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Exact Solutions to Some Nonlinear Time-Fractional Evolution Equations Using the Generalized Kudryashov Method in Mathematical Physics

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Abstract: In this study, we utilize the potent generalized Kudryashov method to address the intricate obstacles presented by fractional differential equations in the field of mathematical physics. Specifically, our focus centers on obtaining novel exact solutions for three pivotal equations: the time-fractional seventh-order Sawada-Kotera-Ito equation, the time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera equation, and the time-fractional seventh-order Kaup–Kupershmidt equation. The generalized Kudryashov method, celebrated for its versatility and efficacy in addressing intricate nonlinear problems, plays a central role in our research. This method not only simplifies the equations but also unveils their inner dynamics, rendering them amenable to meticulous analysis. It is worth noting that our fractional derivatives are defined in the context of the conformable fractional derivative, providing a solid foundation for our mathematical investigations. One notable aspect of our study is the visual representation of our findings. Graphical representations of the yielded solutions enliven intricate mathematical structures, providing a concrete insight into the dynamics and behaviors of said equations. This paper highlights the proficiency of the generalized Kudryashov method in resolving complex issues presented by fractional differential equations. Our study not only broadens the range of mathematical methods but also enhances our comprehension of the intriguing realm of nonlinear physical phenomena.

Keywords: conformable fractional derivative; time-fractional seventh-order Sawada-Kotera-Ito equation; time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera; time-fractional seventh-order Kaup-Kupershmidt equation

1. Introduction

In recent decades, fractional differential equations (FDEs) have garnered significant attention among scientists. FDEs are more versatile than classical differential equations, since they represent a generalization of integer-order differential equations. These equations go beyond traditional derivative equations, allowing for the examination of more complex real-world problems. One of the various application areas of these equations is symmetry. Symmetry refers to the condition where the properties of an object or system remain unchanged. Symmetry analysis holds a substantial role in many academic disciplines, notably within the realms of physics, chemistry, and engineering. These disciplines collectively rely on symmetry considerations to elucidate various phenomena. FDEs are instrumental in a multitude of academic fields, spanning disciplines such as acoustics, control theory, viscoelasticity, electrochemistry, fluid dynamics, rheology, and system identification, among others. Furthermore, nonlinear FDEs offer exact solutions that describe a wide spectrum of intricate nonlinear physical phenomena, enhancing our understanding of complex systems and their behaviors. When fractional differential equations are coupled with symmetry analysis, they enable a better understanding of complex systems and more precise predictions of their behaviors. Hence, fractional differential equations and symmetry form a significant area of research and application in the scientific world. The exact solutions to

nonlinear FDEs describe a wide range of nonlinear physical phenomena. These phenomena intricately depend on both temporal and spatial variables. In order to achieve a comprehensive understanding of the physical implications of FDEs, it is essential to develop effective and robust methods for their solution. Many scientists have explored exact solutions for nonlinear fractional differential equations using various methodologies. Some of these methods include the Adomian decomposition method [1,2], the differential transformation method [3,4], the finite difference method [5], the homogeneous balance method [6], the (G'/G)-expansion method [7–9], the trial function method [10], Jacobi elliptic function expansion [11], the sub-ODE method [12,13], the homotopy analysis method [14], the tanh-function expansion method [15], the sinc-collocation method [16], the exponential function method [17,18], the fractional sub-equation method [19], the generalized Kudryashov method [20], etc.

Within the scope of this research, we undertook a comprehensive analysis encompassing three distinct time-fractional differential equations: specifically, the time-fractional (2+1)-Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation, the seventh-order Sawada–Kotera–Ito (SKI) equation, and the seventh-order Kaup–Kupershmidt (KK) equation. Notably, the CDGSK equation holds a significant position as a constituent within the B-type Kadomtsev–Petviashvili (BKP) integrable hierarchy, further underscoring its importance in the context of our study [21]. This research endeavor aims to elucidate the intricate dynamics and behaviors inherent in these time-fractional differential equations, contributing to a deeper understanding of their mathematical properties and potential applications in diverse scientific domains [22]. In recent years, a significant cohort of researchers have turned their focus towards the investigation of the CDGSK equation, employing a myriad of approaches for its solution. One notable example is the work by Geng et al. [23], wherein they demonstrate the invariance of the transformation of the independent variables pertaining to the CDGSK equation through the strategic application of the Riccati equation. Furthermore, they successfully derive the CDGSK equation by employing the Darboux transformation on the (2+1)-dimensional CDGSK equation. These research endeavors collectively contribute to the expanding body of knowledge surrounding the CDGSK equation, shedding light on its mathematical properties and solution methodologies.

