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Symmetries and Conservation Laws for a Class of Fourth-Order Reaction–Diffusion–Advection Equations

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Abstract: We have studied a class of $(1+1)$-dimensional equations that models phenomena with heterogeneous diffusion, advection, and reaction. We have analyzed these fourth-order partial differential equations within the framework of group methods. In this class, the diffusion coefficient is constant, while the coefficients of advection and the reaction term are assumed to depend on the unknown density $u(t,x)$. We have identified the Lie symmetries extending the Principal Algebra along with all the conservation laws corresponding to the different forms of the coefficients, and have derived several brief applications.

Keywords: heterogenous diffusion; advection–reaction equations; Lie symmetries; conservation laws; first integrals; exact solutions

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1. Introduction

In this paper, we consider the following class of reaction–diffusion–advection equations:

$$u_t + Au_{xxxx} + f(u)u_x + g(u) = 0. \tag{1}$$

We assume a high-order linear diffusion with a constant coefficient A. In the above equation, the coefficient of the advection term $f(u)$ and the reaction term $g(u)$ are arbitrary functions that depend on u.

Equation (1) generalizes

$$w_t = -w_{xxxx} + c(w^q)_x + w(a-w), \tag{2}$$

introduced in [1], where the authors provided an analysis of a Fisher–KPP [2,3] nonlinear reaction equation in a problem with higher-order diffusion and in the presence of an advection term. A more careful study of diffusion based on statistical concepts and a random walk approach can be found in [4]. Moreover, other proposals using the free energy in the Landau–Ginzburg approach to analyze diffusion processes have been followed in [5–7], and can generally reach fourth-order diffusion. The second-order diffusion derived in the case of Fick's law is a special instance of this approach. Higher-order diffusion (i.e., heterogeneous diffusion) may be seen as a perturbation of the standard second-order diffusion; see, e.g., [7–10] for extensions of the Fisher–Kolmogorov-type equation to the fourth order. Actually, the problem of writing a higher-order diffusion equation has been the focus of many scientists for several decades [11–15].

Here, the class (1) is studied in the wide framework of group techniques. This paper falls within a set of papers belonging to a wider project in which we take into considerations the symmetry structure of certain classes of reaction–diffusion–advection equations (RDAEs); see, e.g., [16–20], and for classes of the same type where advection is neglected, see [21] and references within. Moreover a significant contribution has been provided by Cherniha and his co-workers in the field of symmetries applied to RDAEs that model several diffusion phenomena [22–25].

One of the most relevant applications of Lie symmetries concerns the search for solutions that are invariant with respect to these symmetries. This approach provides a methodological way to derive exact solutions. In the case of a $(1+1)$-dimensional PDE, this method leads to the search for solutions of ODEs.

In this paper, starting from the well known Lie invariance criterion, we obtain classifying equations that allow us to find several extensions of the Principal Lie Algebra $\mathcal{L}_{\mathcal{P}}$ [26],which is concerned with special forms of the arbitrary parameters $f(u)$ and $g(u)$ describing class (1). In the analysis of PDEs, conservation laws are essential, as they can detect conserved physical quantities; they are additionally employed to search for solutions, additional symmetries (including potential symmetries [27,28]), and to reach solutions in a numerical way. Here, we determine them using the multipliers method [29,30]. Finally, we derive a first integral and several special exact solutions as brief applications of our previous results. We leave a wider and richer discussion about these results to a future work.

As far as we know, RDAEs (1) have not been previously studied within the framework of group methods. It is worth noting that in this study we have selected different forms of constitutive functions using symmetry classifications, a number of which may have significance in real processes. For example, Equations (2) are invariant only with respect to translations in time and space, while we have found that when the advection coefficient is

$$f(u) = f_1 \ln(u) \tag{3}$$

and the reaction term is

$$g(u) = u(g_0 + g_1 \ln(u)), \tag{4}$$

which is a generalized logistic type [19], the corresponding Equation (1) admits an additional Lie generator that allows exact solutions to be obtained and can be written in the conservative form. Finally, we wish to emphasize that the construction of exact solutions for real problems, the existence and uniqueness of their solutions, and the accuracy and stability of numerical methods, although not the main focus of the present work, can be analyzed with the help of conservation laws.

The rest of this paper is outlined as follows. In Section 2, after having carried out the classifying conditions, a symmetry classification with respect to $f(u)$ and $g(u)$ is obtained. Section 3 is devoted to the derivation of multipliers and their corresponding conserved vectors. In Section 4, we briefly illustrate a number of applications. Finally, our conclusions are presented in Section 5.

2. Invariance and Classifying Conditions

In this section, we look for symmetry operators for Equation (1) in the form

$$X = \xi_1(t,x,u)\partial_t + \xi_2(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{5}$$

To this end, we follow well known monographs in the field [28,31–37]. By applying the Lie infinitesimal criterion, we can write the invariance condition for Equation (1):

$$X^{(4)}[u_t + Au_{xxxx} + f(u)u_x + g(u)]\Big|_{u_t + Au_{xxxx} + f(u)u_x + g(u) = 0} = 0. \tag{6}$$

In this case, the fourth extended operator $X^{(4)}$ is

$$X^{(4)} = X + \zeta_t \partial_{u_t} + \zeta_x \partial_{u_x} + \zeta_{xxxx} \partial_{u_{xxxx}} \tag{7}$$

with

$$\zeta_t = D_t\eta - u_t D_t\xi_1 - u_x D_t\xi_2, \tag{8}$$
$$\zeta_x = D_x\eta - u_t D_x\xi_1 - u_x D_x\xi_2, \tag{9}$$
$$\zeta_{xx} = D_x\zeta_x - u_{tx} D_x\xi_1 - u_{xx} D_x\xi_2, \tag{10}$$
$$\zeta_{xxx} = D_x\zeta_{xx} - u_{txx} D_x\xi_1 - u_{xxx} D_x\xi_2, \tag{11}$$
$$\zeta_{xxxx} = D_x\zeta_{xxx} - u_{txxx} D_x\xi_1 - u_{xxxx} D_x\xi_2, \tag{12}$$

where D_t and D_x represent the total derivatives with respect to t and x, respectively.

