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Remarks on Approximate Solutions to Difference Equations in Various Spaces

Janusz Brzdęk

Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Cracow, Poland; brzdek@agh.edu.pl

Abstract: Quite often (e.g., using numerical methods), we are only able to find approximate solutions of some equations, and it is necessary to know the size of the difference between such approximate solutions and the mappings that satisfy the equation exactly. This issue is the main subject of the theory of Ulam stability, and it is related to other areas of research such as, e.g., shadowing, optimization, and approximation theory. In this expository paper, we present several selected outcomes on Ulam stability of difference equations, show possible extensions of them and indicate further directions for research. We also present and discuss some simple methods that allow improvement of several already known results concerning Ulam stability of some difference equations in normed or metric spaces and extend them to b-metric and 2-normed spaces. Our results show that the noticeable symmetry exists between the outcomes of this type in normed and metric spaces and those obtained by us for other spaces. In particular, we extend the result of Pólya and Szegö concerning the stability of equation $x_{n+m} = x_n + x_m$ for $m, n \in T$, where T means either the set of integers $\mathbb{Z}$ or the set of positive integers $\mathbb{N}$. We also consider the stability of equation $x_{n+p} + a_1 x_{n+p-1} + \cdots + a_p x_n + b_n = 0$ (with a fixed positive integer p) and of two more general difference equations.

Keywords: Ulam stability; difference equation; 2-norm; b-metric; Banach space

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1. Introduction

This is an expository paper presenting some selected results concerning Ulam stability of difference equations, showing (with proof) some extensions of them and indicating directions of further possible investigations. To avoid misunderstanding, let us mention here that it is not a survey (or review) paper, and therefore numerous other outcomes on Ulam stability of difference equations are not mentioned here.

Let us remind that the Ulam stability theory deals with the following problem that concerns solutions to various equations (e.g., difference, differential, integral, functional, etc.) and naturally arises in many areas of scientific investigations: How much an approximate solution to an equation differs from the exact solutions of it?

This inquiry has been inspired by a question raised by Stanisław Ulam in 1940 (concerning the functional equation of group homomorphism) and the first answer to it given (for Banach spaces) by D.H. Hyers in [1]. However, there is an earlier result of this type concerning real sequences that was formulated by Gy. Pólya and G. Szegö in [2]. It can be stated in the following way.

Theorem 1. *Assume that $(r_n)_{n\in\mathbb{N}}$ is a real sequence fulfilling inequality*

$$\sup_{n,m\in\mathbb{N}} |r_{n+m} - r_n - r_m| \le 1.$$

Then,

$$\sup_{n\in\mathbb{N}} |r_n - \omega n| \le 1$$

for some real number ω.

Moreover,

$$\omega = \lim_{n \to \infty} \frac{r_n}{n}.$$

After Hyers' paper [1], some further stability results were formulated by Bourgin [3,4] and Aoki [5] (for some other generalization of the result of Hyers, we refer to [6]). The result of Aoki is included in Theorem 2 (when $s \in [0,1)$), given below.

More information on this subject can be found in [7] (see also [8,9]). For various examples of the Ulam stability outcomes, we refer to, e.g., [10–18]. Problems of such type are quite natural and are related to the subjects studied in the theories of optimization, approximation and shadowing (cf. [19]).

The next theorem can be regarded quite representative for the theory of stability of Ulam type (see, e.g., [20] or Theorem 1 in [21]).

Theorem 2. *Assume that W is a Banach space, V is a normed space, and $V_0 := V \setminus \{0\}$. Let $\eta \geq 0$ and $s \neq 1$ be real numbers, and $h\colon V \to W$ satisfy inequality*

$$\|h(z+w) - h(z) - h(w)\| \leq \eta(\|z\|^s + \|w\|^s), \qquad \forall z, w \in V_0\,. \tag{1}$$

Then, there is a unique mapping $g : V \to W$ that is additive and fulfils inequality

$$\|h(u) - g(u)\| \leq \frac{\eta\|u\|^s}{|1 - 2^{s-1}|}, \qquad \forall u \in V_0. \tag{2}$$

Let us remind that $g : V \to W$ is additive provided

$$g(z+w) = g(z) + g(w), \qquad \forall z, w \in V. \tag{3}$$

If $s = 1$, then a result analogous to Theorem 2 is not valid (see [14]). Next, the constant in (2) is the best possible for $s \geq 0$ (see, e.g., [8]), but for $s < 0$, this is not the case, because then each mapping $h : V \to W$ satisfying condition (1) must be additive even without completeness of space W (see, e.g., [20,21]). Namely, the following complement to Theorem 2 is true (see Theorem 3.1 in [20]).

Theorem 3. *Assume that V and W are normed spaces, $Y \subset V \setminus \{0\}$ is a nonempty set, and $\gamma \geq 0$ and $s < 0$ are fixed real numbers. Next, assume that*

$$-Y := \{-y : y \in Y\} = Y \tag{4}$$

and there is an integer $n_0 > 0$ with

$$nz \in Y, \qquad \forall z \in Y, n \in \mathbb{N}, n \geq n_0.$$

Then, each mapping $h\colon Y \to W$ satisfying inequality

$$\|h(z+w) - h(z) - h(w)\| \leq \gamma(\|z\|^s + \|w\|^s), \qquad \forall z, w \in Y, z+w \in Y$$

must be additive on Y, i.e.,

$$h(z+w) = h(z) + h(w), \qquad \forall z, w \in Y, z+w \in Y. \tag{5}$$

In the case $Y = V \setminus \{0\}$, this outcome can also be easily deduced from Theorem 5 in [22] (concerning a more general monomial functional equation).

The next theorem (see Theorem 3.4 in [20]) complements Theorem 3 and generalizes Theorem 2.

Theorem 4. *Assume that W is a Banach space, V is a normed space, $Y \subset V \setminus \{0\}$ is a nonempty set, $\gamma \geq 0$ and $s \geq 0$, $s \neq 1$. Let one of the subsequent two conditions be valid.*

(i) $s < 1$ *and* $2Y := \{2y : y \in Y\} \subset Y$.

(ii) $s > 1$ *and* $Y \subset 2Y$.

Then, for each $g\colon Y \to W$ with

$$\|g(z+v) - g(z) - g(v)\| \leq \gamma(\|z\|^s + \|v\|^s), \qquad \forall z, v \in Y, z+v \in Y,$$

there exists exactly one mapping $T\colon Y \to W$ that is additive on Y and fulfils inequality

$$\|g(z) - T(z)\| \leq \frac{\gamma}{|1 - 2^{s-1}|}\|z\|^s, \qquad \forall u \in Y.$$

The subsequent stability result concerning the Cauchy Equation (3) (but without assumption (4)) was obtained in [23].

