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Solutions of Magnetohydrodynamics Equation through Symmetries

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Abstract: The magnetohydrodynamics (1 + 1) dimension equation, with a force and force-free term, is analysed with respect to its point symmetries. Interestingly, it reduces to an Abel's Equation of the second kind and, under certain conditions, to equations specified in Gambier's family. The symmetry analysis for the force-free term leads to Euler's Equation and to a system of reduced second-order odes for which singularity analysis is performed to determine their integrability.

Keywords: lie symmetries; invariant functions; MHD equations; fluid equations; singularity analysis

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1. Introduction

In this paper, we focus on the magnetohydrodynamics (MHD) $(1+1)$-dimension equation, as proposed in reference [1]. Fleischer et al. presents the effect of intermittency in the turbulence of plasmas, by specifying a replica of the Burger's equation for the magnetohydrodynamics (MHD) equation. This paper discusses in detail the physical significance of each parameter and the impact on the properties of the $(1+1)$ dimensional MHD under a nonzero external force. It is to be noted here that the model proposed by Fleischer and Diamond is a plainer version of the presence of magnetic field to $(1+1)$-dimensional MHD equation. As suggested in reference [1], it is to be considered as an enhanced model in comparison to the one cited in reference [2]. Later on, Basu et al. [3] discussed in detail the physical significance of the equation and its corresponding similarity to certain well-known equations. Moreover, a comprehensive list of historical details of the equation are provided in references [3–9]. The equation in $(1+1)$ dimension is of the following form:

$$\begin{aligned} u_t + B_0 v_x + u u_x + v v_x &= \nu u_{xx} + f(t,x), \\ v_t + B_0 u_x + (uv)_x &= \mu v_{xx} + g(t,x), \end{aligned} \tag{1}$$

where $f(t,x)$ and $g(t,x)$ are external forces, ν, μ are fluid kinematic viscosity and magnetic diffusivity, respectively. u and v are velocity field and magnetic field, respectively. B_0 is a mean magnetic field (independent of x and t).

In the literature, a comprehensive work has been carried out on the various dimensions of MHD equations. In reference [10], Fuchs discussed the symmetries and the corresponding reductions at a stretch for the three-dimensional MHD equation. The references of the paper also cited some interesting work done by Nucci [11] on the group analysis of $(1+3)$ dimensional MHD equations along with the equations with respect to its magnetic component. Further, we suggest that readers consult references [12,13] for an in-depth study. We also recommend references [14,15] which discuss the impact of symmetries on the stabilitiy of the MHD equations. Also, in references [16,17] the constitution of the MHD equation is specified elaborately.

Our main objective in this work is to compute certain new similarity reductions and solutions for Equation (1) using Lie group analysis. The Lie point symmetry analysis does provide new results for Equation (1). The equation reduces to Abel's equation of the second kind and, under certain conditions, leads to Gambier's equation [18], the classification number of which is presented against each equation. Also, we performed a symmetry analysis of the force-free term. The subsequent reductions leads to an Euler's equation and to a system of second-order odes, with respect to travelling-wave, for which we conduct a singularity analysis.

The application of the symmetry analysis can also be extended to problems consisting of boundary conditions. The main criteria is that the equation under study remains invariant under the transformations and all the properties of the domain are preserved. It is to be noted that, for certain equations, a composition of solutions can be constructed from the derived invariant solutions and that particular composition could possibly satisfy the boundary conditions. Here, it is important to emphasize that the invaraint solutions obtained by the symmetries can be considered to be more general, which remains valid for any bounded or unbounded domain and, hence, it describes the system properly and efficiently. Therefore, our main objective of the computation of solutions through the similarity transformations is more generic, which could possibly satisfy the boundary conditions.

This paper is organised as follows. In Section 2, the preliminaries of the computation of the point symmetries are mentioned. In Section 3, the point symmetries and subsequent computations are done for the external forces. In Section 4, the reduction to the corresponding odes and its symmetries, which are used for further reductions, are mentioned. In Sections 5 and 6, the particular cases of a second-order equation obtained in Section 4 is discussed in detail. Finally, in Section 7, a symmetry analysis of the Equation (1) without an external force is presented and a singularity analysis test for certain reduced equations is elaborated. The conclusion, acknowledgements, and proper references follow henceforth.

2. Preliminaries

Any differential equation is understood to possess a symmetry, provided the equation possesses its basic properties and characteristics, even after undergoing a transformation with respect to its coordinates. Mathematically, it can be explained as follows:

Consider a PDE of the form $F(\hat{x}, u, \hat{u}) = 0$, where $\hat{x}$ represents the array of independent variables, u represents the dependent variable, and $\hat{u}$ represents the array of derivatives of the dependent variable. Our objective is to compute the point symmetries of the PDE. First of all, certain transformations need to be defined as follows:

$$\begin{aligned} \underline{\hat{x}} &= \hat{x} + \epsilon\xi(\hat{x}, u) + \mathcal{O}(\epsilon^2) \\ \underline{u} &= u + \epsilon\eta(\hat{x}, u) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2}$$

The transformations with respect to the each derivative term in $\hat{u}$ can be derived easily using (2) with the aid of the Chain rule. Now, the PDE, to be invariant, need to satisfy the condition

$$F(\hat{x}, u, \hat{u}) = F(\underline{\hat{x}}, \underline{u}, \underline{\hat{u}}).$$

The set of transformations (2) forms a one-parameter Lie group, the generator of which is represented as

$$\Gamma = \xi(\hat{x}, u) + \eta(\hat{x}, u). \tag{3}$$

The operator (3) is the generator of the infinitesimal transformations $\xi(\hat{x}, u)$ and $\eta(\hat{x}, u)$. This opertor is being used to define the characteristics using the method of the associated Lagarange's system and it is to be considered a point symmetry of the PDE under study. Further, these generators are mainly used to reduce the order of the PDE, to compute conservation laws, and, also, it has been used recently to derive nonlocal symmetries. The reader can follow references [19–25] to understand the basics of point symmetries and the procedure of computation.

