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Regarding the Ideal Convergence of Triple Sequences in Random 2-Normed Spaces

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Abstract: In our ongoing study, we explore the concepts of $\mathcal{I}_3$-Cauchy and $\mathcal{I}_3$-Cauchy for triple sequences in the context of random 2-normed spaces (RTNS). Moreover, we introduce and analyze the notions of $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-limit points, and $\mathcal{I}_3$-cluster points for random 2-normed triple sequences. Significantly, we establish a notable finding that elucidates the connection between $\mathcal{I}_3$-convergence and $\mathcal{I}_3$-convergence within the framework of random 2-normed spaces, highlighting their interrelation. Additionally, we provide an illuminating example that demonstrates how $\mathcal{I}_3$-convergence in a random 2-normed space might not necessarily imply $\mathcal{I}_3$-convergence. Our observations underscore the importance of condition (AP3) when examining summability using ideals. Furthermore, we thoroughly investigate the relationship between the properties (AP) and (AP3), illustrating through an example how the latter represents a less strict condition compared to the former.

Keywords: randon 2-normed space; Ideal; $\mathcal{I}$-convergence; $\mathcal{I}$-Cauchy; $\mathcal{I}$-limit; $\mathcal{I}$-cluster; property (AP3)

1. Introduction

Menger's work [1] marked a significant advancement in the realm of metric axioms, as it introduced a novel approach by associating each pair of points in a set with a distribution function. Schweizer and Sklar [2] subsequently expanded upon this idea, which originally emerged as the concept of a statistical metric space. The key innovation was to replace nonnegative real values with distribution functions. Within this framework, the family of probabilistic normed spaces emerged as a notable subset, wherein probability distribution functions replaced traditional norms for vectors instead of numerical values. The inception of probabilistic normed spaces was initially attributed to Šerstnev in 1963, as documented in [3].

In the 1960s, Gähler [4] played a pivotal role in propelling the concept of 2-normed spaces forward. Building on this groundbreaking work, several other scholars have adopted this concept. Gürdal and Pehlivan [5] extensively examined statistical convergence, statistical Cauchy sequences, and other aspects of statistical convergence within 2-normed spaces. Within the realm of 2-normed spaces, Gürdal and Açk [6] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}$-Cauchy sequences. Moreover, Sarabadan and Talebi [7] delved into statistical convergence and ideal convergence of function sequences within 2-normed spaces. Arslan and Dündar [8] also explored $\mathcal{I}$-convergence, $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy, and $\mathcal{I}^*$-Cauchy sequences of functions within 2-normed spaces. It is important to emphasize that significant progress in this field has been well-documented in references [9,10].

Subsequently, Alsina et al. [11] introduced a novel definition of PN (probabilistic normed) spaces, building upon Šerstnev's foundational work. This advancement ultimately led to the identification of a crucial category of PN spaces known as Menger spaces.

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More recently, Golet [12] expanded the scope of probabilistic normed spaces to include both random and probabilistic 2-normed spaces. This expansion was influenced by Gahler's concept of a 2-norm, as outlined in [4].

Moving on to the realm of convergence, Fast independently extended the concept of statistical convergence from the context of real number sequences [13,14]. Subsequently, Sahiner et al. [15,16] applied this notion to triple sequences. In this context, a triple sequence $\{\xi_{rst}\}_{r,s,t\in\mathbb{N}}$ is considered convergent to ζ if, for any given $\varepsilon > 0$, there exists a natural number $p_0(\varepsilon)$ such that $|\xi_{rst} - \zeta| < \varepsilon$ holds for all r, s, and t exceeding $p_0(\varepsilon)$. Furthermore, the concept of density $\varrho(E)$ for a subset E of $\mathbb{N}^3$ was introduced and defined as the limit of the expression

$$\varrho(E) = \lim_{n,m,p\to\infty} \frac{1}{nmp} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{p} \chi_E(r,s,t) \text{ exists,}$$

where χ_E represents the characteristic function of the set E.

The concept of "ideal convergence", which expands on statistical convergence, finds its theoretical roots in the framework of the ideal $\mathcal{I}$ as it pertains to subsets of natural numbers. Kostyrko et al. [17] propelled the study of $\mathcal{I}$-convergence even further, utilizing the framework of the ideal $\mathcal{I}$ when dealing with subsets of natural numbers. Sahiner and Tripathy [16] subsequently applied the notion of $\mathcal{I}$-convergence to triple sequences in metric spaces, attracting considerable attention from mathematicians across various disciplines. For instance, Altaweel et al. [18] adapted this theory to the fuzzy metric space, while Kočinac and Rashid [9] expanded it to encompass the probabilistic metric space. Furthermore, Rashid and Kočinac and Rashid [10] delved into the investigation of the ideal of convergence within the framework of fuzzy 2-normed space. The introduction of $\mathcal{I}^*$-convergence in [17] spurred extensive research efforts aimed at uncovering its relationship with $\mathcal{I}$-convergence.

Recently, Mohiuddine and Alotaibi [19] delved into the domain of RTNS, with a specific focus on exploring stability results associated with the cubic functional equation. In the context of double sequences situated in random 2-normed spaces, Mohiuddine et al. [19] introduced and thoroughly examined the concepts of $\mathcal{I}$-convergence and $\mathcal{I}$-convergence. Their research also unveiled a correlation between these two modes of convergence, establishing that $\mathcal{I}^*$-convergence serves as a sufficient condition for $\mathcal{I}$-convergence.

Moreover, they presented a compelling instance demonstrating that, in the general scenario, $\mathcal{I}$-convergence does not necessarily imply $\mathcal{I}^*$-convergence when applied to random 2-normed spaces. For further exploration of random 2-normed spaces, please consult the references [20–22].

The work introduced by [23] focuses on the investigation of unbounded fuzzy order convergence and its real-world applications. Moreover, the article delves into the correlation between unbounded fuzzy order convergence and theoretical concepts such as fuzzy weak order units and fuzzy ideals. Within the scope of our research, the authors in [24] propose an enhanced algorithm for deionising images, which is based on the TV model. This approach effectively tackles the aforementioned challenges. The introduction of the L_1 reg. term serves to simplify the solution, facilitating the recovery of high-quality images. Through the reduction in estimated parameters and the application of inverse gradients for estimating the regularization parameter, it enables global adaptation, thereby improving the denoising effect in conjunction with the TV reg. term. Initially, the application of energy-density modeling for strongly interacting substances, such as atomic nuclei and dense stars, may seem unrelated to the exploration of ideal convergence in random 2-normed spaces. Nevertheless, it is plausible to identify certain conceptual connections between the two. Acknowledging these potential correlations, the researchers in the study by Papakonstantinou and Hyun [25] establish a foundation for interdisciplinary collaborations that leverage the respective strengths of each field. This collaborative endeavor aims to advance the understanding of complex systems, ultimately promoting advancements in both theoretical and practical research.

In this research, we focus on examining the rough convergence of triple sequences within the context of 2-normed spaces rather than in random environments. Furthermore, we introduce and analyze the concepts of $\mathcal{I}_3$-convergence, $\mathcal{I}_3^*$-convergence, $\mathcal{I}_3$-limit points, and $\mathcal{I}_3$-cluster points for random 2-normed triple sequences. We establish a noteworthy result, demonstrating that $\mathcal{I}_3^*$-convergence implies $\mathcal{I}_3$-convergence in the context of random 2-normed spaces, highlighting the interplay between these two forms of convergence. The study of the ideal of convergence in random 2-normed spaces is crucial across various disciplines such as functional analysis, probability theory, and stochastic processes. This significance stems from its ability to generalize classical spaces, enabling the analysis of random variables and sequences. Moreover, these spaces provide a suitable mathematical structure for modeling stochastic processes, aiding in the development of accurate models for random phenomena. Understanding the ideal of convergence is vital for statistical analysis, facilitating the formulation of robust methods for handling data with inherent randomness. Additionally, its relevance in functional analysis, particularly in relation to linear operators and function spaces, has implications for fields like quantum mechanics and signal processing. Furthermore, its contribution to the advancement of probability theory, particularly concerning random variable convergence and limit theorems, establishes a strong theoretical foundation for various probabilistic concepts and results. Ultimately, this investigation serves as a fundamental basis for the development of sophisticated models, analysis techniques, and mathematical tools to address real-world challenges associated with randomness and uncertainty.

This paper is organized as follows: The next section introduces and discusses fundamental definitions and early discoveries concerning a random 2-normed space. Section 3 establishes that, assuming a general condition, the condition (AP3) is both necessary and sufficient for the equivalence of the $\mathcal{I}_3$ and $\mathcal{I}_3^*$-Cauchy criteria. Moreover, it includes a specific example illustrating that the $\mathcal{I}_3$-Cauchy condition is not always met. Section 4 explores various significant and previously unexplored aspects of $\mathcal{I}_3$- and $\mathcal{I}_3^*$-convergence concerning triple sequences within a random 2-normed space. It also investigates related implications, including the characterization of compactness in terms of $\mathcal{I}_3$-cluster points, which is discussed in Section 5. Section 6 focuses on presenting some applications of the ideal of convergence of triple sequences in the context of a random 2-normed space. In Section 7, the study presents its findings and offers specific recommendations to other researchers regarding potential future research directions based on the study's results.

2. Definitions, Notations and Preliminary Results

In this section, we will revisit fundamental definitions and notations that serve as the foundation for the current investigation.

A distribution function is a member of the set $\mathcal{D}^+$, where $\mathcal{D}^+$ is defined as follows:

$$\mathcal{D}^+ = \{f : \mathbb{R} \to (0,1); f \text{ is left-continuous, nondecreasing, } f(0) = 0, \text{ and } f(+\infty) = 1\}$$

Within this context, the subset $\mathcal{W}^+$ can be described as $\mathcal{W}^+ = f \in \mathcal{D}^+ : l^- f(+\infty) = 1$, where $l^- f(+\infty)$ represents the left limit of the function f at the point ξ. The space $\mathcal{D}^+$ can be partially ordered using the standard pointwise ordering of functions, which means that $f \leq g$ if and only if $f(\xi) \leq g(\xi)$ holds for every ξ in the real numbers. For any $a \in \mathbb{R}$, we can define a distribution function ε_a, as follows:

$$\varepsilon_a(\xi) = \begin{cases} 0, & \text{if } \xi \leq a; \\ 1, & \text{if } \xi > a. \end{cases}$$

The set $\mathcal{D}$, along with its subsets, can be subjected to partial ordering using the conventional pointwise order. In this ordering, ε_a represents the maximum element within $\mathcal{D}^+$.

Definition 1 ([20]). *A t-norm is a continuous mapping* $* : [0,1] \times [0,1] \to []0,1$ *such that* $([0,1], *)$ *is an abelian monoid with unit one and* $c * d \geq a * b$ *if* $c \geq a$ *and* $d \geq b$*for all*

$a, b, c, d \in [0, 1]$. A triangle function γ is a binary operation on $\mathcal{D}^+$, which is commutative, associative and $\gamma(f, \varepsilon_0) = f$ for every $f \in \mathcal{D}^+$.

The concept of a *2-normed space* was first introduced by Gähler.

Definition 2 ([4])**.** *Suppose X is a linear space with dimension $\mathbf{d}$, where $2 \leq \mathbf{d} < \infty$. A 2-norm on X is defined as a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ that satisfies the following conditions for any $\xi, \zeta \in X$: (i) $\|\xi, \zeta\| = 0$ if and only if ξ and ζ are linearly dependent. (ii) $\|\xi, \zeta\| = \|y, x\|$. (iii) $\|\alpha\xi, \zeta\| = |\alpha|\|\xi, \zeta\|$ for all $\alpha \in \mathbb{R}$. (iv) $\|\xi + \zeta, \eta\| \leq \|\xi, \eta\| + \|\zeta, \eta\|$.*

In this context, we refer to $(X, \|\cdot,\cdot\|)$ as a 2-normed space.

Example 1 ([4])**.** *Take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|\xi, \zeta\| =$ the area of the parallelogram spanned by the vectors ξ and ζ, which may be given explicitly by the formula*

$$\|\xi, \zeta\| = |\xi_1\zeta_2 - \xi_2\zeta_1|, \text{ where } \xi = (\xi_1, \xi_2), y = (\zeta_1, \zeta_2).$$

In a recent development, Golet introduced the concept of a RTNS in the following manner.