After deriving the *determining system* for the infinitesimal unknowns ξ_1, ξ_2, and η from (6), these functions are restricted to the forms

$$\xi_1 = 4\phi_1(t), \tag{13}$$
$$\xi_2 = \phi_{1_t} x + \phi_2(t), \tag{14}$$
$$\eta = \phi_3(t)u + \phi_4(t,x), \tag{15}$$

where the functions $\phi_i, i = 1, \ldots, 4$ satisfy the following classifying conditions:

$$(\phi_3 u + \phi_4) f_u + 3\phi_{1_t} f - \phi_{1_{tt}} x - \phi_{2_t} = 0, \tag{16}$$
$$(\phi_3 u + \phi_4) g_u + (4\phi_{1_t} - \phi_3) g + \phi_{4_x} f + A\phi_{4_{xxxx}} + \phi_{3_t} u + \phi_{4_t} = 0. \tag{17}$$

For arbitrary functions f and g, we obtain

$$\phi_{1_t} = 0,\ \phi_{2_t} = 0,\ \phi_3 = 0,\ \phi_4 = 0. \tag{18}$$

We can now affirm the following theorem.

Theorem 1. *For arbitrary functions f and g, Equation (1) admits a two-dimensional Lie algebra spanned by*

$$\mathrm{X}_1 = \partial_t,\ \mathrm{X}_2 = \partial_x. \tag{19}$$

This algebra is called the "principal Lie algebra" and is denoted by $L_{\mathcal{P}}$.

Now, we seek specific forms of the functions $f(u)$ and $g(u)$ such that Equation (1) admits additional Lie symmetries.

From (16), we carry out

$$f_u\phi_{4_{xx}} = 0, \quad \text{and} \quad f_{uu}\phi_{4_x} = 0, \tag{20}$$

bringing us to the following three cases:

1. $f(u) = f_0$
2. $f(u) = f_0 + f_1 u, f_1 \neq 0$
3. $f_{uu} \neq 0$

We analyze these cases separately in the next three subsections.

2.1. Constant Advection Term Coefficient

Here, we consider the advection term coefficient to be constant, that is,

$$f(u) = f_0. \tag{21}$$

We recall that the following invertible change of the independent variable:

$$x \to x + f_0 t \tag{22}$$

maps Equation (1) to

$$u_t + Au_{xxxx} + g(u) = 0. \tag{23}$$

Thus, the special cases of (1) where the advection term does not appear can be recovered from the symmetry analysis of this subsection. However, in this study we choose to retain the advection term in Equation (1).

In this case, from (16), we obtain

$$\phi_1(t) = c_1 t + c_2, \quad \phi_2(t) = 3c_1 f_0 t + c_3. \tag{24}$$

From (17), we obtain

$$\phi_{4_x} g_{uu} = 0, \tag{25}$$

therefore, we need to consider the two subcases $g_{uu} = 0$ and $g_{uu} \neq 0$.

2.1.1. $g_{uu} = 0$

If $g_{uu} = 0$, i.e., $g(u) = g_0 + g_1 u$, Equation (1) becomes

$$u_t + Au_{xxxx} + f_0 u_x + g_0 + g_1 u = 0. \tag{26}$$

In this case, we obtain

$$\phi_3 = -4c_1 g_1 t + c_4, \tag{27}$$

and $\phi_4(t, x)$ is a solution of the equation

$$\phi_{4_t} + A\phi_{4_{xxxx}} + f_0 \phi_{4_x} + \phi_4 g_1 + 4c_1 g_0 + g_0(4c_1 g_1 t - c_4) = 0. \tag{28}$$

Then, the additional generators are

$$X_3 = u\partial_u, \tag{29}$$

$$X_4 = 4t\partial_t + (3f_0 t + x)\partial_x - 4g_1 t u \partial_u, \tag{30}$$

$$X_\phi = \phi_4 \partial_u, \tag{31}$$

where $\phi_4(t, x)$ is a solution of

$$\phi_{4_t} + A\phi_{4_{xxxx}} + f_0 \phi_{4_x} + \phi_4 g_1 = 0. \tag{32}$$

Without loss of generality, we can assume $g_0 = 0$. Indeed, in the case where $g_1 \neq 0$, the simple transformation $u \to u - \frac{g_0}{g_1}$ maps (26) to

$$u_t + Au_{xxxx} + f_0 u_x + g_1 u = 0, \tag{33}$$

while in the case where $g_1 = 0$ the simple transformation $u \to u - g_0 t$ maps (26) with $g_1 = 0$ to

$$u_t + Au_{xxxx} + f_0 u_x = 0. \tag{34}$$

2.1.2. $g_{uu} \neq 0$

If $g_{uu} \neq 0$, we obtain $\phi_4(t, x) = \phi_4(t)$ and condition (17) becomes

$$\phi_3 g_u u + \phi_{3_t} u - \phi_3 g + \phi_4 g_u + 4c_1 g + \phi_{4_t} = 0. \tag{35}$$

To discuss (35), we can distinguish the following two cases.

