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Existence and General Decay of Solutions for a Weakly Coupled System of Viscoelastic Kirchhoff Plate and Wave Equations

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Abstract: In this paper, a weakly coupled system (by the displacement of symmetric type) consisting of a viscoelastic Kirchhoff plate equation involving free boundary conditions and the viscoelastic wave equation with Dirichlet boundary conditions in a bounded domain is considered. Under the assumptions on a more general type of relaxation functions, an explicit and general decay rate result is established by using the multiplier method and some properties of the convex functions.

Keywords: Kirchhoff plate equation; wave equation; weakly coupled equations; asymptotic stability

MSC: 35L05; 93D20

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1. Introduction

In this paper, we consider the following weakly coupled system of Kirchhoff plate and wave equations:

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s)\,ds + \alpha v = 0 & \text{in} \quad \Omega\times(0,\infty) \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + \alpha u = 0 & \text{in} \quad \Omega\times(0,\infty) \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0\times(0,\infty) \\ \mathbf{B}_1 u - \mathbf{B}_1\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & \text{on} \quad \Gamma_1\times(0,\infty) \\ \mathbf{B}_2 u - \gamma\partial_\nu u_{tt} - \mathbf{B}_2\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & \text{on} \quad \Gamma_1\times(0,\infty) \\ v = 0 & \text{on} \quad \Gamma\times(0,\infty) \\ u(0) = u^0, \ u_t(0) = u^1, \ v(0) = v^0, \ v_t(0) = v^1 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where Ω is an open set of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ (class C^4 will be enough), such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, the initial data u^0, u^1, v^0 and v^1 lie in an appropriate Hilbert space; the constant $\gamma > 0$ is the rotational inertia of the plate; and the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient. The boundary operators $\mathbf{B}_1$, $\mathbf{B}_2$ are defined by

$$\mathbf{B}_1 u = \Delta u + (1-\mu)B_1 u,$$

$$\mathbf{B}_2 u = \partial_\nu \Delta u + (1-\mu)B_2 u,$$

and

$$B_1 u = 2\nu_1\nu_2 u_{x_1x_2} - \nu_1^2 u_{x_2x_2} - \nu_2^2 u_{x_1x_1},$$

$$B_2 u = \partial_\tau\Big((\nu_1^2 - \nu_2^2)u_{x_1x_2} + \nu_1\nu_2(u_{x_2x_2} - u_{x_1x_1})\Big),$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ, and $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector.

The coupling parameter α is assumed to satisfy:

$$|\alpha| < \lambda_0 \eta,$$

where λ_0^2 is the first eigenvalue of the operator '$-\Delta$' with Dirichlet boundary conditions, and η^2 is the coercivity constant of the operator $\mathring{A} = \Delta^2$, defined as follows:

$$\mathring{A} : D(\mathring{A}) \subset L^2(\Omega) \to L^2(\Omega),$$

with domain

$$D(\mathring{A}) = \{u \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \Delta u + (1-\mu)B_1 u = \partial_\nu \Delta u + (1-\mu)B_2 u = 0 \text{ on } \Gamma_1\},$$

with $H^2_{\Gamma_0} = V = \{u \in H^2(\Omega) : u = \partial_\nu u = 0 \text{ on } \Gamma_0\}$.

It is clear that $\mathring{A}$ is positive definite and self-adjoint. We also define

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}.$$

We have, for all $u, v \in V$ (see [1]):

$$\langle \mathring{A}^{\frac{1}{2}} u, \mathring{A}^{\frac{1}{2}} v \rangle_{L^2(\Omega)} = \left(\mathring{A} u, v\right)_{V' \times V} = a(u,v),$$

where $a : V \times V \to \mathbb{R}$ is a symmetric bilinear form defined by

$$a(u,v) = \int_\Omega \{u_{x_1x_1} v_{x_1x_1} + u_{x_2x_2} v_{x_2x_2} + 2(1-\mu) u_{x_1x_2} v_{x_1x_2} + \mu(u_{x_1x_1} v_{x_2x_2} + u_{x_2x_2} v_{x_1x_1})\} dx.$$

We first recall the following Green's formula (see [2]):

$$a(u,v) = \int_\Omega \Delta^2 u v dx + \int_\Gamma (\mathcal{B}_1 u \partial_\nu v - \mathcal{B}_2 u v) d\Gamma, \ \ \forall u \in H^4(\Omega), \ \ v \in H^2(\Omega).$$

For further purposes, we need a weaker version of the above. Indeed, as $\mathcal{D}(\overline{\Omega})$ (the space of all functions defined in Ω, which are restrictions to Ω of C^∞ functions with compact support in $\mathbb{R}^2$) is dense in $E(\Delta^2, L^2(\Omega)) := \left\{u \in H^2(\Omega) \mid \Delta^2 u \in L^2(\Omega)\right\}$ equipped with its natural norm, we deduce that $u \in E(\Delta^2, L^2(\Omega))$ (see Theorem 5.6 in [3]) satisfies $\mathcal{B}_1 u \in H^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{B}_2 u \in H^{-\frac{3}{2}}(\Gamma)$ with

$$a(u,v) = \int_\Omega \Delta^2 u v dx + \langle \mathcal{B}_1 u, \partial_\nu v \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \langle \mathcal{B}_2 u, v \rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)}, \ \ \forall\, v \in H^2(\Omega).$$

Now, with the parameter $\gamma > 0$, we define a space $W = H^1_{\Gamma_0,\gamma}(\Omega)$ equivalent to $H^1_{\Gamma_0}(\Omega)$, with its inner product being

$$\langle u_1, u_2 \rangle_{H^1_{\Gamma_0,\gamma}(\Omega)} \equiv \langle u_1, u_2 \rangle_{L^2(\Omega)} + \gamma \langle \nabla u_1, \nabla u_2 \rangle_{L^2(\Omega)} \ \ \forall\, u_1, u_2 \in H^1_{\Gamma_0}(\Omega),$$

and with its dual (pivotal with respect to L_2 inner product) denoted as $H^{-1}_{\Gamma_0,\gamma}(\Omega)$.

When $g_1 = g_2 = \alpha = 0$, the first equation, in system (1), is well known as the Kirchhoff plate equation, while the second equation represents the classical wave equation. We study, in the present paper, a weak coupling of a symmetric type of these two equations (with the presence of memory terms), which means that the equations are coupled by displacements.

Model (1) describes the interaction of a viscoelastic Kirchhoff plate with rotational forces, and a viscoelastic membrane. The plate is clamped along Γ_0, and without bending and twisting moments on Γ_1.

We first recall some results for a single-wave equation and Kirchhoff plate equation. For a viscoelastic wave equation, we refer to [4–8] and references therein, in which the authors proved that the energy decays exponentially if the relaxation function g decays exponentially, and polynomially if g decays polynomially. In [9], Cavalcanti et al. considered the following wave equation:

$$u_{tt} - \kappa_0 \Delta u + \int_0^t div[a(x)g(t-s)\nabla u(s)]ds + f(u) + b(x)h(u_t) = 0,$$

where frictional damping was also considered. They proved an exponential stability result for g decaying exponentially and h having linear and polynomial stability result for g decaying polynomially and h having a polynomial growth near zero. We mention, in the case where $\kappa_0 = 1$ and $f = h = 0$, that the uniform decay of solutions was obtained in [10]. For the viscoelastic Kirchhoff plate equation, in [11], the authors showed the exponential and polynomial decay of the solutions to the viscoelastic plate equation. They considered a relaxation function satisfying

$$-d_0 g(t) \leq g'(t) \leq -d_1 g(t), \quad 0 \leq g''(t) \leq d_2 g(t).$$

For some positive constant $d_i,\ i = 0, 1, 2$. Park et al. [12] obtained a general decay for weak viscoelastic Kirchhoff plate equations with delay boundary conditions. Motivated by the work of Lasiecka and Tatar [13], where a wave equation with frictional damping was considered, another step forward was taken by considering relaxation functions satisfying

$$g'(t) \leq -H(g(t)),$$

where the function $H > 0$ satisfies $H(0) = H'(0) = 0$, and is strictly increasing and strictly convex near the origin. This condition was first introduced by Alabau-Boussouira and Cannarsa [14]. It turned out that the convexity properties can be explored for a general class of dissipative systems [15,16]. We also point out that the importance of the works [15,16] in which simple sharp optimal or quasi-optimal upper energy decay rates have been established.

For a coupled wave system, a general model on coupled wave equations with weak damping is given by:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + h_2(v_t) = f_2(u,v). \end{cases}$$

In [17], Han and Wang established several results related to local existence, global existence and finite time blow-up (the initial energy $E(0) < 0$), by taking $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(v_t) = |v_t|^{r-1}v_t$. Later on, Houari et al. [18] improved the last results by considering a larger class of initial data for which the initial energy can take positive values. Messaoudi and Tatar [13] considered a coupled system only with viscoelastic terms, and proved exponential decay and polynomial decay results. Al-Gharabli and Kafini considered the system in [13] and established a more general decay result; see [19]. Mustafa [20] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u,v) = 0, \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(u,v) = 0, \end{cases}$$

and proved the well-posedness and energy decay result. The decay result was improved by Messaoudi and Hassan in their recent paper [21], where they established a new general decay result for a wider class of relaxation functions. We also mention the work [22], in which the authors proved the global existence and decay rate estimates of solutions for a system of viscoelastic wave equations of the Kirchhoff type with logarithmic nonlinearity.