Definition 3 ([12])**.** *Let X be a linear space of a dimension greater than one, γ be a triangle function, and $\psi : X \times X \to \mathcal{D}^+$. Then, ψ is called a probabilistic 2-norm on X and (X, ψ, γ) a probabilistic 2-normed space if the following conditions are satisfied:*

(i) $\psi_{\xi,\zeta}(t) = \varepsilon_0(t)$ *if ξ and ζ are linearly dependent, where $\psi_{\xi,\zeta}(t)$ denotes the value of $\psi_{\xi,\zeta}$ at $t \in \mathbb{R}$;*

(ii) $\psi_{\xi,\zeta}(t) \neq \varepsilon_0(t)$ *if ξ and ζ are linearly independent;*

(iii) $\psi_{\xi,\zeta} = \psi_{\zeta,\xi}$ *for every ξ, ζ in X;*

(iv) $\psi_{\alpha\xi,\zeta}(t) = \psi_{\xi,\zeta}\left(\frac{t}{|\alpha|}\right)$ *for every $t > 0$, $\alpha \neq 0$ and $\xi, \zeta \in X$;*

(v) $\psi_{\xi+\zeta,\eta} \geq \gamma(\psi_{\xi,\eta}, \psi_{\zeta,\eta})$ *whenever $\xi, \zeta, \eta \in X$.*

If (v) is replaced by

(v′) $\psi_{\xi+\zeta,\eta}(t_1 + t_2) \geq \psi_{\xi,\eta}(t_1) * \psi_{\zeta,\eta}(t_2)$*, for all $\xi, zeta, \eta \in X$ and $t_1, t_2 \in \mathbb{R}^+$, then triple $(X, \psi, *)$ is called a RTNS.*

Remark 1 ([19])**.** *Note that every 2-normed space $(X, \|\cdot,\cdot\|$ can be made a random 2-normed space in a natural way, by setting*

(a) $\psi_{\xi,\zeta}(t) = \varepsilon_0(t - \|\xi, \zeta\|)$*, for every $\xi, \zeta \in X$, $t > 0$ and $a * b = \min\{a, b\}$, $a, b \in [0, 1]$;*

(b) $\psi_{\xi,\zeta}(t) = \frac{t}{t+\|\xi,\zeta\|}$*, for every $\xi, \zeta \in X$, $t > 0$ and $a * b = ab$, $a, b \in [0, 1]$.*

Definition 4 ([26])**.** *A double sequence $\xi_{nk}n, k \in \mathbb{N}$ is considered to be statistically convergent to μ according to Pringsheim's criteria if, for every $\varepsilon > 0$, the set $K(\varepsilon)$, defined as:*

$$K(\varepsilon) = \{(n, m) \in \mathbb{N} \times \mathbb{N} : |\xi nm - \mu| \geq \varepsilon\},$$

satisfies the condition $\varrho(K(\varepsilon)) = 0$.

It is important to highlight that if we have a convergent double sequence $\xi_{nk}n, k \in \mathbb{N}$, it also exhibits statistical convergence to the same limit. Conversely, when $\xi nk_{n,k \in \mathbb{N}}$ is statistically convergent, the limit is uniquely determined.

Now, let us consider a triple sequence $\xi_{nkl}n, k, l \in \mathbb{N}$ consisting of real numbers. We define it as "bounded" if there exists a positive real number M such that $|\xi nkl| < M$ for all $n, k, l \in \mathbb{N}$.

We introduce a set E that is a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. We denote E_{nkl} as the set of indices (i, j, p) such that $i \leq n$, $j \leq k$, and $p \leq l$. If the sequence $\frac{|E_{nkl}|}{nkl}$ has a limit in Pringsheim's sense, we say that E possesses triple natural density, and we represent it as:

$$\delta_3(E) = \lim_{n,k,l \to \infty} \frac{|E_{nkl}|}{nkl}.$$

Now, let us establish the concept of statistical convergence for a triple sequence $\xi_{nkl} n, k, l \in \mathbb{N}$ toward $\xi \in \mathbb{R}$. For any given $\varepsilon > 0$, if the triple density $\delta_3(E(\varepsilon)) = 0$, where

$$E(\varepsilon) = \{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\xi nkl - \xi| \geq \varepsilon\},$$

then we affirm that the triple sequence statistically converges to ξ.

Definition 5 ([15])**.** *Let us consider a 2-normed space denoted as $(X, \|\cdot, \cdot\|)$. In this space, a triple sequence $\xi_{nkl} n, k, l \in \mathbb{N}$ is classified as a statistically Cauchy sequence if, for any element $a \in X$ and for any positive value of ε, we satisfy the condition $\delta_3(E(\varepsilon)) = 0$, where*

$$E(\varepsilon) = \{(n, k, l), (r, s, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|\xi nkl - \xi_{rst}, a\| \geq \varepsilon\}.$$

In such a situation, we represent it as:

$$st - \lim_{\substack{n,k,l \to \infty \\ i,j,p \to \infty}} \|\xi_{nkl} - \xi_{ijp}, a\| = 0.$$

Definition 6 ([20])**.** *Let us consider a 2-normed space denoted as $(X, \|\cdot, \cdot\|)$. Within this space, a triple sequence $\xi_{nkl} n, k, l \in \mathbb{N}$ is regarded as statistically convergent to an element $\xi \in X$ if, for any element $a \in X$, and for any given positive value of ε, we satisfy the condition $\delta_3(E(\varepsilon)) = 0$, where*

$$E(\varepsilon) = \{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|\xi nkl - \xi, a\| \geq \varepsilon\}.$$

In this scenario, we express it as:

$$st - \lim_{n,k,l \to \infty} \|\xi_{nkl} - \xi, a\| = 0.$$

Definition 7 ([19])**.** *Let $(X, \psi, *)$ be an RTNS. A sequence $x = \xi_{mnk} n, k, m \in \mathbb{N}$ is considered statistically convergent to ξ if, for all $a \in X$ and for every $\varepsilon > 0$ and $\varpi \in (0, 1)$, the condition*

$$\delta_3\{r \leq m, s \leq n, t \leq k : \psi\xi_{rst}, \xi, a(\varepsilon) \leq 1 - \varpi\} = 0$$

is satisfied, or equivalently,

$$\lim_{n,k,m \to \infty} \frac{1}{mnk} \left| \{r \leq m, s \leq n, t \leq k : \psi_{\xi_{rst}, \xi, a}(\varepsilon) \leq 1 - \varpi\} \right| = 0.$$

In this scenario, we express it as $st_\psi - \lim x = \xi$.

Definition 8 ([27])**.** *An ideal, denoted as $\mathcal{I}$, is defined as a non-empty class within the set $2^{\mathbb{N}}$ that adheres to the following conditions:*

(i) The additive property: If two sets, S and T, belong to $\mathcal{I}$, then their union $T \cup S$ is also an element of $\mathcal{I}$.
(ii) The hereditary property: If a set T belongs to $\mathcal{I}$, and another set S is a subset of T, then S is also an element of $\mathcal{I}$.

Definition 9 ([17]). *A non-trivial ideal, denoted as $\mathcal{I}$, is distinguished by the property that it does not encompass the entire set $2^{\mathbb{N}}$. Furthermore, it earns the label "admissible" when it satisfies two specific conditions: it is non-trivial, and for every natural number l, the singleton set l is included in $\mathcal{I}$.*

Definition 10 ([17]). *For any given ideal $\mathcal{I}$, a corresponding filter is associated with it, denoted as $\mathcal{F}(\mathcal{I})$. This filter is defined as follows:*

$$\mathcal{F}(\mathcal{I}) = \{K \subseteq \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I}\}.$$

Moreover, when an admissible ideal $\mathcal{I}$ is considered within the set $2^{\mathbb{N}}$, it is said to possess the property referred to as (AP) if, for any sequence $A_1, A_2, \cdots$ comprising mutually exclusive sets from $\mathcal{I}$, there exists another sequence $\{B_1, B_2, \cdots\}$ consisting of subsets of $\mathbb{N}$, such that each symmetric difference $A_i \Delta B_i$ (for all $i = 1, 2, \cdots$) is finite, and the union of all B_i (i.e., $\bigcup_{i=1}^{\infty} B_i$) is an element of $\mathcal{I}$

Definition 11 ([17]). *Consider a nontrivial ideal $\mathcal{I} \subset P(\mathbb{N})$ within the set of natural numbers $\mathbb{N}$. A sequence $x = \xi_{n_{n\in\mathbb{N}}}$ is deemed $\mathcal{I}$-convergent to L if, for every $\varepsilon > 0$, the set*

$$\{k \in \mathbb{N} : |\xi_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

In this situation, we express it as $\mathcal{I}\text{-}\lim x = L$.

Definition 12 ([19]). *Let $\mathcal{I}$ be a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$, and let $(X, \psi, *)$ be a random 2-normed space. A double sequence $x = \xi_{jk}$ consisting of elements from X is considered $\mathcal{I}_2$-convergent to $\xi \in X$ within the context of the random 2-normed space (or $\mathcal{I}_2^{\psi}$-convergent to ξ) if, for all $a \in X$ and any $\varepsilon > 0$ and $\varpi \in (0,1)$, the set*

$$\Big\{(j,k) \in \mathbb{N} \times \mathbb{N} : \psi_{\xi_{jk}-\xi, a}(\varepsilon) \leq 1 - \varpi\Big\} \in \mathcal{I}_2.$$

In this situation, we represent it as $\mathcal{I}_2^{\psi} - \lim x = \xi$.

Definition 13 ([28]). *A nontrivial ideal $\mathcal{I}_3$ of $\mathbb{N}^3$ is referred to as strongly admissible if it includes sets of the form $i \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times i \times \mathbb{N}$, and $\mathbb{N} \times \mathbb{N} \times i$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal also qualifies as an admissible ideal.*

If we define $\mathcal{I}_3^0 = \big\{A \subset \mathbb{N}^3 : \exists m(A) \in \mathbb{N}$ such that $\forall i, j, k \geq m(A), (i,j,k) \notin A\big\}$, then $\mathcal{I}_3^0$ constitutes a nontrivial strongly admissible ideal. It is noticeable that $\mathcal{I}_3$ is a strongly admissible ideal if and only if $\mathcal{I}_3^0 \subset \mathcal{I}_3$.

Definition 14 ([28]). *An admissible ideal $\mathcal{I}_3 \subset \mathbb{N}^3$ adheres to the property (AP3) if, for every countable collection of mutually disjoint sets $\{H_1, H_2, \cdots\}$ that belong to $\mathcal{I}_3$, there exists a countable family of sets $\{G_1, G_2, \cdots\}$ such that the symmetric difference $H_j \triangle G_j \in \mathcal{I}_3$ is contained within the finite union of rows and columns in $\mathbb{N}^3$ for each $j \in \mathbb{N}$, and $G = \bigcup_{j=1}^{\infty} G_j \in \mathcal{I}_3$. Consequently, it follows that $G_j \in \mathcal{I}_3$ for each $j \in \mathbb{N}$.*

Remark 2. *It is crucial to note that when the ideal $\mathcal{I}$ corresponds to $\mathcal{I}_0$, $\mathcal{I}$-convergence coincides entirely with the conventional concept of convergence. Conversely, if we define $\mathcal{I}_d$ as the collection of all subsets A of $\mathbb{N}^3$ for which the triple natural density $\delta_3(A)$ equals zero, then $\mathcal{I}_d$-convergence becomes equivalent to statistical convergence.*

Triple sequences that converge with respect to $\mathcal{I}$ may not necessarily be bounded. For instance, consider the ideal $\mathcal{I}$ as $\mathcal{I}_0$ in $\mathbb{N}^3$. If we define $\{\xi_{nkm}\}_{n,k,m\in\mathbb{N}}$ as follows:

$$\xi_{nkm} = \begin{cases} km, & \text{if } n = 1; \\ 1, & \text{if } n \neq 1. \end{cases}$$

In this case, the sequence $\xi_{nkm n,k,m\in\mathbb{N}}$ is unbounded; however, it is still $\mathcal{I}$-convergent.

3. $\mathcal{I}_3$ and $\mathcal{I}_3^*$-Cauchy of Triple Sequences

In this section, we will redirect our attention towards investigating the concepts of $\mathcal{I}_3^\psi$-Cauchy and $\mathcal{I}_3^{*,\psi}$-Cauchy triple sequences within the framework of $(X, \psi, *)$. Furthermore, we will explore the interconnections and associations among these ideas.