Theorem 5. *Assume that W is a Banach space, V is a normed space, $Y \subset V \setminus \{0\}$ is a nonempty set, $\gamma \geq 0$, $s < 0$, and there is an integer $n_0 > 0$ with*

$$ky \in Y, \qquad \forall y \in Y, k \in \mathbb{N}, k > n_0.$$

Let $g : Y \to W$ be a mapping such that

$$\|g(z+v) - g(z) - g(v)\| \leq \gamma(\|z\|^s + \|v\|^s), \qquad \forall z, v \in Y, z+v \in Y.$$

Then, there is exactly one mapping $T : Y \to W$, which is additive on Y and fulfils the subsequent inequality:

$$\|g(u) - T(u)\| \leq \gamma\|u\|^s, \qquad \forall u \in Y.$$

Of course, instead of (1), some other inequalities, of the form

$$\|g(z+v) - g(z) - g(v)\| \leq \Phi(z, v),$$

can be considered and, for instance, condition

$$\|g(z+v) - g(z) - g(v)\| \leq \xi\|z\|^p\|v\|^q, \qquad \forall z, v \in V \setminus \{0\}$$

was studied in [24,25] with fixed $p, q \in \mathbb{R}$ and $\xi > 0$.

Also, from Theorem 9 in [26] (see the proof of it) the next two more precise results can be derived.

Theorem 6. *Let V be a normed space and $d : V^2 \to \mathbb{R}$ be such that*

$$\begin{aligned} d(u+t, u+t) - d(u,u) - d(t,t) \\ = d(2u, 2t) - 2d(u,t), \qquad \forall u, t \in V. \end{aligned}$$

Let $Y \neq \emptyset$ be a subset of $V \setminus \{0\}$, $2Y \subset Y$, $\chi, \nu, s \in \mathbb{R}$, $s < 1$, $\chi \leq \nu$, and $\phi\colon Y \to \mathbb{R}$ satisfy

$$\begin{aligned} \chi(\|z\|^s + \|v\|^s) &\leq \phi(z+v) - \phi(z) - \phi(v) - d(z,v) \\ &\leq \nu(\|z\|^s + \|v\|^s), \qquad \forall z, v \in Y, z+v \in Y. \end{aligned} \tag{6}$$

Then, there exists exactly one mapping $\Phi\colon Y \to \mathbb{R}$ with

$$\Phi(z+v) = \Phi(z) + \Phi(v) + d(z,v), \qquad \forall z, v \in Y, z+v \in Y, \tag{7}$$

$$\frac{\chi}{1-2^{s-1}}\|z\|^s \leq \Phi(z)-\phi(z) \leq \frac{\nu}{1-2^{s-1}}\|z\|^s, \qquad \forall z \in Y.$$

Theorem 7. *Let V, d and χ, ν be as in Theorem 6 and $s > 1$. Let $Y \subset V \setminus \{0\}$ be nonempty, $Y \subset 2Y$, and $\phi\colon Y \to \mathbb{R}$ satisfy (6). Then, there exists exactly one mapping $\Phi\colon Y \to \mathbb{R}$ such that (7) is fulfilled and*

$$\frac{\chi}{2^{s-1}-1}\|z\|^s \leq \phi(z)-\Phi(z) \leq \frac{\nu}{2^{s-1}-1}\|z\|^s, \qquad \forall z \in Y.$$

Clearly, in a similar way, the stability of many other equations can be studied. Further information on this subject, references and examples can be found in [7,8]. In particular, some authors (see, e.g., [27–32]) studied the stability of various particular cases of the following quite general functional equation

$$\sum_{i=1}^{m} C_i g\Big(\sum_{j=1}^{n} c_{ij} z_j\Big) = d(z_1,\ldots,z_n) \tag{8}$$

for mappings g from a module M over a commutative ring $\mathbb{P}$ into a Banach space B over the field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, where mapping $d : M^n \to B$ is given and satisfies some additional assumptions, $C_1,\ldots,C_m \in \mathbb{K} \setminus \{0\}$, and $c_{ij} \in \mathbb{P}$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Clearly, functional Equation (3) is a special case of Equation (8). Information on various other particular cases of (8) can be found, e.g., in [7,8,33–36].

Plainly, the closeness of two mappings and the notion of an approximate solution can be understood in many ways (see, e.g., [35–40]). Therefore, Ulam stability can also be considered with respect to some nonstandard ways of measuring distance. For instance, in recent years, Ulam stability in 2-normed spaces has been investigated in numerous papers (see, e.g., [35,41–48]), and in survey paper [21] a discussion of such results and further references can be found. Also, Ulam stability with respect to quasi-norms and b-metrics has been studied (see [37]).

This paper shows that it is possible to easily obtain many quite general stability results for difference equations with respect to b-metrics and 2-norms by deriving them from some already known outcomes proved for normed spaces. In this way, it is demonstrated that a significant symmetry exists between such results in classical normed and metric spaces and those obtained by us for other spaces.

2. Auxiliary Results

In this section, several examples of results on Ulam stability obtained for difference equations are presented. They are used in the next sections.

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote, as usual, the sets of positive integers, nonnegative integers, integers, reals and complex numbers, respectively (also in all the next sections). Let $T \in \{\mathbb{N}_0,\mathbb{Z}\}$, $\mathbb{K}$ be either the field of reals $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, X be a nontrivial normed space over $\mathbb{K}$, $\mathbb{S} := \{a \in \mathbb{C} : |a| = 1\}$, $p \in \mathbb{N}$, $a_1,\ldots,a_p \in \mathbb{K}$, $(b_n)_{n\in T}$ be a sequence in X, and $r_1,\ldots,r_p \in \mathbb{C}$ denote all the roots of equation

$$r^p - \sum_{i=1}^{p} a_i r^{p-i} = 0. \tag{9}$$

The next theorem can be easily derived from [11] (for some related results, see also [38,49,50]) and concerns the Ulam stability of difference equation

$$x_{n+p} = a_1 x_{n+p-1} + \ldots + a_p x_n + b_n, \qquad \forall n \in T. \tag{10}$$

Theorem 8. *Let X be a Banach space, $\delta > 0$ and $r_1, \ldots, r_p \in \mathbb{C} \setminus \mathbb{S}$. Let $(y_n)_{n\in T}$ be a sequence of elements of space X with*

$$\|y_{n+p} - a_1 y_{n+p-1} - \ldots - a_p y_n - b_n\| \le \delta, \qquad \forall n \in T. \tag{11}$$

Then, there exists a sequence $(x_n)_{n\in T}$ in X satisfying (10) and the subsequent condition

$$\|y_n - x_n\| \le \frac{\delta}{|1 - |r_1|| \cdot \ldots \cdot |1 - |r_p||}, \qquad \forall n \in T. \tag{12}$$

Moreover, the next three statements hold.

(a) *$(x_n)_{n\in T}$ is unique if and only if the following condition is satisfied:*

$$T = \mathbb{Z} \quad \text{or} \quad |r_k| > 1, \quad \forall k \in \{1, \ldots, p\}. \tag{13}$$

(b) *If (13) is valid, then $(x_n)_{n\in T}$ is the only sequence in X fulfilling Equation (10) and such that*

$$\sup_{n\in T} \|x_n - y_n\| < \infty.$$

(c) *If (13) is not satisfied, then the cardinality of the family of all sequences $(x_n)_{n\in T}$ in X that fulfill Equation (10) and Inequality (12) is the same as the cardinality of space X.*

The next theorem also follows from [11] and concerns difference Equation (10), but depicts the situations when the stability does not occur (we have non-stability).