The SKI equation serves as a pivotal model utilized to elucidate the dynamics of long waves in the presence of gravitational effects in shallow water, as well as within the realm of nonlinear optical modeling. To further expound upon this, Koonprasert et al. [24] made notable contributions by successfully solving the seventh-order fractional Sawada–Kotera equation, employing a combination of the Riccati equation and the mapping method. Additionally, Naher et al. [25] significantly advanced our understanding by deriving traveling wave solutions for the seventh-order Sawada–Kotera equation, applying the exp-function method. These collective research endeavors have played a crucial role in enhancing our comprehension of the mathematical properties and practical applications of the seventh-order time-fractional SKI equation [26–28].

The time-fractional CDGSK equation is given by

$$36D_t^\alpha u + u_{5x} + 15(uu_{xx})_x + 45u_x u^2 - 5u_{xxy} - 15uu_y - 15u_x \partial_x^{-1} u_y - 5\partial_x^{-1} u_{yy} = 0, \quad (1)$$

where α is the fractional derivative in the sense of conformable derivative.

We can give a seventh-order time-fractional KdV equation as follows [27,29]:

$$D_t^\alpha u + 252\, u^3 u_x + 63\, u_x^3 + 378\, u_x u_{xx} + 126\, u^2 u_{3x} + 63\, u_{2x} u_{3x} \\ + 42\, u_x u_{4x} + 21\, u\, u_{5x} + u_{7x} = 0, \quad (2)$$

where α denotes the fractional derivative and is taken into account in the sense of conformable fractional derivative. Here, we exploit the generalized Kudryashov method for finding the exact traveling wave solution of Equations (1) and (2).

This study can be succinctly summarized as follows: In Section 2, we provide a concise exposition of the conformable fractional derivative and its associated properties. Section 3

is dedicated to elucidating the generalized Kudryashov method, a pivotal technique employed for tackling fractional partial differential equations, which constitutes the central focus of this paper. In Section 4, we leverage the power of the generalized Kudryashov method to obtain precise solutions for select fractional partial differential equations. Finally, we present a comprehensive discussion of our findings and offer insights into potential avenues for future research. In particular, we include graphical representations in the continuation of the solutions, showing the solutions to the final problem set, which also increases the depth of our analysis and adds a visual dimension to our contributions.

2. Brief Idea of the Conformable Fractional Derivative

Here, we provide a succinct introduction to the conformable fractional derivative and highlight some of its key properties.

Definition 1. *Let $\alpha \in (0,1]$ and $\omega : \mathbb{R}^+ \cup \{0\} \to R$ be given. The conformable fractional derivative of ω of order α is defined as follows:*

$$(T_\alpha \omega)(t) = \lim_{\varepsilon \to 0} \frac{\omega(t + \varepsilon t^{1-\alpha}) - \omega(t)}{\varepsilon} \quad (t > 0).$$

Theorem 1. *Let $\alpha \in (0,1]$, $t > 0$ and ω, ψ be α-differentiable. Then, we can write the following properties:*

- $T_\alpha(k\omega + s\psi) = k(T_\alpha \omega) + s(T_\alpha \psi)$, *for all $k, s \in \mathbb{R}$;*
- $T_\alpha(t^m) = mt^{m-\alpha}$ *for all $m \in \mathbb{R}$;*
- $T_\alpha(\lambda) = 0$, *for all constant functions $\omega(t) = \lambda$;*
- $T_\alpha(\omega\psi) = \omega(T_\alpha \psi) + \psi(T_\alpha \omega)$;
- $T_\alpha\left(\frac{\omega}{\psi}\right) = \frac{\psi(T_\alpha \omega) - \omega(T_\alpha \psi)}{\psi^2}$;
- *If, in addition, ω is differentiable, then $(T_\alpha \omega)(t) = t^{1-\alpha} \frac{d\omega}{dt}$.*

The derivative of order α for a constant is zero [30].

3. The Generalized Kudryashov Method

The generalized Kudryashov method represents a powerful approach in the realm of mathematical physics and is particularly renowned for its applicability in solving nonlinear fractional differential equations. One of its prominent advantages lies in its effectiveness in obtaining analytical solutions to complex and nonlinear equations. What sets it apart is its versatility and adaptability, making it a valuable tool for researchers seeking to obtain analytical solutions, even in cases involving fractional derivative equations.