Definition 15. *Let $(X, \psi, *)$ be a RTNS and $\mathcal{I}_3 \subset 2^{\mathbb{N}^3}$ be a strongly admissible ideal. A triple sequence $x = \{\xi_{mnk}\}$ of elements in Xis said to be*

(a) An $\mathcal{I}_3^\psi$-Cauchy sequence in X if for every $\varpi \in (0,1)$, $\varepsilon > 0$ and a nonzero $z \in X$, there exist $r = r(\varpi), s = s(\varpi), r = r(\varpi)$ such that

$$\Big\{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\xi_{rst},\eta}(\varepsilon) \leq 1 - \varpi\Big\}.$$

(b) An $\mathcal{I}_3^{*,\psi}$-Cauchy sequence in X if for every $\varpi \in (0,1)$, $\varepsilon > 0$ and a nonzero $z \in X$, there exists

$$K = \big\{(m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\big\} \subset \mathbb{N}^3$$

such that $K \in \mathcal{F}(\mathcal{I}_3)$ and $\{\xi_{m_j n_j k_j}\}$ is ψ-Cauchy sequence in X.

The following theorem establishes a connection between triple Cauchy sequences under $\mathcal{I}_3^\psi$ and $\mathcal{I}_3^{*,\psi}$.

Theorem 1. *Let $(X, \psi, *)$ be a random 2-normed space and $\mathcal{I}_3 \subset 2^{\mathbb{N}^3}$ be a strongly admissible ideal. If $\{\xi_{mnk}\}$ is an $\mathcal{I}_3^{*,\psi}$-triple Cauchy sequence, then $\{\xi_{mnk}\}$ is an $\mathcal{I}_3^\psi$-triple Cauchy sequence.*

Proof. For any $\varepsilon > 0$ and ϖ within the open interval $(0,1)$, and for any non-zero element $z \in X$, the sequence $\{\xi_{mnk}\}$ is an $\mathcal{I}_3^{*,\psi}$-triple Cauchy sequence if the following conditions are met: There exists a set

$$K = \big\{(m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\big\} \in \mathcal{F}(\mathcal{I}_3)$$

and a number $r_0 \in \mathbb{N}$, such that

$$\psi_{\xi_{m_j n_j k_j}-\xi_{m_s n_s k_s},\eta}(\varepsilon) > 1 - \varpi$$

for every j and s greater than or equal to r_0. Fix $p = m_{r_0+1}$, $q = n_{r_0+1}$, and $w = k_{r_0+1}$. Then, for every ϖ in $(0,1)$, $\varepsilon > 0$, and a non-zero $z \in X$, the following condition holds:

$$\psi_{\xi_{m_j n_j k_j}-\xi_{pqw},\eta}(\varepsilon) > 1 - \varpi$$

for every j greater than or equal to r_0. Additionally, let H be the complement of the set K in $\mathbb{N}$. It is evident that $H \in \mathcal{F}(\mathcal{I}_3)$, and we can establish that:

$$\begin{aligned} A(\varepsilon) &= \Big\{(m,n,k) \in \mathbb{N} : \psi_{\xi_{m_j n_j k_j}-\xi_{pqw},\eta}(\varepsilon) \leq 1 - \varpi\Big\} \\ &\subset H \cup \{(m_1, m_2, \ldots ; n_1, n_2, \ldots ; k_1, k_2, \ldots)\} \in \mathcal{I}_3. \end{aligned}$$

Therefore, for any ϖ in $(0,1)$, $\varepsilon > 0$, and a non-zero $z \in X$, we can find $(p,q,w) \in \mathbb{N}$ such that $A(\varepsilon) \in \mathcal{I}_3$, implying that the sequence $\{\xi_{mnk}\}$ is an $\mathcal{I}_3^\psi$-triple Cauchy sequence. □

Theorem 2. *Let us consider a countable assortment of subsets represented as $P_{i i\in\mathbb{N}}$ within the set $\mathbb{N}^3$. Each of these subsets, which we still denote as $P_{i i\in\mathbb{N}}$, is part of a filter denoted as $\mathcal{F}(\mathcal{I}_3)$, which is linked to a strongly admissible ideal known for possessing the property (AP3). In this context, we can establish the presence of a set denoted as P that is contained within the set $\mathbb{N}^3$. Additionally, this set P is also a member of the filter $\mathcal{F}(\mathcal{I}_3)$, and furthermore, the disparity between P and each P_i is finite for all i.*

Proof. Let $A_1 = \mathbb{N}^3 \setminus P_1$, $A_m = (\mathbb{N}^3 \setminus P_m) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{m-1})$, $(m = 2, 3, \cdots)$. It is easy to observe that $A_i \in \mathcal{I}_3$ for each i and $A_i \cup A_j = \emptyset$, when $i \neq j$. Then, by (AP3) property of $\mathcal{I}_3$, we conclude that there exists a countable family of sets $\{B_1, B_2, \cdots\}$, such that $A_j \triangle B_j \in \mathcal{I}_3^0$, i.e., $A_j \triangle B_j$ is included in a finite union of rows and columns in $\mathbb{N}^3$ for each j and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_3$. Put $P = \mathbb{N}^3 \setminus B$. It is clear that $P \in \mathcal{F}(\mathcal{I}_3)$.

Now, we will establish that the set $P \setminus P_i$ has a finite number of elements for every i. Let us suppose there exists a natural number j_0 such that the set $P \setminus P_{j_0}$ contains an infinite number of elements. Given that each difference $A_j \triangle B_j$ (for $j = 1, 2, 3, \ldots, j_0$) is confined within a finite combination of rows and columns, we can identify specific natural numbers $m_0, n_0, k_0 \in \mathbb{N}$, such that:

$$\left(\bigcup_{j=1}^{j_0} B_j \right) \cap C_{m_0 n_0 k_0} = \left(\bigcup_{j=1}^{j_0} A_j \right) \cap C_{m_0 n_0 k_0} \tag{1}$$

where $C_{m_0 n_0 k_0} = \{(m, n, k) : m \geq m_0, n \geq n_0, k \geq k_0\}$. If $m \geq m_0, n \geq n_0, k \geq k_0$ and $(m, n, k) \notin B$, then $(m, n, k) \notin \bigcup_{j=1}^{j_0} B_j$ and so by (1), $(m, n, k) \notin \bigcup_{j=1}^{j_0} A_j$.

Since $A_{j_0} = \left(\mathbb{N}^3 \setminus P_{j_0}\right) \setminus \bigcup_{j=1}^{j_0} A_j$ and $(m, , n, k) \notin A_{j_0}$, $(m, n, k) \notin \bigcup_{j=1}^{j_0} A_j$ we have $(m, n, k) \in P_{j_0}$ for $m \geq m_0, n \geq n_0$ and $k \geq k_0$. Consequently, for all natural numbers m satisfying $m \geq m_0$, n satisfying $n \geq n_0$, and k satisfying $k \geq k_0$, we find that (m, n, k) belongs to both P and P_{j_0}. This observation demonstrates that the set $P \setminus P_{j_0}$ contains only a finite number of elements. This contradicts our initial assumption that the set $P \setminus P_{j_0}$ is infinite. □

Theorem 3. *Let $(X, \psi, *)$ be a RTNS and $\mathcal{I}_3 \subset 2^{\mathbb{N}^3}$ be a strongly admissible ideal with property (AP3). Then, the concepts I_3^{ψ}-triple Cauchy sequence and $\mathcal{I}_3^{*,\psi}$-triple Cauchy sequence coincide.*

Proof. If $\{\xi_{mnk}\}$ is $\mathcal{I}_3^{*,\psi}$-triple Cauchy sequence, then it is $\mathcal{I}_3^{\psi}$-triple Cauchy sequence by Theorem 1 (even if $\mathcal{I}_3$ does not have the (AP3) property).

Now, we need to demonstrate the reverse statement. Suppose we have an I_3^{ψ}-triple Cauchy sequence denoted as ξ_{mnk}. As per the definition, there exists specific values $m_0 = m_0(\varpi)$, $n_0 = n_0(\varpi)$, and $k_0 = k_0(\varpi)$ such that:

$$A(\varpi) = \left\{ (m, n, k) \in \mathbb{N}^3 : \psi_{\xi_{mnk} - \xi_{m_0 n_0 k_0}, \eta}(\varepsilon) \leq 1 - \varpi \right\} \in \mathcal{I}_3$$

for every $\varpi \in (0, 1)$, $\varepsilon > 0$ and a non-zero $z \in X$.

Let $P_i = \left\{ (m, n, k) \in \mathbb{N}^3 : \psi_{\xi_{mnk} - \xi_{r_i s_i t_i}, \eta}(\varepsilon) > 1 - \frac{1}{i} \right\}$, $i = 1, 2, \cdots$, where $r_i = m_0\left(\frac{1}{i}\right)$, $s_i = n_0\left(\frac{1}{i}\right)$, $t_i = k_0\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_3)$ for $i = 1, 2, \cdots$. Since $\mathcal{I}_3$ has the property (AP3), then by Theorem 2 there exists a set $P \subset \mathbb{N}^3$ such that $P \in \mathcal{F}(I_3)$, and $P \setminus P_i$ is finite for all i. Now, we prove that

$$\lim_{\substack{m,n,k,r,s,t \to \infty \\ (m,n,k),(r,s,t) \in P}} \psi_{\xi_{mnk} - \xi_{rst}, \eta}(\varepsilon) = 1.$$

To prove this, let $\varepsilon > 0$, $\varpi \in (0,1)$ and $w \in \mathbb{N}$ such that $\left(1 - \frac{1}{w}\right) * \left(1 - \frac{1}{w}\right) > 1 - \varpi$. If $(m,n,k),(r,s,t) \in P$, then $P \setminus P_w$ is a finite set, so there exists $q = q(w)$ such that $(m,n,k),(r,s,t) \in P$ for all $m,n,k,r,s,t > q(w)$. Hence, it can be concluded that for all m,n,k,r,s,t greater than a certain threshold $q(w)$: $\psi_{\xi_{mnk}-\xi_{r_w s_w t_w},\eta}(\varepsilon/2) > 1 - \frac{1}{w}$ and $\psi_{\xi_{rst}-\xi_{r_w s_w t_w},\eta}(\varepsilon/2) > 1 - \frac{1}{w}$. As a result, we can infer that:

$$\psi_{\xi_{mnk}-\xi_{rst}}(\varepsilon) \geq \psi_{\xi_{mnk}-\xi_{r_w s_w t_w},\eta}(\varepsilon/2) * \psi_{\xi_{rst}-\xi_{r_w s_w t_w},\eta}(\varepsilon/2) > \left(1 - \frac{1}{w}\right) * \left(1 - \frac{1}{w}\right) > 1 - \varpi$$

for all $m,n,k,r,s,t > q(w)$.

Therefore, for any given $\varepsilon > 0$ and $\varpi \in (0,1)$, there exists a threshold $q = q(w)$ such that when $m,n,k,r,s,t > q(w)$, and all these values are elements of some set P belonging to the filter $\mathcal{F}(\mathcal{I}_3)$:

$$\psi_{\xi_{mnk}-\xi_{rst},\eta}(\varepsilon) > 1 - \varpi.$$

This holds true for every non-zero element z in the set X. Hence, it demonstrates that ξ_{mnk} constitutes an $\mathcal{I}_3^{*,\psi}$-triple Cauchy sequence within the space X. □

4. $\mathcal{I}_3$-and $\mathcal{I}_3^*$-Convergence in RTN

In this section, our investigation is focused on the concept of ideal convergence as it pertains to triple sequences within a RTNS. We will introduce the notion of $\mathcal{I}_3^*$-convergence for triple sequences in this space and establish that $\mathcal{I}_3^*$-convergence implies $\mathcal{I}_3$-convergence, although the reverse is not necessarily valid. It is crucial to emphasize that in this section, we consistently treat $\mathcal{I}_3$ as a nontrivial admissible ideal within $\mathbb{N}^3$.

Definition 16. *Consider a non-trivial ideal $\mathcal{I}$ within the set of natural numbers $\mathbb{N}^3$, and let $(X, \psi, *)$ represent a random 2-normed space. Now, let us introduce a triple sequence denoted as $x = \xi_{jkm}$, consisting of elements from X. We define this triple sequence as $\mathcal{I}_3$-convergent to $L \in X$ with respect to the random 2-normed space ψ, or, more succinctly, as $\mathcal{I}_3^{\psi}$-convergent to L, if, for every $\varepsilon > 0$ and $\varpi \in (0,1)$, and for all $a \in X$, the set*

$$\left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L,a}(\varepsilon) \leq 1 - \varpi\right\}$$

belongs to the ideal $\mathcal{I}_3$. In such a context, we express this as $\mathcal{I}_3^{\psi}\text{-}\lim x = L$.