3. Lie Symmetries of the Magnetohydrodynamics Equation

Corresponding to the differential Equation (1), we write a general element of the symmetry vector field of Equation (1), in the form

$$V = \xi^1\partial_t + \xi^2\partial_x + \eta_1\partial_u + \eta_2\partial_v. \tag{4}$$

where ξ^1, ξ^2, η_1, and η_2 are functions of t, x, u, and v.

Requiring that the second prolongation $Pr^{(1)}V$ of Equation (4) should annihilate (1) on the solution set, we obtain the determining equations for the coefficients ξ^1,ξ^2,η_1, and η_2. The solution of the determining equations are given by

$$\xi^1 = a_1(t) \tag{5}$$

$$\xi^2 = a_2(t) + \frac{x\dot{a}_1}{2} \tag{6}$$

$$\eta_1 = \dot{a_2} + \frac{x\ddot{a}_1}{2} - \frac{u\dot{a}_1}{2} \tag{7}$$

$$\eta_2 = -\frac{1}{2}(v + B_0)\dot{a}_1 \tag{8}$$

Provided $f(t,x)$ and $g(t,x)$ satisfies the following linear partial differential equations,

$$2a_1 f_t + (2a_2 + x\dot{a}_1)f_x + 3f\dot{a}_1 - 2\ddot{a}_2 - x\dddot{a}_1 = 0 \tag{9}$$

$$2a_1 g_t + (2a_2 + x\dot{a}_1)g_x + 3g\dot{a}_1 = 0 \tag{10}$$

where a_1 and a_2 are arbitrary functions of t. Then, the Lie symmetry algebra of Equation (1) is

$$\Gamma = a_1(t)\partial_t + \left(a_2(t) + \frac{x\dot{a}_1}{2}\right)\partial_x + \left(\dot{a_2} + \frac{x\ddot{a}_1}{2} - \frac{u\dot{a}_1}{2}\right)\partial_u + \left(-\frac{1}{2}(v + B_0)\dot{a}_1\right)\partial_v \tag{11}$$

Since (11) is a first-order partial differential equation, the equation of the characteristic curve may be expressed invariantly using the Lagrange–Charpit method. Equation (11) can be rewritten in the Lagrange form.

$$\frac{dt}{a_1} = \frac{dx}{\frac{1}{2}\dot{a}_1 x + a_2} = \frac{df}{-\frac{3}{2}\dot{a}_1 f + \frac{1}{2}x\dddot{a}_1 + \ddot{a}_2}. \tag{12}$$

The characteristic systems are

$$\frac{dx}{dt} - \frac{\dot{a}_1}{2a_1}x - \frac{a_2}{a_1} = 0 \tag{13}$$

$$\frac{df}{dt} + \frac{3\dot{a}_1}{2a_1}f - \frac{\dddot{a}_1}{2a_1}x - \frac{\ddot{a}_2}{a_1} = 0. \tag{14}$$

The first equation gives a characteristic $z = \frac{(x-\alpha)}{\rho} = constant$, where $a_1 = \rho^2$ and $\alpha = \rho\int\frac{a_2}{\rho^3}\,dt$.

Then, the Equation (14) becomes

$$\frac{df}{dt} + \frac{3\dot{a}_1}{2a_1}f - \frac{1}{2}(a_1^{\frac{1}{2}}z + \alpha)\frac{\dddot{a}_1}{a_1} - \frac{\ddot{a}_2}{a1} = 0 \tag{15}$$

and gives the characteristic

$$H_1(z) = f\rho^3 - \rho^2\dddot{\rho}x - (\ddot{\alpha}\rho - \alpha\ddot{\rho})\rho^2 = constant.$$

Then, the solution of Equation (9) is

$$f(x,t) = \ddot{\rho}\left(\frac{x-\alpha}{\rho}\right) + \ddot{\alpha} + \frac{1}{\rho^3}H_1\left(\frac{x-\alpha}{\rho}\right), \tag{16}$$

where H is an arbitrary function of its argument.

Similarly,

$$g(x,t) = \frac{1}{\rho^3}H_2\left(\frac{x-\alpha}{\rho}\right), \tag{17}$$

is the solution of Equation (10), where H_2 is an arbitrary function of its argument.

It is important to note, here, that the analysis of the linear counterpart of Equation (1) leads to certain important observations. The linear counterpart of Equation (1) is obtained by considering the terms uu_x, vv_x, and $(uv)_x$ equal to zero. The subsequent search for symmetries lists certain trivial and nontrivial symmetries and certain equations similar to (9) and (10). The explicit symmetries are as follows:

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_x \\ \Gamma_3 &= u\partial_u + v\partial_v. \end{aligned}$$

Now, along with these symmetries, certain arbitrary functions representing infinite-dimensional symmetries are also obtained, which can be used productively to obtain the reduced ordinary differential equations of the subsequent linear counterpart of Equation (1). It is also important to note here that the symmetries mentioned as Γ_1, Γ_2, and Γ_3 are the trivial symmetries inherent to any linear equations and, hence, forms the main difference between the linear and nonlinear equations. The reduced equations, using these symmetries, may lead to certain trivial equations which are easily solvable and, hence, the plot of these solutions can be easily understood.