For indirect stabilization, Alabau et al. [23] considered the stabilization of an abstract system of two coupled second-order evolution equations, wherein only one of the equations is stabilized and showed that the energy decays polynomially. Recently, Hajej et al. [24] studied the indirect stabilization (only one equation of the coupled system is damped) of a coupled wave equation and Kirchhoff plate equation without viscoelastic terms ($g_1 = g_2 = 0$), and with frictional damping, the polynomial decay was derived. Motivated by these works, in this paper, we study the stability of this coupled system but only with the presence of viscoelastic terms in the two equations with a wider class of relaxation functions. We establish a very general energy decay result of the system by the general approach in [14].

Hereinafter, we assume that

(A1): $g_i : [0, +\infty) \to (0, +\infty)$ (for $i = 1, 2$) are two non-increasing C^1 functions such that:

$$1 - \int_0^\infty g_i(\tau)\, d\tau = l_i > 0.$$

(A2): There exists a positive C^1 function $Q : (0, +\infty) \to (0, +\infty)$, where Q is linear or a strictly increasing and strictly convex C^2 function on $(0, r], (r < 1)$, with $Q(0) = Q'(0) = 0$, such that

$$g_i'(t) \le -\xi_i(t) Q(g_i(t)), \quad \forall\, t \ge 0,$$

where ξ_1 and ξ_2 are positive non-increasing differentiable functions.

Remark 1. *The function Q, defined in assumption (A2), was introduced by Alabau-Boussouira and Cannarsa [14].*

To simplify calculations in our analysis, we introduce the following notations:

$$(g_1 \Box u)(t) = \int_0^t g_1(t-s) a(u(t) - u(s), u(t) - u(s)) ds,$$

$$(g_2 \circ v)(t) = \int_0^t g_2(t-s) \|v(t) - v(s)\|^2 ds.$$

We will use C and c, throughout this paper, to denote generic positive constants.

The paper is organized as follows. The well-posedness of the problem, that is, the existence of a global weak solution, is proved in Section 2. In Section 3, we state and establish the general decay result of the energy by using the perturbed energy method, which introduces a new Lyapunov function.

2. Global Existence

This section deals with the existence and uniqueness of a global weak solution. In fact, we start by proving the existence and uniqueness of a unique local weak solution by using the Faedo–Galerkin approach, and afterward, show that this solution is global. This means that our system is well-posed.

We start this section by presenting the definition of a weak solution of the problem (1).

Definition 1. *Let $T > 0$. A pair of functions (u, v) such that*

$$u \in C([0, T], V) \cap C^1([0, T], W), \; v \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)),$$

is called a weak solution of the problem (1) *if*

$$\int_\Omega u_{tt} w dx + \gamma \int_\Omega \nabla u_{tt} \nabla w dx + a(u,w) - \int_0^t g_1(t-s) a(u(s),w) ds + \alpha \int_\Omega v w dx = 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

and

$$\int_\Omega v_{tt} y dx + \int_\Omega \nabla v \nabla y dx - \int_0^t g_2(t-s) \int_\Omega \nabla v(s) \nabla y dx ds + \alpha \int_\Omega u y dx = 0,$$
$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x)$$

for all test functions $w \in V, y \in H_0^1$ *and almost all* $t \in [0,T]$.

Now, we state the local existence theorem.

Theorem 1. *Suppose (A1) holds and let* $(u_0,u_1) \in V \times W$ *and* $(v_0,v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then, problem (1) has a unique local weak solution on* $[0,T]$, *for any* $T > 0$.

Proof. The existence is proven using the Faedo–Galerkin method. In order to do so, let $\{w_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ be a basis of V and H_0^1, respectively. Define $V_m = span\{w_1, w_2, \dots, w_m\}$ and $Y_m = span\{y_1, y_2, \dots, y_m\}$. The projection of the initial data on the finite dimensional subspaces V_m and Y_m is given by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j, \quad u_1^m(x) = \sum_{j=1}^m b_j w_j, \quad v_0^m(x) = \sum_{j=1}^m c_j y_j, \quad v_1^m(x) = \sum_{j=1}^m d_j y_j,$$

such that

$$(u_0^m, v_0^m) \to (u_0, v_0) \ \text{ in } \ V \times H_0^1(\Omega), \quad \text{and} \ \ (u_1^m, v_1^m) \to (u_1, v_1) \text{ in } \ W \times L^2(\Omega). \tag{2}$$

We search a solution of the form

$$u^m(x,t) = \sum_{j=1}^m h_j(t) w_j(x), \qquad v^m(x,t) = \sum_{j=1}^m k_j(t) y_j(x),$$

which satisfy the approximate problem in V_m and Y_m, respectively:

$$\begin{aligned}
&\int_\Omega u_{tt}^m w dx + \gamma \int_\Omega \nabla u_{tt}^m \nabla w dx + a(u^m,w) - \int_0^t g_1(t-s) a(u^m(s),w) ds + \alpha \int_\Omega v^m w dx = 0, \\
&\int_\Omega v_{tt}^m y dx + \int_\Omega \nabla v^m \nabla y dx - \int_0^t g_2(t-s) \int_\Omega \nabla v^m(s) \nabla y dx ds + \alpha \int_\Omega u^m y dx = 0, \\
&u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m, \quad v^m(0) = v_0^m, \quad v_t^m(0) = v_1^m.
\end{aligned} \tag{3}$$

This system leads to a system of ODEs for unknown functions $h_j(t)$ and $k_j(t)$. Based on the standard existence theory for ODE, one can conclude the existence of a solution (u^m, v^m) of (3) on a maximal interval $[0, t_m), 0 < t_m \le T$ for each $m \ge 1$. In fact, $t_m = T$, and the local solution is uniformly bounded independent of m and t. To show this, we take $w = u_t^m$ in the first equation of (3) and $y = v_t^m$ in the second one. By summing up the resulting equations and integrating by parts over Ω, we obtain

$$\frac{d}{dt} E^m(t) = \frac{1}{2}(g_1{}' \Box u^m)(t) - \frac{1}{2} g_1(t) a(u^m, u^m) + \frac{1}{2}(g_2' \circ \nabla v^m)(t) - \frac{1}{2} g_2(t) \|\nabla v^m(t)\|^2, \tag{4}$$

where

$$\begin{aligned} E^m(t) &= \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)a(u^m,u^m)+\frac{1}{2}\|u_t^m\|^2+\frac{\gamma}{2}\|\nabla u_t^m\|^2+\frac{1}{2}(g_1\Box u^m)(t) \\ &+ \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\|\nabla v^m\|^2+\frac{1}{2}(g_2\circ\nabla v^m)(t)+\frac{1}{2}\|v_t^m\|^2+\alpha\int_\Omega u^m v^m dx. \end{aligned}$$

Notably, by (2), that (u_0^m, v_m^0) and (u_1^m, v_1^m) are bounded, respectively, in $V\times H_0^1(\Omega)$ and $W\times L^2(\Omega)$, we integrate (4) over $(0,t), 0<t<t_m$, to obtain

$$E^m(t)\le E^m(0)\le M,$$

where M is a positive constant independent of t and m. Thus, we can extend t_m to T and, in addition, we have

$$\begin{cases} (u^m) \text{ is a bounded sequence in } L^\infty(0,T;V), \\ (u_t^m) \text{ is a bounded sequence in } L^\infty(0,T;W), \\ (v^m) \text{ is a bounded sequence in } L^\infty(0,T;H_0^1(\Omega)), \\ (v_t^m) \text{ is a bounded sequence in } L^\infty(0,T;L^2(\Omega)). \end{cases}$$

Therefore, there exists a subsequence of (u^m) and (v^m), still denoted by (u^m) and (v^m), respectively, such that

$$\begin{cases} u^m \rightharpoonup u \text{ weakly star in } L^\infty(0,T;V) \text{ and weakly in } L^2(0,T;V), \\ u_t^m \rightharpoonup u_t \text{ weakly star in } L^\infty(0,T;W) \text{ and weakly in } L^2(0,T;W), \\ v^m \rightharpoonup v \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)) \text{ and weakly in } L^2(0,T;H_0^1(\Omega)), \\ v_t^m \rightharpoonup v_t \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{5}$$

Now, integrate (3) over $(0,t)$ to obtain

$$\begin{aligned} &\int_\Omega u_t^m w dx+\gamma\int_\Omega \nabla u_t^m\nabla w dx+\int_0^t a(u^m,w)ds-\int_0^t\int_0^s g_1(s-\zeta)a(u^m(\zeta),w)d\zeta ds \\ &+\alpha\int_0^t\int_\Omega v^m w dx ds=\int_\Omega u_1^m w dx+\gamma\int_\Omega \nabla u_1^m\nabla w dx. \\ &\int_\Omega v_t^m y dx+\int_0^t\int_\Omega \nabla v^m\nabla y dx ds-\int_0^t\int_0^s g_2(s-\zeta)\int_\Omega \nabla v^m(\zeta)\nabla y dx d\zeta ds \\ &+\alpha\int_0^t\int_\Omega u^m y dx ds=\int_\Omega v_1^m y dx. \end{aligned}$$