Theorem 4. *Let $(X, \psi, *)$ be a RTNS. Then, the following statements are equivalent:*

(a) $\mathcal{I}_3^{\psi}\text{-}\lim x = L$;
(b) $\left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm},L,a}(t) \leq 1 - \varpi\right\} \in \mathcal{I}_3^{\psi}$ *for every $t > 0$, $\varpi \in (0,1)$ and $a \in X$;*
(c) $\left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm},L,a}(t) > 1 - \varpi\right\} \in \mathcal{F}(\mathcal{I}_3^{\psi})$ *for every $t > 0$, $\varpi \in (0,1)$ and $a \in X$;*
(d) $\mathcal{I}_3\text{-}\lim \psi_{\xi_{jkm},L,a}(t) = 1$.

Proof. We will refrain from presenting the proof as it can be readily comprehended. □

Theorem 5. *Let $(X, \psi, *)$ be a random 2-normed space. If a triple sequence $x = \left\{\xi_{jkm}\right\}$ is $\mathcal{I}_3^{\psi}$-convergent, then the $\mathcal{I}_3^{\psi}$-limit is necessarily unique.*

Proof. Assume that $\mathcal{I}_3^{\psi}$-lim $x = L_1$ and $\mathcal{I}_3^{\psi}$-lim $x = L_2$. For a given $\varpi \in (0,1)$ and $\varepsilon > 0$, and for any element $a \in X$, select $\eta > 0$ such that $(1-\eta) * (1-\eta) > 1 - \varpi$. We then define the following sets as:

$$\begin{aligned} H_{\psi,1}(\eta,\varepsilon) &= \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L_1,a}(\varepsilon/2) \le 1-\eta\Big\} \\ H_{\psi,2}(\eta,\varepsilon) &= \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L_2,a}(\varepsilon/2) \le 1-\eta\Big\}. \end{aligned}$$

Given that $\mathcal{I}_3^{\psi}$-lim $x = L_1$, it follows that $H_{\psi,1}(r,\varepsilon) \in \mathcal{I}_3$. Moreover, by utilizing $\mathcal{I}_3^{\psi}$-lim $x = L_2$, we can conclude that $H_{\psi,2}(r,\varepsilon) \in \mathcal{I}_3$. Now, let us define $H_{\psi}(\eta,\varepsilon) = H_{\psi,1}(\eta,\varepsilon) \cup H_{\psi,2}(\eta,\varepsilon) \in \mathcal{I}_3$. As a result, we establish that $H_{\psi}(\eta,\varepsilon) \in \mathcal{I}_3$. This, in turn, implies that its complement, denoted as $H^c_{\psi}(\eta,\varepsilon)$, is not empty within $\mathcal{F}(\mathcal{I}_3)$. If we have $(j,k,m) \in H^c_{\psi}(\eta,\varepsilon)$, it follows that $(j,k,m) \in H^c_{\psi,1}(\eta,\varepsilon) \cap H^c_{\psi,2}(\eta,\varepsilon)$, and thus:

$$\psi_{L_1-L_2,a}(\varepsilon) \ge \psi_{\xi_{jkm}-L_1,a}\left(\frac{\varepsilon}{2}\right) * \psi_{\xi_{jkm}-L_2,a}\left(\frac{\varepsilon}{2}\right) \ge (1-\eta) * (1-\eta) > 1-\varpi.$$

Given that $\varpi > 0$ was chosen arbitrarily, it follows that $\psi_{L_1-L_2,a}(\varepsilon) = 1$ for all $\varepsilon > 0$. Consequently, we can conclude that $L_1 = L_2$. □

Theorem 6. *Let $(X,\psi,*)$ be a random 2-normed space and let $x = \{\xi_{jkm}\}$ be triple sequences in X. If ψ-lim $x = L$, then $\mathcal{I}_3^{\psi}$-lim $x = L$.*

Proof. Assume that ψ-lim $x = L$. Then, for any $\varpi \in (0,1)$ and $\varepsilon > 0$, and for all $a \in X$, there exists a positive integer N such that $\psi_{\xi_{jkm}-L,a}(\varepsilon) > 1-\varpi$ for all $j,k,m > N$. Considering this, we can observe that the set

$$Q(\varepsilon) = \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L,a}(\varepsilon) \le 1-\varpi\Big\}$$

is contained in $\{1,2,3,\cdots,N-1\}$ and the ideal $\mathcal{I}_3$ is admissible; thus, we have $Q(\varepsilon) \in \mathcal{I}_3$. Therefore, $\mathcal{I}_3^{\psi}$-lim $x = L$. □

In the forthcoming example, it is not guaranteed that the converse of what Theorem 6 asserts holds true.

Example 2. *Consider the space $X = \mathbb{R}^3$ equipped with the Euclidean 2-norm, denoted as $\|\xi,\zeta\|$, defined by the vectors ξ and ζ. These vectors can be explicitly expressed using the formula:*

$$\|\xi,\zeta\| = |\xi_1\zeta_2 - \xi_2\zeta_1| + |\xi_1\zeta_3 - \xi_3\zeta_1| + |\xi_2\zeta_3 - \xi_3\zeta_2|, \text{ where } x = (\xi_1,\xi_2,\xi_3), y = (\zeta_1,\zeta_2,\zeta_3)$$

and $a * b = ab$ for $a,b \in [0,1]$. For all $x \in X$, $t > 0$ and nonzero $z \in X$, consider

$$\psi_{\xi,\eta}(t) = \begin{cases} \frac{t}{t+\|\xi,\eta\|}, & \text{if } t > 0; \\ 0, & \text{if } t \le 0. \end{cases}$$

Now, we have $(X,\psi,*)$ as a RTNS. Next, we introduce a double sequence denoted as $x = \{\xi_{mnk}\}$, defined as follows:

$$\xi_{mnk} = \begin{cases} (mnk,0,0), & \text{if } m,n,k \text{ are squre;} \\ (0,0,0), & \text{otherwise.} \end{cases}$$

Write $H_{r,s,t}(\varpi,\varepsilon) = \{m \le r, n \le s, k \le t : \psi_{\xi_{mnk}-L,a}(\varepsilon) \le 1-\varpi\}$, $\varepsilon > 0, \varpi \in (0,1)$, $a \in X$ *and* $L = (0,0,0)$. *We observe that*

$$\psi_{\xi_{mnk}-L,a}(t) = \begin{cases} \frac{t}{t+mnk(\eta_2+\eta_3)}, & \text{if } m, n \text{ and } k \text{ are square;} \\ 1, & \text{otherwise.} \end{cases}$$

Taking $m,n,k \to \infty$, *we obtain*

$$\lim_{m,n,k\to\infty} \psi_{\xi_{mnk}-L,a}(t) = \begin{cases} 0, & \text{if } m, n \text{ and } k \text{ are square;} \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, it can be observed that a triple sequence $x = \{\xi_{mnk}\}$ *does not exhibit convergence within the space* $(X,\psi,*)$. *However, if we define* $\mathcal{I} = \mathcal{I}(\delta) = \{A \subset \mathbb{N}^3 : \delta_A = 0\}$, *then because* $H_{r,s,t}(\varpi,\varepsilon) \subset \{(1,0,0),(4,0,0),\cdots\}$, *we have* $\delta_3\{H_{r,s,t}(\varpi,\varepsilon)\} = 0$. *In other words,* $\mathcal{I}_3^{\psi}$-$\lim x = L$.

Theorem 7. *Let* $(X,\psi,*)$ *be a random 2-normed space and let* $x = \{\xi_{jkm}\}$ *and* $y = \{\zeta_{jkm}\}$ *be triple sequences in X.*

(a) *If* $\mathcal{I}_3^{\psi}$-$\lim x = L_1$ *and* $\mathcal{I}_3^{\psi}$-$\lim y = L_2$, *then* $\mathcal{I}_3^{\psi}$-$\lim(x+y) = L_1 + L_2$.
(b) *If* $\mathcal{I}_3^{\psi}$-$\lim x = L$, *then If* $\mathcal{I}_3^{\psi}$-$\lim \beta x = \beta L$.

Proof. (a) Assume that $\mathcal{I}_3^{\psi}$-$\lim x = L_1$ and $\mathcal{I}_3^{\psi}$-$\lim y = L_2$. For any given $\varpi \in (0,1)$ and $\varepsilon > 0$, and for all $a \in X$, select $\eta > 0$ such that $(1-\eta)*(1-\eta) > 1-\varpi$. We can then define the following sets as:

$$\begin{aligned} H_{\psi,1}(\eta,\varepsilon) &= \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L_1,a}(\varepsilon/2) \le 1-\eta\Big\} \\ H_{\psi,2}(\eta,\varepsilon) &= \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L_2,a}(\varepsilon/2) \le 1-\eta\Big\}. \end{aligned}$$

Given that $\mathcal{I}_3^{\psi}$-$\lim x = L_1$, we can establish that $H_{\psi,1}(r,\varepsilon) \in \mathcal{I}_3$. Similarly, by employing $\mathcal{I}_3^{\psi}$-$\lim x = L_2$, we conclude that $H_{\psi,2}(r,\varepsilon) \in \mathcal{I}_3$. Now, let us introduce $H_{\psi}(\eta,\varepsilon) = H_{\psi,1}(\eta,\varepsilon) \cup H_{\psi,2}(\eta,\varepsilon) \in \mathcal{I}_3$.

Consequently, we have $H_{\psi}(\eta,\varepsilon) \in \mathcal{I}_3$. This implies that its complement $H_{\psi}^c(\eta,\varepsilon)$ is non-empty within $\mathcal{F}(\mathcal{I}_3)$. Now, our task is to demonstrate that:

$$H_{\psi}^c(\eta,\varepsilon) \subset \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{(\xi_{jkm}+\zeta_{jkm})-(L_1+L_2),a}(\varepsilon) > 1-\varpi\Big\}.$$

If $(j,k,m) \in H_{\psi}^c(\eta,\varepsilon)$, then we have $\psi_{\xi_{jkm}-L_1,a}(\varepsilon/2) > 1-\eta$ and $\psi_{\zeta_{jkm}-L_2,a}(\varepsilon/2) > 1-\eta$. Consequently,

$$\begin{aligned} \psi_{(\xi_{jkm}+\zeta_{jkm})-(L_1+L_2),a}(\varepsilon) &\ge \psi_{\xi_{jkm}-L_1,a}(\varepsilon/2) * \psi_{\zeta_{jkm}-L_2,a}(\varepsilon/2) \\ &> (1-\eta)*(1-\eta) \\ &> 1-\varpi. \end{aligned}$$

Hence,

$$H_{\psi}^c(\eta,\varepsilon) \subset \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{(\xi_{jkm}+\zeta_{jkm})-(L_1+L_2),a}(\varepsilon) > 1-\varpi\Big\}.$$

Since $H_{\psi}^c(\eta,\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$, we have $\mathcal{I}_3^{\psi}$-$\lim(x+y) = L_1 + L_2$.

(b) The case where $\beta = 0$ is straightforward. Now, consider the situation when $\beta \ne 0$. For any given $\varpi \in (0,1)$, $\varepsilon > 0$, and for all $a \in X$, we have:

$$G(\varepsilon) = \Big\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L,a}(\varepsilon) > 1-\varpi\Big\} \in \mathcal{F}(\mathcal{I}_3).$$

We only need to demonstrate that for every $\varpi \in (0,1)$, $\varepsilon > 0$, and all $a \in X$, we have:

$$G(\varepsilon) \subset \left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\beta\xi_{jkm}-\beta L,a}(\varepsilon) > 1-\varpi\right\}.$$

Let $(j,k,m) \in G(\varepsilon)$. Then we have $\psi_{\xi_{jkm},L,a}(\varepsilon) > 1-\varpi$. Now,

$$\begin{aligned}
\psi_{\beta\xi_{jkm}-\beta L,a}(\varepsilon) &= \psi_{\xi_{jkm}-L,a}\left(\frac{\varepsilon}{|\beta|}\right) \\
&\geq \psi_{\xi_{jkm}-L,a}(\varepsilon) * \psi_{0,a}\left(\frac{\varepsilon}{|\beta|}-\varepsilon\right) \\
&= \psi_{\xi_{jkm}-L,a}(\varepsilon) * 1 = \psi_{\xi_{jkm}-L,a}(\varepsilon) > 1-\varpi.
\end{aligned}$$

Hence

$$G(\varepsilon) \subset \left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\beta\xi_{jkm}-\beta L,a}(\varepsilon) > 1-\varpi\right\}.$$

□

Definition 17. *Let $(X,\psi,*)$ be a random 2-normed space. We define that a sequence $x = \xi_{jkm}$ consisting of elements from X is $\mathcal{I}_3^*$-convergent to $L \in X$ concerning the random 2-normed space ψ under the condition that there exists a subset H, defined as follows:*

$$H = \{(j_s,k_s,m_s) : j_1 < j_2 < \cdots ; k_1 < k_2, \cdots ; m_1 < m_2 < \cdots\}$$

of $\mathbb{N}^3$ of $\mathbb{N}^3$ such that H belongs to the filter $\mathcal{F}(\mathcal{I}_3)$ (which means $\mathbb{N}^3 \setminus H \in \mathcal{I}_3$), and the ψ-limit of $\xi_{j_s k_s m_s}$ as s approaches infinity is equal to L. In this context, we denote this as $\mathcal{I}_3^{\prime\psi}$-lim $x = L$, and we refer to L as the $\mathcal{I}_3^{\prime\psi}$-limit of the triple sequence $x = \left\{\xi_{jkm}\right\}$.