4. Reduction to an Ordinary Differential Equation

The characteristics corresponding to the generator (4) are found by solving the equation

$$\frac{dt}{a_1} = \frac{dx}{\frac{1}{2}\dot{a}_1 x + a_2} = \frac{du}{\frac{1}{2}\ddot{a}_1 x - \frac{1}{2}\dot{a}_1 u + \dot{a}_2} = \frac{dv}{-\frac{1}{2}\dot{a}_1(v + B_0)}. \tag{18}$$

The first characteristic for the Equation (18) is $z = \frac{(x-\alpha)}{\rho}$, which is the new similarity variable, where α and ρ are defined as $a_1 = \rho^2$ and $\alpha = \rho \int \frac{a_2}{\rho^3}\, dt$. The second characteristic can be obtained by solving the equation

$$\frac{du}{dt} + \frac{\dot{a}_1}{2a_1}u - \frac{\ddot{a}_1}{2a_1}(\rho z + \alpha) - \frac{\dot{a}_2}{a_1} = 0. \tag{19}$$

The solution is

$$G(z) = \rho u - \rho\dot{\rho}z - \rho\dot{\alpha}. \tag{20}$$

From the above solution, u can be written as

$$u = \dot{\rho}z + \dot{\alpha} + \frac{G(z)}{\rho}. \tag{21}$$

The third characteristic can be obtained by solving the equation

$$\frac{dv}{dt} + \frac{\dot{a}_1}{2a_1}v - B_0\frac{\ddot{a}_1}{2a_1} = 0. \tag{22}$$

The solution is

$$F(z) = \rho v + \rho B_0. \tag{23}$$

From the above solution, v can be written as

$$v = \frac{F(z)}{\rho} - B_0. \tag{24}$$

In the above, $G(z)$ and $F(z)$ are similarity functions, when one substitutes the values of u, v, f, and g into Equation (1), we have

$$\begin{aligned} H_1(z) + G'(z)G(z) + F'(z)F(z) - \nu G''(z) = 0 \\ -H_2(z) + F(z)G'(z) + G(z)F'(z) - \mu F''(z) = 0 \end{aligned} \tag{25}$$

Letting $H_1(z) = \mathcal{H}_1'(z)$ and $H_2(z) = \mathcal{H}_2'(z)$ and integrating (25) with respect to z, we have a system of first-order ordinary differential equations:

$$\frac{G(z)^2}{2} + \frac{F(z)^2}{2} + \mathcal{H}_1(z) - \nu G'(z) \quad = \quad 0 \tag{26}$$

$$-F(z)G(z) + \mathcal{H}_2(z) + \mu F'(z) \quad = \quad 0 \tag{27}$$

We can find the value of $G(z)$ from Equation (27) and, substituting into Equation (26), we get the second-order ordinary differential equation:

$$F'' = \left(\frac{\mu}{2\nu} + 1\right)\frac{F'^2}{F} + \left(\frac{1}{\nu} + \frac{1}{\mu}\right)\mathcal{H}_2\frac{F'}{F} + \frac{1}{2\nu\mu}F^3 + \frac{1}{\nu\mu}\mathcal{H}_1 F - \frac{1}{\mu}\mathcal{H}_2' + \frac{\mathcal{H}_2^2}{2\mu\nu F} \tag{28}$$

Now, apply the transformation $F(z) = \frac{1}{w(z)}$ and the above equation becomes

$$w'' = \left(1 - \frac{\mu}{2\nu}\right)\frac{w'^2}{w} + \left(\frac{1}{\nu} + \frac{1}{\mu}\right)\mathcal{H}_2 w w' - \frac{\mathcal{H}_2^2}{2\nu\mu}w^3 + \frac{1}{\mu}\mathcal{H}_2'^2 - \frac{\mathcal{H}_1}{\mu\nu}w - \frac{1}{2\mu\nu w} \tag{29}$$

The general case does not fit into the known equation. However, we perform another symmetry reduction and identify it to an Abel's Equation of the second kind.

$$F'' = \left(1 + \frac{\mu}{2\nu}\right)\frac{F'^2}{F} + \left(\frac{1}{\mu} + \frac{1}{\nu}\right)H_2\frac{F'}{F} + \frac{F^3}{2\mu\nu} + \frac{H_1 F}{\mu\nu} - \frac{H_2'}{\mu} + \frac{H_2^2}{2\mu\nu F} \tag{30}$$

The infinitesimal generator is

$$\Gamma = \left(p_2 - \frac{p_1 z\mu}{\mu - \nu}\right)\partial_z + F\left(\frac{p_1\mu}{\mu - \nu}\right)\partial_F, \tag{31}$$

provided $H_1(z) = kH_2(z)$ and $H_2(z) = \frac{h_1}{(p_1 z\mu + p_2(\nu-\mu))^2}$.

The canonical coordinates corresponding to the vector field (31) are

$$r = -F(p_1 z\mu + p_2(\nu - \mu)) \ , \ s = \frac{(\mu - \nu)\log(p_1 z\mu + p_2(\nu - \mu))}{p_1\mu}. \tag{32}$$

Then, it is easy to see that

$$\frac{ds}{dr} = \frac{\nu - \mu}{(p_1 z\mu + p_2(\nu - \mu))(-p_1\mu F - (p_1 z\mu + p_2(\nu - \mu))F')}.$$

If we choose v to be $\left(\frac{ds}{dr}\right)^{-1}$

$$\frac{dv}{dr} = \frac{p_1^2\mu^2 F + (p_1 z\mu + p_2(\nu-\mu))(3p_1\mu F' + (p_1 z\mu + p_2(\nu-\mu))F'')}{(\nu-\mu)(p_1\mu F + (p_1 z\mu + p_2(\nu-\mu))F')} \tag{33}$$

$$\begin{aligned} &r^2\left(4\mu^3(\mu+6\nu)+r^2\right)+2r(2\mu(\mu-\nu)+5r)+4\mu(\mu-\nu)v(r) \\ &\quad\left(\mu+\nu+2\mu^2 r(\mu+5\nu)+\mu\nu r(\mu-\nu)\frac{dv}{dr}\right)+4\mu^3(\mu-\nu)^2(\mu+2\nu)v(r)^2+1=0 \end{aligned} \tag{34}$$

5. Particular Cases of Equation (28)

5.1. Case 1 (Gambier.B 19)

Let $\mu = -\nu = \frac{i}{2\sqrt{2}}, \mathcal{H}_2 = 0, \mathcal{H}_1 = -2\nu^2$ and apply the transformation $z \to x$, $F \to y$, we obtain

$$y'' = \frac{y'^2}{y} + 4y^3 + 2y,$$

with the analytic solution

$$\int \frac{dy}{\sqrt{4y^2 + 4\ln(y) + y_0 y}} = x - x_0.$$

The numerical evolution of the latter equation is presented in Figure 1.