Theorem 9. *Suppose that there is $j \in \{1, \ldots, p\}$ with $|r_j| = 1$. Then, for each $\delta > 0$, there is a sequence $(y_n)_{n\in T}$ of elements of X such that (11) holds and*

$$\sup_{n\in T} \|y_n - x_n\| = \infty$$

for each sequence $(x_n)_{n\in T}$ in X that is a solution to difference Equation (10).

Moreover, if $r_1, \ldots, r_p \in \mathbb{K}$ or there exists a sequence $(x_n)_{n\in T}$ in X that is bounded and satisfies (10), then there exists such sequence $(y_n)_{n\in T}$ that is unbounded.

This section is concluded with two results from Theorems 2.1 and 2.3 in [51]. It is assumed that (M, ρ) is a complete metric space, $\mathbb{J} \in \{\mathbb{N}, \mathbb{Z}\}$, $p \in \mathbb{N}$, and $T_n \colon M^p \to M$ for $n \in \mathbb{J}$. The subsequent two theorems concern stability of difference equations

$$u_{n+p} = T_n(u_n, u_{n+1}, \ldots, u_{n+p-1}), \qquad \forall n \in \mathbb{J}, \tag{14}$$
$$u_n = T_n(u_{n+1}, u_{n+2}, \ldots, u_{n+p}), \qquad \forall n \in \mathbb{J}, \tag{15}$$

for sequences $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ ($M^{\mathbb{J}}$ denotes the family of all sequences $(u_n)_{n\in\mathbb{J}}$ in M).

Stability of particular cases of difference Equations (14) and (15) was studied earlier in [49,50].

Theorem 10. *Let $\delta_n \in \mathbb{R}_0^+$ (positive reals) and $\Theta_n : \mathbb{R}_+^p \to \mathbb{R}_+$ for $n \in \mathbb{J}$ be such that*

$$\rho(T_n(\overline{y}), T_n(\overline{w})) \le \Theta_n(\rho(y_1, w_1), \ldots, \rho(y_p, w_p)), \tag{16}$$
$$\forall \overline{y} = (y_1, \ldots, y_p), \overline{w} = (w_1, \ldots, w_p) \in M^p, n \in \mathbb{J},$$
$$\sup_{i\in\mathbb{J}} \frac{\Theta_i(b_i, \ldots, b_{i+p-1})}{\delta_{p+i}} < \vartheta \sup_{i\in\mathbb{J}} \frac{b_i}{\delta_i}, \qquad \forall (b_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}},$$

with some $\vartheta \in (0,1)$. Let $(z_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfy inequality

$$\rho(z_{n+p}, T_n(z_n, z_{n+1}, \dots, z_{n+p-1})) \leqslant \delta_{n+p}, \qquad \forall n \in \mathbb{J}.$$

Then, there exists a sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfying Equation (14) and such that

$$\sup_{n\in\mathbb{J}} \frac{\rho(z_n, u_n)}{\delta_n} \leqslant \frac{1}{1-\vartheta}. \tag{17}$$

Furthermore, if $\mathbb{J} = \mathbb{Z}$, then such sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ is unique.

Theorem 11. *Let $\vartheta \in (0,1)$ and $\delta_n \in \mathbb{R}_0^+$, $\Theta_n : \mathbb{R}_+^p \to \mathbb{R}_+$ for $n \in \mathbb{J}$ be such that (16) holds and*

$$\sup_{i\in\mathbb{J}} \frac{\Theta_i(b_{i+1}, \dots, b_{i+p})}{\delta_i} \leqslant \vartheta \sup_{i\in\mathbb{J}} \frac{b_i}{\delta_i}, \qquad \forall (b_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}.$$

Let $(z_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfy inequality

$$\rho(z_n, T_n(z_{n+1}, z_{n+2}, \dots, z_{n+p})) \leq \delta_n, \qquad \forall n \in \mathbb{J}.$$

Then, there is exactly one sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ fulfilling difference Equation (15) and Inequality (17).

3. Auxiliary Information

In this section, some auxiliary information on 2-norms, b-metrics and quasi-norms is provided, which is necessary in the further parts of this paper.

3.1. 2-Normed Spaces

Assume that V is a linear space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and the dimension of V is greater than one. Let us start with the following definition (cf. [52–54]).

Definition 1. *Mapping $\|\cdot,\cdot\| : V^2 \to \mathbb{R}_+$ is a 2-norm if, for all $v_1, v_2, v_3 \in V$ and $\beta \in \mathbb{K}$, the subsequent conditions are fulfilled:*

(*a*) $\|v_1, v_2\| = 0$ *if and only if vectors v_1 and v_2 are linearly dependent;*
(*b*) $\|v_1, v_2\| = \|v_2, v_1\|$;
(*c*) $\|v_1, v_2 + v_3\| \leq \|v_1, v_2\| + \|v_1, v_3\|$;
(*d*) $\|\beta v_1, v_2\| = |\beta|\|v_1, v_2\|$.

Further, let $\|\cdot,\cdot\| : V^2 \to \mathbb{R}_+$ be a 2-norm. Then, pair $(V, \|\cdot,\cdot\|)$ is called a 2-normed space. Next, sequence $(v_n)_{n\in\mathbb{N}}$ in V is a 2-Cauchy sequence if

$$\lim_{m,n\to\infty} \|v_m - v_n, w_i\| = 0, \qquad i = 1, 2$$

for some linearly independent vectors $w_1, w_2 \in V$. Sequence $(v_n)_{n\in\mathbb{N}}$ in V is 2-convergent if there is $v \in V$ such that $\lim_{n\to\infty} \|v_n - v, w\| = 0$ for each $w \in V$; such vector v must be unique. It is called here a limit of $(v_n)_{n\in\mathbb{N}}$ and denoted by $\lim_{n\to\infty} v_n$. The 2-norm $\|\cdot,\cdot\|$ (and also the 2-normed space $(V, \|\cdot,\cdot\|)$) is complete if all 2-Cauchy sequences are 2-convergent.

Let $\langle\cdot,\cdot\rangle$ be a real inner product in V. Then, a 2-norm in V can be defined by formula

$$\|u, w\| := \sqrt{\|u\|^2\|w\|^2 - \langle u, w\rangle^2}, \qquad \forall u, w \in V. \tag{18}$$

In the case where $(V, \langle\cdot,\cdot\rangle)$ is a Hilbert space, from Proposition 2.3 in [55], it results that the 2-norm defined by (18) is complete. In $\mathbb{R}^2$, with the usual inner product given by $\langle(v_1, v_2), (w_1, w_2)\rangle = v_1w_1 + v_2w_2$, the 2-norm given by (18) has the following form:

$$\|(v_1, v_2), (w_1, w_2)\| := |v_1w_2 - v_2w_1|, \qquad \forall (v_1, v_2), (w_1, w_2) \in \mathbb{R}^2.$$

Finally, let $\|\cdot,\cdot\|_1 : V \times V \to \mathbb{R}_+$ and $\|\cdot,\cdot\|_2 : V \times V \to \mathbb{R}_+$ be 2-norms. Fix $\beta_1, \beta_2 \in (0, \infty)$. Then, the next two expressions also define 2-norms:

(A) $\max\{\beta_1\|\cdot,\cdot\|_1, \beta_2\|\cdot,\cdot\|_2\}$;

(B) $\beta_1\|\cdot,\cdot\|_1 + \beta_2\|\cdot,\cdot\|_2$.