1. $\phi_3 = c_5$
   Here, $\phi_4 = c_4$ and (35) becomes

$$c_5 g_u u - c_5 g + c_4 g_u + 4c_1 g = 0, \tag{36}$$

   which implies that

$$g(u) = g_0 e^{g_1 u} \quad \text{or} \quad g(u) = g_0 (u + g_1)^{g_2}. \tag{37}$$

(a) If $g(u) = g_0 e^{g_1 u}$, we obtain $c_5 = 0$ and $c_4 = -4\frac{c_1}{g_1}$ from (36). Then, for $g(u) = g_0 e^{g_1 u}$ we obtain the following additional generator:

$$X_3 = 4t\partial_t + (3f_0 t + x)\partial_x - \frac{4}{g_1}\partial_u. \tag{38}$$

(b) If $g(u) = g_1(u + g_0)^{g_2}$, then in the same way we obtain

$$c_1 = c_5 \frac{1 - g_2}{4}, \quad c_4 = c_5 g_0 \tag{39}$$

from (36) and

$$X_3 = 4t(1 - g_2)\partial_t + (1 - g_2)(3f_0 t + x)\partial_x + 4(u + g_0)\partial_u. \tag{40}$$

It is be useful to recall here that we can assume $g_0 = 0$ through the transformation $u \to u - g_0$.

2. $\phi_3 \neq const$
By differentiating Equation (35) with respect to t and u, we obtain

$$\phi_{3_t} u g_{uu} + \phi_{3_{tt}} + \phi_{4_t} g_{uu} = 0. \tag{41}$$

From this equation, we can derive

$$\left(\frac{\phi_{4_t}}{\phi_{3_t}}\right)_t = 0 \ \Rightarrow \phi_4(t) = c_4 \phi_3(t) + c_5. \tag{42}$$

Substituting this into (41), we can carry out the following additional restriction on ϕ_3:

$$\frac{\phi_{3_{tt}}}{\phi_{3_t}} = c_6 = constant. \tag{43}$$

In order to find additional generators, it must be the case that $c_6 \neq 0$, which implies that $\phi_3(t) = c_8 + c_7 e^{c_6 t}$ with $c_6 c_7 \neq 0$. In this case, it is possible to determine that $g(u)$ must be of the form

$$g(u) = (u + g_2)(g_0 + g_1 ln(u + g_2)), \quad g_1 \neq 0, \tag{44}$$

and that

$$c_4 = g_2, \ c_6 = -g_1, \ c_1 = c_5 = c_8 = 0. \tag{45}$$

Then, for g of the form (44), the following generator exists:

$$X_3 = e^{-g_1 t}(g_2 + u)\partial_u. \tag{46}$$

Of course, in this case we can assume that $g_2 = 0$.

2.2. Linear Advection Term Coefficient

Here, we consider the advection term coefficient in the linear form:

$$f(u) = f_0 + f_1 u \quad \text{with} \quad f_1 \neq 0. \tag{47}$$

In this case, from (16), we obtain

$$\phi_3(t) = -3\phi_{1_t}, \quad \phi_4(t, x) = \frac{1}{f_1}(x\phi_{1_{tt}} + \phi_{2_t} - 3f_0\phi_{1_t}). \tag{48}$$

From (17), we need to consider the following three subcases: $g(u)$ nonlinear, $g(u)$ linear but non-constant, and $g(u)$ constant.

2.2.1. $g_{uu} \neq 0$

In this case, from (17), we immediately obtain

$$\phi_1(t) = c_1 t + c_2, \quad \phi_2(t) = c_3 t + c_4 \tag{49}$$

and find extensions of $L_{\mathcal{P}}$ if

$$g(u) = g_0(g_1 + u)^{\frac{7}{3}}, \quad g_0 \neq 0, \tag{50}$$

and $c_3 = 3c_1(f_0 - f_1 g_1)$. Then, Equation (1) becomes

$$u_t + Au_{xxxx} + (f_0 + f_1 u)u_x + g_0(g_1 + u)^{\frac{7}{3}} = 0, \tag{51}$$

and admits the following additional Lie symmetry:

$$X_3 = 4t\partial_t + (x + 3(f_0 - f_1 g_1)t)\partial_x - 3(g_1 + u)\partial_u. \tag{52}$$

Without loss of generality, we can assume that $f_0 = 0$ and that $g_1 = 0$. This can be proved using the following transformation:

$$x \to x + (f_0 - f_1 g_1)t, \quad t \to t, \quad u \to u - g_1, \tag{53}$$

which maps (51) to

$$u_t + Au_{xxxx} + f_1 u u_x + g_0 u^{\frac{7}{3}} = 0. \tag{54}$$

Equation (54) admits the additional Lie symmetry

$$X_3 = 4t\partial_t + x\partial_x - 3u\partial_u. \tag{55}$$

2.2.2. $g(u) = g_0 + g_1 u$, with $g_1 \neq 0$

In this case, we have

$$\phi_1(t) = c_1, \; \phi_2(t) = c_3 e^{-g_1 t} + c_2, \; \phi_3(t) = 0, \; \phi_4(t) = -c_3 \frac{g_1}{f_1} e^{-g_1 t}. \tag{56}$$