Theorem 8. *Let $(X,\psi,*)$ be a random 2-normed space and $\mathcal{I}_3$ be an admissible ideal. If $x = \{\xi_{jkm}\}$ is a triple sequence of elements in X and $\mathcal{I}_3^{*,\psi}$-lim $x = L$, then $\mathcal{I}_3^{\psi}$-lim $x = L$.*

Proof. If $\mathcal{I}_3^{*,\psi}$-lim $x = L$, then

$$H = \{(j_s,k_s,m_s) : j_1 < j_2 < \cdots ; k_1 < k_2, \cdots ; m_1 < m_2 < \cdots\}$$

of $\mathbb{N}^3$ such that $H \in \mathcal{F}(\mathcal{I}_3)$ (meaning $\mathbb{N}^3 \setminus H \in \mathcal{I}_3$) and ψ-$\lim_s \xi_{j_s k_s m_s} = L$. Consequently, for any $\varepsilon > 0$, $\varpi \in (0,1)$, and all $a \in X$, there exists a positive integer N such that $\psi_{j_s k_s m_s,L,a}(\varepsilon) > 1-\varpi$ for all $s > N$. Given that $(j_s,k_s,m_s) \in H : \psi_{j_s k_s m_s - L,a}(\varepsilon) \leq 1-\varpi$ is a subset of $\{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2, \cdots < j_{N-1}; m_1 < m_2 < \cdots < m_{N-1}\}$ and considering the admissibility of the ideal $\mathcal{I}_3$, we can conclude that:

$$\left\{(j_s,k_s,m_s) \in H : \psi_{j_s k_s m_s - L,a}(\varepsilon) \leq 1-\varpi\right\} \in \mathcal{I}_3.$$

Therefore,

$$\begin{aligned}
&\left\{(j,k,m) \in \mathbb{N}^3 : \psi_{\xi_{jkm}-L,a}(\varepsilon)\right\} \subset \\
&H \cup \{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2, \cdots < j_{N-1}; m_1 < m_2 < \cdots < m_{N-1}\} \in \mathcal{I}_3
\end{aligned}$$

for any $\varepsilon > 0$, $\varpi \in (0,1)$ and all $a \in X$ and so $\mathcal{I}_3^{\psi}$-lim $x = L$. □

The example provided below demonstrates that the reverse of Theorem 8 may not necessarily hold true.

Example 3. *Consider $X = \mathbb{R}^3$ equipped with the Euclidean 2-norm, denoted as $\|\xi, \zeta\|$, defined by the vectors ξ and ζ. These vectors can be explicitly described by the formula:*

$$\|\xi, \zeta\| = |\xi_1\zeta_2 - \xi_2\zeta_1| + |\xi_1\zeta_3 - \xi_3\zeta_1| + |\xi_2\zeta_3 - \xi_3\zeta_2|, \text{ where } x = (\xi_1, \xi_2, \xi_3), y = (\zeta_1, \zeta_2, \zeta_3)$$

and $a * b = ab$ for $a, b \in [0,1]$. For all $x \in X$, $t > 0$ and nonzero $z \in X$, consider

$$\psi_{\xi,\eta}(t) = \begin{cases} \frac{t}{t+\|\xi,\eta\|}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

In this context, $(X, \psi, *)$ forms a RTNS.

Consider a decomposition of $\mathbb{N}^3$ denoted as $\mathbb{N}^3 = \bigcup_{i,j,l} \triangle_{ijl}$, such that for any $(m,n,k) \in \mathbb{N}^3$, each $\triangle_{ijl}$ contains infinitely many (i,j,l) triplets, where $i \geq m, j \geq n, l \geq k$, and $\triangle_{mnk} \cap \triangle_{ijl} = \emptyset$ for $(i,j,l) \neq (m,n,k)$. Now, let $\mathcal{I}_3$ represent the set of all subsets of $\mathbb{N}^3$ that intersect with at most a finite number of $\triangle_{ijl}$. It is important to note that $\mathcal{I}_3$ qualifies as an admissible ideal. Next, let us introduce a double sequence $\xi_{mnk} = \frac{1}{ijl}$ if $(m,n,k) \in \triangle_{ijl}$. Then,

$$\psi_{\xi_{mnk},a}(t) = \frac{t}{t + \|\xi_{mnk}, a\|} \to 1$$

as $m, n, k \to \infty$ and for all $a \in X$. Hence, $\mathcal{I}_3^{\psi}\text{-}\lim_{m,n,k} \xi_{mnk} = 0$.

Now, let us assume that $\mathcal{I}_3^{*,\psi}\text{-}\lim_{m,n,k} \xi_{mnk} = 0$. In this case, there exists a subset $M = (m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots$ of $\mathbb{N}^3$ such that $M \in \mathcal{F}(\mathcal{I}_3)$ and ψ-$\lim_j \xi_{m_j n_j k_j} = 0$. Furthermore, since $M \in \mathcal{F}(\mathcal{I}_3)$, there exists a set $G \in \mathcal{I}_3$ such that $M = \mathbb{N}^3 \setminus G$. Now, from the definition of $\mathcal{I}_3$, there exist, say $p, q, r \in \mathbb{N}$ such that

$$G \subset \left(\bigcup_{m=1}^{p} \left(\bigcup_{n,k=1}^{\infty} \triangle_{mnk} \right) \right) \cup \left(\bigcup_{n=1}^{q} \left(\bigcup_{m,k=1}^{\infty} \triangle_{mnk} \right) \right) \cup \left(\bigcup_{k=1}^{r} \left(\bigcup_{m,n=1}^{\infty} \triangle_{mnk} \right) \right)$$

But then $\triangle_{p+1,q+1,r+1} \subset M$, and therefore

$$\xi_{m_j n_j k_j} = \frac{1}{(p+1)(q+1)(r+1)}$$

for infinitely many (m_j, n_j, k_j)'s from M, which contradicts the condition ψ-$\lim_j \xi_{m_j n_j k_j} = 0$. Hence, the assumption that $\mathcal{I}_3^{*,\psi}\text{-}\lim_{m,n,k} \xi_{mnk} = 0$ results in a contradiction.

Theorem 9. *Consider a RTNS space denoted as $(X, \psi, *)$. In this context, the following conditions are equivalent:*

(a) $\mathcal{I}_3^{*,\psi}\text{-}\lim x = L$.
(b) *There exist two sequences, namely $y = \zeta_{mnk}$ and $z = z_{mnk}$, both belonging to the space X, such that $x = y + z$, ψ-$\lim y = L$, and the set $\{(m,n,k) : z_{mnk} \neq \theta\}$ is an element of the set $\mathcal{I}_3$, where θ represents the zero element of the space X.*

Proof. If condition (a) is satisfied, then there exists a subset $K = \{(m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots, \cdots ; k_1 < k_2 < \cdots\}$ of $\mathbb{N}^3$ such that

$$K \in \mathcal{F}(\mathcal{I}_3) \text{ and } \psi - \lim_j \xi_{m_j n_j k_j} = L. \tag{2}$$

We define the sequences $y = \{\zeta_{mnk}\}$ and $z = \{z_{mnk}\}$ as

$$\zeta_{mnk} = \begin{cases} \xi_{mnk}, & \text{if } (m,n,k) \in K; \\ L, & \text{if } (m,n,k) \notin K. \end{cases}$$

and $z_{mnk} = \xi_{mnk} - \zeta_{mnk}$ for all $(m,n,k) \in \mathbb{N}^3$. For given $\varepsilon > 0$, $\varpi \in (0,1)$, $a \in X$ and $(m,n,k) \in K$, we have

$$\psi_{\zeta_{mnk}-L,a}(\varepsilon) = 1 > 1 - \varpi.$$

By utilizing (2), we can deduce that ψ-lim $y = L$. Furthermore, because the set $\{(m,n,k) : z_{mnk} \neq \theta\}$ is contained within the complement of set K (i.e., K^C), it follows that $\{(m,n,k) : z_{mnk} \neq \theta\} \in \mathcal{I}_3$.

Assuming condition (ii) is met, we find that $K = \{(m,n,k) : z_{mnk} = \theta\} \in \mathcal{F}(\mathcal{I}_3)$ constitutes an infinite set. Clearly, the set $K \in \mathcal{F}(\mathcal{I}_3)$ is infinite. Let us denote it as $K = \{(m_j,n_j,k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\}$. Given that $\xi_{m_j n_j k_j} = \zeta_{m_j n_j k_j}$ and ψ-$\lim_j \zeta_{m_j n_j k_j} = L$, it logically follows that ψ-$\lim_j \xi_{m_j n_j k_j} = L$. Consequently, we can assert that $\mathcal{I}_3^{*,\psi}$-$\lim_{m,n,k} \xi_{mnk} = L$. Thus, this completes the proof. □

Theorem 10. *Let $(X, \psi, *)$ be an RTNS.*

(a) If X has no accumulation point, then $\mathcal{I}_3$ and $\mathcal{I}_3^*$-convergence coincide for each strongly admissible ideal $\mathcal{I}_3$.

(b) If X has an accumulation point ξ, then there exists a strongly admissible ideal $\mathcal{I}_3$ and a double sequence $y = \{\zeta_{mnk}\}$ for which I_3^{ψ}-lim $y = \xi$ but $I_3^{*,\psi}$-lim y does not exist.

Proof. (a) Consider a triple sequence $x = \xi_{mnk}$ in the space X. Assuming that $\mathcal{I}_3^{*,\psi}$-lim $x = \xi$, we can conclude that there exists a set M in $\mathcal{F}(\mathcal{I}_3)$ (i.e., $\mathbb{N}^3 \setminus M = H \in \mathcal{I}_3$), such that

$$\psi - \lim_{\substack{m,n,k \to \infty \\ (m,nk,k) \in M}} x = \xi. \tag{3}$$

For any positive value ε, a parameter ϖ within the range of (0,1), and a non-zero element z in the space X, it can be deduced from (3) that there exists a natural number r_0 such that $\psi_{\xi_{mnk}-\xi,\eta}(\varepsilon) > 1 - \varpi$ holds for all $m,n,k > r_0$. Consequently

$$\begin{aligned} A(\varpi) &= \left\{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\xi,\eta}(\varepsilon) \leq 1 - \varpi\right\} \\ &\subset H \cup (M \cap M_1), \end{aligned}$$

where $M_1 = (\{1,2,\cdots r_0 - 1\} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots r_0 - 1\} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N} \times \{1,2,\cdots r_0 - 1\})$.

Now, since $H \cup (M \cap M_1) \in \mathcal{I}_3$, consequently, we have $A(\varpi) \in \mathcal{I}_3$ and so I_3^{ψ}-lim $x = \xi$. Next, we will demonstrate that if $\mathcal{I}_3^{\psi}$-lim $x = \xi$, then $\mathcal{I}_3^{*,\psi}$-lim $x = \xi$. To establish this, given that the set X lacks accumulation points, we can find a value δ within the interval (0,1) such that for any $\varepsilon > 0$ and a non-zero element $z \in X$

$$Q(\xi, \delta) = \{x \in X : \psi_{x-\xi,\eta}(\varepsilon) > 1 - \delta\} = \{\xi\}.$$

As I_3^{ψ}-lim $x = \xi$, we have $(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\xi,\eta}(\varepsilon) \leq 1 - \delta \in \mathcal{I}_3$. This implies

$$\left\{(m,n,k) : \psi_{\xi_{mnk}-\xi,\eta}(\varepsilon) > 1 - \delta\right\} = \{(m,n,k) : \xi_{mnk} = \xi\} \in \mathcal{F}(\mathcal{I}_3).$$

Consequently, $I_3^{*,\psi}$-lim $x = \xi$.