In this study, we present the generalized Kudryashov method, which establishes stable and explicit soliton solutions to FDEs and considers a general form for nonlinear evolution equations as follows:

$$P(u, D_t^\alpha u, u_x, u_y, D_t^{2\alpha} u, D_t^\alpha u_x, D_t^\alpha u_y, \dots) = 0, \tag{3}$$

where α denotes the conformable fractional derivative and P is a polynomial in a $u(x, y, t)$ unknown function, with its various derivatives. The generalized Kudryashov method is designed to generate characteristic and broad-spectral soliton solutions for nonlinear FDEs with respect to time variables [31]. The generalized Kudryashov method can be summarized in the following steps:

Step 1: We introduce a new wave variable, ξ, and apply the following transformation:

$$u(x, y, t) = q(\xi), \qquad \xi = x + y + \frac{k}{\Gamma(1 + \alpha)} t^\alpha, \tag{4}$$

where k is a parameter. The transformation in Equation (4) reduces Equation (3) into the following nonlinear ordinary differential equation (NODE):

$$F\left(q, kq', \ q', \ q', \ k^2q'', k(q')^2, k(q')^2, \ \ldots \right) = 0, \tag{5}$$

where F is a polynomial of q, with its derivatives in terms of ξ. Equation (5) is integrated one or more times, and the integral constants are set to zero.

Step 2: We assume the following expression as the solution of Equation (5):

$$q(\xi) = \frac{a_0 + \sum\limits_{i=1}^{m} a_i U^i(\xi)}{b_0 + \sum\limits_{j=1}^{n} b_j U^j(\xi)}, \tag{6}$$

where $a_i(i = 0, 1, 2, 3, \cdots, m)$ and $b_j(j = 0, 1, 2, 3, \cdots, n)$ are the constants to be determined later, with $a_m \neq 0$ and $b_n \neq 0$, and

$$U(\xi) = \frac{1}{1 + \lambda \exp(\xi)},$$

is the general solution of the following Riccati equation:

$$U'(\xi) = U^2(\xi) - U(\xi), \tag{7}$$

where λ is the integration constant of the solution and the prime denotes the ordinary derivative in terms of ξ.

Step 3: The values of m and n are to be determined using homogeneous balancing, which involves considering the terms with the highest-order derivatives and the highest-order nonlinear term in Equation (5). Substituting the expression in Equation (6) into Equation (5) along with Equation (7) and setting each coefficient, including the powers of $U(\xi)$, to zero yield a system of algebraic equations.

Step 4: When we solve these algebraic equations using mathematical software programs such as Maple, we determine the values of the unknown constants $a_i(i = 0, 1, \ldots, n)$, $b_j(j = 0, 1, \ldots, m), k$, and λ. Subsequently, by substituting the values of a_i, b_j, and λ into Equation (6), we effectively finalize the solution for the nonlinear evolution equation represented in Equation (5).

4. Application of the Method

This research endeavor encompasses the application of the generalized Kudryashov method to tackle the solution of three notable equations. Specifically, we focus on solving the (2+1) conformable time-fractional CDGSK equation, the seventh-order conformable time-fractional SKI equation, and the seventh-order conformable time-fractional KK equation. By applying this method to these intricate equations, we aim to unveil insightful solutions and gain a deeper understanding of the underlying dynamics and behaviors described by these conformable time-fractional differential equations.

4.1. The Time-Fractional Caudrey–Dodd–Gibbon–Sawada–Kotera Equation

We investigate the precise traveling wave solutions to the (2+1) conformable time-fractional CDGSK equation employing the generalized Kudryashov method. The CDGSK equation, originally discovered independently by Sawada and Kotera [32], as well as by Caudrey, Dodd, and Gibbon [33,34], constitutes the equation of interest in this work. The CDGSK equation is

$$36D_t^\alpha u + u_{5x} + 15(uu_{xx})_x + 45u_x u^2 - 5u_{xxy} - 15uu_y - 15u_x \partial_x^{-1} u_y - 5\partial_x^{-1} u_{yy} = 0, \tag{8}$$

where α is fractional derivative for interval of $[0, 1]$, $u(x, y, t)$ is a differentiable function, and ∂_x^{-1} shows the integration in terms of x.

Equation (8) stands as one of the most pivotal integrable equations within the realm of nonlinear dynamics. This equation finds its significance in describing a broad spectrum of nonlinear dispersive physical phenomena and holds many applications in the field of nonlinear sciences. For instance, it plays a critical role in modeling the conservative flow of the Liouville equation, the two-dimensional gauge field theory of quantum gravity, and the theory of conformal field, among others.