Equation (1) assumes the form

$$u_t + Au_{xxxx} + (f_0 + f_1 u)u_x + g_0 + g_1 u = 0 \tag{57}$$

and admits the additional Lie symmetry generator

$$X_3 = e^{-g_1 t}\partial_x - \frac{g_1}{f_1} e^{-g_1 t}\partial_u. \tag{58}$$

Without loss of generality, we can assume that $f_0 = g_0 = 0$. Indeed, the transformation

$$x \to x + \frac{f_0 g_1 - f_1 g_0}{g_1} t, \quad t \to t, \quad u \to u - \frac{g_0}{g_1}, \tag{59}$$

maps (57) to

$$u_t + Au_{xxxx} + f_1 u u_x + g_1 u = 0 \tag{60}$$

and admits the additional Lie symmetry generator (58).

2.2.3. $g(u) = g_0$

In this case, we obtain

$$\phi_1(t) = c_1 t + c_2,\ \phi_2(t) = -\frac{7}{2} g_2 f_1 c_1 t^2 + c_3 t + c_4, \tag{61}$$

$$\phi_3(t) = -3c_1,\ \phi_4(t) = \frac{1}{f_1}(c_3 - c_1 7 g_2 f_1 t - 3c_1 f_0). \tag{62}$$

Equation (1) assumes the form

$$u_t + A u_{xxxx} + (f_0 + f_1 u) u_x + g_0 = 0 \tag{63}$$

and admits the additional Lie symmetry generators

$$X_3 = 4t\partial_t + \left(x - \frac{7}{2} g_0 f_1 t^2\right)\partial_x - \left(3u + 7g_0 t + 3\frac{f_0}{f_1}\right)\partial_u, \tag{64}$$

$$X_4 = t\partial_x + \frac{1}{f_1}\partial_u. \tag{65}$$

Without loss of generality, we can assume that $f_0 = 0$; indeed, in this case, the simple transformation $x \to x + f_0 t$ maps (63) to

$$u_t + A u_{xxxx} + f_1 u u_x + g_0 = 0 \tag{66}$$

and admits the additional Lie symmetry generators (64) and (65) with $f_0 = 0$.

2.3. Nonlinear Advection Term Coefficient

When $f_{uu} \neq 0$, Equation (1) has a nonlinear advection term coefficient. In this case, we immediately obtain

$$\phi_4(t, x) = \phi_4(t), \quad \phi_1(t) = c_1 t + c_2 \tag{67}$$

and (16) becomes

$$(\phi_3 u + \phi_4) f_u + 3c_1 f - \phi_{2_t} = 0. \tag{68}$$

Observing that if we want to obtain extensions of $L_{\mathcal{P}}$, then $\phi_3(t)$ and $\phi_4(t)$ cannot both be zero, we can distinguish the following three subcases depending on the function f:

- $f(u) = f_0 + f_1(u + f_2)^{f_3}$ with $f_1 f_3 \neq 0$, $f_3 \neq 1$
- $f(u) = f_0 + f_1 \ln(u + f_2)$ with $f_1 \neq 0$
- $f(u) = f_0 + f_1 e^{f_2 u}$ with $f_1 f_2 \neq 0$

2.3.1. $f(u) = f_0 + f_1(u + f_2)^{f_3}$ with $f_1 f_3 \neq 0,\ f_3 \neq 1$

In this case, we have

$$\phi_3(t) = -\frac{3c_1}{f_3},\ \phi_4(t) = -\frac{3c_1 f_2}{f_3},\ \phi_2(t) = 3c_1 f_0 t + c_3. \tag{69}$$

From (17), we obtain extensions of $L_{\mathcal{P}}$ if

$$g(u) = g_0 (u + f_2)^{\frac{4}{3} f_3 + 1}. \tag{70}$$

Then, the equation

$$u_t + A u_{xxxx} + (f_0 + f_1(u + f_2)^{f_3}) u_x + g_0 (u + f_2)^{\frac{4}{3} f_3 + 1} = 0 \tag{71}$$

admits the following additional Lie symmetry:

$$X_3 = 4t\partial_t + (x + 3f_0 t)\partial_x - \frac{3(u + f_2)}{f_3}\partial_u. \tag{72}$$

Without loss of generality, we can assume that $f_0 = f_2 = 0$. This is proved using the transformation

$$x \to x + f_0 t, \quad t \to t, \quad u \to u - f_2, \tag{73}$$

which maps Equation (71) to

$$u_t + Au_{xxxx} + f_1 u^{f_3} u_x + g_0 u^{\frac{4}{3}f_3+1} = 0, \tag{74}$$

admitting the additional Lie symmetry in (72) with $f_0 = f_2 = 0$.

It can be observed that if we remove the conditions $f_3 \neq 1$ and $f_1 \neq 0$ from this case we can obtain two previous results. In fact, by choosing $f_1 = 0$, $f_2 = g_0$, and $f_3 = \frac{3}{4}(g_2 - 1)$, we obtain case $1(b)$ from Section 2.1.2, while if we choose $f_0 = \bar{f}_0 - f_1 g_1$, $f_2 = g_1$, and $f_3 = 1$ we obtain the result from Section 2.2.1.