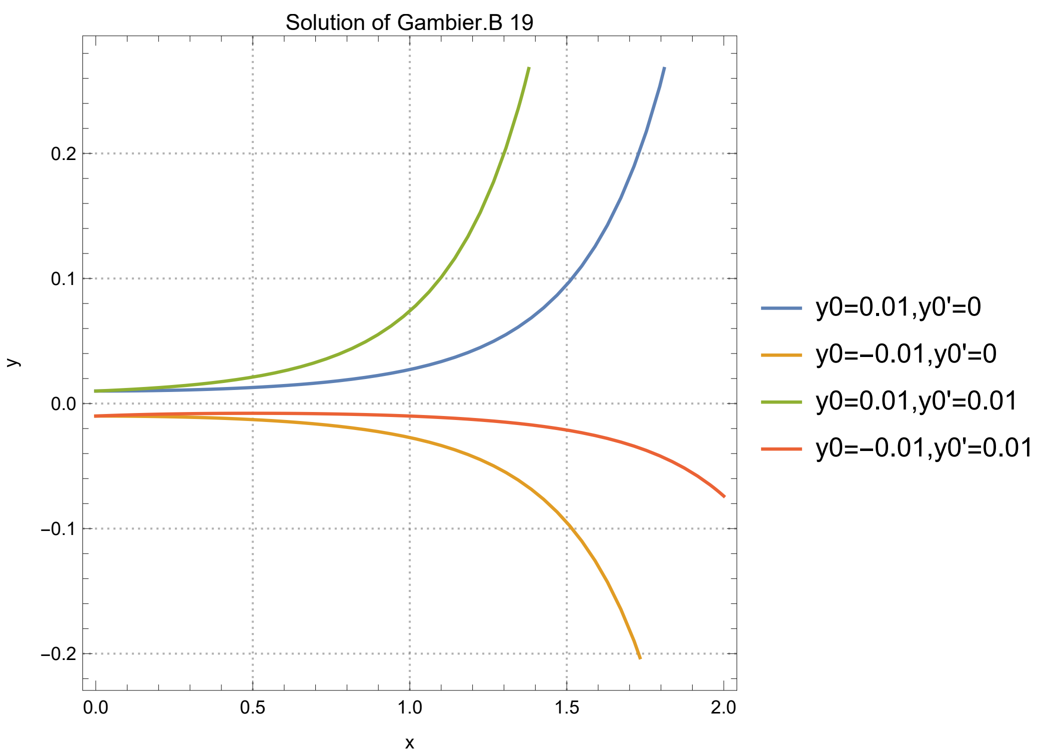

Figure 1. Qualitative evolution of the Gambier.B 19 equation.

5.2. Case 2 (Gambier.B 29)

Let $\mu = -\nu = \frac{i}{\sqrt{3}}, \mathcal{H}_1 = \mathcal{H}_2 = 0$, and apply the transformation $z \to x$, $F \to y$, we obtain

$$y'' = \frac{y'^2}{y} + \frac{3}{2}y^3,$$

with the analytic solution

$$y(x) = \frac{8y_0 e^{-\frac{\sqrt{2}}{2y_1}x}}{(y_0)^2 e^{-\frac{\sqrt{2}}{y_1}x} - 48} \text{ and } y(x) = \frac{8y_0 e^{\frac{\sqrt{2}}{2y_1}x}}{(y_0)^2 e^{\frac{\sqrt{2}}{y_1}x} - 48}.$$

The numerical evolution of the latter equation is presented in Figure 2.

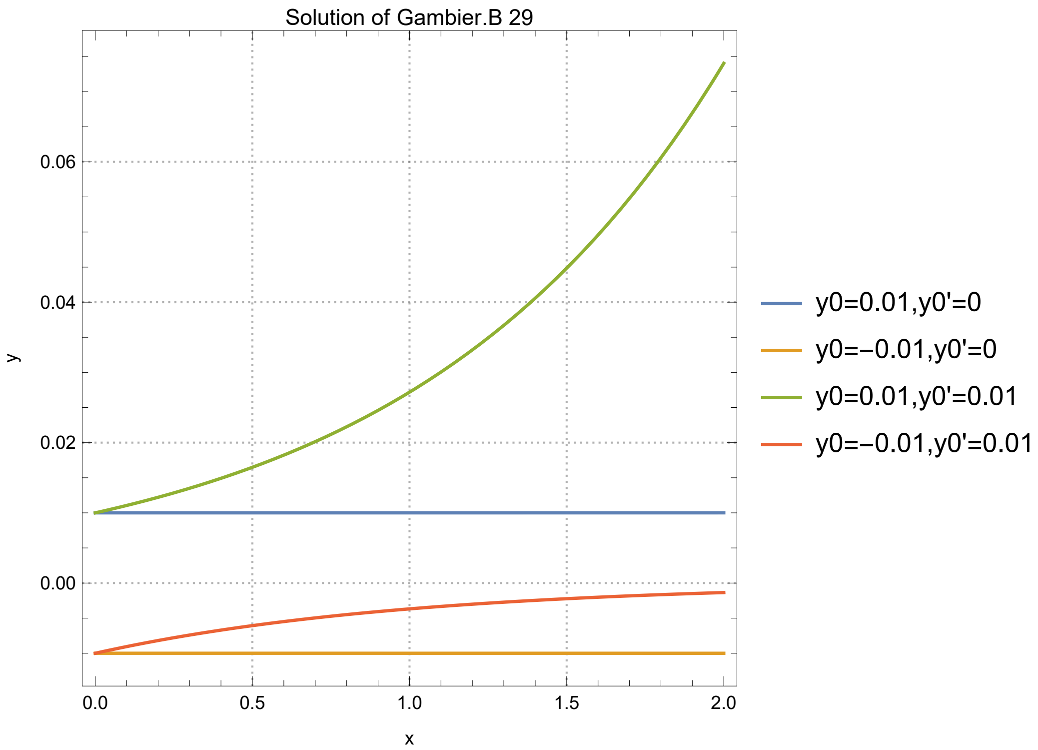

Figure 2. Qualitative evolution of the Gambier.B 29 equation.