Using (5) and letting $m\to\infty$, we obtain for all $w\in V$ and $y\in H_0^1$

$$\begin{aligned} &\int_\Omega u_t w dx+\gamma\int_\Omega \nabla u_t\nabla w dx-\int_\Omega u_1 w dx-\gamma\int_\Omega \nabla u_1\nabla w dx \\ &=-\int_0^t a(u,w)ds+\int_0^t\int_0^s g_1(s-\zeta)a(u^m(\zeta),w)d\zeta ds-\alpha\int_0^t\int_\Omega v w dx ds \\ &\int_\Omega v_t y dx-\int_\Omega v_1 y dx=-\int_0^t\int_\Omega \nabla v\nabla y dx ds \\ &+\int_0^t\int_0^s g_2(s-\zeta)\int_\Omega \nabla v(\zeta)\nabla y dx d\zeta ds-\alpha\int_0^t\int_\Omega u y dx ds. \end{aligned} \tag{6}$$

Using the fact that the right-hand side of the first equation and the second one in (6) is an absolutely continuous function—hence, it is differentiable almost everywhere—we obtain

$$\int_\Omega u_{tt} w dx + \gamma \int_\Omega \nabla u_{tt} \nabla w dx + a(u,w) - \int_0^t g_1(t-s) a(u(s),w) ds + \alpha \int_\Omega v w dx = 0, \ \forall\, w \in V,$$
$$\int_\Omega v_{tt} y dx + \int_\Omega \nabla v \nabla y dx - \int_0^t g_2(t-s) \int_\Omega \nabla v(s) \nabla y dx ds + \alpha \int_\Omega u y dx = 0, \ \forall\, y \in H_0^1(\Omega).$$

Regarding the initial conditions, we can also use (6) to verify that

$$u(0) = u_0, \ \ u_t(0) = u_1, \ \ v(0) = v_0, \ \ v_t(0) = v_0.$$

For uniqueness, let us assume that $(u_1, v_1), (u_2, v_2)$ are two weak solutions of (1). Then, $(p, q) = (u_1 - u_2, v_1 - v_2)$ satisfies

$$\begin{aligned} & p_{tt} - \gamma \Delta p_{tt} + \Delta^2 p - \int_0^t g_1(t-s) \Delta^2 p(s) ds + \alpha q = 0, \ \text{in } L^2(0,T;V'), \\ & q_{tt} - \Delta q + \int_0^t g_2(t-s) \Delta q(s) ds + \alpha p = 0, \ \text{in } L^2(0,T;H^{-1}(\Omega)), \\ & p(0) = p_t(0) = q(0) = q_t(0) = 0. \end{aligned} \tag{7}$$

We shall use the Visik–Ladyzenskaya method. We consider, for each $s \in [0, T]$, the following functions:

$$\psi(t) = \begin{cases} -\int_t^s p(\zeta) d\zeta, & 0 \le t \le s, \\ 0, & s \le t \le T, \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} -\int_t^s q(\zeta) d\zeta, & 0 \le t \le s, \\ 0, & s \le t \le T. \end{cases}$$

The derivatives (in the distributions sense) of ψ and φ are given by

$$\psi'(t) = \begin{cases} p(t), & 0 \le t \le s, \\ 0, & s \le t \le T, \end{cases} \quad and \quad \varphi'(t) = \begin{cases} q(t), & 0 \le t \le s, \\ 0, & s \le t \le T. \end{cases}$$

It is clear that

$$\psi, \psi' \in L^\infty(0,T;V) \quad and \quad \varphi, \varphi' \in L^\infty(0,T;H_0^1(\Omega)),$$

which implies that

$$\psi \in C^0([0,T];V) \quad and \quad \varphi \in C^0([0,T];H_0^1(\Omega)).$$

By composing the first equation in (7) using ψ and the second equation using φ, we obtain

$$\begin{aligned} & \int_0^s ((I - \gamma\Delta) p_{tt}, \psi(t))_{V' \times V} dt + \int_0^s (\Delta^2 p, \psi(t))_{V' \times V} dt - \int_0^s \int_0^t g_1(t-\zeta)(\Delta^2 p(\zeta), \psi(t))_{V' \times V} d\zeta dt \\ & \quad + \alpha \int_0^s (q, \psi(t))_{V' \times V} dt = 0, \\ & \int_0^s (q_{tt}, \varphi(t))_{H^{-1}(\Omega) \times H_0^1(\Omega)} dt - \int_0^s (\Delta q, \varphi(t))_{H^{-1}(\Omega) \times H_0^1(\Omega)} dt \\ & \quad + \int_0^s \int_0^t g_2(t-\zeta)(\Delta q(\zeta), \varphi(t))_{H^{-1}(\Omega) \times H_0^1(\Omega)} d\zeta dt + \alpha \int_0^s (p, \varphi(t))_{H^{-1}(\Omega) \times H_0^1(\Omega)} dt = 0. \end{aligned}$$

Using the fact that $\psi(s) = \varphi(s) = p_t(0) = q_t(0) = 0$, $\psi'(t) = p(t)$ and $\varphi'(t) = q(t)$ in $[0,s]$, we integrate by parts and add the resulting equations to obtain

$$
\begin{aligned}
&-\int_0^s (p_t, p(t))_{W'\times W}dt + \int_0^s a(\psi'(t),\psi)dt - \int_0^s\int_0^t g_1(t-\zeta)a(\psi'(\zeta),\psi(t))d\zeta dt \\
&-\int_0^s (q_t, q(t))_{H^{-1}(\Omega)\times H_0^1(\Omega)}dt + \int_0^s (\varphi'(\zeta),\varphi)_{H_0^1(\Omega)}dt \\
&-\int_0^s\int_0^t g_2(t-\zeta)(\varphi'(\zeta),\varphi(t))_{H_0^1(\Omega)}d\zeta dt + \alpha\int_0^s \Big((q(t),\psi(t))_{L^2(\Omega)} + (p(t),\varphi(t))_{L^2(\Omega)}\Big)dt = 0,
\end{aligned}
$$

which, by using (11), results in: and (12)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{ -\int_0^s \|p\|_W^2 dt + \int_0^s \Big(1-\int_0^t g_1(\zeta)d\zeta\Big)a(\psi,\psi)dt + \int_0^s (g_1\Box\psi)(t)dt - \int_0^s \|q\|^2_{H_0^1(\Omega)}dt \\
&+\int_0^s\Big(1-\int_0^t g_2(\zeta)d\zeta\Big)\|\varphi(t)\|^2_{H_0^1(\Omega)}dt\Big\} \\
&+\frac{1}{2}\int_0^s g_1(t)a(\psi,\psi)dt - \frac{1}{2}\int_0^s (g_1'\Box\psi)(t)dt + \frac{1}{2}\int_0^s g_1(t)\|\varphi\|^2_{H_0^1(\Omega)}dt - \frac{1}{2}\int_0^s (g_2'\circ\nabla\varphi)(t)dt \\
&= -\alpha\int_0^s\Big((q(t),\psi(t))_{L^2(\Omega)} + (p(t),\varphi(t))_{L^2(\Omega)}\Big)dt.
\end{aligned}
$$

Now, using the fact that $g_i, -g_i', a(\psi,\psi) \geq 0$ for $i = 1,2$, and $W, H_0^1 \subset L^2(\Omega)$, we obtain the existence of a positive constant C such that

$$
\begin{aligned}
&\frac{1}{2}\|p(s)\|^2 + \frac{1}{2}\|\psi(0)\|^2 + \frac{1}{2}\|q(s)\|^2 + \frac{1}{2}\|\varphi(0)\|^2 \\
&\leq C\int_0^s (\|q(t)\|\|\psi(t)\| + \|p(t)\|\|\varphi(t)\|)dt.
\end{aligned}
\tag{8}
$$