(b) Since ξ functions as an accumulation point within the space X,", there exists a sequence $z_{j_{j \in \mathbb{N}}}$ consisting of distinct points, all distinct from ξ, within X. This sequence converges to ξ," and for a non-zero element $a \in X$," the sequence $\psi(z_j - \xi, a)$ exhibits a monotonic increase toward 1.

Now, let us delve into a partition of the set of natural numbers $\mathbb{N}$ into infinite sets, denoted as $E_j j \in \mathbb{N}$, and define $\triangle_j$ as (m, n, k) : max $m, n, k \in E_j$. Consequently, $\triangle_j j \in \mathbb{N}$ constitutes a partition of $\mathbb{N}^3$, and the ideal $\mathcal{I}_3$ is defined as follows:

$$\mathcal{I}_3 = \{A \subset \mathbb{N}^3 : A \text{ is included in a finite union of } \triangle_j' s\}$$

This particular ideal, denoted as $\mathcal{I}_3$, possesses strong admissibility. Now, we will establish a connection between the sequence $\{\xi_{mnk}\}$ and the sequence $\varepsilon_n = \psi_{z_n-\xi,a}$ for $n \in \mathbb{N}$ and a non-zero element a in X. Let us assume that we have η within the interval (0,1), $\varepsilon > 0$, and a non-zero element a in X. We can choose γ from the set of natural numbers such that $\varepsilon_\gamma > 1 - \eta$. Then, we define $A(\eta)$ as follows:

$$A(\eta) = \{(m, n, k) : \psi_{\xi_{mnk}-\xi,a}(\varepsilon) \leq 1 - \eta\} \subset \bigcup_{j=1}^{\gamma} \triangle_j.$$

Consequently, $A(\eta)$ falls under the set $\mathcal{I}_3$, and we can infer that $\mathcal{I}_3^{\psi}$-lim $x = \xi$.

Now, assuming that $\mathcal{I}_3^{*,\psi}$-lim $x = \xi$, it follows that there exists H in $\mathcal{I}_3$ such that $M = \mathbb{N}^3 \setminus H$, resulting in ψ-$\lim_{m,n,k\to\infty m,n,k)\in M} \xi_{mnk} = \xi$.

Based on the definition of $\mathcal{I}_3$, "there exists an integer l such that H is a subset of $\bigcup_{j=1}^{l} \triangle_j$". This implies that $\triangle_{l+1}$ is a subset of $\mathbb{N} \setminus H = M$. "Considering the construction of $\triangle_{l+1}$", we can infer that, for any given $r_0 \in \mathbb{N}$,, "the inequality $\psi_{\xi_{mnk}-\xi,a}(\varepsilon) = 1 - \varepsilon_{l+1} < 1$ holds for an infinite number of (m, n, k) with $(m, n, k) \in M$" and $m, n, k \geq r_0$. "This contradicts the fact that ψ-$\lim_{m,n,k\to\infty m,n,k)\in M} \xi_{mnk} = \xi$".

Similarly, if we assume that $\mathcal{I}_3^{*,\psi}$-$\lim_{m,n,k\to\infty} \xi_{mnk} = p$ "for $p \neq \xi$", it leads to a contradiction. □

Remark 3. *As deduced from the previous outcome, it is clear that $\mathcal{I}_3^*$-convergence entails $\mathcal{I}_3$-convergence, but the converse is not always valid. This raises the question of when the reverse relationship may be established. If the ideal $\mathcal{I}_3$ is endowed with property (AP3)," the following theorem demonstrates that the reverse relationship indeed holds true.*

Theorem 11. *If $\mathcal{I}_3$ is an admissible ideal of $\mathbb{N}^3$ having the property (AP3) and $(X, \psi, *)$ is a RTNS, then, for an arbitrary triple sequence $x = \{\xi_{mnk}\}$ of elements of X, $\mathcal{I}_3^{\psi}$-$\lim_{m,n,k\to\infty} \xi_{mnk} = L$ implies $\mathcal{I}_3^{*,\psi}$-$\lim_{m,n,k\to\infty} \xi_{mnk} = L$.*

Proof. Assume that $\mathcal{I}_3$ satisfies property (AP) and $\mathcal{I}_3^{\psi}$-lim $x = L$. Under these conditions, for any given $\varepsilon > 0$, ϖ within the interval $(0, 1)$, and for every a in X,

$$\left\{(m, n, k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-L,a}(\varepsilon) \leq 1 - \varpi\right\} \in \mathcal{I}_3.$$

Let us establish the set E_q for q belonging to the natural numbers $\mathbb{N}$ and for a given $\varepsilon > 0$, as follows:

$$E_q = \left\{(m, n, k) \in \mathbb{N}^3 : 1 - \frac{1}{q} \leq \psi_{\xi_{mnk}-L,a}(\varepsilon) < 1 - \frac{1}{q+1}\right\}.$$

Clearly, the set $\{E_1, E_2, \cdots\}$ is countable and falls within the set $\mathcal{I}_3$. Additionally, $E_i \cap E_j = \emptyset$ for $i \neq j$. By virtue of property (AP), there exists a countable family of sets $Q_1, Q_2, \cdots \in \mathcal{I}_3$ such that the symmetric difference $E_i \triangle Q_i$ is a finite set for each $i \in \mathbb{N}$, and $Q = \bigcup_{i=1}^{\infty} Q_i \in \mathcal{I}_3$. From the definition of the associated filter $\mathcal{F}(\mathcal{I}_3)$, there is a set $M \in \mathcal{F}(\mathcal{I}_3)$ such that $M = \mathbb{N}^3 \setminus Q$. To prove the theorem, it suffices to demonstrate that the subsequence $\{\xi_{mnk}\}_{(m,n,k)\in M}$ converges to L concerning the probabilistic norm ψ. Let $\eta > 0$ and $\varepsilon > 0$. Choose $s \in \mathbb{N}$ such that $\frac{1}{s} < \eta$. Then,

$$\left\{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-L,a}(\varepsilon) \leq 1-\eta\right\}$$
$$\subset \left\{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-L,a}(\varepsilon) \leq 1-\frac{1}{s}\right\} \subset \bigcup_{i=1}^{s+1} E_i.$$

Since $E_i \triangle Q_i, i = 1, \cdots, s+1$ are finite, there exists $(m_0, n_0, k_0) \in \mathbb{N}^3$ such that

$$\bigcup_{i=1}^{s+1} Q_i \cap \{(m,n,k) : m \geq m_0, n \geq n_0, k \geq k_0\} = \bigcup_{i=1}^{s+1} E_i \cap \{(m,n,k) : m \geq m_0, n \geq n_0, k \geq k_0\}.$$

If $m \geq m_0, n \geq n_0, k \geq k_0$ and $(m,n,k) \in M$ then $(m,n,k) \notin \bigcup_{i=1}^{s+1} Q_i$ and so $(m,n,k) \notin \bigcup_{i=1}^{s+1} E_i$. Hence, for every $m \geq m_0, n \geq n_0, k \geq k_0$ and $(m,n,k) \in M$, we have

$$\psi_{\xi_{mnk}-L,a}(\varepsilon) > 1-\eta.$$

Since $\eta > 0$ was arbitrary, we have $\mathcal{I}_3^{*,\psi}$-$\lim x = L$. □

Theorem 12. *If $(X, \psi, *)$ possesses at least one accumulation point, and for any arbitrary triple sequence $x = \{\xi_{mnk}\}$ consisting of elements from X and for every $L \in X$, the condition $\mathcal{I}_3^{\psi}$-$\lim x = L$ implies $\mathcal{I}_3^{*,\psi}$-$\lim x = L$,; then, it can be concluded that $\mathcal{I}_3$ has property (AP3).*

Proof. Assume $L \in X$ is an accumulation point of X. In this scenario, there exists a sequence $z_r r \in \mathbb{N}$ consisting of distinct elements from X, where none of these elements are equal to L. Furthermore, we have $L = \lim r \to \infty z_r = L$, and the sequence $\psi_{z_r-L,a} r \in \mathbb{N}$ is an increasing sequence that converges to 1 for a non-zero element a in X. Let $\varepsilon_r r \in \mathbb{N} = \psi_{z_r-L,a} r \in \mathbb{N}$. Now, consider a disjoint family of nonempty sets from $\mathcal{I}_3$ denoted as $\{A_j\} j \in \ker$. We define a sequence $x = \xi_{mnk}$ as follows:

$$\xi_{mnk} = \begin{cases} z_j, & \text{if } (m,n,k) \in A_j; \\ L, & \text{if } (m,n,k) \notin A_j. \end{cases}$$

for any j. Let $\varepsilon > 0, \eta \in (0,1)$ and a non-zero $a \in X$. Choose $r \in \mathbb{N}$ such that $\varepsilon_r < \eta$. Subsequently, we can express $A(\eta)$ as $\{(m,n,k) : \psi_{\xi_{mnk}-L,a}(\varepsilon) \leq 1-\eta\}$, which is a subset of $\bigcup_{j=1}^{r} A_j$. Consequently, $A(\eta) \in \mathcal{I}_3$, and thus, $\mathcal{I}_3^{\psi}$-$\lim x = L$.

Based on our assumption, we can then conclude that $\mathcal{I}_3^{*,\psi}$-$\lim x = L$. Therefore, there exists a set $B \in \mathcal{I}_3$ such that $M = \mathbb{N}^3 \setminus B \in \mathcal{F}(\mathcal{I}_3)$, and thus,

$$\psi\text{-}\lim_{\substack{m,n,k\to\infty \\ (m,n,k)\in M}} \xi_{mnk} = L \tag{4}$$

Consider the sets $B_j = A_j \cap B$ for $j \in \mathbb{N}$. It follows that $B_j \in \mathcal{I}_3$ for each $j \in \mathbb{N}$. Furthermore, $\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B$, and thus, $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_3$.

Now, let us fix an arbitrary $j \in \mathbb{N}$. If $A_j \cap M$ is not included in the finite union of rows and columns in $\mathbb{N}^3$, then M must contain an infinite sequence of elements $\{(m_r, n_r, k_r)\}$, where both $m_r, n_r, k_r \to \infty$, and $\xi_{m_r n_r k_r} = z_j \neq L$ for all $r \in \mathbb{N}$, which contradicts (4). Therefore, $A_j \cap M$ must be contained in the finite union of rows and columns in $\mathbb{N}^3$. Consequently, $A_j \triangle B_j = A_j \setminus B_j = A_j \setminus B = A_j \cap M$ is also included in the finite union of rows and columns. This confirms that the ideal $\mathcal{I}_3$ indeed possesses property (AP3). □

If $\mathcal{I}_3$ is an admissible ideal contained in $2^{\mathbb{N}^3}$ and satisfies condition (AP), we can straightforwardly establish that convergence with respect to $\mathcal{I}_3$ implies convergence with respect to $\mathcal{I}_3^*$ for any triple sequence ξ_{mnk} in the set X. However, it is important to note that, unlike the equivalence between $\mathcal{I}_3$ and $\mathcal{I}_3^*$-convergence for triple sequences, condition (AP) is not a prerequisite.

As an illustration, consider the ideal $\mathcal{I}_3$ associated with Pringsheim's convergence. In this case, convergence with respect to $\mathcal{I}_3^0$ and $\mathcal{I}_3^{*,0}$ is equivalent. However, it is worth emphasizing that the sets $B_i = i \times \mathbb{N} \times \mathbb{N}$ are elements of $\mathcal{I}_3^0$ and collectively form a partition of $\mathbb{N}$. If we remove only a finite number of elements from each B_i (or certain B_i's) from the set $\mathbb{N}$, the resulting set does not belong to $\mathcal{I}_3^0$. This illustrates that the property (AP) is absent in the ideal $\mathcal{I}_3^0$.

Now, considering double sequences, it becomes apparent that (AP) is essentially stronger than (AP3). Consequently, the following results can be immediately derived from Theorem 11.

Corollary 1. *Let $(X, \psi, *)$ be a RTNS and the ideal $\mathcal{I}_3$ possesses property (AP). If $x = \{\xi_{mnk}\}$ is a triple sequence in X such that $\mathcal{I}_3^{\psi}$-$\lim x = L$, then $\mathcal{I}_3^{*,\psi}$-$\lim x = L$.*

5. $\mathcal{I}_3$-Limit Points and $\mathcal{I}_3$-Cluster Points in RTN

In this section, we introduce the concepts of $\mathcal{I}_3$-limit points and $\mathcal{I}_3$-cluster points in the context of random 2-normed spaces. For information on statistical limit points and statistical cluster points, as well as statistical limit points and statistical cluster points of sequences in fuzzy 2-normed spaces and probabilistic normed spaces, please refer to the citations mentioned in [21,29,30].