2.3.2. $f(u) = f_0 + f_1 \ln(u + f_2)$ with $f_1 \neq 0$

In this case, we obtain

$$c_1 = 0,\ \phi_4(t) = \frac{\phi_{2_t} f_2}{f_1},\ \phi_3(t) = \frac{\phi_{2_t}}{f_1}. \tag{75}$$

From (17), we obtain extensions of $L_{\mathcal{P}}$ if

$$g(u) = (u + f_2)(g_0 + g_1 \ln(u + f_2)); \tag{76}$$

moreover, if $g_1 \neq 0$, then $\phi_2(t) = c_4 e^{-g_1 t} + c_3$, while if $g_1 = 0$, then $\phi_2(t) = c_4 t + c_3$. Then,

$$u_t + Au_{xxxx} + (f_0 + f_1 \ln(u + f_2))u_x + (u + f_2)(g_0 + g_1 \ln(u + f_2)) = 0 \tag{77}$$

admits a third Lie symmetry. If $g_1 \neq 0$, this is

$$X_3 = e^{-g_1 t}(f_1\partial_x - g_1(u + f_2)\partial_u), \tag{78}$$

while if $g_1 = 0$ (that is, if $g(u)$ is linear), it is

$$X_3 = tf_1\partial_x + (u + f_2)\partial_u. \tag{79}$$

In this case, without loss of generality, we can assume that $f_0 = f_2 = 0$, because with the transformation

$$x \to x + f_0 t, \quad t \to t, \quad u \to u - f_2 \tag{80}$$

we obtain

$$u_t + Au_{xxxx} + f_1 \ln(u)u_x + u(g_0 + g_1 \ln(u)) = 0, \tag{81}$$

from which the additional generators can be obtained from (78) and (79) by setting $f_2 = 0$.

It can be observed that if we remove the condition $f_1 \neq 0$, by choosing $f_1 = 0$ and $f_2 = g_2$ we obtain the same result as in the second case in Section 2.1.2.

2.3.3. $f(u) = f_0 + f_1 e^{f_2 u}$ with $f_1 f_2 \neq 0$

In this case, we obtain

$$\phi_4(t) = -\frac{3c_1}{f_2},\ \phi_3(t) = 0,\ \phi_2(t) = 3c_1 f_0 t + c_3. \tag{82}$$

From (17), we obtain extensions of $L_{\mathcal{P}}$ if

$$g(u) = g_0 e^{\frac{4}{3} f_2 u}. \tag{83}$$

Then, equation

$$u_t + Au_{xxxx} + \left(f_0 + f_1 e^{f_2 u}\right) u_x + g_0 e^{\frac{4}{3} f_2 u} = 0 \tag{84}$$

admits the following additional Lie symmetry:

$$X_3 = 4t\partial_t + \frac{x + 3f_0 t}{f_2}\partial_x - \frac{3}{f_2}\partial_u. \tag{85}$$

Without loss of generality, we can assume that $f_0 = 0$ (in this case, it is enough to use the transformation $x \to x + f_0 t$). Equation (1) assumes the form

$$u_t + Au_{xxxx} + f_1 e^{f_2 u} u_x + g_0 e^{\frac{4}{3} f_2 u} = 0 \tag{86}$$

and admits the following additional Lie symmetry:

$$X_3 = 4t\partial_t + x\partial_x - \frac{3}{f_2}\partial_u. \tag{87}$$

If we remove the condition $f_1 \neq 0$, by choosing $f_1 = 0$ and $f_2 = \dfrac{2}{4} g_1$ we obtain the same result as in case $1(a)$ in Section 2.1.2.

Our results are summarized in Table 1, which lists the functions f and g (in their equivalent simplified form) for which Equation (1) admits additional generators with respect to the principal Lie algebra, along with their corresponding generators.

Thus, we have proved the following theorem.

Theorem 2. *The functions f and g for which the Lie algebra in Equation* (1) *admits extension with respect to the principal Lie algebra are shown in Table 1 along with their corresponding additional generators.*

Table 1. Functions f and g and their corresponding additional generators with respect to the principal Lie algebra.

$\mathbf{f(u)}$	$\mathbf{g(u)}$	**Additional Generators**
$f_1 u^{f_3}, \quad f_3 \neq 0$	$g_0 u^{\frac{4}{3} f_3 + 1}$	$X_3 = 4t\partial_t + x\partial_x - \frac{3}{f_3} u\partial_u$
$f_1 \ln(u)$	$u(g_0 + g_1 ln(u)), \quad g_1 \neq 0$	$X_3 = e^{-g_1 t}(f_1\partial_x - g_1 u\partial_u)$
$f_1 \ln(u)$	$g_0 u$	$X_3 = t f_1 \partial_x + u\partial_u$
$f_1 e^{f_2 u}, \quad f_2 \neq 0$	$g_0 e^{4 f_2 u/3}$	$X_3 = 4t\partial_t + x\partial_x - \frac{3}{f_2}\partial_u$
$f_1 u, \quad f_1 \neq 0$	$g_1 u$	$X_3 = e^{-g_1 t}\left(\partial_x - \frac{g_1}{f_1}\partial_u\right)$
$f_1 u, \quad f_1 \neq 0$	g_0	$X_3 = t\partial_x + \frac{1}{f_1}\partial_u$ $X_4 = 4t\partial_t + \left(x - \frac{7}{2} g_0 f_1 t^2\right)\partial_x - \left(3u + 7g_0 t + 3\frac{f_0}{f_1}\right)\partial_u$
0	$g_1 u$	$X_3 = 4t\partial_t + x\partial_x - 4g_1 t u\partial_u$, $X_4 = u\partial_u$, $X_\phi = \phi(u)\partial_u$ with $\phi(u)$ solution of the eq. $\phi_t + A\phi_{xxxx} + g_1\phi = 0$

3. Multipliers and Conservation Laws

In this section, we look for local conservation laws of class (1) and provide a brief overview of key related concepts (see, e.g., [29] and references therein).