Finally, let $p_1(t) = \int_0^t p(\zeta)d\zeta$ and $q_1(t) = \int_0^t q(\zeta)d\zeta$. We have , for all $t\in[0,s]$

$$\psi(t) = p_1(t) - p_1(s),\ \ \psi(0) = -p_1(s),$$

and

$$\varphi(t) = q_1(t) - q_1(s),\ \ \varphi(0) = -q_1(s).$$

Consequently, (8) becomes

$$
\begin{aligned}
&\frac{1}{2}\|p(s)\|^2 + \frac{1}{2}\|p_1(s)\|^2 + \frac{1}{2}\|q(s)\|^2 + \frac{1}{2}\|q_1(s)\|^2 \\
&\leq C\int_0^s (\|q(t)\|\|p_1(t)-p_1(s)\| + \|p(t)\|\|q_1(t)-q_1(s)\|)dt \\
&\leq C\Big\{\int_0^s \|q(t)\|\|p_1(t)\|dt + \int_0^s \|q(t)\|\|p_1(s)\|dt \\
&\qquad + \int_0^s \|p(t)\|\|q_1(t)\|dt + \int_0^s \|p(t)\|\|q_1(s)\|dt\Big\} \\
&\leq C\Big\{\int_0^s \|q(t)\|\|p_1(t)\|dt + \int_0^s \sqrt{2Cs}\|q(t)\|\frac{1}{\sqrt{2Cs}}\|p_1(s)\|dt \\
&\qquad + \int_0^s \|p(t)\|\|q_1(t)\|dt + \int_0^s \sqrt{2Cs}\|p(t)\|\frac{1}{\sqrt{2Cs}}\|q_1(s)\|dt\Big\} \\
&\leq \frac{C}{2}\int_0^s \|p(t)\|^2dt + \frac{C}{2}\int_0^s \|p_1(t)\|^2dt + TC^2\int_0^s \|p^2(t)\|^2dt + \frac{1}{4}\|p_1(s)\|^2 \\
&\quad + \frac{C}{2}\int_0^s \|q(t)\|^2dt + \frac{C}{2}\int_0^s \|q_1(t)\|^2dt + TC^2\int_0^s \|q^2(t)\|^2dt + \frac{1}{4}\|q_1(s)\|^2,
\end{aligned}
$$

which implies that

$$\frac{1}{4}\|p(s)\|^2+\frac{1}{4}\|p_1(s)\|^2+\frac{1}{4}\|q(s)\|^2+\frac{1}{4}\|q_1(s)\|^2$$
$$\leq C\int_0^s\Big(\|p(t)\|^2+\|p_1(t)\|^2+\|q(t)\|^2+\|q_1(t)\|^2\Big)dt.$$

By using Gronwall's Lemma, we deduce that

$$\frac{1}{4}\|p(s)\|^2+\frac{1}{4}\|p_1(s)\|^2+\frac{1}{4}\|q(s)\|^2+\frac{1}{4}\|q_1(s)\|^2\leq 0.$$

Then, we can determine that

$$p(s)=q(s)=0,\ \text{in } L^2(\Omega),\ \forall\, s\in(0,T),$$

and since $p(0)=q(0)=0$, we obtain

$$p(s)=q(s)=0,\ \text{in } L^2(\Omega),\ \forall\, s\in[0,T],$$

which means that $(u_1,v_1)=(u_2,v_2)$.

Consequently, the proof of the local existence of a weak solution is complete. Furthermore, it is easy to see that

$$l_1a(u,u)+\|u_t\|^2+\gamma\|\nabla u_t\|^2+l_2\|\nabla v\|^2+\|v_t\|^2\leq 2E(t)\leq 2E(0),$$

which shows that the solution is bounded and global in time.

This completes the proof. □

We also need the following regularity result. Indeed, in some parts of the paper, we multiply the first equation by u_t and the second one by v_t. This is only possible if we are working with regular solutions. For this reason, we will introduce a theorem for regular solutions as well. Thus, it is enough to work with regular solutions all time. The decay rate estimates for weak solutions are obtained using standard density arguments. But, before performing this, we present the definition of regular solutions in our case, which was introduced in Definition 2 [11].

Definition 2. *We previously stated that (u_0,u_1) is 2-regular if*

$$u_j\in H^{4-j}(\Omega)\cap V,\ j=0,\dots,2;\qquad u_3\in W,$$

where u_j is obtained by the following recursive formulas:

$$\begin{aligned}&u_{j+2}-\gamma\Delta u_{j+2}=\Delta^2u_j,\\&u_j=\partial_\nu u_j=0,\ \text{on } \Gamma_0,\\&\mathbf{B}_1u_j=0,\ \text{on } \Gamma_1, &&\forall\, j=1,2.\\&\mathbf{B}_2u_j=\gamma\partial_\nu u_{j+2},\ \text{on } \Gamma_1.\end{aligned}$$

Now, we present our regularity result.

Theorem 2. *Suppose (A1) holds and suppose that (u_0,u_1) is 2-regular and $(v_0,v_1)\in\big(H^2(\Omega)\cap H_0^1(\Omega)\big)\times H_0^1(\Omega)$. Then, the solution of problem (1) satisfies*

$$u\in C([0,T],V\cap H^4(\Omega))\cap C^1([0,T],W\cap H^3(\Omega)),$$

$$v\in L^\infty([0,T],H^2(\Omega)\cap H_0^1(\Omega))\cap W^{1,\infty}([0,T],H_0^1(\Omega))\cap W^{2,\infty}([0,T],L^2(\Omega)).$$

Proof. The proof can be performed by combining the arguments used, for example, in [9,11]. □

3. General Decay

In this section, we will present and establish our principal theorem of this paper, which states the general decay of the energy of our system. This will be conducted by the help of the perturbed energy method. First, we introduce the energy functional by

$$E(t) = \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)a(u,u) + \frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}(g_1\Box u)(t) + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\|\nabla v\|^2$$
$$+\frac{1}{2}(g_2 \circ \nabla v)(t) + \frac{1}{2}\|v_t\|^2 + \alpha\int_\Omega uvdx,$$

which satisfies the following dissipation identity:

Proposition 1. *Under the hypothesis of Theorem 2, the following identity holds:*

$$E'(t) = \frac{1}{2}(g_1'\Box u)(t) - \frac{1}{2}g_1(t)a(u,u) + \frac{1}{2}(g_2' \circ \nabla v)(t) - \frac{1}{2}g_2(t)\|\nabla v(t)\|^2 \leq 0. \quad (9)$$

Proof. In (1), upon multiplying the first equation by u_t and the second one by v_t, add the resulting equations and integrate by parts over Ω to obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}a(u,u) + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \alpha\int_\Omega uvdxdy\right\}$$
$$-\int_0^t g_1(t-s)a(u(s),u_t)ds - \int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v_t(t)dxds = 0 \quad (10)$$

By the virtue of Lemma 2.1 in [11], we have

$$a\left(\int_0^t g_1(t-s)u(s)ds, u_t\right) = -\frac{1}{2}g_1(t)a(u,u) - \frac{1}{2}\frac{d}{dt}\left\{(g_1\Box u)(t) - \left(\int_0^t g_1(s)ds\right)a(u,u)\right\}$$
$$+ \frac{1}{2}(g_1'\Box u)(t), \quad (11)$$

For any $u \in C^1(0,T;H^2(\Omega))$.
Besides, a direct computation shows that

$$\int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v_t(t)dxds = \frac{1}{2}(g_2' \circ \nabla v)(t) - \frac{1}{2}g_2(t)\|\nabla v(t)\|^2$$
$$- \frac{1}{2}\frac{d}{dt}\left\{(g_2 \circ \nabla v)(t) - \left(\int_0^t g_2(s)ds\right)\|\nabla v(t)\|^2\right\}. \quad (12)$$

□

By replacing (11) and (12) in (10), we obtain the desired result.
The main result of this paper reads as follows.

Theorem 3. *Suppose that (u_0,u_1) is 2-regular and $(v_0,v_1) \in \left(H^2(\Omega)\cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. Assume that (A1) and (A2) hold. Then, there exist positive constants β_1 and β_2, such that the energy $E(t)$ satisfies for any $t > g^{-1}(r)$*

$$E(t) \leq \beta_2 Q_1^{-1}\left(\beta_1\int_{g^{-1}(r)}^t \xi(s)ds\right), \quad (13)$$

where $\xi(t) = \min\{\xi_1(t),\xi_2(t)\}$, $g(t) = \max\{g_1(t),g_2(t)\}$ and $Q_1(t) = \int_t^r \frac{1}{sQ'(s)}ds$.

Remark 2 ([25]).

1. *The following Jensen's inequality is critical to prove our main result. Let G be a convex increasing function on* $[a,b]$*,* $h:\Omega\to[a,b]$ *and m is the integrable function on* Ω*, such that* $m(x)\geq 0$ *and* $\int_\Omega m(x)\,dx = k>0$*, then Jensen's inequality states that*

$$G\Big[\frac{1}{k}\int_\Omega h(x)m(x)\,dx\Big] \leq \frac{1}{k}\int_\Omega G[h(x)]m(x)\,dx.$$

2. *From (A2), we infer that* $\lim_{t\to+\infty} g_i(t)=0$*. Then, there exists some large enough* $t_1\geq 0$*, such that*

$$g_i(t_1)=r \Rightarrow g_i(t)\leq r, \quad \forall\, t\geq t_1. \tag{14}$$

Since Q is a positive continuous function and g_i,ξ_i *are positive non-increasing continuous functions, we can obtain for every* $t\in[0,t_1]$*,*

$$0<g_i(t_1)\leq g_i(t)\leq g_i(0) \;\text{ and }\; 0<\xi_i(t_1)\leq \xi_i(t)\leq \xi_i(0),\;\; i=1,2,$$

which implies for some positive constants a_i *and* b_i*:*

$$a_i \leq \xi_i(t)Q(g_i(t))\leq b_i,\;\; i=1,2.$$

This shows that for every $t\in[0,t_1]$*,*

$$g_i'(t)\leq -\xi_i(t)Q(g_i(t)) \leq -\frac{a_i}{g_i(0)}g_i(0)\leq -\frac{a_i}{g_i(0)}g_i(t),\;\; i=1,2. \tag{15}$$