3.2. *b-Metrics*

If M is a nonempty set, then mapping $d\colon M \times M \to \mathbb{R}_+$ is said to be a b–metric (in M) if there is a real constant $\mu \geq 1$ such that, for all $u, z, w \in M$, the next three conditions are valid:

(a) $d(u, z) = 0$ if and only if $u = z$;

(b) $d(u, z) = d(z, u)$;

(c) $d(u, z) \leq \mu\big(d(u, w) + d(w, z)\big)$.

If conditions (a)–(c) are fulfilled, then (M, d, μ) is called a b–metric space.

b–metric spaces also have been called quasi-metric spaces (e.g., in [56]). Actually, this name better corresponds to the notion of quasi-norms (cf. the comments after Theorem 12), but the name b-metric seems to be less ambiguous, because there are also other meanings of the term quasi-metric (see, e.g, [57,58]); for example, in [57], it is a mapping d fulfilling only conditions (a) and (c) for $\mu = 1$ (without condition (b)).

The notion of b-metric has been introduced in [59] with $\mu = 2$ and later used in [60] for $\mu \geq 1$.

Let us also recall that, if (M, d, μ) is a b-metric space, then

- sequence $(u_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ is convergent to an element $u \in M$ if $\lim_{n\to\infty} d(u, u_n) = 0$ (then, we say that u is a limit of the sequence and denote it by $x = \lim_{n\to\infty} u_n$; such limit must be unique);
- sequence $(u_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ is Cauchy if $\lim_{n,m\to\infty} d(u_m, u_n) = 0$;
- (M, d, μ) is said to be complete if all Cauchy sequences in M are convergent to some elements of M.

Remark 1. *Let $s \in (1, \infty)$ and ρ be a metric in a set $M \neq \emptyset$. Since for all $a, b \in \mathbb{R}_+$, $(a + b)^s \leq 2^{s-1}(a^s + b^s)$, the following is also true*

$$\rho(x, y)^s \leq \big(\rho(x, z) + \rho(z, y)\big)^s \leq 2^{s-1}\big(\rho(x, z)^s + \rho(z, y)^s\big)$$

for all $x, y, z \in M$. This means that $(M, d_s, 2^{s-1})$ is a b-metric space, with $d_s(z, w) := \rho(z, w)^s$ for all $z, w \in M$.

Next, if $n \in \mathbb{N}$, $c_1, \ldots, c_n \in (0, \infty)$ and $d_1, \ldots, d_n$ are b-metrics in a set $M \neq \emptyset$, then it is easy to check that d and d_0 also are b-metrics in M, where

$$d(x, y) = \sum_{i=1}^{n} c_i d_i(x, y), \qquad d_0(x, y) = \max_{i=1,\ldots,n} c_i d_i(x, y), \qquad \forall x, y \in M.$$

The following result from [61] (Proposition , p. 4308) is also needed.

Theorem 12. *Assume that (M, d, μ) is a b-metric space and*

$$D_d(z, w) = \inf\Big\{\sum_{i=1}^{n} d^{\xi}(u_i, u_{i+1}) : u_2, ..., u_n \in M,\ n \in \mathbb{N},\ u_1 = z,\ u_{n+1} = w\Big\}$$

for every $z, w \in M$, where $\xi := \log_{2\mu} 2$ and $d^{\xi}(z,w) = \big(d(z,w)\big)^{\xi}$ for all $z, w \in M$. Then, D_d is a metric on M with

$$\frac{1}{4} d^{\xi}(z,w) \le D_d(z,w) \le d^{\xi}(z,w), \qquad \forall z, w \in M. \tag{19}$$

Moreover, if d is a metric, then $D_d = d$.

Let Y be a real or a complex vector space. Mapping $\|\cdot\| : Y \to \mathbb{R}_+$ is a quasi-norm if there exists $\mu \in [1, \infty)$ such that, for every $z, w \in Y$ and every scalar β,

(a1) $\|w\| = 0$ if and only if $w = 0$;
(b1) $\|\beta w\| = |\beta| \, \|w\|$;
(c1) $\|z + w\| \le \mu\big(\|z\| + \|w\|\big)$.

$(Y, \|\cdot\|, \mu)$ is said to be a quasi–normed space if (a1)–(c1) are valid.

Note that if $(Y, \|\cdot\|, \mu)$ is a quasi–normed space, then (Y, d, μ) is a b-metric space, with mapping $d : Y^2 \to \mathbb{R}_+$ given by $d(z,w) := \|z - w\|$ for $z, w \in Y$.

In a similar way as in Remark 1, we can obtain the following two examples (cf. Examples 1.1 and 1.2 in [62]).

Example 1. Let Y be a Banach space and $p \in (0,1)$. Let

$$\ell_p(Y) := \Big\{ (u_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}} : \sum_{n=1}^{\infty} \|u_n\|^p < \infty \Big\}$$

and define $\|\cdot\|_p : \ell_p(Y) \to \mathbb{R}_+$ by

$$\|u\|_p := \Big(\sum_{n=1}^{\infty} \|u_n\|^p \Big)^{1/p}, \qquad \forall u = (u_n)_{n \in \mathbb{N}} \in \ell_p(Y).$$

Then, $\big(\ell_p(Y), \|\cdot\|_p, 2^{(1-p)/p}\big)$ is a quasi-normed space.

Example 2. Let $p \in (0,1)$,

$$L_p[0,1] := \Big\{ y : [0,1] \to \mathbb{R} : y \text{ is continuous and } \int_0^1 |y(t)|^p dt < 1 \Big\},$$

and $\|\cdot\|_p : L_p[0,1] \to \mathbb{R}_+$ be given by

$$\|x\|_p := \Big(\int_0^1 |x(t)|^p dt \Big)^{1/p}, \qquad \forall x \in L_p[0,1].$$

Then, $\big(L_p[0,1], \|\cdot\|_p, 2^{(1-p)/p}\big)$ is a quasi-normed space.

According to the Aoki–Rolewicz Theorem (see, e.g., Theorem 1 in [63]), every quasi-norm is equivalent to a p-norm. However, there exist p-norms that are not equivalent to any norm (see, e.g., Examples 1 and 2 in [63]). Let us reiterate here that mapping $\|\cdot\|$ from a real or complex linear space Y into $\mathbb{R}_+$ is a p-norm (with a real $p > 0$) if conditions (a1) and (b1) are fulfilled and the following inequality is valid:

(c1′) $\|z + w\|^p \le \|z\|^p + \|w\|^p$ for every $z, w \in Y$.

4. Extensions of Theorem 1

Write $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$. If in Theorems 3–5 $V = \mathbb{R}$ and $Y \in \{\mathbb{N}, \mathbb{Z}_0\}$ are taken, then the following extensions of Theorem 1 are obtained.

Theorem 13. *Let W be a normed space, $T \in \{\mathbb{N}, \mathbb{Z}\}$, $T_0 := T \setminus \{0\}$, and $\gamma \geq 0$ and $s < 1$ be real numbers. Let $(x_n)_{n \in T}$ be a sequence in W with*

$$\|x_{n+m} - x_n - x_m\| \leq \gamma(|n|^s + |m|^s), \qquad \forall n, m \in T_0. \tag{20}$$

Then, the subsequent three statements are true.