5.3. Case 3 (Gambier.B 30)

Let $\mu = -\nu = \frac{i}{\sqrt{3}}, \mathcal{H}_1 = 2\beta\mu\nu, \mathcal{H}_2 = \gamma\nu$, and apply the transformation $z \to x$, $F \to y$, we obtain

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 2\beta y - \frac{\gamma^2}{2y}$$

where the solution is given in terms of the elliptic integral

$$\int \frac{2dy}{\sqrt{4y^2 y_0 + 6y^4 + 16y^2\beta \ln a + 2\gamma^2}} = x - x_0.$$

This can be considered a variant of Gambier.B 30 for $\alpha = 0$,

$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4\alpha y^2 + 2\beta y - \frac{\gamma^2}{2y}.$$

The numerical evolution of the latter equation is presented in Figure 3.

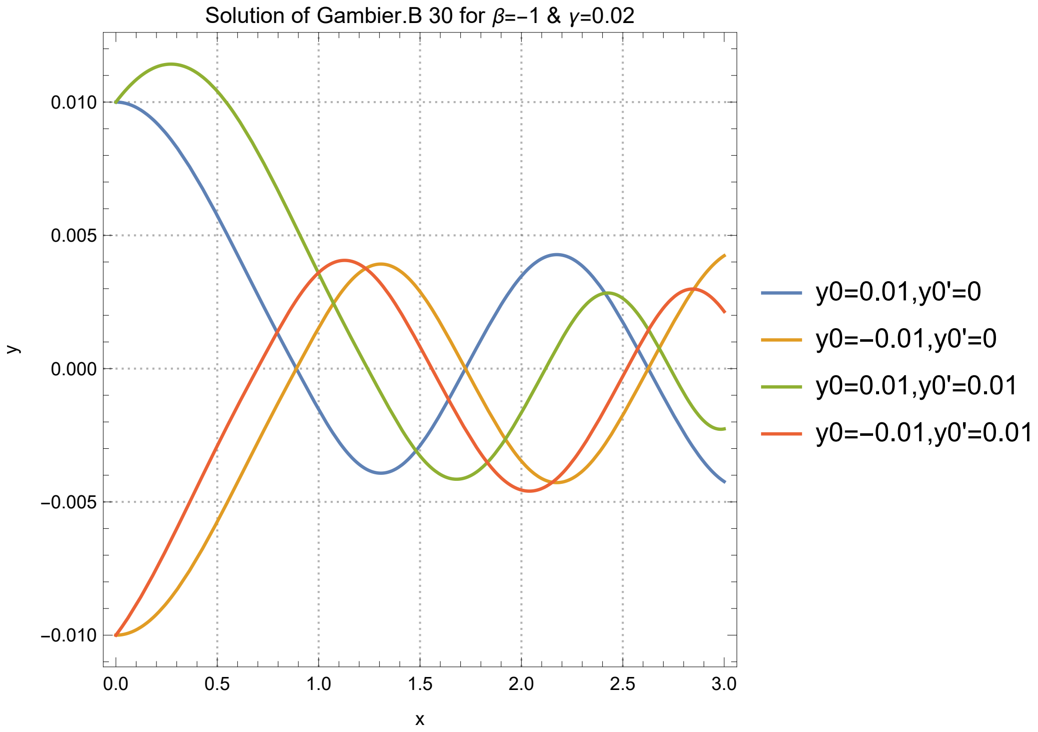

Figure 3. Qualitative evolution of the Gambier.B 30 equation.

5.4. *Case 4 (Second Painlevé Transcendent)*

Let $\mu = -2\nu = \frac{i}{\sqrt{2}}, \mathcal{H}_2 = 0,\ \mathcal{H}_1 = -2\nu^2 z$ and apply the transformation $F \to w,\ z \to x$. We obtain the second Painlevé transcendent:

$w''^3 + xw = 0.$

The numerical evolution of the latter equation is presented in Figure 4.