3. *If different functions* Q_1 *and* Q_2 *have the properties mentioned in (A2), such that* $g_1'(t)\leq -Q_1(g_1(t))$ *and* $g_2'(t)\leq -Q_2(g_2(t))$*, then there exists* $r<\min\{r_1,r_2\}$ *small enough so that, e.g.,* $Q_1(t)\leq Q_2(t)$ *on the interval* $(0,r]$*. Thus, the function* $Q(t)=Q_1(t)$ *satisfies (A2) for both functions* g_1 *and* g_2 *for all* $t\geq t_1$*.*

We will work with regular solutions; by standard density arguments, the decay result remains valid for weak solutions as well. In order to prove the main Theorem (3), we need to introduce several lemmas. To this end, let us introduce the functionals

$$I(t)=\int_\Omega (uu_t+\gamma\nabla u_t\nabla u)\,dx+\int_\Omega vv_t\,dx, \tag{16}$$

and

$$\begin{aligned} K(t) &= -\int_\Omega u_t\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx-\gamma\int_\Omega \nabla u_t\int_0^t g_1(t-s)\nabla(u(t)-u(s))\,ds\,dx \\ &- \int_\Omega v_t\int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx. \end{aligned} \tag{17}$$

Lemma 1. *Assume that (A1) and (A2) hold. Then, the functional* $I(t)$ *introduced in (16) satisfies (along the solution) the estimate*

$$\begin{aligned} I'(t) &\leq \int_\Omega |u_t|^2\,dx+\gamma\int_\Omega|\nabla u_t|^2\,dx-\frac{l_1}{2}a(u,u)+\frac{1-l_1}{2l_1}(g_1\Box u)(t) \\ &+ \int_\Omega |v_t|^2\,dx-\frac{l_2}{2}\int_\Omega|\nabla v|^2\,dx+\frac{1-l_2}{2l_2}(g_2\circ\nabla v)(t)-2\alpha\int_\Omega uv\,dx. \end{aligned} \tag{18}$$

Proof. Direct differentiation of I, using (1), yields

$$
\begin{aligned}
I'(t) &= \int_\Omega |u_t|^2\,dx + \int_\Omega uu_{tt}\,dx + \gamma\int_\Omega |\nabla u_t|^2\,dx + \gamma\int_\Omega \nabla u\nabla u_{tt}\,dx + \int_\Omega |v_t|^2\,dx + \int_\Omega vv_{tt}\,dx \\
&= \int_\Omega |u_t|^2\,dx + \gamma\int_\Omega |\nabla u_t|^2\,dx - a(u,u) + \int_0^t g_1(t-s)a(u(s),u(t))\,ds \\
&\quad + \int_\Omega |v_t|^2\,dx - \int_\Omega |\nabla v|^2\,dx + \int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)\,dx\,ds - 2\alpha\int_\Omega uv\,dx. \qquad (19)
\end{aligned}
$$

By using the Cauchy–Schwarz inequality, Young's inequality and the fact that $\int_0^t g_1(s)ds \leq \int_0^{+\infty} g_1(s)ds = 1-l_1$, we obtain

$$
\begin{aligned}
&\int_0^t g_1(t-s)a(u(t),u(s))\,ds \\
&= \int_0^t g_1(t-s)a(u(s)-u(t),u(t))\,ds + \int_0^t g_1(t-s)a(u(t),u(t))\,ds \\
&\leq \int_0^t g_1(t-s)\{a(u(s)-u(t),u(s)-u(t))\}^{\frac{1}{2}}\{a(u(t),u(t))\}^{\frac{1}{2}}\,ds + \left(\int_0^t g_1(s)ds\right)a(u(t),u(t)) \\
&\leq \frac{l_1}{2}a(u(t),u(t)) + \frac{1}{2l_1}\left(\int_0^t \sqrt{g_1(t-s)}\Big\{g_1(t-s)a(u(s)-u(t),u(s)-u(t))\Big\}^{\frac{1}{2}}\,ds\right)^2 \\
&\quad +(1-l_1)a(u(t),u(t)) \qquad (20) \\
&\leq \left(1-\frac{l_1}{2}\right)a(u(t),u(t)) + \frac{1-l_1}{2l_1}(g_1\Box u)(t).
\end{aligned}
$$

Furthermore, we have (see, for example, [5]) that

$$
\begin{aligned}
&\int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v(t)\,dx\,ds \\
&\leq (1-\frac{l_2}{2})\int_\Omega |\nabla v|^2\,dx + \frac{1-l_2}{2l_2}(g_2\circ\nabla v)(t). \qquad (21)
\end{aligned}
$$

Inserting (20) and (21) in (19), the assertion of the lemma is established. □

Lemma 2. *Assume that (A1) and (A2) hold. Then, the functional $K(t)$ introduced in (17) satisfies, along the solution, the estimate*

$$
\begin{aligned}
K'(t) \leq{} & \left(\delta - \int_0^t g_1(s)ds\right)\left(\int_\Omega |u_t|^2dx\right) \\
&+\left\{\delta(C+1) + \delta(1-l_1)(\delta+2(1-l_1)) - \int_0^t g_1(s)ds\right\}a(u,u) \\
&+\gamma\left(\delta - \int_0^t g_1(s)ds\right)\left(\int_\Omega |\nabla u_t|^2dx\right) \\
&+\left\{\frac{(1-l_1)(C+2)}{4\delta} + \delta(1-l_1) + (1-l_1)^2\right\}(g_1\Box u)(t) \\
&-\frac{Cg_1(0)}{2\delta}(g_1'\Box u)(t) + \left(\delta - \int_0^t g_2(s)ds\right)\left(\int_\Omega |v_t|^2dx\right) \\
&+\delta\left(C+1+2(1-l_2)^2\right)\left(\int_\Omega |\nabla v|^2dx\right) \\
&+\left\{\frac{(1-l_2)(C+1)}{4\delta} + (2\delta+\frac{1}{4\delta})(1-l_2)\right\}(g_2\circ\nabla v)(t) \\
&-\frac{Cg_2(0)}{4\delta}(g_2'\circ\nabla v)(t), \quad \forall\,\delta>0. \qquad (22)
\end{aligned}
$$

Proof. By exploiting equation (1) and integrating by parts, we have

$$\begin{aligned}
K'(t) &= \left(-\int_0^t g_1(s)ds\right)\left(\int_\Omega |u_t|^2dx\right) - \int_\Omega u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&\quad - \int_\Omega u_{tt} \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&\quad -\gamma \int_\Omega \nabla u_{tt} \int_0^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\
&\quad -\gamma \int_\Omega \nabla u_t \int_0^t g_1'(t-s)\nabla(u(t)-u(s))dsdx - \gamma\left(\int_0^t g_1(s)ds\right)\left(\int_\Omega |\nabla u_t|^2dx\right) \\
&\quad - \int_\Omega v_{tt} \int_0^t g_2(t-s)(v(t)-v(s)dsdx \\
&\quad - \int_\Omega v_t \int_0^t g_2'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g_2(s)ds\right)\left(\int_\Omega |v_t|^2dx\right) \\
&= \left(-\int_0^t g_1(s)ds\right)\left(\int_\Omega |u_t|^2dx\right) - \int_\Omega u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&\quad +a\left(u, \int_0^t g_1(t-s)(u(t)-u(s))ds\right) \\
&\quad - \int_0^t g_1(t-s)a\left(u(s), \int_0^t g_1(t-s)(u(t)-u(s))ds\right)ds \\
&\quad +\alpha \int_\Omega v \int_0^t g_1(t-s)(u(t)-u(s))dsdx
\end{aligned}$$

$$\begin{aligned}
&-\gamma \int_\Omega \nabla u_t \int_0^t g_1'(t-s)\nabla(u(t)-u(s))dsdx - \gamma\left(\int_0^t g_1(s)ds\right)\left(\int_\Omega |\nabla u_t|^2dx\right) \\
&+ \int_\Omega \nabla v \int_0^t g_2(t-s)\nabla(v(t)-v(s))dsdx \\
&- \int_\Omega\left(\int_0^t g_2(t-s)\nabla v(s)ds\right)\left(\int_0^t g_2(t-s)\nabla(u(t)-u(s))ds\right)dx \\
&+\alpha \int_\Omega u \int_0^t g_2(t-s)(v(t)-v(s))dsdx - \left(\int_0^t g_2(s)ds\right)\left(\int_\Omega |v_t|^2dx\right) \\
&- \int_\Omega v_t \int_0^t g_2'(t-s)(u(t)-u(s))dsdx.
\end{aligned} \tag{23}$$

Using Young's inequality and Cauchy–Schwarz's inequality , we obtain for any $\delta > 0$

$$\begin{aligned}
-\int_\Omega u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx &\le \delta \int_\Omega |u_t|^2 + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)^2dx \\
&\le \delta \int_\Omega |u_t|^2 - \frac{g_1(0)}{4\delta}\int_0^t g_1'(t-s)\int_\Omega |u(t)-u(s)|^2dxds \\
&\le \delta \int_\Omega |u_t|^2 - \frac{Cg_1(0)}{4\delta}(g_1'\Box u)(t),
\end{aligned} \tag{24}$$

and

$$a\left(u, \int_0^t g_1(t-s)(u(t)-u(s))ds\right) = \int_0^t g_1(t-s)a(u(t), u(t)-u(s))ds$$
$$\leq \int_0^t g_1(t-s)[a(u(t),u(t))]^{\frac{1}{2}}[a(u(t)-u(s), u(t)-u(s))]^{\frac{1}{2}}ds$$
$$\leq \delta a(u,u) + \frac{1}{4\delta}\left\{\int_0^t g_1(t-s)[a(u(t)-u(s), u(t)-u(s))]^{\frac{1}{2}}ds\right\}^2$$
$$\leq \delta a(u,u) + \frac{1-l_1}{4\delta}(g_1\square u)(t). \tag{25}$$