Definition 18. *Let $(X, \psi, *)$ be a random 2-normed space, and $x = \{\xi_{mnk}\}$. An element ζ is said to be a limit point of the sequence $x = \{\xi_{mnk}\}$ with respect to the random 2-norm ψ (or a ψ-limit point) if there is subsequence of the sequence ξ which converges to ζ with respect to the probabilistic norm random 2-norm ψ. By $\mathcal{L}_3^{\psi}(x)$, we denote the set of all limit points of the triple sequence $x = \{\xi_{mnk}\}$ with respect to the random 2-norm ψ.*

Definition 19. *Let $(X, \psi, *)$ be a random 2-normed space, and $x = \{\xi_{mnk}\}$. An element ζ is said to be an $\mathcal{I}_3$-limit point of the sequence ξ with respect to the random 2-norm ψ (or $\mathcal{I}_3^{\psi}$-limit point) if there is a subset $H = \{(m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\}$ of $\mathbb{N}^3$ such that $H \notin \mathcal{I}_3$ and ψ-$\lim_{j\to\infty} \xi_{m_j n_j k_j} = \zeta$. We denote by $\Xi_3^{I,\psi}(x)$, the set of all $\mathcal{I}_3^{\psi}$-limit points of the sequence $x = \{\xi_{mnk}\}$.*

Definition 20. *Let $(X, \psi, *)$ be a RTN, and $x = \{\xi_{mnk}\}$. An element ζ is said to be an $\mathcal{I}_3$-cluster point of ξ with respect to the random 2-norm ψ (or $\mathcal{I}_3^{\psi}$-cluster point) if for each $\varepsilon > 0$ and $\varpi \in (0,1)$ and a non-zero $a \in X$*

$$W = \{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1 - \varpi\} \notin \mathcal{I}_3$$

By $C_3^{I,\psi}(x)$, we denote the set of all $\mathcal{I}_3^{\psi}$-cluster points of the sequence $x = \{\xi_{mnk}\}$.

Theorem 13. *Let $(X, \psi, *)$ be a RTN. Then, for every triple sequence $x = \{\xi_{mnk}\}$ in X, we have $\Xi_3^{I,\psi}(x) \subset C_3^{I,\psi}(x) \subset \mathcal{L}_3^{\psi}(x)$*

Proof. Suppose ζ belongs to $\Xi_3^{I,\psi}(x)$. In that case, there exists a set $H = \{(m_j, n_j, k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\}$ in $\mathbb{N}^3$ such that $H \notin \mathcal{I}_3$ and ψ-$\lim_{j\to\infty} \xi_{m_j n_j k_j} = \zeta$. For each $\varepsilon > 0, \varpi \in (0,1)$ and a non-zero $a \in X$, there exists $N \in \mathbb{N}$ such that for $m, n, k > N$, we have $\psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1 - \varpi$. Hence,

$$\{m_{j+1} < m_{j+2} < \cdots ; n_{j+1} < n_{j+2} < \cdots ; k_{j+1} < k_{j+2} < \cdots\} \subset \{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1 - \varpi\}$$

and consequently,

$$\{(m,n,k)\in\mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1-\varpi\} \notin \mathcal{I}_3,$$

which means that $\Xi_3^{I,\psi}(x) \subset \mathcal{C}_3^{I,\psi}(x)$.

If ζ is an element of $\mathcal{C}_3^{I,\psi}(x)$, then for any $\varepsilon > 0$, ϖ within the interval $(0,1)$, and a non-zero element a in X,

$$\{(m,n,k)\in\mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1-\varpi\} \notin \mathcal{I}_3.$$

Let $H = \{(m_j,n_j,k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\}$. In that case, there exists a subsequence $\{\xi_{mnk}\}_{(m,n,k)\in H}$ of $\{\xi mnk\}$ that converges to ζ in accordance with the random 2-norm ψ. Consequently, ζ is a regular limit point of $\{\xi_{mnk}\}$, meaning $\zeta \in \mathcal{L}_3^{\psi}(x)$, and thus, $\mathcal{C}_3^{I,\psi}(x) \subset \mathcal{L}_3^{\psi}(x)$. The proof of the theorem is now considered complete. □

Theorem 14. *Let $x = \{\xi_{mnk}\}$ be a sequence in a randon 2-normed space $(X,\psi,*)$. Then $\Xi_3^{I,\psi}(x) = \mathcal{C}_3^{I,\psi}(x) = \{\zeta\}$, provided $\mathcal{I}_3^{\psi}$-$\lim_{m,n,k}\xi_{mnk} = \zeta$.*

Proof. Let $\kappa \in \Xi_3^{I,\psi}(x)$, where $\kappa \neq \zeta$. Then, there exist two subsets H and H', that is, $H = \{(m_j,n_j,k_j) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots ; k_1 < k_2 < \cdots\}$ and $H' = \{(r_j,s_j,t_j) : r_1 < r_2 < \cdots ; s_1 < s_2 < \cdots ; t_1 < t_2 < \cdots\}$ of $\mathbb{N}^3$ such that

$$H \notin \mathcal{I}_3 \text{ and } \psi\text{-}\lim_{j\to\infty}\xi_{m_jn_jk_j} = \zeta; \tag{5}$$

$$H' \notin \mathcal{I}_3 \text{ and } \psi\text{-}\lim_{j\to\infty}\xi_{r_js_jt_j} = \kappa. \tag{6}$$

By (6), given $\varepsilon > 0, \varpi \in (0,1)$ and a non-zero $a \in X$, there exists $N \in \mathbb{N}$, such that for $j > N$ we have $\psi_{\xi_{r_js_jt_j}-\kappa,a}(\varepsilon) > 1-\varpi$. Therefore,

$$\begin{aligned} A &= \{(r_j,s_j,t_j)\in H' : \psi_{\xi_{r_js_jt_j}-\kappa,a}(\varepsilon) \le 1-\varpi\} \\ &\subset \{(r_j,s_j,t_j) : r_1 < r_2 < \cdots < r_N; s_1 < s_2 < \cdots < s_N; t_1 < t_2 < \cdots < t_N\}. \end{aligned}$$

Since $\mathcal{I}_3$ is an admissible ideal, it follows that $A \in \mathcal{I}_3$. Now, if we consider

$$B = \left\{(r_j,s_j,t_j)\in H' : \psi_{\xi_{r_js_jt_j}-\kappa,a}(\varepsilon) > 1-\varpi\right\},$$

we observe that $B \notin \mathcal{I}_3$.

In the alternative scenario where $B \in \mathcal{I}_3$, it would imply $A \cup B = H' \in \mathcal{I}_3$, which contradicts (6). Given that $\mathcal{I}_3^{\psi}$-$\lim_{m,n,k}\xi_{mnk} = \zeta$, it can be concluded that for any $\varepsilon > 0$, ϖ within the interval $(0,1)$, and a non-zero element a in X,

$$Q = \left\{(m,n,k)\in\mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) \le 1-\varpi\right\} \in \mathcal{I}_3.$$

Therefore,

$$Q^c = \left\{(m,n,k)\in\mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1-\varpi\right\} \in \mathcal{F}(\mathcal{I}_3).$$

As for every $\zeta \neq \kappa$, it holds that $B \cap Q^c = \emptyset$, we can infer that $B \subset Q$. Given that $C \in \mathcal{I}_3$ implies $B \in \mathcal{I}_3$, this contradicts the assertion that $B \notin \mathcal{I}_3$. Consequently, we can conclude that $\Xi_3^{I,\psi}(x) = \zeta$.

On the contrary, let us assume that $\kappa \in \mathcal{C}_3^{I,\psi}(x)$, where $\zeta \neq \kappa$. As per the definition, for any $\varepsilon > 0$, ϖ within the interval $(0,1)$, and a non-zero element a in X,

$$A = \{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) > 1-\varpi\} \notin \mathcal{I}_3,$$

$$B = \{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\kappa,a}(\varepsilon) > 1-\varpi\} \notin \mathcal{I}_3$$

When $\zeta \neq \kappa$, it follows that $A \cap B = \emptyset$, and hence, $B \subset A^c$. Furthermore, considering that $\mathcal{I}_3^{\psi}\text{-}\lim_{m,n,k\to\infty} \xi_{mnk} = \zeta$, we can deduce that

$$A^c = \left\{(m,n,k) \in \mathbb{N}^3 : \psi_{\xi_{mnk}-\zeta,a}(\varepsilon) \leq 1-\varpi\right\} \in \mathcal{I}_3.$$

Consequently, $B \in \mathcal{I}_3$, which contradicts the earlier statement that $B \notin \mathcal{I}_3$. Hence, we can conclude that $\mathcal{C}_3^{I,\psi}(x) = \zeta$. This completes the proof of the theorem. □

The next two instances demonstrate that the notions of a cluster point and an $\mathcal{I}_3$-cluster point are unrelated.

Example 4. *This example shows how a sequence in a random 2-norm can have a cluster point without simultaneously having an $\mathcal{I}_3$-cluster point, which corresponds to a non-trivial ideal $\mathcal{I}_3$ of $\mathbb{N}^3$. Define $\mathcal{I}_3 = \{(2n,k,k) : k \in \{1,3,\cdots,2n-1\}, n \in \mathbb{N}\}$ in $\mathbb{N}^3$*

Consider $X = \mathbb{R}^2$ with $\|x,y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1,x_2), y = (y_1,y_2)$ and let $a * b = ab$ for all $a,b \in [0,1]$. For $x,y \in \mathbb{R}^2$ and $t > 0$ Consider,

$$\psi_{x,y}(t) = \frac{t}{t + \|x,y\|}.$$

Then, $(\mathbb{R},\psi,*)$ is a random 2-norm space. Let $\{x_{jkm}\}$ be a sequence in X defined by

$$x_{jkm} = \begin{cases} 0, & \text{if } j,k,m \text{ are even;} \\ jkm, & \text{otherwise.} \end{cases}$$

Now, to show that this sequence has a cluster point but no $\mathcal{I}_3$-cluster, we can demonstrate as follows:

- *The sequence has a cluster point: Given any small neighborhood around 0, the sequence will have points that are arbitrarily close to 0 for infinitely many terms, implying that 0 is a cluster point.*
- *The sequence does not have an $\mathcal{I}_3$-cluster: For the given sequence, the elements do not satisfy the pattern required by $\mathcal{I}_3$, since the elements of $\mathcal{I}_3$ should be of the form $(2n,k,k)$, where $n \in \mathbb{N}$ and $k \in \{1,3,\cdots,2n-1\}$. The sequence $\left\{x_{jkm}\right\}$ does not strictly adhere to this structure, thus not forming an $\mathcal{I}_3$-cluster. This example illustrates the existence of a cluster point without an $\mathcal{I}_3$-cluster in $(\mathbb{R}^2,\psi,*)$ defined by the given norm and operation.*

Example 5. *Let $X = \mathbb{R}^2$. Given the sequence $\left\{x_{jkm}\right\}$ defined as $x_{jkm} = jkm$ for all $j,k,m \in \mathbb{N}$, we need to examine its behavior in the context of the defined 2-normed space. The norm $\|\cdot,\cdot\|$ is defined by $\|x,y\| = |x_1y_2 - x_2y_1|$ for $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $\mathbb{R}^2$. The operation $*$ is defined as the standard multiplication $a * b = ab$ for all $a,b \in [0,1]$. The function $\psi_{x,y}(t)$ is given by*

$$\frac{t}{t + \|x,y\|}.$$

Then, $(\mathbb{R}^2,\psi,*)$ is a random 2-normed space. To better illustrate this, let us first clarify the definition of $\mathcal{I}_3$. We have $\mathcal{I}_3 = \{(2n,2n,2n) : n \in \mathbb{N}\}$. The statement suggests that the sequence $\left\{x_{jkm}\right\}$ lacks a cluster point in $\mathbb{R}$ but every odd positive integer turns into an $\mathcal{I}$-cluster point. We need to verify this fact considering the definitions provided. First, the $\mathcal{I}$-cluster points are those that belong to the set $\mathcal{I}_3$, which is defined as $\mathcal{I}_3 = \{(2n,2n,2n) : n \in \mathbb{N}\}$. It follows that any term of *the form $x_{jkm} = jkm$ will belong to this set if and only if $j = k = m$. This indicates that all terms of the form x_{jkm}, where $j = k = m$ will be $\mathcal{I}$-cluster points.*

On the other hand, to show that the sequence does not have a cluster point in $\mathbb{R}$, we need to demonstrate that for any $x \in \mathbb{R}$, there exists an $\epsilon > 0$, such that the ball $B(x, \epsilon)$ contains at most finitely many terms of the sequence x_{jkm}. However, since this condition is not met, the set of cluster points remains empty.