Taking into account that the independent variables are the time t and spatial variable x, a conservation law [34] for an equation $\mathcal{F}$ belonging to class (1) is a divergence expression of a vector $T \equiv (T_1, T_2)$

$$div\ T \equiv D_t(T_1) + D_x(T_2) = 0,$$

which holds true for all solutions of the equation $\mathcal{F}$. Here, T is called a conserved vector associated with this conservation law.

A conserved vector T is said to be trivial if its divergence vanishes identically. Otherwise, it is called nontrivial. If two conserved vectors differ by a trivial conserved vector, they are called equivalent.

From now on, we call a conservation law of the equation $\mathcal{F}$ an equivalence class of conserved vectors of $\mathcal{F}$.

The order of a conserved vector T is the maximal order of derivatives that explicitly appear in T. The order of a conservation law is the minimum of the order of conserved vectors T over all conserved vectors belonging to the same class.

A multiplier is a function of the independent variables t and x along with the dependent variable u and its derivatives up to a certain order, which multiplies the equation $\mathcal{F}$ to transform it into a conservation law.

It can be observed that each equation of class (1) can be expressed in Cauchy–Kovalevskaya form with respect to the independent variable t, implying [30] that all of its nontrivial (up to equivalence) local conservation laws arise from multipliers; moreover, for each multiplier there exists a nontrivial (up to equivalence) local conservation law such that

$$D_t(T_1) + D_x(T_2) = (u_t + Au_{xxxx} + f(u)u_x + g(u))Q. \tag{88}$$

3.1. Multipliers

In order to find conservation laws, we begin by calculating the multipliers, which are non-zero functions Q that identically satisfy the condition

$$\frac{\delta}{\delta u}[(u_t + Au_{xxxx} + f(u)u_x + g(u))Q] = 0, \tag{89}$$

where $\frac{\delta}{\delta u}$ represents the Euler operator, which in this case is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\left(\frac{\partial}{\partial u_t}\right) - D_x\left(\frac{\partial}{\partial u_x}\right) + D_x^2\left(\frac{\partial}{\partial u_{xx}}\right) - D_x^3\left(\frac{\partial}{\partial u_{xxx}}\right) + D_x^4\left(\frac{\partial}{\partial u_{xxxx}}\right). \tag{90}$$

Moreover, (1) consists of even-order evolution equations, and we only need to consider multipliers Q that depend on t, x, u, and derivatives of u with respect to x of an order not greater than the order of equation [33].

We can apply (89) to look for multipliers:

$$Q = Q(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}). \tag{91}$$

Then, condition (89) becomes

$$\begin{aligned}(f_u u_x + g_u)Q - D_t(Q) - D_x[Q_{u_x}(u_t + Au_{xxxx} + fu_x + g) + fQ] + \\ + D_x^2[Q_{u_{xx}}(u_t + Au_{xxxx} + fu_x + g)] - D_x^3[Q_{u_{xxx}}(u_t + Au_{xxxx} + fu_x + g)] + \\ + D_x^4[Q_{u_{xxxx}}(u_t + Au_{xxxx} + fu_x + g) + AQ] = 0.\end{aligned} \tag{92}$$

This condition splits with respect to any derivatives of u that do not appear in Q. From the coefficient of the eighth-order derivative of u with respect to x, we obtain

$$2AQ_{u_{xxxx}} = 0, \tag{93}$$

that is,
$$Q = Q(t, x, u, u_x, u_{xx}, u_{xxx}). \tag{94}$$
Substituting (94) into (92), we obtain
$$(f_u u_x + g_u)Q - D_t(Q) - D_x[Q_{u_x}(u_t + Au_{xxxx} + fu_x + g) + fQ] + \\ + D_x^2[Q_{u_{xx}}(u_t + Au_{xxxx} + fu_x + g)] - D_x^3[Q_{u_{xxx}}(u_t + Au_{xxxx} + fu_x + g)] + D_x^4(AQ) = 0. \tag{95}$$
From the coefficient of u_{txxx}, we obtain
$$-2Q_{u_{xxx}} = 0, \tag{96}$$
that is,
$$Q = Q(t, x, u, u_x, u_{xx}). \tag{97}$$
Substituting (97) into (95), we have
$$(f_u u_x + g_u)Q - D_t(Q) - D_x[Q_{u_x}(u_t + Au_{xxxx} + fu_x + g) + fQ] + \\ + D_x^2[Q_{u_{xx}}(u_t + Au_{xxxx} + fu_x + g)] + D_x^4(AQ) = 0. \tag{98}$$
From the coefficient of the sixth-order derivative of u with respect to x, we obtain
$$2AQ_{u_{xx}} = 0, \tag{99}$$
that is,
$$Q = Q(t, x, u, u_x). \tag{100}$$
Substituting (100) into (98), we obtain
$$(f_u u_x + g_u)Q - D_t(Q) - D_x[Q_{u_x}(u_t + Au_{xxxx} + fu_x + g) + fQ] + D_x^4(AQ) = 0. \tag{101}$$
From the coefficient of u_{tx}, we obtain
$$-2Q_{u_x} = 0, \tag{102}$$
that is,
$$Q = Q(t, x, u). \tag{103}$$
Substituting (103) into (101), we have
$$(f_u u_x + g_u)Q - D_t(Q) - D_x(fQ) + D_x^4(AQ) = 0. \tag{104}$$
From the coefficient of u_{xxxx}, we obtain
$$2AQ_u = 0, \tag{105}$$
that is,
$$Q = Q(t, x). \tag{106}$$
Finally, substituting (106) into (104), we obtain
$$g_u Q - Q_t - fQ_x + AQ_{xxxx} = 0. \tag{107}$$
The solution $Q(t, x)$ depends on the functions f and g. We find solutions different from zero only in the following cases:

1. If $f(u) = f_0$ and $g(u) = g_0 + g_1 u$, then any solution of the linear equation
$$g_1 Q - Q_t - f_0 Q_x + A Q_{xxxx} = 0 \tag{108}$$
will be a multiplier (without loss of generality, we can take $g_0 = 0$ and $f_0 = 0$).
2. If $f_u \neq 0$ and $g(u) = g_0 + g_1 u$ (without loss of generality, we can take $g_0 = 0$), then from (107) we obtain
$$Q(t,x) = e^{g_1 t}. \tag{109}$$
3. If $f_u \neq 0$ and $g_u = g_0 + g_1 f$ with $g_1 \neq 0$, then from (107) we obtain
$$Q(t,x) = e^{(g_0 + A g_1^4)t + g_1 x}. \tag{110}$$

3.2. Conservation Laws

In this subsection, we find conservation laws for equations of class (1) that admit multipliers. For each multiplier admitted by the special Equation (1), a corresponding conserved vector can be derived by integrating the divergence condition (88). We reach the following results:

1. Considering $f(u) = f_0$ and $g(u) = g_1 u$, we obtain the multiplier $Q(t,x)$ as a solution of Equation (108). In this case, Equation (1) is linear and can be written as follows:
$$u_t + A u_{xxxx} + f_0 u_x + g_1 u = 0. \tag{111}$$
As a result, we obtain the following conserved vector:
$$T_1 = Q u, \tag{112}$$
$$X_1 = A(Q u_{xxx} - Q_x u_{xx} + Q_{xx} u_x - Q_{xxx} u) - Q f_0 u, \tag{113}$$
where the function Q satisfies condition (108).
2. If we consider $f_u \neq 0$ and $g(u) = g_1 u$, Equation (1) becomes
$$u_t + A u_{xxxx} + f(u) u_x + g_1 u = 0. \tag{114}$$
It admits the multiplier $Q = e^{g_1 t}$, and by substituting this into (88) we are able to find the following conserved vector:
$$T_2 = e^{g_1 t} u, \tag{115}$$
$$X_2 = e^{g_1 t}(A u_{xxx} + F(u)), \tag{116}$$
where the function $F(u)$ satisfies the condition
$$F_u(u) = f(u). \tag{117}$$
3. Considering $f_u \neq 0$ and $g_u = g_1 f + g_0$ with $g_1 \neq 0$, we can write Equation (1) in the following form:
$$u_t + A u_{xxxx} + \frac{g_u - g_0}{g_1} u_x + g(u) = 0. \tag{118}$$
We obtain the multiplier $Q = e^{(g_0 + A g_1^4)t + g_1 x}$, and by substituting it into (88) we obtain the following conserved vector:
$$T_3 = e^{(g_0 + A g_1^4)t + g_1 x} u, \tag{119}$$
$$X_3 = e^{(g_0 + A g_1^4)t + g_1 x} \left(A u_{xxx} - g_1 A u_{xx} + g_1^2 A u_x + \frac{1}{g_1} g - \frac{A g_1^4 + g_0}{g_1} u \right). \tag{120}$$

It is straightforward to verify that Equation (81) falls into this case. In fact, the functions f and g take forms (3) and (4), respectively, satisfying the condition

$$g_u = g_1 f + g_0. \tag{121}$$

We can now summarize our results about conservation laws in the following theorem.

Theorem 3. *The equations of class (1) that admit a conservative form (up to equivalence) are (111), (114), and (118). The corresponding conservative forms are*

$$D_t(Qu) + D_x(A(Qu_{xxx} - Q_x u_{xx} + Q_{xx}u_x - Q_{xxx}u) - Qf_0 u) = 0 \tag{122}$$

$$D_t(e^{g_1 t}u) + D_x(e^{g_1 t}(Au_{xxx} + F(u))) = 0 \tag{123}$$

$$D_t\left(e^H u\right) + D_x\left(e^H\left(Au_{xxx} - g_1 Au_{xx} + g_1^2 Au_x + \frac{1}{g_1}g - \frac{Ag_1^4 + g_0}{g_1}u\right)\right) = 0, \tag{124}$$

where the function $Q(t,x)$ satisfies condition (108) and $H(t,x) = (g_0 + Ag_1^4)t + g_1 x$.

4. Applications

In this section, we show examples od how these results can be applied. We obtain the first integral of the conserved vectors under the time–space group invariant (see, e.g., [38] and references therein) and exact invariant solutions.

4.1. First Integrals

For arbitrary forms of the functions f and g, we find that Equation (1) is invariant under translations in both time and space. The invariance of (1) under a combination of generators (19), specifically,

$$X = \lambda\partial_x + \partial_t, \tag{125}$$

provides the invariant solution

$$u(t,x) = v(\sigma), \tag{126}$$

where $\sigma = x - \lambda t$ and λ is a constant. Substituting a traveling wave solution (126) into Equation (1), we obtain the reduced fourth-order ODE:

$$-\lambda v' + Av^{iv} + f(v)v' + g(v) = 0. \tag{127}$$

We can find a first integral of this equation if we have a conserved vector of Equation (1) that is invariant with respect to (125).