Furthermore, we have

$$- \int_0^t g_1(t-s)a\left(u(s), \int_0^t g_1(t-s)(u(t)-u(s))ds\right)ds$$
$$\leq \left(\int_0^t g_1(t-s)[a(u(s),u(s))]^{\frac{1}{2}}ds\right)\left(\int_0^t g_1(t-s)[a(u(t)-u(s), u(t)-u(s))]^{\frac{1}{2}}ds\right)$$
$$\leq \delta\left(\int_0^t g_1(t-s)[a(u(s),u(s))]^{\frac{1}{2}}ds\right)^2 + \frac{1}{4\delta}\left(\int_0^t g_1(t-s)[a(u(t)-u(s), u(t)-u(s))]^{\frac{1}{2}}ds\right)^2$$
$$\leq \delta\left(\int_0^t g_1(t-s)[a(u(s),u(s))]^{\frac{1}{2}}ds\right)^2 + \frac{1-l_1}{4\delta}(g_1\square u)(t). \tag{26}$$

Now, we will estimate the term $\left(\int_0^t g_1(t-s)[a(u(s),u(s))]^{\frac{1}{2}}ds\right)^2$. We have

$$\left(\int_0^t g_1(t-s)[a(u(s),u(s))]^{\frac{1}{2}}ds\right)^2$$
$$\leq (1-l_1)\int_0^t g_1(t-s)a(u(s),u(s))ds$$
$$= (1-l_1)\int_0^t g_1(t-s)(a(u(t)-u(s), u(t)-u(s)) + 2a(u(t),u(s)) - a(u(t),u(t)))ds$$
$$= (1-l_1)(g_1\square u)(t) - (1-l_1)\left(\int_0^t g_1(s)ds\right)a(u,u) + 2(1-l_1)\int_0^t g_1(t-s)a(u(t),u(s))ds$$
$$\leq (1-l_1)(g_1\square u)(t) - (1-l_1)\left(\int_0^t g_1(s)ds\right)a(u,u) + \delta(1-l_1)a(u,u)$$
$$+\frac{(1-l_1)^2}{\delta}(g_1\square u)(t) + 2(1-l_1)^2 a(u,u)$$
$$= (1-l_1)\left(\delta + 2(1-l_1) - \int_0^t g_1(s)ds\right)a(u,u) + \left(1-l_1+\frac{(1-l_1)^2}{\delta}\right)(g_1\square u)(t). \tag{27}$$

Inserting (27) in (26), we obtain

$$-\int_0^t g_1(t-s)a\left(u(s), \int_0^t g_1(t-s)(u(t)-u(s))ds\right)ds$$
$$\leq \delta(1-l_1)\left(\delta + 2(1-l_1) - \int_0^t g_1(s)ds\right)a(u,u)$$
$$+\left(\delta(1-l_1) + (1-l_1)^2 + \frac{(1-l_1)}{4\delta}\right)(g_1\square u)(t). \tag{28}$$

Next, we have

$$\alpha \int_{\Omega} v \int_0^t g_1(t-s)(u(t)-u(s))dsdx$$
$$\leq \delta \int_{\Omega} |v|^2 dx + \frac{C(1-l_1)}{4\delta}(g_1 \square u)(t)$$
$$\leq C\delta \int_{\Omega} |\nabla v|^2 dx + \frac{C(1-l_1)}{4\delta}(g_1 \square u)(t). \tag{29}$$

The term $-\gamma \int_{\Omega} \nabla u_t \int_0^t g_1'(t-s)\nabla(u(t)-u(s))dsdx$ can be estimated as follows:

$$-\gamma \int_{\Omega} \nabla u_t \int_0^t g_1'(t-s)\nabla(u(t)-u(s))dsdx$$
$$\leq \gamma\delta \int_{\Omega} |\nabla u_t|^2 - \frac{Cg(0)}{4\delta}(g_1' \square u)(t). \tag{30}$$

Furthermore, we determine that

$$\alpha \int_{\Omega} u \int_0^t g_2(t-s)(v(t)-v(s))dsdx$$
$$\leq C\delta a(u,u) + \frac{C(1-l_2)}{4\delta}(g_2 \circ \nabla v)(t). \tag{31}$$

The remaining terms can be estimated as, for example, in [5] (see estimates (3.14)–(3.16) in the mentioned paper).

$$\int_{\Omega} \nabla v \int_0^t g_2(t-s)\nabla(v(t)-v(s))dsdx$$
$$\leq \delta \int_{\Omega} |\nabla v|^2 dx + \frac{1-l_2}{4\delta}(g_2 \circ \nabla v)(t), \tag{32}$$

$$-\int_{\Omega}\left(\int_0^t g_2(t-s)\nabla v(s)ds\right)\left(\int_0^t g_2(t-s)\nabla(v(t)-v(s))ds\right)dx$$
$$\leq (2\delta + \frac{1}{4\delta})(1-l_2)(g_2 \circ \nabla v)(t) + 2\delta(1-l_2)^2 \int_{\Omega} |\nabla v|^2 dx, \tag{33}$$

and

$$-\int_{\Omega} v_t \int_0^t g_2'(t-s)(u(t)-u(s))dsdx$$
$$\leq \delta \int_{\Omega} |v_t|^2 dx - \frac{Cg_2(0)}{4\delta}(g_2' \circ \nabla v)(t). \tag{34}$$

By combining (23) and (34), we achieve the desired estimate. □

Now, we define the functional $F(t)$. The idea is to construct a new Lyapunow function, equivalent to the energy quantity, that will satisfy an "appropriate" inequality. Let

$$F(t) = NE(t) + N_1 I(t) + N_2 K(t),$$

where N, N_1 and N_2 are positive constants that will be chosen later. It is easy to verify that for a large enough N, we have $F \sim E$, i.e.,

$$c_1 E(t) \leq F(t) \leq c_2 E(t),$$

for some $c_1, c_2 > 0$.

Lemma 3. *The functional F satisfies*

$$\begin{aligned} F'(t) &\leq -\left(\int_\Omega |u_t|^2\,dx + \int_\Omega |v_t|^2\,dx + \gamma\int_\Omega |\nabla u_t|^2\,dx\,dy\right) \\ &- 4(1-l)\left(a(u,u) + \int_\Omega |\nabla v|^2\,dx\right) + c((g_1\Box u)(t) + (g_2\circ\nabla v)(t)) - 2\alpha N_1\int_\Omega uv\,dx, \quad \forall\, t\geq t_1, \end{aligned} \tag{35}$$

where t_1 was introduced in (14) *and $c>0$.*

Proof. Let

$$g_0 = \min\left\{\int_0^{t_1} g_1(s)ds, \int_0^{t_1} g_2(s)ds\right\} > 0, \qquad \text{and} \qquad l = \min\{l_1, l_2\}.$$

By using (9), (18) and (22), we obtain for any $t \geq t_1$

$$\begin{aligned} F'(t) &= NE'(t) + N_1 I'(t) + N_2 K'(t) \\ &\leq -(N_2(g_0-\delta) - N_1)\left(\int_\Omega |u_t|^2\,dx + \int_\Omega |v_t|^2\,dx + \gamma\int_\Omega |\nabla u_t|^2\,dx\right) \\ &- \left(\frac{N_1 l}{2} - N_2\delta\left(C+1+2(1-l)^2\right) - N_2\delta^2(1-l) + N_2 g_0\right)a(u,u) \\ &- \left(\frac{N_1 l}{2} - N_2\delta\left(C+1+2(1-l)^2\right)\right)\int_\Omega |\nabla v|^2\,dx \\ &+ \left(\frac{N_1(1-l)}{2l} + N_2\left(\frac{(C+2)(1-l)}{4\delta} + \delta(1-l) + (1-l)^2\right)\right)(g_1\Box u)(t) \\ &+ \left(\frac{N_1(1-l)}{2l} + N_2\left(\frac{(C+1)(1-l)}{4\delta} + (2\delta + \frac{1}{4\delta})(1-l)\right)\right)(g_2\circ\nabla v)(t) \\ &+ \left(\frac{N}{2} - \frac{CN_2 g_0}{2\delta}\right)\{(g_1'\Box u)(t) + (g_2'\circ\nabla v)(t)\} - 2\alpha N_1\int_\Omega uv\,dx. \end{aligned} \tag{36}$$