When $u_y = 0$, Equation (8) reduces it to the following time-fractional SK equation [32]:

$$36 D_t^\alpha u + u_{5x} + 15(u u_{xx})_x + 45 u_x u^2 = 0. \tag{9}$$

Now, we apply the generalized Kudryashov method to Equation (8). Substituting Equation (4) into Equation (8) reduces it to the nonlinear ODE

$$(36k - 5)q + q^{(4)} + 15 q q'' + 15 q^3 - 5 q'' - 15 q^2 = 0, \tag{10}$$

where $q' = \frac{dq}{d\xi}$. Using the homogeneous balance method, that is, balancing the $q^{(4)}$ term and the q^3 term in Equation (10), we find $n = 3$, $m = 1$. Hence, from Equation (6), we have

$$q(\xi) = \frac{a_0 + a_1 U(\xi) + a_2 U^2(\xi) + a_3 U^3(\xi)}{b_0 + b_1 U(\xi)}. \tag{11}$$

Next, we substitute Equation (11) into Equation (10) and organize all the terms so that all coefficients of $U^i(\xi)$ $(i = 0, 1, \ldots, 11)$ to zero acquire a set of equations. By solving these equations with the help of a mathematical software program, we deduce a set of solutions for $k, b_0, b_1, a_i (i = 0, 1, 2, 3)$.

Case 1:

$$a_0 = \mp \frac{\sqrt{30}}{15} b_0, \ a_1 = 2b_0, \ a_2 = -2b_0, \ a_3 = 0, \ b_1 = 0, \ k = \frac{\mp\sqrt{30} + 3}{36}.$$

By substituting these values into Equation (11), we obtain the solution to Equation (10):

$$q(\xi) = \frac{2\lambda[\sinh(\xi) + \cosh(\xi)]}{[1 + \lambda \sinh(\xi) + \lambda \cosh(\xi)]^2} \mp \frac{\sqrt{30}}{15}, \tag{12}$$

where $\xi = x + y + \left(\frac{\mp\sqrt{30} + 3}{36} \right) \frac{t^\alpha}{\Gamma(1+\alpha)}$. If we substitute ξ in Equation (12),

$$u(x, y, t) = \frac{2\lambda e^{x + y + \left(\frac{\sqrt{30} + 3}{36} \right) \frac{t^\alpha}{\Gamma(1+\alpha)}}}{\left(1 + \lambda e^{x + y + \left(\frac{\sqrt{30} + 3}{36} \right) \frac{t^\alpha}{\Gamma(1+\alpha)}} \right)^2} + \frac{\sqrt{30}}{15}. \tag{13}$$

The graphical representation of the Equation (13) is as shown in Figure 1.

Case 2:

$$a_0 = 0, \ a_1 = 4b_0, a_2 = -4b_0 + 4b_1, \ a_3 = -4b_1, \ k = \frac{1}{4}.$$

By substituting these values into Equation (11), we obtain the solution to Equation (10):

$$q(\xi) = \frac{4\lambda[\sinh(\xi) + \cosh(\xi)]}{[1 + \lambda \sinh(\xi) + \lambda \cosh(\xi)]^2}, \tag{14}$$

where $\xi = x + y + \frac{t^\alpha}{4\,\Gamma(1+\alpha)}$. The graphical representation of Equation (14) is similar to that in Figure 1.
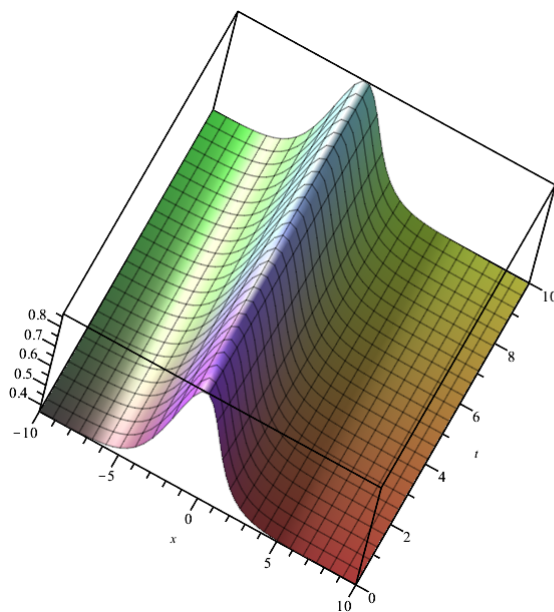
Figure 1. Three-dimensional plots of time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera Equation (13) with $\alpha = 0.5$ for $\lambda = 1$.