(i) *If $s < 0$ and $T = \mathbb{Z}$, then*

$$x_n = n x_1, \qquad \forall n \in T. \tag{21}$$

(ii) *If W is complete, $s < 0$ and $T = \mathbb{N}$, then there exists exactly one $z_0 \in W$ with*

$$\|x_n - n z_0\| \leq \gamma\, n^s, \qquad \forall n \in T. \tag{22}$$

Moreover,

$$z_0 = \lim_{n \to \infty} \frac{1}{n}\, x_n. \tag{23}$$

(iii) *If W is complete and $s \geq 0$, then there is exactly one $z_0 \in W$ with*

$$\|x_n - n z_0\| \leq \frac{\gamma\, |n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0. \tag{24}$$

Moreover, (23) holds.

Proof. First, assume that $s < 0$ and $T = \mathbb{Z}$. Then, by Theorem 3 with $V = \mathbb{R}$, $Y = T$ and $h(n) = x_n$ for $n \in T$,

$$x_{n+m} = x_n + x_m, \qquad \forall n, m \in T_0, n + m \neq 0. \tag{25}$$

By induction, it is easy to show that, in view of (25),

$$x_n = n x_1, \qquad \forall n \in \mathbb{N}. \tag{26}$$

Further,

$$x_n = x_{n+m-m} = x_{n+m} + x_{-m} = x_n + x_m + x_{-m}, \qquad \forall n, m \in \mathbb{N},$$

whence

$$x_{-m} = -x_m, \qquad \forall m \in \mathbb{N},$$

and consequently (in view of (26)),

$$x_n = n x_1, \qquad \forall n \in T_0. \tag{27}$$

Finally, from (20) and (27), obtain

$$\|x_0\| = \|x_0 - x_n - x_{-n}\| \leq 2\gamma\, |n|^s, \qquad \forall n \in T_0,$$

which (with $n \to \infty$) yields $x_0 = 0$. This completes the proof of (21).

Now, assume that W is complete, $s < 0$ and $T = \mathbb{N}$. Then, by Theorem 5 with $V = \mathbb{R}$, $Y = T$ and $h(n) = x_n$ for $n \in T$, there is sequence $(y_n)_{n \in T}$ in W with

$$y_{n+m} = y_n + y_m, \qquad \forall n, m \in \mathbb{N}, \tag{28}$$

and

$$\|x_n - y_n\| \leq \gamma\, n^s, \qquad \forall n \in T.$$

Clearly, (28) implies that $y_n = ny_1$ for $n \in \mathbb{N}$, whence

$$\|x_n - ny_1\| \leq \gamma\, n^s, \qquad \forall n \in T. \tag{29}$$

Further, from (29), obtain

$$\left\|\frac{1}{n}x_n - y_1\right\| \leq \gamma\, n^{s-1}, \qquad \forall n \in T,$$

which, with $n \to \infty$, yields

$$y_1 = \lim_{n\to\infty} \frac{1}{n}\, x_n. \tag{30}$$

Thus, statement (ii) is proven (with $z_0 = y_1$).

It is still necessary to consider the case where W is complete and $s \geq 0$. Similarly as above, by Theorem 4 (with $V = \mathbb{R}$, $Y = T$ and $h(n) = x_n$ for $n \in T$), there is sequence $(y_n)_{n\in T}$ in W with

$$y_{n+m} = y_n + y_m, \qquad \forall n, m \in T_0, n+m \neq 0, \tag{31}$$

and

$$\|x_n - y_n\| \leq \frac{\gamma\, |n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0. \tag{32}$$

In the same way as above (in the case of (25)), it can be shown that (31) implies $y_n = ny_1$ for $n \in T$. Hence

$$\|x_n - ny_1\| \leq \frac{\gamma\, |n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0.$$

Now, as before, it can be shown that (30) holds. This ends the proof of statement (iii). □

Note that Theorem 1 results from Theorem 13 (with $s = 0$).

Remark 2. *Let $T = \mathbb{Z}$, $\gamma \neq 0$ and $s \geq 0$ (in Theorem 13). Let $w_0 \in W$ be such that $\|w_0\| = \gamma$. If $x_n = w_0$ for $n \in T_0$ and $x_0 = 3w_0$, then inequality (20) is fulfilled. This simple example shows that, in the case $T = \mathbb{Z}$ and $s \geq 0$, the inequality in (24) does not need to hold for $n = 0$.*

Remark 3. *It seems to be interesting whether an outcome similar to Theorem 13 can also be obtained for $s > 1$.*

Remark 4. *Estimations (22) and (24) are optimal in the general situation. To notice this, in the case of (22), it is enough to take $x_n = n^s w_0$ for $n \in \mathbb{N}$, with any fixed $w_0 \in W$ such that $\|w_0\| = \gamma$. Then, we have equality in (22) with $z_0 = 0$. Moreover, for every $n, m \in \mathbb{N}$,*

$$\|x_{n+m} - x_n - x_m\| = \gamma\big(n^s + m^s - (n+m)^s\big) \leq \gamma\big(n^s + m^s\big),$$

which means that (20) holds.

In the case of (24), $x_n = sign(n)|n|^s w_0$ can be taken for $n \in T$ (sign means the signum function that returns the sign of a real number) with any fixed $w_0 \in W$ such that

$$\|w_0\| = \frac{\gamma}{1 - 2^{s-1}}.$$

Then, (24) becomes equality with $z_0 = 0$. Moreover, Theorem 2.10 in [8] shows that (20) is fulfilled.

From Theorem 6, the subsequent finer outcome for real sequences can also be obtained.

Theorem 14. *Let $T \in \{\mathbb{N}, \mathbb{Z}\}$, $T_0 := T \setminus \{0\}$, $\chi, \nu, s \in \mathbb{R}$, $\chi \leq \nu$, $s < 1$, and $d : T^2 \to \mathbb{R}$ be such that there is a real sequence $(y_n)_{n \in T}$ with*

$$y_{n+m} = y_n + y_m + d(n,m), \qquad \forall n, m \in T_0. \tag{33}$$

Let $(x_n)_{n \in T}$ be a real sequence satisfying inequality

$$\begin{aligned} \chi(|n|^s + |m|^s) &\leq x_{n+m} - x_n - x_m - d(n,m) \\ &\leq \nu(|n|^s + |m|^s), \qquad \forall n, m \in T_0. \end{aligned} \tag{34}$$

Then, there exists exactly one $z_0 \in \mathbb{R}$ such that

$$\frac{\chi\,|n|^s}{1 - 2^{s-1}} \leq y_n + nz_0 - x_n \leq \frac{\nu\,|n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0. \tag{35}$$

Moreover,

$$z_0 = \lim_{n \to \infty} \frac{x_n - y_n}{n}.$$

Proof. Write $u_n := x_n - y_n$ for $n \in T$. Then, by (33) and (34),

$$\begin{aligned} \chi(|n|^s + |m|^s) &\leq u_{n+m} - u_n - u_m \\ &\leq \nu(|n|^s + |m|^s), \qquad \forall n, m \in T_0. \end{aligned} \tag{36}$$