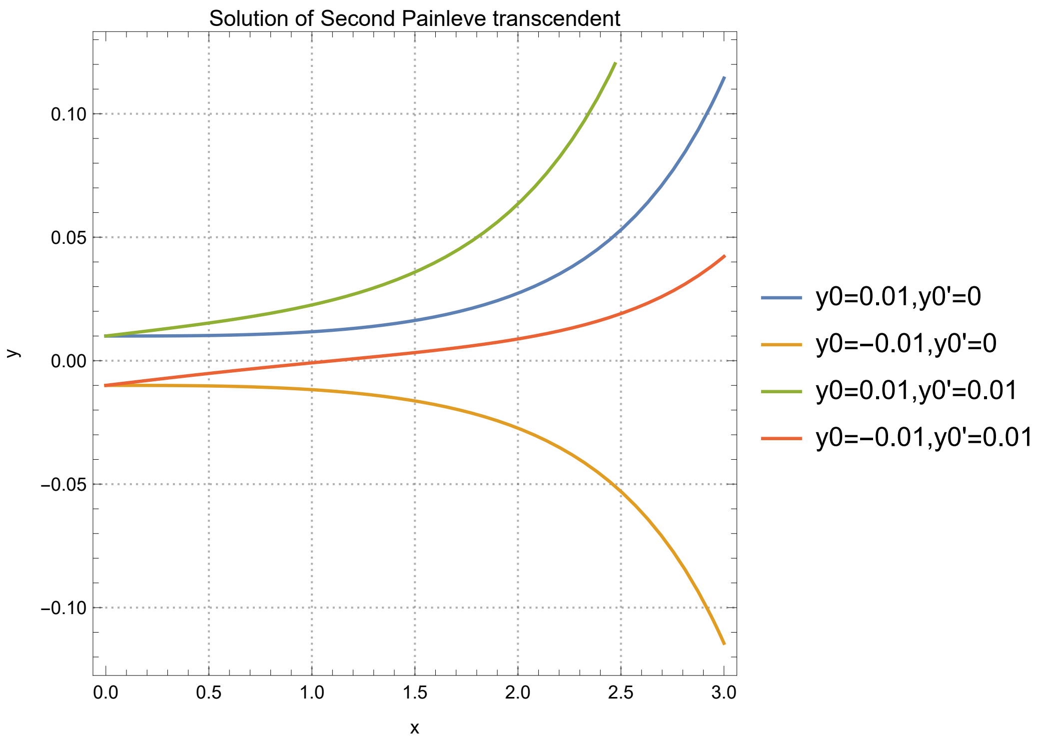

Figure 4. Qualitative evolution of the second Painlevé transcendent equation.

5.5. Case 5 (Kummer–Schwarz Equation)

If $\mu = \nu$ and $\mathcal{H}_2 = 0$, and apply the transformation $z \to x$, $F \to y$, we obtain the second-order Kummer–Schwarz equation

$F'' = \frac{3}{2}\frac{F'^2}{F} + \frac{1}{2\mu^2}F^3 + \frac{\mathcal{H}_1}{\mu^2}F.$

The numerical evolution of the latter equation is presented in Figure 5.

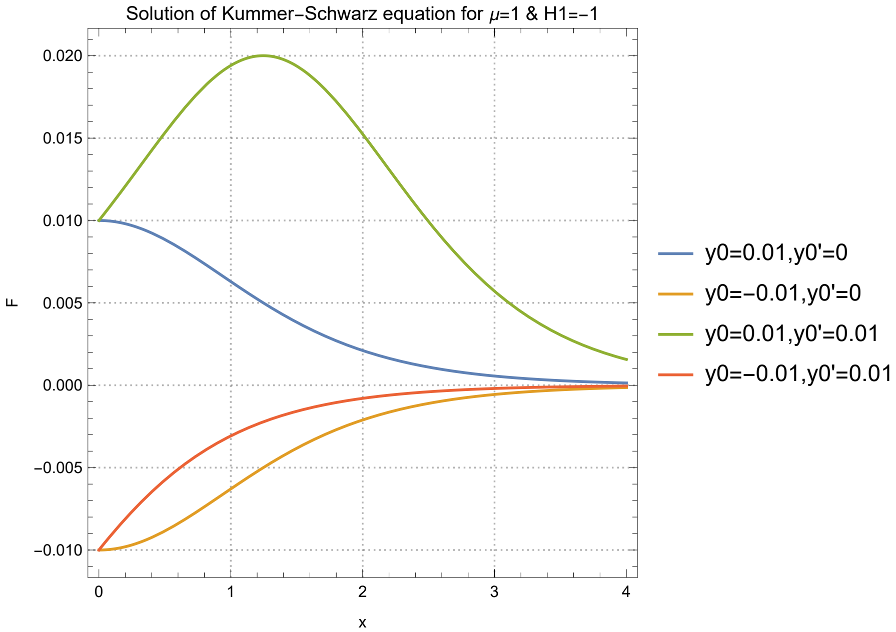

Figure 5. Qualitative evolution of the Kummer–Schwarz equation.

5.6. Case 6 (Duffing Equation)

If $\mathcal{H}_1$ is constant $\mathcal{H}_2 = 0$ and $\mu = -2\nu$, and apply the transformation $z \to x$, $F \to y$, we obtain the undamped and unforced Duffing equation:

$F'' + \frac{1}{4\nu^2}F^3 + \frac{1}{2\nu^2}\mathcal{H}_1 F = 0$

The numerical evolution of the latter equation is presented in Figure 6.

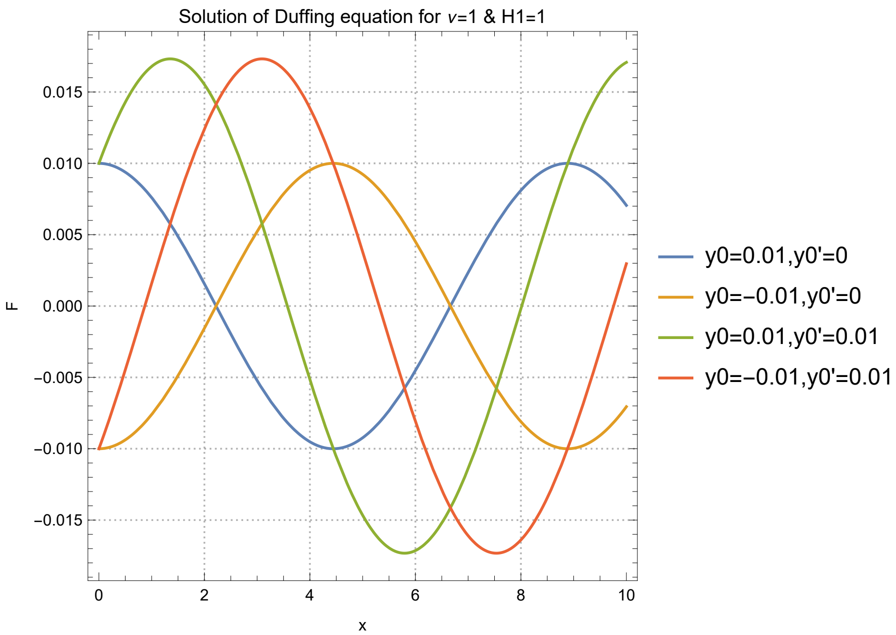

Figure 6. Qualitative evolution of the Duffing equation.