Case 3:

$$a_0 = -\frac{2b_0}{15} \frac{\left[\left(\mp 2\sqrt{30} - 15\right)a_1 + \left(30 \pm \sqrt{30}\right)b_0\right]}{\left[\left(\mp\sqrt{30} - 4\right)a_1 + \left(2 \pm 2\sqrt{30}\right)b_0\right]},$$

$$a_2 = \mp\sqrt{30}a_1 - \left(\mp 2\sqrt{30} + 2\right)b_0, \ a_3 = \mp\sqrt{30}(2b_0 - a_1),$$

$$b_1 = \mp\frac{\sqrt{30}}{2}(a_1 - 2b_0),$$

$$k = -\frac{1}{36} \frac{\left[\left(\mp 11\sqrt{30} - 51\right)a_1^2 + \left(\mp 50\sqrt{30} + 66\right)b_0^2 + \left(96 \pm 38\sqrt{30}\right)a_1 b_0\right]}{\left[\left(\mp 4\sqrt{30} - 23\right)a_1^2 + \left(\mp 4\sqrt{30} - 62\right)b_0^2 + \left(68 \pm 10\sqrt{30}\right)a_1 b_0\right]}.$$

By substituting these values into Equation (11), we obtain the solution to Equation (10):

$$q(\xi) = \frac{-\frac{2b_0}{15} \frac{\left[\left(\mp 2\sqrt{30} - 15\right)a_1 + \left(30 \pm \sqrt{30}\right)b_0\right]}{\left[\left(\mp\sqrt{30} - 4\right)a_1 + \left(2 \pm 2\sqrt{30}\right)b_0\right]} \mp \sqrt{30}a_1 U^2(\xi)}{b_0 \mp \frac{\sqrt{30}}{2}(a_1 - 2b_0)U(\xi)}$$

$$- \frac{\left(\mp 2\sqrt{30} + 2\right)b_0 U^2(\xi) \mp \sqrt{30}(2b_0 - a_1)U^3(\xi)}{b_0 \mp \frac{\sqrt{30}}{2}(a_1 - 2b_0)U(\xi)}, \tag{15}$$

where $U(\xi) = \frac{1}{1 + \lambda \exp(\xi)}$ and $\xi = x + y + \frac{kt^\alpha}{\Gamma(1+\alpha)}$. The graphical representation of the Equation (15) is as shown in Figure 2.
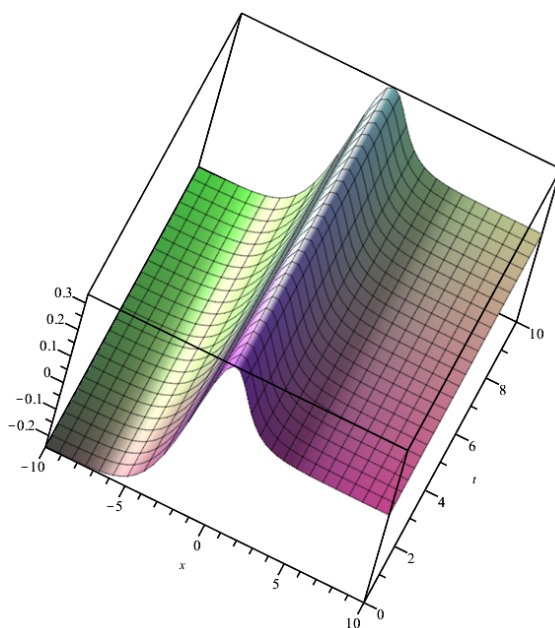
Figure 2. Three-dimensional plots of time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera Equation (15) with $\alpha = 0.5$ for $\lambda = 1$, $a_1 = 1$, $b_0 = 1$.

Case 4:

$$
a_0 = \frac{\left(\mp\sqrt{30} - 4\right)b_1}{45}, a_1 = \frac{2}{3}b_1 \pm \frac{\sqrt{30}}{45}b_1, a_2 = \frac{4}{3}b_1 \mp \frac{4\sqrt{30}}{45}b_1, \ a_3 = -2b_1,
$$

$$
b_0 = \frac{1}{3}b_1 \pm \frac{2\sqrt{30}}{45}b_1, \ k = \frac{1}{36}\left(\frac{51 \pm 11\sqrt{30}}{23 \pm 4\sqrt{30}}\right).
$$

By substituting these values into Equation (11), we obtain the solution to Equation (10):

$$
q(\xi) = \frac{\frac{\left(\mp\sqrt{30}-4\right)}{45} + \left(\frac{2}{3} \pm \frac{\sqrt{30}}{45}\right)U(\xi) + \left(\frac{4}{3} \mp \frac{4\sqrt{30}}{45}\right)U^2(\xi) - 2U^3(\xi)}{\left(\frac{1}{3} \pm \frac{2\sqrt{30}}{45}\right) + U(\xi)}, \tag{16}
$$

where $U(\xi) = \frac{1}{1 + \lambda \exp(\xi)}$ and $\xi = x + y + \frac{1}{36}\left(\frac{51 \pm 11\sqrt{30}}{23 \pm 4\sqrt{30}}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}$. The graphical representation of Equation (16) is similar to that in Figure 2.