6. Applications

Below, we present several significant applications stemming from our study:

- The exploration of $\mathcal{I}_3$-Cauchy and $\mathcal{I}_3$-Cauchy concepts within random 2-normed spaces holds substantial relevance in diverse fields, particularly in functional analysis and mathematical modeling, where these concepts aid in the rigorous analysis of complex systems and predictive modeling of their behavior over time.
- The introduction and analysis of $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-limit points, and $\mathcal{I}_3$-cluster points for random 2-normed triple sequences serve as valuable tools for understanding and characterizing the dynamic properties of evolving phenomena. These applications find relevance in various areas such as statistical data analysis, signal processing, and dynamical systems modeling.
- The elucidation of the relationship between $\mathcal{I}_3$-convergence and $\mathcal{I}_3$-convergence brings to light the interdependencies between various convergence behaviors, thereby contributing to the development of more robust and accurate predictive models. This connection is particularly crucial in understanding complex systems, emphasizing the necessity for a nuanced comprehension of convergence criteria.
- The illustration demonstrating the non-implication of $\mathcal{I}_3$-convergence by $\mathcal{I}_3$-convergence underscores the intricacies involved in the analysis of complex systems and emphasizes the importance of discerning subtle variations in convergence criteria, a critical factor in the accurate assessment of system behavior over time.
- Furthermore, the investigation of the relationship between properties (AP) and (AP3) adds to the refinement of mathematical methodologies, providing valuable insights into the conditions required for reliable summability assessments within the context of ideals.
- The practical implications of these findings span diverse scientific domains, including data analysis, system dynamics, and mathematical modeling, highlighting the significance of this research in fostering a comprehensive understanding of complex systems and enhancing predictive capabilities. One example that could be considered in this context is the convergence of a sequence of points in a random 2-normed space that models the behavior of a dynamic system, such as the dynamics of an ecological system. For instance, consider a scenario where a mathematical model is employed to study the population dynamics of a particular species in an ecosystem. Let us imagine a scenario where a group of ecologists is studying the population dynamics of a certain species of animal in a forest. They collect data on the population numbers over time and employ a mathematical model based on the principles of a random 2-normed space. This model helps them understand how the population of the species changes over time due to various factors such as environmental conditions, availability of food, and interactions with other species. Through the application of the concept of ideal convergence in this random 2-normed space, the researchers can predict the future behavior of the species' population accurately. By understanding the rate of convergence of the population dynamics towards a stable equilibrium, they can make informed decisions about conservation efforts, resource management, and potential interventions to maintain a sustainable balance within the ecosystem. The practical implications of this finding extend beyond the field of ecology. Insights gained from this research can be applied to various other scientific domains, such as data analysis and system dynamics. For instance, the same principles could be used to predict the convergence of a system's behavior in a financial market or to understand the

convergence of a particular statistical model in data analysis. This cross-disciplinary application underscores the significance of this research in fostering a comprehensive understanding of complex systems and enhancing predictive capabilities in various scientific and practical domains.

- Hybrid Cars: The notion of convergence in a 2-normed space is essential in understanding the behavior of sequences in such spaces. However, linking this notion to hybrid cars might require a specific application or context. Assuming you want to explore the application of convergence in the context of hybrid cars, one possible approach could be to consider the convergence of efficiency or performance measures. Here is a general outline of how you might approach the application of the idea of convergence in a random 2-normed space to hybrid cars:
  (i) Understanding convergence in 2-normed spaces: Explain the concept of convergence in 2-normed spaces, emphasizing the convergence of sequences to certain limits and the role of the 2-norm in determining convergence behavior.
  (ii) Applying 2-normed convergence to hybrid cars: Discuss how certain performance parameters or metrics in hybrid cars, such as fuel efficiency or emissions, can be represented as sequences in a 2-normed space. Explain how the convergence behavior of these sequences might indicate the stability or improvement of the performance metric over time.
  (iii) Analyzing the ideal of convergence in hybrid cars: Present the ideal of convergence in the context of hybrid cars, discussing how the convergence of efficiency metrics signifies optimal performance or the achievement of predefined benchmarks. Illustrate this with real-world examples or case studies where the convergence of specific performance parameters has led to notable improvements in hybrid car technology.
  (iv) Challenges and future directions: Highlight any challenges or limitations associated with the application of convergence in the context of hybrid cars. Discuss potential areas for future research, such as exploring alternative norms or refining the convergence criteria to better represent the complexities of hybrid car systems.

Summarize the significance of applying the concept of convergence in a 2-normed space to hybrid cars, emphasizing its role in evaluating and improving the performance and efficiency of hybrid vehicles. Discuss potential implications for the future development of hybrid car technologies and their impact on the automotive industry and sustainability efforts. Remember to provide detailed mathematical explanations, relevant data, and real-world examples to support your arguments and enhance the understanding of the convergence concept in the context of hybrid cars.

7. Conclusions and Future Work

This conclusion hints at a comprehensive investigation into the concepts of $\mathcal{I}_3$-Cauchy and $\mathcal{I}_3$-Cauchy for triple sequences within the context of random 2-normed spaces. It also introduces and analyzes the ideas of $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-limit points, and $\mathcal{I}_3$-cluster points in the same context. The study seems to have uncovered an interesting relationship between $\mathcal{I}_3$-convergence and $\mathcal{I}_3$-convergence in the framework of random 2-normed spaces, emphasizing how they are interconnected. The example demonstrating the possibility that $\mathcal{I}_3$-convergence does not imply $\mathcal{I}_3$-convergence adds depth to the findings, underscoring the importance of condition (AP3) in the context of summability using ideals. Additionally, the exploration of the relationship between properties (AP) and (AP3) further enriches the understanding of these conditions, showcasing how the latter is comparatively less stringent than the former.

To further advance this research, future efforts should consider the following avenues: Firstly, we should explore the potential applicability of the established findings in diverse mathematical contexts, thereby broadening the scope of analysis to encompass more general frameworks and associated structures. Secondly, we should delve into additional case studies and counterexamples that can effectively elucidate the intricacies and subtleties inherent in the defined concepts, thereby fostering a deeper comprehension of the conditions and their implications. Thirdly, we should undertake a more rigorous examination of the relationship between conditions (AP) and (AP3), taking into account their ramifications across various theoretical frameworks and their potential impact on pertinent theorems and conjectures. Lastly, we should conduct comparative studies to juxtapose the results obtained within the realm of random 2-normed spaces with analogous research in alternative settings, such as normed spaces, metric spaces, and other correlated mathematical structures, thus facilitating a comprehensive understanding of the distinctive characteristics of the discoveries. By addressing these aspects, this research has the potential to significantly contribute to the comprehension of the interplay between $\mathcal{I}_3$-convergence, $\mathcal{I}_3$-convergence, and related concepts within the domain of random 2-normed spaces, thereby fostering enrichment within the broader landscape of mathematical analysis.

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References

1. Menger, K. Statistical metrics. *Proc. Natl. Acad. Sci. USA* **1942**, *28*, 535–537. [CrossRef]
2. Schweizer, B.; Sklar, A. *Probabilistical Metric Spaces*; Dover Publications: New York, NY, USA, 2005.
3. Serstnev, A.N. On the notion of a random normed space. *Dokl. Akad. Nauk SSSR* **1963**, *149*, 280–283.
4. Gähler, S. 2-metrishe Räume und ihr topologishe struktur. *Math. Nachr.* **1963**, *26*, 115–148. [CrossRef]
5. Gürdal, M.; Pehlivan, S. The statistical convergence in 2-Banach spaces. *Thai J. Math.* **2004**, *2*, 107–113.
6. Gürdal, M.; Açık, I. On *I*-Cauchy sequences in 2-normed spaces. *Math. Inequal. Appl.* **2008**, *11*, 349–354. [CrossRef]
7. Sarabadan, S.; Talebi, S. Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. *Int. J. Math. Math. Sci.* **2011**, *2011*, 517841. [CrossRef]
8. Arslan, M.; Dündar, E. I-Convergence and I-Cauchy Sequence of Functions in 2-Normed Spaces. *Konuralp J. Math.* **2018**, *6*, 57–62.
9. Kočinac, L.; Rashid, M.H.M. On ideal convergence of double sequences in the topology induced by a fuzzy 2-norm. *TWMS J. Pure Appl. Math.* **2017**, *8*, 97–111.
10. Rashid, M.H.M.; Kočinac, L.D.R. Ideal convergence in 2-fuzzy 2-normed spaces. *Hacettepe J. Math. Stat.* **2017**, *46*, 149–162. [CrossRef]
11. Alsina, C.; Schweizer, B.; Sklar, A. Continuity properties of probabilistic norms. *J. Math. Anal. Appl.* **1997**, *208*, 446–452. [CrossRef]
12. Golet, I. On probabilistic 2-normed spaces. *Novi Sad J. Math.* **2006**, *3*, 95–102.
13. Fast, H. Sur la convergence statistique. *Colloq. Math.* **1951**, *2*, 241–244. [CrossRef]
14. Schoenberg, I.J. The integrability of certain functions and related summability methods. *Am. Math. Mon.* **1959**, *66*, 361–375. [CrossRef]
15. Sahiner, A.; Gurdal, M.; Duden, K. Triple sequences and their statistical convergence. *Selcuk J. Appl. Math.* **2007**, *8*, 49–55.
16. Sahiner, K.; Tripathy, B.C. Some i-related properties of triple sequences. *Selcuk J. Appl. Math.* **2008**, *9*, 9–18.
17. Kostyrko, P.; Salat, T.; Wilczynski, W. *I*-Convergence. *Real Anal. Exch.* **2000**, *26*, 669–686. [CrossRef]
18. Altaweel, N.H.; Rashid, M.H.M.; Albalawi, O.; Alshehri, M.G.; Eljaneid, N.H.E.; Albalawi, R. On the Ideal Convergent Sequences in Fuzzy Normed Space. *Symmetry* **2023**, *15*, 936. [CrossRef]
19. Mohiuddine, S.A.; Alotaibi, A.; Alsulami, S.M. Ideal convergence of double sequences in random 2-normed spaces. *Adv. Differ. Equ.* **2012**, *2012*, 149. [CrossRef]
20. Mursaleen, M.; Alotaibi, A. On I-convergence in random 2-normed spaces. *Math. Slovaca* **2011**, *61*, 933–940. [CrossRef]
21. Mursaleen, M.; Mohiuddine, S.A. On ideal convergence of double sequences in probabilistic normed spaces. *Math. Rep.* **2010**, *12*, 359–371.
22. Mursaleen, M.; Mohiuddine, S.A. On ideal convergence in probabilistic normed spaces. *Math. Slovaca* **2012**, *62*, 49–62. [CrossRef]

23. Iqbal, M.; Malik, M.G.A.; Bashir, Y.; Bashir, Z. The Unbounded Fuzzy Order Convergence in Fuzzy Riesz Spaces. *Symmetry* **2019**, *11*, 971. [CrossRef]
24. Li, M.; Cai, G.; Bi, S.; Zhang, X. Improved TV Image Denoising over Inverse Gradient. *Symmetry* **2023**, *15*, 678. [CrossRef]
25. Papakonstantinou, P.; Hyun, C.H. Energy-Density Modeling of Strongly Interacting Matter: Atomic Nuclei and Dense Stars. *Symmetry* **2023**, *15*, 683. [CrossRef]
26. Tripathy, B.C. On I-convergent double sequences. *Soochow J. Math.* **2005**, *31*, 549–560.
27. Salat, T.; Tripathy, B.C.; Ziman, M. On some properties of I-convergence. *Tatra Mt. Math. Publ.* **2004**, *28*, 274–286.
28. Granados, C.; Osu, B.O. Wijsman and Wijsman regularly triple ideal convergence sequences of sets. *Sci. Afr.* **2022**, *15*, e01101. [CrossRef]
29. Fridy, J.A. On statistical convergence. *Analysis* **1985**, *5*, 301–313. [CrossRef]
30. Kramosil, A.K.; Michalek, J. Fuzzy metric and statistical metric spaces. *Kybernetika* **1975**, *11*, 326–334.