Taking $\delta = \frac{l}{4N_2(C+1+2(1-l)^2)}$, (36) becomes

$$\begin{aligned} F'(t) &\leq -\left(N_2 g_0 - \frac{l}{4(C+1+2(1-l)^2)} - N_1\right)\left(\int_\Omega |u_t|^2\,dx + \int_\Omega |v_t|^2\,dx + \gamma\int_\Omega |\nabla u_t|^2\,dx\right) \\ &- \left(\frac{N_1 l}{2} - \frac{l}{4} - \frac{l^2(1-l)}{16N_2(C+1+2(1-l)^2)^2} + N_2 g_0\right)a(u,u) \\ &- \left(\frac{N_1 l}{2} - \frac{l}{4}\right)\int_\Omega |\nabla v|^2\,dx \\ &+ \Big(\frac{N_1(1-l)}{2l} + \frac{N_2^2(C+2)(1-l)(C+1+2(1-l)^2)}{l} \\ &+ \frac{l(1-l)}{4(C+1+2(1-l)^2)} + N_2(1-l)^2\Big)(g_1\Box u)(t) \\ &+ \left(\frac{N_1(1-l)}{2l} + \frac{N_2^2(C+2)(1-l)(C+1+2(1-l)^2)}{l} + \frac{l(1-l)}{2(C+1+2(1-l)^2)}\right)(g_2\circ\nabla v)(t) \\ &+ \left(\frac{N}{2} - \frac{2Cg_0 N_2^2(C+1+2(1-l)^2)}{l}\right)\{(g_1'\Box u)(t) + (g_2'\circ\nabla v)(t)\} - 2\alpha N_1\int_\Omega uv\,dx. \end{aligned}$$

At this point, we choose N_1 that is large enough, so that

$$\frac{N_1 l}{2} - \frac{l}{4} > 4(1-l),$$

and then N_2 that is large enough, such that

$$N_2 g_0 - \frac{l}{4(C+1+2(1-l)^2)} - N_1 > 1,$$

and

$$N_2 g_0 - \frac{l^2(1-l)}{16N_2(C+1+2(1-l)^2)^2} > 0.$$

Now, we choose N that is large enough, such that

$$\frac{N}{2} - \frac{2Cg_0N_2^2(C+1+2(1-l)^2)}{l} > 0.$$

Thus, (35) is established. □

Now, we are in a position to prove our main result.

Proof of Theorem (3). Taking into account (9) and (15), we obtain that for any $t \geq t_1$

$$\begin{aligned}&\int_0^{t_1} g_1(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\ &\leq -\frac{g_1(0)}{a_1}\int_0^{t_1} g_1'(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \leq -cE'(t),\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t_1} g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2dxds \\ &\leq -\frac{g_2(0)}{a_2}\int_0^{t_1} g_2'(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2dxds \leq -cE'(t).\end{aligned}$$

Therefore, (35) yields for some $m > 0$ and all $t \geq t_1$,

$$\begin{aligned}F'(t) &\leq -mE(t) + c(g_1\Box u)(t) + c(g_2\circ\nabla v)(t) \\ &\leq -mE(t) - cE'(t) + c\int_{t_1}^t g_1(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\ &\quad + c\int_{t_1}^t g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2dxds. \qquad (37)\end{aligned}$$

Denote $\mathcal{L}(t) = F(t) + cE(t)$. Clearly, $\mathcal{L}(t)$ is equivalent to $E(t)$. It follows from (37) that

$$\begin{aligned}\mathcal{L}'(t) &\leq -mE(t) + c\int_{t_1}^t g_1(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\ &\quad + c\int_{t_1}^t g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2dxds. \qquad (38)\end{aligned}$$

Next, the following two cases are considered.
Case 1. The function $Q(t)$ is linear.
We multiply (38) by $\xi(t)$ and use Assumption (A2) and (9) to obtain

$$\begin{aligned}
\xi(t)\mathcal{L}'(t) &\leq -m\xi(t)E(t) + c\xi(t)\int_{t_1}^{t} g_1(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\
&\quad + c\xi(t)\int_{t_1}^{t} g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2 dxds \\
&\leq -m\xi(t)E(t) + c\int_{t_1}^{t} \xi_1(s)g_1(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\
&\quad + c\int_{t_1}^{t} \xi_2(s)g_2(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2 dxds \\
&\leq -m\xi(t)E(t) - c\int_{t_1}^{t} g_1'(s)a(u(t)-u(t-s),u(t)-u(t-s))ds \\
&\quad - c\int_{t_1}^{t} g_2'(s)\int_\Omega |\nabla v(t)-\nabla v(t-s)|^2 dxds \\
&\leq -m\xi(t)E(t) - cE'(t). \qquad (39)
\end{aligned}$$

Denote $\mathcal{F}(t) = \xi(t)\mathcal{L}(t) + cE(t) \sim E(t)$. Then, we have, from (39) and the fact that ξ is non-increasing, that for any $t \geq t_1$:

$$\mathcal{F}'(t) \leq -m\xi(t)E(t).$$

Using the fact that $\mathcal{F} \sim E$, we obtain

$$\mathcal{F}'(t) \leq -c_1\mathcal{F}(t),$$

for some positive constant c_1. By applying Gronwall's Lemma, we obtain the existence of a constant $c_2 > 0$ such that

$$\mathcal{F}(t) \leq c_2 e^{-c_1\int_{t_1}^{t}\xi(s)\,ds},$$

which yields to

$$E(t) \leq c_3 e^{-c_1\int_{t_1}^{t}\xi(s)\,ds},$$

for some constant $c_3 > 0$.

Case 2: Q is nonlinear. First, we define the following quantities

$$I_1(t) = \frac{\kappa}{t}\int_0^t a(u(t)-u(t-s),u(t)-u(t-s))ds, \quad t > 0$$

and

$$I_2(t) = \frac{\kappa}{t}\int_0^t \int_\Omega |\nabla v(t)-\nabla v(t-s)|^2 dxds, \quad t > 0.$$

Then, we have

$$\begin{aligned}
I_1(t) &\leq \frac{2\kappa}{t}\int_0^t [a(u(t),u(t)) + a(u(t-s),u(t-s))]ds \\
&\leq \frac{4\kappa}{lt}\left(\int_0^t (E(t)+E(t-s))ds\right) \\
&\leq \frac{8\kappa}{lt}\int_0^t E(s)ds \\
&\leq \frac{8\kappa}{lt}\int_0^t E(0)ds = \frac{8\kappa}{l}E(0) < +\infty,
\end{aligned}$$

and likewise, we have

$$I_2(t) \leq \frac{8\kappa}{l} E(0) < +\infty.$$

Thus, choosing $0 < \kappa < 1$ that is small enough so that, for all $t > 0$:

$$I_i(t) < 1, \quad \text{for} \quad i = 1, 2. \tag{40}$$

Also, we define $\lambda_1(t)$ and $\lambda_2(t)$ by

$$\lambda_1(t) = -\int_0^t g_1'(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds,$$

and

$$\lambda_2(t) = -\int_0^t g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dx ds.$$

It is obvious that $\lambda_i(t) \leq -cE'(t), \ \ i = 1, 2.$

Noting Q is strictly convex on $(0, r]$ and $Q(0) = 0$, then $Q(\theta x) \leq \theta Q(x)$, provided that $0 \leq \theta \leq 1$ and $x \in (0, r]$. This, together with (A1), (40) and Jensen's inequality, leads to

$$\begin{aligned}
\lambda_1(t) &= \frac{1}{\kappa I_1(t)} \int_0^t I_1(t)(-g_1'(s)) \kappa a(u(t) - u(t-s), u(t) - u(t-s)) ds \\
&\geq \frac{1}{\kappa I_1(t)} \int_0^t I_1(t) \xi_1(s) Q(g_1(s)) \kappa a(u(t) - u(t-s), u(t) - u(t-s)) ds \\
&\geq \frac{\xi_1(t)}{\kappa I_1(t)} \int_0^t Q(I_1(t) g_1(s)) \kappa a(u(t) - u(t-s), u(t) - u(t-s)) ds \\
&\geq \frac{\xi_1(t)}{\kappa} Q\left(\frac{1}{I_1(t)} \int_0^t I_1(t) g_1(s) \kappa a(u(t) - u(t-s), u(t) - u(t-s)) ds \right) \\
&= \frac{\xi_1(t)}{\kappa} Q\left(\kappa \int_0^t g_1(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \right) \\
&= \frac{\xi_1(t)}{\kappa} \overline{Q}\left(\kappa \int_0^t g_1(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \right),
\end{aligned}$$

where $\overline{Q}$ is an extension of Q such that $\overline{Q}$ is strictly increasing and a strictly convex C^2 function on $(0, +\infty)$. This implies that

$$\int_0^t g_1(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds \leq \frac{1}{\kappa} \overline{Q}^{-1}\left(\frac{\kappa \lambda_1(t)}{\xi_1(t)} \right).$$

Similarly, we have

$$\int_0^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dx ds \leq \frac{1}{\kappa} \overline{Q}^{-1}\left(\frac{\kappa \lambda_2(t)}{\xi_2(t)} \right).$$

We infer from (38) that for any $t \geq t_1$

$$\mathcal{L}'(t) \leq -mE(t) + c\overline{Q}^{-1}\left(\frac{\kappa \lambda_1(t)}{\xi_1(t)} \right) + c\overline{Q}^{-1}\left(\frac{\kappa \lambda_2(t)}{\xi_2(t)} \right). \tag{41}$$