4.2. The Time-Fractional Seventh Order Sawada–Kotera–Ito Equation

Pomeau et al. [35] conducted a study on the seventh-order Korteweg–de Vries (KdV) equation to investigate the structural stability of the KdV equation under singular perturbations. The generalization of the seventh-order conformable time-fractional modified KdV equation is as follows [29]:

$$
D_t^\alpha u + a\, u^3 u_x + b\, u_x^3 + c\, u_x u_{xx} + d\, u^2 u_{3x} + e\, u_{xx} u_{3x} + f\, u_x u_{4x} + g\, u\, u_{5x} + u_{7x} = 0. \tag{17}
$$

If we choose special values for the constants in Equation (17), we obtain the seventh-order conformable time-fractional SKI equation

$$
\begin{aligned}
D_t^\alpha u + 252\, u^3 u_x + 63\, u_x^3 + 378\, u_x u_{xx} + 126\, u^2 u_{3x} + 63\, u_{2x} u_{3x} + 42\, u_x u_{4x} \\
+ 21\, u\, u_{5x} + u_{7x} = 0.
\end{aligned} \tag{18}
$$

Now, we apply the generalized Kudryashov method to Equation (18). Suppose that $u(x,t) = q(\xi)$.

$$\xi = x + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}, \tag{19}$$

where k is wave velocity. If we apply Equation (19) to Equation (18), Equation (18) reduces it to the following nonlinear ODE:

$$\begin{aligned} kq' + 252q^3q' + 63(q')^3 + 378q'q'' + 126q^2q''' \\ + 63q''q''' + 42q'q^{(4)} + 21qq^{(5)} + q^{(7)} = 0, \end{aligned} \tag{20}$$

where $q' = \frac{dq}{d\xi}$. Using the homogeneous balance method, that is, balancing the $q^{(7)}$ term and the q^3q' term in Equation (20), we find $n = 3$, $m = 1$. Hence, from Equation (6), we have

$$q(\xi) = \frac{a_0 + a_1 U(\xi) + a_2 U^2(\xi) + a_3 U^3(\xi)}{b_0 + b_1 U(\xi)}. \tag{21}$$

Next, we substitute Equation (21) into Equation (20) and organize all terms so that all coefficients of $U^i(\xi)$ $(i = 0, 1, \ldots, 17)$ to zero acquire a set of equations. Solving these equations with the help of a mathematical software program, we deduce a set of solutions for $k, b_0, b_1, a_i (i = 0, 1, 2, 3)$.

Case 1:

$$a_0 = -\frac{1}{3}b_0, \; a_1 = 4b_0 - \frac{1}{3}b_1, \; a_2 = -4b_0 + 4b_1, \; a_3 = -4b_1, \; b_1 = 0, \; k = \frac{4}{3}.$$

By substituting these values into Equation (21), we obtain the solution to Equation (20):

$$q(\xi) = \frac{-\frac{1}{3}b_0 + \left(4b_0 - \frac{1}{3}b_1\right)U(\xi) + (-4b_0 + 4b_1)U^2(\xi) - 4b_1 U^3(\xi)}{b_0 + b_1 U(\xi)}, \tag{22}$$

where $U(\xi) = \frac{1}{1+\lambda \exp(\xi)}$ and $\xi = x + \left(\frac{4}{3}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$. The graphical representation of the Equation (22) is as shown in Figure 3.
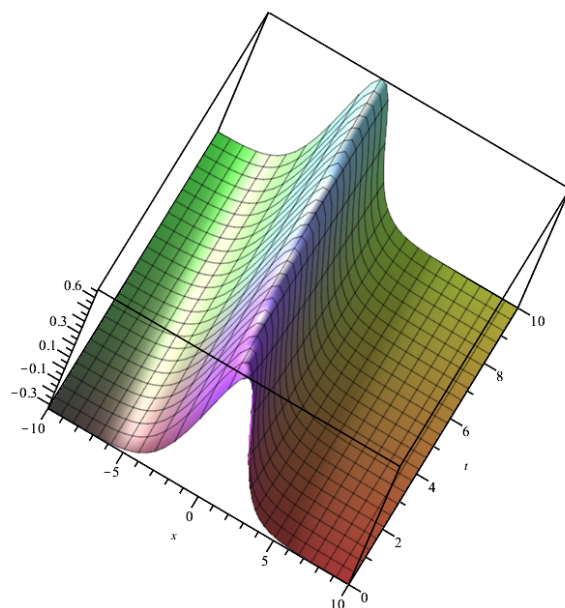

Figure 3. Three-dimensional plots of time-fractional seventh-order Sawada–Kotera–Ito Equation (22) with $\alpha = 0.5$ for $\lambda = 1$, $b_0 = 1$.