In fact, if we have a conservation law

$$D_t T + D_x X = 0 \tag{128}$$

with the conserved vector (T, X) being invariant with respect to (125), then we can write it in terms of the transformed variables v and σ as follows:

$$D_t T + D_x X \equiv -\lambda D_\sigma \bar{T} + D_\sigma \bar{X} = 0, \tag{129}$$

where $\bar{T}$ and $\bar{X}$ are T and X, respectively, as written in the new variables v and σ. The conservation law (129) provides the first integral

$$-\lambda\bar{T} + \bar{X} = K. \tag{130}$$

In the previous section, if functions f and g are linked by the condition $g_u = g_1 f + g_0$ with $g_1 \neq 0$ and $f_u \neq 0$, then we obtain the multiplier (110) and the conserved vectors (119) and (120), which are invariant with respect to (125) if we choose

$$\lambda = -\frac{g_0 + A g_1^4}{g_1}. \tag{131}$$

Then, if we reduce equation

$$u_t + A u_{xxxx} + \frac{g_u - g_0}{g_1} u_x + g(u) = 0 \tag{132}$$

by using the traveling wave solutions (126) with λ as provided by (131), we obtain the reduced fourth-order ODE:

$$-\lambda v' + A v^{iv} + \frac{g_v - g_0}{g_1} v' + g(v) = 0. \tag{133}$$

By writing T_3 and X_3 in terms of the new variables $v(\sigma)$ and $\sigma = x + \dfrac{g_0 + A g_1^4}{g_1} t$, we obtain

$$\bar{T}_3 = e^{g_1 \sigma} v, \tag{134}$$

$$\bar{X}_3 = e^{g_1 \sigma} \left(A v''' - g_1 A v'' + g_1^2 A v' + \frac{1}{g_1} g - \frac{A g_1^4 + g_0}{g_1} v \right). \tag{135}$$

Thus, a first integral (130) of Equation (133) is

$$e^{g_1 \sigma} \left(A v''' - g_1 A v'' + g_1^2 A v' + \frac{1}{g_1} g \right) = K. \tag{136}$$

4.2. Invariant Solutions

By solving (127), we can obtain traveling wave solutions. However, we may find other invariant solutions using different symmetry generators as well, if they exist. In the following, we provide several examples.

1. For Equation (81), using generator (78), we obtain

$$u(t,x) = v(t) e^{-\frac{g_1}{f_1} x}, \tag{137}$$

where $v(t)$ is a solution of the following ODE:

$$v' + \left(\frac{g_1^4}{f_1^4} A + g_0 \right) v = 0, \tag{138}$$

which can be solved as follows:

$$v(t) = v_0 e^{g_0 + \frac{g_1^4}{f_1^4} A t}. \tag{139}$$

2. For Equation (81), in the case where $g_1 = 0$, when using generator (79) we obtain

$$u(t,x) = v(t) e^{\frac{x}{f_1 t}}, \tag{140}$$

where $v(t)$ is a solution of the following ODE:

$$v' + \left(\frac{A}{f_1^4 t^4} + \frac{\ln v}{t} + g_0 \right) v = 0, \tag{141}$$

that is,

$$v(t) = e^{\left(\frac{A}{2t^3 f_1^4} - \frac{t g_0}{2} + \frac{v_0}{t} \right)}. \tag{142}$$

3. For Equation (66), using generator (65) we obtain

$$u(t,x) = v(t) + \frac{x}{f_1 t}, \tag{143}$$

where $v(t)$ is a solution of the following ODE:

$$v' + \frac{v}{t} = 0, \tag{144}$$

that is,

$$v(t) = \frac{v_0}{t}. \tag{145}$$

5. Conclusions

In this paper, we have used group techniques to study a class of type (1) heterogeneous RDAEs. We have derived extensions of the principal Lie algebra concerned with several couples of the function $f(u)$ and $g(u)$ that describe class (1). This offers a large number of possibilities for further applications in the studies of real phenomena modeled by RDAEs.

We stress that the previous results derived from our analysis of the theoretical structure of class (1) could allow special sets of constitutive parameters $f(u)$ and $g(u)$ to be identified that could represent a better fit in simulations of real phenomena.

Moreover, it is a simple matter to ascertain that it is possible to characterize forms of the reaction term $g(u)$ such that they are of the generalized logistic function type. In addition to its theoretical interest, the search for symmetries offers a powerful tool to find solutions through reduction techniques.

The multipliers method has been applied to search for conservation laws. We observed that equations of type (1) are of even order and admit a Cauchy–Kovalevskaya form. Consequently, all nontrivial (up to equivalence) of local conservation laws arise from multipliers depending on t, x, u, and derivatives of u with respect to x of order not greater than four. Therefore, we have identified all the equations in the considered class that can be in a conservative form, and have additionally written the concerned conserved vectors. As is well known, conservative forms allow numerical techniques to be applied so in order to find additional symmetries.

Finally, it is of interest to determine whether it is possible to obtain additional results about the structure of equations using both conservation laws and symmetries. In fact, in Section 4 we have illustrated how to obtain both first integral and exact invariant solutions.

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