Therefore, by Theorem 6 (with $d(n,m) \equiv 0$), there exists a real sequence $(w_n)_{n \in T}$ such that

$$w_{n+m} = w_n + w_m, \qquad \forall n, m \in T_0, \tag{37}$$

and

$$\frac{\chi\,|n|^s}{1 - 2^{s-1}} \leq w_n - u_n \leq \frac{\nu\,|n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0. \tag{38}$$

Since (37) implies that $w_n = nw_1$ for $n \in T$ (see the proof of Theorem 13), from (38), the following can be obtained:

$$\frac{\chi\,|n|^s}{1 - 2^{s-1}} \leq nw_1 - u_n \leq \frac{\nu\,|n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0, \tag{39}$$

whence

$$\frac{\chi\,|n|^s}{1 - 2^{s-1}} \leq y_n + nw_1 - x_n \leq \frac{\nu\,|n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0, \tag{40}$$

and

$$\frac{\chi\,|n|^{s-1}}{1 - 2^{s-1}} \leq w_1 - \frac{1}{n}u_n \leq \frac{\nu\,|n|^{s-1}}{1 - 2^{s-1}}, \qquad \forall n \in T_0.$$

The last inequality (with $n \to \infty$) means that

$$w_1 = \lim_{n \to \infty} \frac{1}{n} u_n.$$

Clearly, (40) is (35) (with $z_0 = w_1$). □

5. Stability in 2-Normed Spaces

In this section, some of the theorems presented in the previous sections are extended to the case of 2-normed spaces. In what follows, W always stands for a 2-normed space with the 2-norm denoted by $\|\cdot,\cdot\|$.

The following simple lemma is needed.

Lemma 1. *Let $k, l \in \mathbb{N}$ and $z, w \in W$. Assume that z and w are linearly independent and write*

$$\|x\|_{k,l} := \frac{1}{l}\|x, z\| + \frac{1}{k}\|x, w\|, \qquad \forall x \in W. \tag{41}$$

Then, $\|\cdot\|_{k,l}$ is a norm in W. Moreover, if the 2-norm in W is complete, then norm $\|\cdot\|_{k,l}$ is complete.

Proof. It is easy to check that $\|\cdot\|_{k,l}$ is a norm in W. Therefore, assume that 2-norm $\|\cdot,\cdot\|$ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in W with respect to norm $\|\cdot\|_{k,l}$. Then

$$\lim_{m,n\to\infty} \|x_n - x_m, u\| = 0$$

for $u \in \{z, w\}$. Consequently, $(x_n)_{n\in\mathbb{N}}$ is a 2-Cauchy sequence. Hence, there exists $x_0 \in W$ with

$$\lim_{n\to\infty} \|x_n - x_0, u\| = 0, \qquad \forall u \in W.$$

This means that

$$\lim_{n\to\infty} \|x_n - x_0\|_{k,l} = 0.$$

Thus, $\|\cdot\|_{k,l}$ is complete. □

Let us start with an analogue of Theorem 13. In this section, Y is a subset of W that contains at least two linearly independent vectors.

Theorem 15. *Let $T \in \{\mathbb{N}, \mathbb{Z}\}$, $T_0 := T \setminus \{0\}$, $s < 1$ be a fixed real number and $\Gamma : Y \to \mathbb{R}_+$. Let $(x_n)_{n\in T}$ be a sequence in W satisfying inequality*

$$\|x_{n+m} - x_n - x_m, u\| \le \Gamma(u)(|n|^s + |m|^s), \qquad \forall n, m \in T_0, u \in Y. \tag{42}$$

Then, the following three statements are valid:

(i) If $s < 0$ and $T = \mathbb{Z}$, then (21) holds.

(ii) If the 2-norm in W is complete, $s < 0$ and $T = \mathbb{N}$, then there exists exactly one $z_0 \in W$ such that

$$\|x_n - nz_0, u\| \le \Gamma(u)\; n^s, \qquad \forall n \in T, u \in Y. \tag{43}$$

Moreover, if $Y = W$, then (with respect to the 2-norm in W)

$$z_0 = \lim_{n\to\infty} \frac{1}{n}\, x_n. \tag{44}$$

(iii) If the 2-norm in W is complete and $s \ge 0$, then there exists exactly one $z_0 \in W$ such that

$$\|x_n - nz_0, u\| \le \frac{\Gamma(u)\; |n|^s}{1 - 2^{s-1}}, \qquad \forall n \in T_0, u \in Y.$$

Moreover, in the case $Y = W$, (44) holds (with respect to the 2-norm in W).

Proof. Fix $k, l \in \mathbb{N}$ and linearly independent $z, w \in Y$. Let $\| \cdot \|_{k,l}$ be a norm in W defined by (41). Then, by (42),

$$\|x_{n+m} - x_n - x_m\|_{k,l} \leq \Big(\frac{1}{l}\Gamma(z) + \frac{1}{k}\Gamma(w)\Big)(|n|^s + |m|^s), \qquad \forall n, m \in T_0.$$

Therefore, in the case $s < 0$ and $T = \mathbb{Z}$, by Theorem 13 with

$$\gamma := \frac{1}{l}\Gamma(z) + \frac{1}{k}\Gamma(w),$$

(21) holds.

Now, assume that the 2-norm in W is complete. Then, in view of Lemma 1, norm $\| \cdot \|_{k,l}$ is complete. Let $s < 0$ and $T = \mathbb{N}$. Then, by Theorem 13, there exists exactly one $z_{k,l} \in W$ such that

$$\|x_n - nz_{k,l}\|_{k,l} \leq \gamma\, n^s, \qquad \forall n \in T. \tag{45}$$

Note that $\|x\|_{k,l} \geq \|x\|_{k+1,l}$ and $\|x\|_{k,l} \geq \|x\|_{k,l+1}$ for every $k, l \in \mathbb{N}$ and $x \in W$. Therefore, the uniqueness of $z_{k,l}$ means that $z_0 := z_{k,l} = z_{k+1,l} = z_{k,l+1}$ for each $k, l \in \mathbb{N}$. Hence, first with $k \to \infty$ and $l = 1$ and next with $k = 1$ and $l \to \infty$, from (45), obtain

$$\|x_n - nz_0, u\| \leq \Gamma(u)\, n^s, \qquad \forall n \in T, u \in \{z, w\}.$$

This completes the proof of (43).

Suppose that $v_0 \in W$ is such that

$$\|x_n - nv_0, u\| \leq \Gamma(u)\, n^s, \qquad \forall n \in T, u \in Y.$$

Then,

$$\|nv_0 - nz_0, u\| \leq \|x_n - nz_0, u\| + \|x_n - nv_0, u\| \leq 2\Gamma(u)\, n^s, \qquad \forall n \in T, u \in Y,$$

and consequently

$$\|v_0 - z_0, u\| \leq 2\Gamma(u)\, n^{s-1}, \qquad \forall n \in T, u \in Y,$$

which with $n \to \infty$ yields $z_0 = v_0$. This shows that z_0 is the unique vector in W satisfying (43).

Further, (43) yields

$$\Big\|\frac{1}{n}\, x_n - z_0, u\Big\| \leq \Gamma(u)\, n^{s-1}, \qquad \forall n \in T, u \in Y,$$

whence with $n \to \infty$, in the case $Y = W$, (44) is obtained (with respect to the 2-norm in W).