For $\varepsilon_0 < r$, using (41) and the fact that $E' \leq 0$, $\overline{Q}' > 0$, $\overline{Q}'' > 0$, we find that the functional $\mathcal{K}_1$, defined by

$$\mathcal{K}_1(t) = \overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)} \right) \mathcal{L}(t) + E(t)$$

is equivalent to $E(t)$ and satisfies

$$\begin{aligned}\mathcal{K}_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} \overline{Q}''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + \overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}'(t) + E'(t) \\ &\leq -mE(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\overline{Q}^{-1}\left(\frac{\kappa\lambda_1(t)}{\xi_1(t)}\right) + c\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\overline{Q}^{-1}\left(\frac{\kappa\lambda_2(t)}{\xi_2(t)}\right). \end{aligned} \tag{42}$$

Now, let $\overline{Q}^*$ be the convex conjugate of $\overline{Q}$ in the sense of Young (see [26]). Then,

$$\overline{Q}^*(s) = s(\overline{Q}')^{-1}(s) - \overline{Q}((\overline{Q}')^{-1}(s)), \tag{43}$$

which satisfies

$$AB_i \leq \overline{Q}^*(A) + \overline{Q}(B_i), \ i = 1, 2, \tag{44}$$

with $A = \overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B_i = \overline{Q}^{-1}\left(\frac{\kappa\lambda_i(t)}{\xi_i(t)}\right)$, $i = 1, 2$.

It is inferred from (42)–(44) that

$$\mathcal{K}_1'(t) \leq -mE(t)\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\kappa\left(\frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)}\right)$$

is obtained by multiplying the last inequality by $\xi(t)$ and using the fact that, as $\varepsilon_0 \frac{E(t)}{E(0)} < r$, $\overline{Q}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = Q'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $\lambda_i(t) \leq -cE'(t)$ (for $i = 1, 2$), that

$$\xi(t)\mathcal{K}_1'(t) \leq -mE(t)\xi(t)Q'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)}\xi(t)Q'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t).$$

Consequently, by letting $\mathcal{K}_2 = \xi\mathcal{K}_1 + cE$, we have: $\alpha_1\mathcal{K}_2(t) \leq E(t) \leq \alpha_2\mathcal{K}_2(t)$, for some $\alpha_1, \alpha_2 > 0$.

Hence, we conclude that, for some constant $\beta_1 > 0$ and for all $t \geq t_1$

$$\mathcal{K}_2'(t) \leq -\beta_1\xi(t)\frac{E(t)}{E(0)}Q'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) := -\beta_1\xi(t)Q_2\left(\frac{E(t)}{E(0)}\right), \tag{45}$$

where $Q_2(t) = tQ'(\varepsilon_0 t)$. Since $Q_2'(t) = Q'(\varepsilon_0 t) + \varepsilon_0 tQ''(\varepsilon_0 t)$, then, using the strict convexity of Q on $(0, r]$, we reach that $Q_2'(t), Q_2(t) > 0$ on $(0, 1]$. Thus, with $H(t) = \frac{\alpha_1\mathcal{K}_2(t)}{E(0)}$ and using the fact that $\mathcal{K}_2 \sim E$ and (45), we have

$$H(t) \sim E(t), \tag{46}$$

and for some $\beta_2 > 0$,

$$H'(t) \leq -\beta_2\xi(t)Q_2(H(t)), \forall\, t \geq t_1.$$

Integrating the latter over (t_1, t) yields

$$\int_{t_1}^{t} \frac{-H'(s)}{Q_2(H(s))}ds \geq \beta_2 \int_{t_1}^{t} \xi(s)ds,$$

which leads to

$$\frac{1}{\varepsilon_0}\int_{\varepsilon_0 H(t)}^{\varepsilon_0 H(t_1)} \frac{1}{sQ_2'(s)}ds \geq \beta_2 \int_{t_1}^{t} \xi(s)ds.$$

Lastly, since the function Q_1 given by $Q_1(t) = \int_t^r \frac{1}{sQ'(s)} ds$ is strictly decreasing on $(0, r]$ and $\lim_{t\to 0} Q_1(t) = +\infty$, we deduce that

$$H(t) \le \frac{1}{\varepsilon_0} Q_1^{-1}\left(\beta_1 \int_{t_1}^{t} \xi(s) ds\right).$$

Combining the latter with (46), one can claim that (13) holds. □

In the following remark, we may extend our previous results in the case where we take nonlinear coupling terms instead of the linear ones used in system (1) and also for a quasi-linear version, where the material densities vary according to the velocity.

Remark 3.

1. *We consider system* (1) *with* $f_1(u,v)$ *(respectively,* $f_2(u,v)$*) instead of* αv *(respectively,* αu*), that is*

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s)\,ds + f_1(u,v) = 0 & in \quad \Omega \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + f_2(u,v) = 0 & in \quad \Omega \times (0,\infty) \\ u = \partial_\nu u = 0 & on \quad \Gamma_0 \times (0,\infty) \\ \mathbf{B}_1 u - \mathbf{B}_1\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & on \quad \Gamma_1 \times (0,\infty) \\ \mathbf{B}_2 u - \gamma\partial_\nu u_{tt} - \mathbf{B}_2\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & on \quad \Gamma_1 \times (0,\infty) \\ v = 0 & on \quad \Gamma \times (0,\infty) \\ u(0) = u^0, \ u_t(0) = u^1, \ \ v(0) = v^0, \ \ v_t(0) = v^1 & in \quad \Omega, \end{cases}$$

where $f_i, i = 1,2$*, satisfy.*
$f_i : \mathbb{R}^2 \to \mathbb{R}$ *(for* $i = 1,2$*) are* C^1 *functions and there exists a positive function* F *such that*

$$f_1(x_1,x_2) = \frac{\partial F}{\partial x_1}, \ \ f_2(x_1,x_2) = \frac{\partial F}{\partial x_2}, \ x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2) - F(x_1,x_2) \ge 0,$$

and

$$\left|\frac{\partial f_i}{\partial x_1}(x_1,x_2)\right| + \left|\frac{\partial f_i}{\partial x_2}(x_1,x_2)\right| \le d(1 + |x_1|^{\beta_{i1}-1} + |x_2|^{\beta_{i2}-1}), \ \forall\ (x_1,x_2) \in \mathbb{R}^2,$$

for some constant $d > 0$ *and* $\beta_{ij} \ge 1$ *for* $i,j = 1,2$*.*
By using the same method derived here, we may prove that the above system is well-posed and a general decay rate can be established, as in (13).

2. *By following the same approaches as in Sections* 3 *and* 4*, we shall prove that the following quasi-linear coupled system*

$$\begin{cases} |u_t|^\rho u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u - \int_0^t g_1(t-s)\Delta^2 u(s)\,ds + f_1(u,v) = 0 & in \quad \Omega \times (0,\infty) \\ |v_t|^\rho v_{tt} - \Delta v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + f_2(u,v) = 0 & in \quad \Omega \times (0,\infty) \\ u = \partial_\nu u = 0 & on \quad \Gamma_0 \times (0,\infty) \\ \mathbf{B}_1 u - \mathbf{B}_1\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & on \quad \Gamma_1 \times (0,\infty) \\ \mathbf{B}_2 u - \gamma\partial_\nu u_{tt} - \mathbf{B}_2\left\{\int_0^t g_1(t-s)u(s)\,ds\right\} = 0 & on \quad \Gamma_1 \times (0,\infty) \\ v = 0 & on \quad \Gamma \times (0,\infty) \\ u(0) = u^0, \ u_t(0) = u^1, \ \ v(0) = v^0, \ \ v_t(0) = v^1 & in \quad \Omega, \end{cases}$$

with $\rho > 0$, possess at least a weak solution $u \in C([0,T], V) \cap C^1([0,T], W)$, $v \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], H_0^1(\Omega))$; and moreover, we shall establish a general decay rate of energy as in (13).

4. Conclusions

This paper focuses on the existence and the asymptotic stability of solutions for a system of two coupled Kirchhoff plate and wave equations in a bounded domain of $\mathbb{R}^2$, subject only to viscoelasticity dissipative terms and with the presence of rotational forces (in the Kirchhoff plate equation). It should be noted that this model takes the memory effects into account, which may exist in some materials, particularly in low temperature. The first equation, in system (1), describes the motion of a plate, which is clamped along one portion of its boundary and has free vibrations on the other portion of the boundary, whereas the second one models the motion of a membrane. This work is motivated by previous results concerning coupled wave equations [13,18–22] and coupled wave–plate equations [24].

By using the Faedo–Galerkin method, we proved the existence of a unique global weak solution. Furthermore, by constructing an appropriate Lyapunov function, we showed the general decay of the energy associated with the system (1). As a future work, we aim to change the type of damping by considering, for example, the Balakrishnan–Taylor damping (of the form $(\nabla y, \nabla y_t)\Delta y$), strong damping (of the form $\Delta^2 y_t$) or past history terms in the Kirchhoff plate equation (of the form $\int_0^\infty g_1(s)\Delta^2(x, t-s)ds$) or/and in the wave equation (of the form $\int_0^\infty g_2(s)\Delta(x, t-s)ds$).

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