Case 2:

$$
\begin{aligned}
a_1 &= 2b_0, \; a_2 = -2b_0, \; a_3 = 0, \; b_1 = 0, \\
k &= -\frac{(b_0^3 + 21a_0 b_0^2 + 126a_0^2 b_0 + 252a_0^3)}{b_0^3}.
\end{aligned}
$$

By substituting these values into Equation (21), we obtain the solution to Equation (20):

$$
q(\xi) = \frac{2\lambda[\sinh(\xi) + \cosh(\xi)]}{[1 + \lambda \sinh(\xi) + \lambda \cosh(\xi)]^2} + \frac{a_0}{b_0}, \tag{23}
$$

where $\xi = x - \left(\frac{b_0^3 + 21a_0 b_0^2 + 126a_0^2 b_0 + 252a_0^3}{b_0^3}\right) \frac{t^\alpha}{\Gamma(1+\alpha)}$. The graphical representation of Equation (23) is similar to that in Figure 3.

Case 3:

$$
\begin{aligned}
a_0 &= -\frac{b_0(2b_0 - a_1)}{b_1}, \; a_2 = -2b_0 + 2b_1, \; a_3 = -2b_1, \\
k &= s\left(\frac{2016b_0^3 + 1512a_1^2 b_0 - 252a_1^3 - 3024a_1 b_0^2}{b_1^3}\right) \\
&\quad + \left(\frac{504a_1 b_0 - 126a_1^2 - 504b_0^2}{b_1^2}\right) - \left(\frac{21a_1 - 42sb_0}{b_1}\right).
\end{aligned}
$$

By substituting these values into Equation (21), we obtain the solution to Equation (20):

$$
q(\xi) = \frac{-\frac{b_0(2b_0 - a_1)}{b_1} + (-2b_0 + 2b_1)U^2(\xi) - 2b_1 U^3(\xi)}{b_0 + b_1 U(\xi)}, \tag{24}
$$

where $U(\xi) = \frac{1}{1 + \lambda \exp(\xi)}$ and $\xi = x + \frac{kt^\alpha}{\Gamma(1+\alpha)}$. The graphical representation of Equation (24) is similar to that in Figure 3.

4.3. The Seventh-Order Time-Fractional Kaup–Kupershmidt Equation

Similarly, if we select distinct special values for the constants in Equation (17), we obtain the conformable time-fractional seventh-order Kaup–Kupershmidt equation

$$
\begin{aligned}
D_t^\alpha u &+ 2016\, u^3 u_x + 630\, u_x^3 + 2268\, u_x u_{xx} + 504\, u^2 u_{3x} \\
&+ 252\, u_{2x} u_{3x} + 147\, u_x u_{4x} + 42\, u\, u_{5x} + u_{7x} = 0.
\end{aligned} \tag{25}
$$

Now, we use the generalized Kudryashov method to solve Equation (25). Substituting Equation (19) into Equation (25) reduces it to following nonlinear ODE:

$$
\begin{aligned}
kq' &+ 2016q^3 q' + 630(q')^3 + 2268q' q'' + 504q^2 q''' \\
&+ 252q'' q''' + 147q' q^{(4)} + 42qq^{(5)} + q^{(7)} = 0,
\end{aligned} \tag{26}
$$

where $q' = \frac{dq}{d\xi}$. Using the homogeneous balance method, that is, balancing the $q^{(7)}$ term and the $q^3 q'$ term in Equation (26), we find $n = 3$, $m = 1$. Hence, from Equation (6), we have

$$
q(\xi) = \frac{a_0 + a_1 U(\xi) + a_2 U^2(\xi) + a_3 U^3(\xi)}{b_0 + b_1 U(\xi)}. \tag{27}
$$

Next, we substitute Equation (27) into Equation (26) and organize all terms so that all coefficients of $U^i(\xi)$ ($i = 0, 1, \ldots, 17$) to zero acquire a set of equations. Solving these

equations with the help of a mathematical software program, we deduce a set of solutions for $k, b_0, b_1, a_i (i = 0, 1, 2, 3)$:

$$a_0 = -\frac{1}{24}b_0, \ a_1 = \frac{1}{2}b_0 - \frac{1}{24}b_1, \ a_2 = -\frac{1}{2}b_0 + \frac{1}{2}b_1, \ a_3 = -\frac{1}{2}b_1, \ k = \frac{1}{48}.$$