The proof of (iii) is analogous. □

The following partial analogue of Theorem 8 was proven in [64].

Theorem 16. *Let $\|\cdot,\cdot\|$ be complete and $T \in \{\mathbb{N}, \mathbb{Z}\}$. Let $p \in \mathbb{N}$, $\mu : Y \to \mathbb{R}_+$,*

$$C_0 := \prod_{i=1}^{p} \big|1 - |r_i|\big| \neq 0$$

and $(b_n)_{n\in T}$, $(y_n)_{n\in T}$ be sequences in W with

$$\|y_{n+p} + a_1 y_{n+p-1} + \ldots + a_p y_n + b_n, z\| \leq \mu(z), \qquad \forall n \in T,\ z \in Y.$$

Assume that $T = \mathbb{Z}$ or (13) is valid. Then, there is exactly one sequence $(x_n)_{n\in T}$ in W such that (10) is fulfilled and

$$\|y_n - x_n, z\| \le C_0^{-1}\mu(z), \qquad \forall n \in T,\ z \in Y.$$

It would be very desirable to obtain a fuller analogue of Theorem 8 for 2-normed spaces and an analogue of Theorem 9 (at least of the first part of it concerning the lack of stability).

The subsequent extension of Theorem 10 is also true.

Theorem 17. *Let $p \in \mathbb{N}$, $\|\cdot,\cdot\|$ be complete and $\vartheta \in (0,1)$. Let $T_n\colon W^p \to W$, $\epsilon_n \in (0,\infty)$ and $\Phi_n \in \mathbb{R}_+$ for $n \in \mathbb{Z}$ be such that*

$$\sup_{i\in\mathbb{Z}} \frac{\Phi_i(a_i + \ldots + a_{i+p-1})}{\epsilon_{p+i}} < \vartheta \sup_{i\in\mathbb{Z}} \frac{a_i}{\epsilon_i}, \qquad \forall (a_n)_{n\in\mathbb{Z}} \in \mathbb{R}_+^{\mathbb{Z}},$$

$$\begin{aligned}\|T_n(\overline{y}) - T_n(\overline{w}), u\| \le \Phi_n(\|y_1 - w_1, u\| + \ldots + \|y_p - w_p, u\|),\\ \forall \overline{y} = (y_1,\ldots,y_p), \overline{w} = (w_1,\ldots,w_p) \in W^p, n \in \mathbb{Z}, u \in Y.\end{aligned} \tag{46}$$

Let $\xi : Y \to \mathbb{R}_+$ and $(z_n)_{n\in\mathbb{Z}} \in W^{\mathbb{Z}}$ satisfy inequality

$$\|z_{n+p} - T_n(z_n,\ldots,z_{n+p-1}), u\| \le \xi(u)\epsilon_{n+p}, \qquad \forall n \in \mathbb{Z}, u \in Y. \tag{47}$$

Then, there is sequence $(u_n)_{n\in\mathbb{Z}} \in W^{\mathbb{Z}}$ such that (14) holds and

$$\sup_{n\in\mathbb{Z}} \frac{\|z_n - u_n, u\|}{\epsilon_n} \le \frac{\xi(u)}{1-\vartheta}, \qquad \forall u \in Y. \tag{48}$$

Proof. Let $z, w \in Y$ be linearly independent and fix $k, l \in \mathbb{N}$. Define norm $\|\cdot\|_{k,l}$ in W by (41). According to Lemma 1, the norm is complete. Therefore, by (47),

$$\|z_{n+p} - T_n(z_n, z_{n+1},\ldots,z_{n+p-1}))\|_{k,l} \le \Big(\frac{\xi(z)}{k} + \frac{\xi(w)}{l}\Big)\epsilon_{n+p} := \delta_{n+p}, \qquad \forall n \in \mathbb{Z}.$$

Further,

$$\begin{aligned}\|T_n(\overline{y}) - T_n(\overline{w})\|_{k,l} \le \Phi_n(\|y_1 - w_1\|_{k,l} + \ldots + \|y_p - w_p\|_{k,l}),\\ \forall \overline{y} = (y_1,\ldots,y_p), \overline{w} = (w_1,\ldots,w_p) \in W^p, n \in \mathbb{Z}.\end{aligned}$$

Hence, by Theorem 10 (with $M = W$, $\Theta_n(a_1,\ldots,a_p) = \Phi_n(a_1 + \ldots + a_p)$ and $\rho(x,y) = \|x-y\|_{k,l}$), there exists exactly one $(u_n(k,l))_{n\in\mathbb{Z}} \in W^{\mathbb{Z}}$ such that (14) holds (with $u_n = u_n(k,l)$) and

$$\sup_{n\in\mathbb{Z}} \frac{\|z_n - u_n(k,l)\|_{k,l}}{\delta_n} \le \frac{1}{1-\vartheta}. \tag{49}$$

Note that $\|x\|_{k,l} \ge \|x\|_{k+1,l}$ and $\|x\|_{k,l} \ge \|x\|_{k,l+1}$ for every $k, l \in \mathbb{N}$ and $x \in W$. Therefore, the uniqueness of $(u_n(k,l))_{n\in\mathbb{Z}}$ means that

$$(u_n)_{n\in\mathbb{Z}} := (u_n(k,l))_{n\in\mathbb{J}} = (u_n(k+1,l))_{n\in\mathbb{J}} = (u_n(k,l+1))_{n\in\mathbb{Z}}$$

for each $k, l \in \mathbb{N}$.

Hence, first with $k \to \infty$ and $l = 1$ and next with $k = 1$ and $l \to \infty$, from (49), obtain

$$\sup_{n\in\mathbb{Z}} \frac{\|z_n - u_n, u\|}{\epsilon_n} \le \frac{\xi(u)}{1-\vartheta}, \qquad \forall u \in \{z, w\}.$$

This completes the proof of (48). □

In a very similar way, the following analogue of Theorem 11 can be obtained.

Theorem 18. *Let $p \in \mathbb{N}$, $\|\cdot,\cdot\|$ be complete, $\vartheta \in (0,1)$ and $\mathbb{J} \in \{\mathbb{N}, \mathbb{Z}\}$. Let $T_n \colon W^p \to W$, $\epsilon_n \in (0,\infty)$ and $\Phi_n \in \mathbb{R}_+$ for $n \in \mathbb{J}$ be such that (46) holds with $\mathbb{Z}$ replaced by $\mathbb{J}$ and*

$$\sup_{i\in\mathbb{J}} \frac{\Phi_i(a_{i+1} + \ldots + a_{i+p})}{\epsilon_i} \le \vartheta \sup_{i\in\mathbb{J}} \frac{a_i}{\epsilon_i}, \qquad \forall (a_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}.$$

Let $\xi : Y \to \mathbb{R}_+$ and $(z_n)_{n\in\mathbb{J}} \in W^{\mathbb{J}}$ satisfy inequality

$$\|z_n - T_n(z_{n+1}, \ldots, z_{n+p}), u\| \le \xi(u)\epsilon_n, \qquad \forall n \in \mathbb{J}, u \in Y.$$

Then, there is sequence $(u_n)_{n\in\mathbb{J}} \in W^{\mathbb{J}}$ satisfying difference Equation (15) such that

$$\sup_{n\in\mathbb{J}} \frac{\|z_n - u_n, v\|}{\epsilon_n} \le \frac{\xi(v)}{1-\vartheta}, \qquad \forall v \in Y.$$

6. Stability in *b*-Metric Spaces

In this section, simplified analogues of Theorems 10 and 11 for *b*-metric spaces are presented.

