} \frac{q^N}{(q;q)_{t_1}(q;q)_{t_2}\cdots(q;q)_{t_N}}.$$

This concludes the proof. □

Relevant to Theorem 11, it would be very appealing to have combinatorial interpretations for

$$\sum_{t_1+2t_2+\cdots+nt_n=n} \frac{q^n}{(q;q)_{t_1}(q;q)_{t_2}\cdots(q;q)_{t_n}}$$

in terms of plane partitions.

Author Contributions: Conceptualization, M.M. and I.-I.R.; Writing–original draft, M.M. and I.-I.R.; Writing–review and editing, M.M. and I.-I.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Andrews, G.E. *The Theory of Partitions*; Cambridge Mathematical Library, Cambridge University Press: Cambridge, UK, 1998; Reprint of the 1976 original.
2. MacMahon, P.A. Memoir on the theory of the partition of numbers, I. *Lond. Phil Trans. A* **1897**, *187*, 619–673.
3. MacMahon, P.A. *Combinatory Analysis*; Cambridge University Press: Cambridge, UK, 1916; Volume 2, reprinted by Chelsea, New York, 1960.
4. Macdonald, I.G. *Symmetric Functions and Hall Polynomials*, 2nd ed.; Oxford University Press: New York, NY, USA; London, UK, 1995.
5. Azarian, M.K. A Generalization of the Climbing Stairs Problem II. *Mo. J. Math. Sci.* **2004**, *16*, 12–17. [CrossRef]
6. Gordon, B.; Houten, L. Notes on plane partitions, II. *J. Combin Theory* **1968**, *4*, 81–99. [CrossRef]
7. Gordon, B. Notes on plane partitions, V. *J. Combin Theory Ser. B* **1971**, *11*, 157–168. [CrossRef]
8. Johnson, W.P. *An Introduction to q-Analysis*; American Mathematical Society: Providence, RI, USA, 2020.