We plug these values into Equation (27); hence, the solution to Equation (26) is

$$q(\xi) = \frac{-\frac{1}{24}b_0 + \left(\frac{1}{2}b_0 - \frac{1}{24}b_1\right)U(\xi) + \left(-\frac{1}{2}b_0 + \frac{1}{2}b_1\right)U^2(\xi) - \frac{1}{2}b_1 U^3(\xi)}{b_0 + b_1 U(\xi)}, \qquad (28)$$

where $U(\xi) = \frac{1}{1 + \lambda \exp(\xi)}$ and $\xi = x + \left(\frac{1}{48}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}$. The graphical representation of Equation (28) is similar to that in Figure 4.
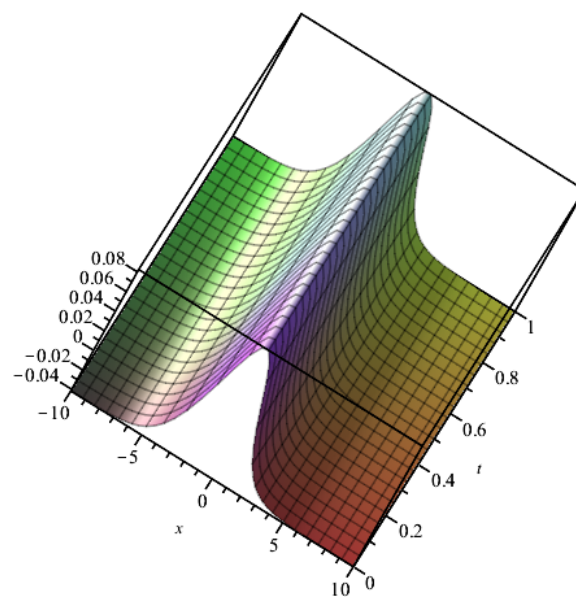

Figure 4. Three-dimensional plots of seventh-order time-fractional Kaup–Kupershmidt Equation (28) with $\lambda = 1$ for $\alpha = 0.5$, $b_0 = 1$, $b_1 = 1$.

5. Conclusions

In the realm of mathematical physics, the pursuit of analytical solutions to nonlinear differential equations represents a central challenge and a milestone in advancing our understanding of complex physical phenomena. Our research endeavors have culminated in a significant accomplishment, as we have successfully derived analytical solutions for three intricate and demanding equations: the seventh-order (2+1) conformable time-fractional CDGSK equation, the seventh-order conformable time-fractional SKI equation, and the seventh-order conformable time-fractional KK equation. Our achievement underscores the effectiveness of analytical methodologies in unveiling the hidden dynamics of nonlinear time-fractional equations, shedding light on their intrinsic behaviors, and offering valuable insights into the underlying mathematical structures. The analytical solutions we present not only expand the boundaries of mathematical physics but also invite further exploration and application within the scientific community. By presenting these findings, we urge our colleagues to embark on a path of continued examination to progress our scientific knowledge. One of the crowning achievements of our study was the prolific discovery of numerous exact solutions, complemented by the unveiling of fresh hyperbolic solutions. These findings, standing as testament to the efficacy of the Kudryashov method, promise to enrich our understanding of nonlinear physical phenomena, casting a bright light on their underlying intricacies.

We navigated the treacherous waters of nonlinear fractional partial differential equations with the aid of the powerful nonlinear fractional transformation, also known as the fractional complex transformation. This approach smoothly translated the complex domain of fractional partial differential equations into more manageable ordinary differential equations with integer orders. Consequently, we were able to express solutions for time-fractional nonlinear evolution equations in the elegant form of polynomials in $U(\xi)$. Our research also provides visual representations of our results through graphs. These graphical depictions illustrate the precise solutions obtained and offer a tangible glimpse into the dynamics of the equations under scrutiny.

In summation, this study underscores the generalized Kudryashov method's role as a potent tool for unraveling the complexities of fractional differential equations. Our findings not only expand the repertoire of mathematical techniques but also deepen our understanding of the world of nonlinear physical phenomena.

Moving forward, we propose the exploration of more intricate fractional differential equations and the application of advanced mathematical methodologies. Our proposed methodology has demonstrated its ease of use, efficiency, and effectiveness, establishing itself as a valuable and potent approach for addressing systems of fractional differential equations. Through our research, we aim to not only expand the toolbox of mathematical methods but also shed light on the intricate dynamics of nonlinear systems.

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