In what follows, $p \in \mathbb{N}$, (M, d, μ) is a complete *b*-metric space, $\xi := \log_{2\mu} 2$ and $d^{\xi}(x,y) = \big(d(x,y)\big)^{\xi}$ for $x, y \in M$. Next, a mapping $\Psi : \mathbb{R}_+^p \to \mathbb{R}$ is nondecreasing if

$$\Psi(y_1, \ldots, y_p) \le \Psi(z_1, \ldots, z_p)$$

for every $y_1, \ldots, y_p, z_1, \ldots, z_p \in \mathbb{R}_+$ with $y_i \le z_i$ for $i = 1, \ldots, p$.

Let us start with a partial analogue of Theorem 10.

Theorem 19. *Let $\mathbb{J} \in \{\mathbb{N}, \mathbb{Z}\}$, $\chi \in (0,1)$ and $\Psi_n : \mathbb{R}_+^p \to \mathbb{R}_+$ for $n \in \mathbb{J}$ be nondecreasing. Let $\epsilon_n \in (0,\infty)$ and $T_n \colon M^p \to M$ for $n \in \mathbb{J}$ be such that*

$$d(T_n(\overline{y}), T_n(\overline{w})) \le \Psi_n\Big(\frac{1}{4} d^{\xi}(y_1, w_1), \ldots, \frac{1}{4} d^{\xi}(y_p, w_p)\Big), \qquad (50)$$
$$\forall \overline{y} = (y_1, \ldots, y_p), \overline{w} = (w_1, \ldots, w_p) \in M^p, n \in \mathbb{J},$$

$$\sup_{i\in\mathbb{J}} \frac{\Psi_i\big(a_i^{\xi}, \ldots, a_{i+p-1}^{\xi}\big)}{\epsilon_{p+i}} < \chi \sup_{i\in\mathbb{J}} \frac{a_i}{\epsilon_i}, \qquad \forall (a_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}.$$

Let $(z_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfy inequality

$$d\big(z_{n+p}, T_n(z_n, z_{n+1}, \ldots, z_{n+p-1})\big) \le \epsilon_{n+p}, \qquad \forall n \in \mathbb{J}. \qquad (51)$$

Then, there is sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ such that (14) holds and

$$\sup_{n\in\mathbb{J}} \frac{d^{\xi}(z_n, u_n)}{\epsilon_n^{\xi}} \le \frac{4}{1-\chi^{\xi}}.$$

Proof. In view of Theorem 12, there exists metric D_d in M such that

$$\frac{1}{4}d^{\xi}(x,y) \le D_d(x,y) \le d^{\xi}(x,y), \qquad \forall x,y \in M. \tag{52}$$

Since d is complete, inequalities (52) imply that metric D_d is also complete.

Note that, by (50),

$$\begin{aligned} D_d(T_n(\overline{y}), T_n(\overline{w})) &\le d^{\xi}(T_n(\overline{y}), T_n(\overline{w})) \\ &\le \Psi_n^{\xi}\Big(\frac{1}{4}d^{\xi}(y_1,w_1),\dots,\frac{1}{4}d^{\xi}(y_p,w_p)\Big) \\ &\le \Psi_n^{\xi}\Big(D_d(y_1,w_1),\dots,D_d(y_p,w_p)\Big) \end{aligned}$$

for every $\overline{y} = (y_1,\dots,y_p), \overline{w} = (w_1,\dots,w_p) \in M^p$ and $n \in \mathbb{J}$. Next, (51) and (52) yield

$$\begin{aligned} D_d(z_{n+p}, T_n(z_n, z_{n+1},\dots,z_{n+p-1})) &\le d^{\xi}(z_{n+p}, T_n(z_n, z_{n+1},\dots,z_{n+p-1})) \\ &\le \epsilon_{n+p}^{\xi}, \qquad \forall n \in \mathbb{J}. \end{aligned}$$

Now, observe that the assumptions of Theorem 10 are fulfilled with $\vartheta = \chi^{\xi}$, $\delta_n = \epsilon_n^{\xi}$, $\rho = D_d$ and $\Theta_n = \Psi_n^{\xi}$. Consequently, there is sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ such that (14) holds and

$$\sup_{n\in\mathbb{J}} \frac{D_d(z_n,u_n)}{\delta_n} \le \frac{1}{1-\vartheta},$$

whence

$$\sup_{n\in\mathbb{J}} \frac{d^{\xi}(z_n,u_n)}{\epsilon_n^{\xi}} \le \frac{4}{1-\chi^{\xi}}.$$

□

In an analogous way, the following (complementary to Theorem 19) partial extension of Theorem 11 can be obtained.

Theorem 20. *Let $\mathbb{J} \in \{\mathbb{N}, \mathbb{Z}\}$, $\chi \in (0,1)$ and $\Psi_n : \mathbb{R}_+^p \to \mathbb{R}_+$ be nondecreasing for $n \in \mathbb{J}$. Let $T_n \colon M^p \to M$ and $\epsilon_n \in (0,\infty)$ for $n \in \mathbb{J}$ be such that (50) holds and*

$$\sup_{i\in\mathbb{J}} \frac{\Psi_i(a_{i+1}^{\xi},\dots,a_{i+p}^{\xi})}{\epsilon_i} \le \chi \sup_{i\in\mathbb{J}} \frac{a_i}{\epsilon_i}, \qquad \forall (a_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}.$$

Let $(z_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfy inequality

$$d(z_n, T_n(z_{n+1}, z_{n+2},\dots,z_{n+p})) \le \epsilon_n, \qquad \forall n \in \mathbb{J}.$$

Then, there is sequence $(u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ satisfying difference Equation (15) such that

$$\sup_{n\in\mathbb{J}} \frac{d^{\xi}(z_n,u_n)}{\epsilon_n^{\xi}} \le \frac{4}{1-\chi^{\xi}}.$$

7. Conclusions

An equation is Ulam stable if each mapping, fulfilling the equation approximately, is somehow close to an exact solution of the equation.

Since the notions of approximate solutions and the closeness of two mappings may be understood in different ways (depending on a situation that we study), it makes sense to consider Ulam stability in various spaces.

In this paper, we showed ways to obtain some general Ulam stability outcomes for difference equations with respect to the norms, 2-norms and b-metrics. In particular, we demonstrated in this way that some symmetries occur between such outcomes in classical normed spaces, 2-normed spaces and b-metric spaces.

We also mentioned some issues that can be studied further. Moreover, it would be interesting to improve and complement the results presented here, but also investigate similar outcomes for other equations (including differential, functional and integral equations). In connection with this last issue, we would like to draw the attention of interested readers to the methods presented in [64] and to the outcomes in publications [10,27–32] (that were proven mainly in the case of normed spaces).

Finally, it also would be interesting to extend the methods presented here to the n-normed and quasi-normed spaces. For the necessary information on Ulam stability in n-normed spaces, we refer the readers to [21,65–69].

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