6. General Cases of Equation (29)

Equation (29) can be considered as a particular case of the following two Gambier's equations:

6.1. Case 7 Gambier.B 28

$$y'' = \frac{y'^2}{2y} - yy' + qy' + \frac{y^3}{2} - 2qy^2 + 3(q' + \frac{q^2}{2})y - \frac{72r^2}{y}$$

6.2. Case 8 Gambier.B 27

$$y'' = (1 - \frac{1}{n})\frac{y'^2}{2y} + f_n(q,r)yy' + \phi_n(q,r)y' - \frac{n-2}{n}\frac{y'}{y} + \frac{nf_n^2}{(n+2)^2}y^3 + \frac{n(f_n' + f_n\phi_n}{(n+2)}y^2 + \psi_n(q,r)y$$
$$+ \phi_n + \frac{1}{ny}$$

7. The Case $f = 0$, $g = 0$

The equation has the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_x \\ \Gamma_3 &= t\partial_x + \partial_u \end{aligned}$$

In the subsequent subsections, we study the reductions with respect to $\Gamma_1, \Gamma_2, \Gamma_3$, and the travelling-wave. In Section 7.2, the singularity analysis of the reduced ode is also mentioned explicitly.

7.1. Case 6a.

The reduction, with respect to Γ_1, leads to the reduced ode

$$\begin{aligned} F_1(x)F_1'(x) + B_0G_1'(x) + G_1(x)G_1'(x) &= \nu F_1''(x), \\ B_0F_1'(x) + G_1(x)F_1'(x) + F_1(x)G_1'(x) &= \mu G_1''(x), \end{aligned} \tag{35}$$

where $u(t,x) = F_1(x)$ and $v(t,x) = G_1(x)$. The point symmetries of Equation (35) are

$$\begin{aligned} \Gamma_{1a} &= \partial_x, \\ \Gamma_{2a} &= x\partial_x - F_1\partial_{F_1} - (B_0 + G_1)\partial_{G_1}. \end{aligned} \tag{36}$$

The reductions with respect to Γ_{1a} and Γ_{2a} does not yield any positive result. The singularity analysis of the Equation (35), too, does not provide any fruitful result.

Next, we look for the reduction with respect to Γ_2, which leads to the simpler first-order ode $F_2'(t) = 0$, and $G_2'(t) = 0$, where $u(t,x) = F_2(t)$ and $v(t,x) = G_2(t)$.

Further, the reduction with respect to Γ_3, leads to an Euler's equation, which is

$$\begin{aligned} F_3(t) + tF_3'(t) &= 0, \\ B_0 + G_3(t) + tG_3'(t) &= 0, \end{aligned} \tag{37}$$

where $u(t,x) = \frac{x}{t} + F_3(t)$ and $v(t,x) = G_3(t)$.

7.2. Case 6b.

In this subsection, we perform the reductions with respect to the travelling-wave. The reduction leads to a system of two second-order odes, which are

$$\begin{aligned} B_0F_1'(z) + F_1(z)F_1'(z) - cG_1'(z) + G_1(z)G_1'(z) &= \nu G_1''(z), \\ -cF_1'(z) + G_1(z)F_1'(z) + B_0G_1'(z) + F_1(z)G_1'(z) &= \mu F_1''(z), \end{aligned} \tag{38}$$

where the similarity variables are

$$\begin{aligned} x-ct &= z, \\ u(t,x) &= G_1(z), \\ v(t,x) &= F_1(z). \end{aligned}$$

The Lie point symmetries of equation (38) are

$$\begin{aligned} \Gamma_{1b} &= \partial_z, \\ \Gamma_{2b} &= z\partial_z - (B_0+F_1)\partial_{F_1} + (c-G_1)\partial_{G_1}. \end{aligned}$$

The reduction, with respect to Γ_{1b} and Γ_{2b}, does not provide any fruitful result, similar to Equation (35). Therefore, we dwell on the singularity analysis to comprehend its integrability.

Singularity Analysis of Equation (38).

We substitute $F_1 \to \alpha z^p$ and $G_1 \to \beta z^q$, in Equation (38). The substitution leads to

$$\begin{aligned} B_0pz^{-1+p}\alpha + pz^{-1+2p}\alpha^2 - cqz^{-1+q}\beta + qz^{-1+2q}\beta^2 + qz^{-2+q}\beta\nu - q^2z^{-2+q}\beta\nu &= 0, \\ -cpz^{-1+p}\alpha + B_0qz^{-1+q}\beta + pz^{-1+p+q}\alpha\beta + qz^{-1+p+q}\alpha\beta + pz^{-2+p}\alpha\mu - p^2z^{-2+p}\alpha\mu &= 0, \end{aligned} \tag{39}$$

One of the possible values for the exponents p and q is -1. After substituting the values of p and q in Equation (39), we use the dominant terms to compute the leading-order coefficients α and β. The terms are

$$\begin{aligned} -\tfrac{\alpha^2}{z^3} - \tfrac{\beta^2}{z^3} - \tfrac{2\beta\nu}{z^3}, \\ -\tfrac{2\alpha\beta}{z^3} - \tfrac{2\alpha\mu}{z^3} \end{aligned} \tag{40}$$

Solving Equation (40) for α and β leads to the following possible values

$$\begin{aligned} &(\alpha \to 0, \beta \to 0), \left(\alpha \to -\sqrt{2\mu\nu-\mu^2}, \beta \to -\mu\right), \\ &\left(\alpha \to \sqrt{2\mu\nu-\mu^2}, \beta \to -\mu\right), (\alpha \to 0, \beta \to -2\nu). \end{aligned} \tag{41}$$

Next, to compute the resonance, we substitute $F_1 \sim \alpha z^p + m_1 z^{p+s}$, $G_1 \sim \beta z^q + m_2 z^{q+s}$ in Equation (38). The substitution leads to

$$\begin{aligned} 0 &= -B_0m_1z^{-2+s} + cm_2z^{-2+s} + B_0m_1sz^{-2+s} - cm_2sz^{-2+s} - m_1^2z^{-3+2s} - m_2^2z^{-3+2s} + m_1^2sz^{-3+2s} + m_2^2sz^{-3+2s} \\ &\quad -\tfrac{B_0\alpha}{z^2} - 2m_1z^{-3+s}\alpha + m_1sz^{-3+s}\alpha - \tfrac{\alpha^2}{z^3} + \tfrac{c\beta}{z^2} - 2m_2z^{-3+s}\beta + m_2sz^{-3+s}\beta - \tfrac{\beta^2}{z^3} - 2m_2z^{-3+s}\nu \\ &\quad +3m_2sz^{-3+s}\nu - m_2s^2z^{-3+s}\nu - \tfrac{2\beta\nu}{z^3}, \\ 0 &= cm_1z^{-2+s} - B_0m_2z^{-2+s} - cm_1sz^{-2+s} + B_0m_2sz^{-2+s} - 2m_1m_2z^{-3+2s} + 2m_1m_2sz^{-3+2s} + \tfrac{c\alpha}{z^2} \\ &\quad -2m_2z^{-3+s}\alpha + m_2sz^{-3+s}\alpha - \tfrac{B_0\beta}{z^2} - 2m_1z^{-3+s}\beta + m_1sz^{-3+s}\beta - \tfrac{2\alpha\beta}{z^3} - 2m_1z^{-3+s}\mu + 3m_1sz^{-3+s}\mu \\ &\quad -m_1s^2z^{-3+s}\mu - \tfrac{2\alpha\mu}{z^3}. \end{aligned} \tag{42}$$

We select the linear terms with respect to m_1 and m_2, corresponding to various powers of z. For z^{-2+s}, we have

$$\begin{aligned} (-B_0+B_0s)m_1 + (c-cs)m_2 &= 0, \\ (c-cs)m_1 + (B_0s-B_0)m_2 &= 0. \end{aligned} \tag{43}$$

For system (43) to possess a non-trivial solution, the necessary requirement is

$$\begin{vmatrix} B_0(s-1) & c(1-s) \\ c(1-s) & B_0(s-1) \end{vmatrix} = 0.$$

The possible values of $s = 1(2)$. Similarly, for z^{-3+s}, we have

$$\begin{array}{rcl} -2m_1\alpha + m_1 s\alpha - 2m_2\beta + m_2 s\beta - 2m_2\nu + 3m_2 s\nu - m_2 s^2\nu & = & 0, \\ -2m_2\alpha + m_2 s\alpha - 2m_1\beta + m_1 s\beta - 2m_1\mu + 3m_1 s\mu - m_1 s^2\mu & = & 0 \end{array} \tag{44}$$

Similarly to Equation (43), for the existence of a non-trivial solution, the system

$$(s-2)\begin{vmatrix} -\alpha & -(\beta+\nu-s\nu) \\ -\alpha & -(\beta\mu - s\mu) \end{vmatrix} = 0,$$

leads to the following possible values of s.

$$s \to 2, s \to \frac{\beta\mu + \beta\nu + 2\mu\nu - \sqrt{\beta^2\mu^2 + 4\alpha^2\mu\nu - 2\beta^2\mu\nu + \beta^2\nu^2}}{2\mu\nu}, s \to \frac{\beta\mu + \beta\nu + 2\mu\nu + \sqrt{\beta^2\mu^2 + 4\alpha^2\mu\nu - 2\beta^2\mu\nu + \beta^2\nu^2}}{2\mu\nu}.$$

Now, for $(\alpha, \beta) = (0, -2\nu)$, the possible values of s are

$$\begin{array}{l} s \to -\frac{\nu}{\mu} - \sqrt{4\mu^2\nu^2 - 8\mu\nu^3 + 4\nu^4}2\mu\nu, \\ s \to -\frac{\nu}{\mu} + \sqrt{4\mu^2\nu^2 - 8\mu\nu^3 + 4\nu^4}2\mu\nu. \end{array} \tag{45}$$

For $\left(\alpha \to -\sqrt{2\mu\nu - \mu^2}, \beta \to -\mu\right)$, the possible values of s are

$$\begin{array}{l} s \to \frac{1}{2} - \frac{\mu}{2\nu} - \frac{\sqrt{\mu^4 - 2\mu^3\nu + \mu^2\nu^2 + 4\mu\nu(-\mu^2 + 2\mu\nu)}}{2\mu\nu}, \\ s \to \frac{1}{2} - \frac{\mu}{2\nu} + \frac{\sqrt{\mu^4 - 2\mu^3\nu + \mu^2\nu^2 + 4\mu\nu(-\mu^2 + 2\mu\nu)}}{2\mu\nu}. \end{array} \tag{46}$$

Similar results are obtained for $\left(\alpha \to \sqrt{2\mu\nu - \mu^2}, \beta \to -\mu\right)$. The generic value of the resonance s being -1 is not obtained for both the computations, and, hence, as cited in references [26–29], the integrability of Equation (38) cannot be determined.

Still, a possibility exists when ν, fluid kinematic viscosity, and μ magnetic diffusivity are equal. This leads to $s = -1$. Therefore, under a special circumstance of $\nu = \mu$, we have $s = -1$, and from the analysis of Equation (44) s also takes the value 2. The readers can easily verify the consistency test and, hence, we conclude that the Laurent expansion leads to a right Painlevé series.

8. Conclusions

In this work, reductions concerned with that of the arbitrary functions are deduced for the magnetohydrodynamics(MHD) $(1+1)$-dimension equation. The reduction leads to a certain well-known family of equations which are connected to Gambier's, Abel's, and Kummer–Schwarz equations. The reductions of the force-free equation are generally trivial but the traveling-wave reduction leads to equations which are devoid of point symmetries. Hence, the computation of a series solution using singularity analysis leads to a right Painlevé series, provided the kinematic viscosity and magnetic diffusivity parameters are equal. The physical significance of such a situation is yet to be ascertained and forms the basis of our future work.

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