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Euclidean Jordan Algebras, Symmetric Association Schemes, Strongly Regular Graphs, and Modified Krein Parameters of a Strongly Regular Graph

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Abstract: In this paper, in the environment of Euclidean Jordan algebras, we establish some inequalities over the Krein parameters of a symmetric association scheme and of a strongly regular graph. Next, we define the modified Krein parameters of a strongly regular graph and establish some admissibility conditions over these parameters. Finally, we introduce some relations over the Krein parameters of a strongly regular graph.

Keywords: Euclidean Jordan algebras; graph theory; strongly regular graphs

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1. Introduction

This paper is an extended version of the work "An Euclidean Jordan Algebra of Symmetric Matrices Closed for the Schur Product of Matrices" presented in Congress Circuits, Systems, Communications, and Computers 2023 [1]. For a good understanding of the theory of Euclidean Jordan algebras we refer to the book Analysis on Symmetric Cones (see [2]) For a good survey about association schemes we refer to the texts presented in References [3,4]. To apply matrix theory to engineering, mathematics, and cryptography, we refer to References [5,6]. Several mathematicians and engineers have developed their investigation into several science areas of mathematics, working in the environment of Euclidean Jordan algebras (see, for instance, References [7–9]). Euclidean Jordan algebras have also become a good tool for analyzing discrete structures' eigenvalues like strongly regular graphs (see [10]). We must also say that other authors extended the properties of the spectrum of a symmetric matrix to simple Euclidean Jordan algebras (see, for example, [11,12]). Euclidean Jordan algebras have also become an excellent environment to analyze the spectrum of symmetric association schemes. We now describe the plan of the paper. In Section 2, we describe the principal concepts of real finite-dimensional Jordan algebras and real finite-dimensional Euclidean Jordan algebras that one needs to understand in the next sections of this paper. In Section 3, we present a description of some properties of symmetric association schemes and some examples. In Section 4, we define the Krein parameters of a Euclidean Jordan algebra associated with a symmetric association scheme, and next, we deduce some admissibility conditions over these Krein parameters. In Section 5, we present some theory about strongly regular graphs. Next, in Section 6, we define the modified Krein parameters of a strongly regular graph and establish some inequalities over these type parameters. And, we define in this section some new inequalities over the Krein parameters of a strongly regular graph. Finally, in the last section, we present some considerations about the Krein parameters of a symmetric association scheme and of a strongly regular graph.

2. Some Theory about Jordan Algebras and Euclidean Jordan Algebras

Herein, we will describe only the more relevant concepts about the theory of finite-dimensional real Euclidean Jordan algebras that we will use in this paper.

Let us consider a real finite-dimensional vector space $\mathcal{A}$ equipped with a vector multiplication of vectors $\bigstar$. Then, $\mathcal{A}$ is a real Jordan algebra if it is a real commutative algebra such that, for any of its elements a and b, we have $a^{2\bigstar}\bigstar(a\bigstar b) = a\bigstar(a^{2\bigstar}\bigstar b)$, where $a^{2\bigstar} = a\bigstar a$. And, for any natural number l, the powers of order l are defined in the following way: $a^{0\bigstar} = e, a^{1\bigstar} = a, a^{l\bigstar} = a\bigstar a^{(l-1)\bigstar}, l \geq 2$, where e is the unit element of the vector multiplication of the finite-dimensional real Jordan algebra $\mathcal{A}$. One says that $\mathcal{A}$ is a real Euclidean Jordan algebra if $\mathcal{A}$ is equipped with an inner product $\bullet|\bullet$ such that, for any elements x, y, and z of $\mathcal{A}$, we have $(x\bigstar y)|z = y|(x\bigstar z)$. In the following text of this paper, we will use the abbreviation RFEJA to designate a real finite-dimensional Euclidean Jordan algebra. We are only interested in an RFEJA with a unit, which we will denote always by **e**. And, we will use the abbreviation EJA to designate an Euclidean Jordan algebra.

Let us consider an RFEJA $\mathcal{B}$ equipped with the operation of the multiplication of its vectors $\bigstar$, the inner product of its vectors $\bullet|\bullet$, and with the multiplication unit **e**. The rank of the element h in $\mathcal{B}$ is the smallest natural number t such that $\{e, h^{1\bigstar}, \ldots, h^{t\bigstar}\}$ is a linearly dependent set, and we write $\text{rank}(h) = t$. Since $\mathcal{B}$ is an RFEJA, then for any $h \in \mathcal{B}$, we have $\text{rank}(h) \leq \dim(\mathcal{B})$. We define $\text{rank}(\mathcal{B}) = \max\{\text{rank}(h) : h \in \mathcal{B}\}$. An element $h \in \mathcal{B}$ is an idempotent if $h^{2\bigstar} = h$. The idempotent h and g are orthogonal if $h\bigstar g = 0$. We say that the set of non-null vectors of $\mathcal{B}$, $\{h_1, h_2, \ldots, h_t\}$ is a complete system of orthogonal idempotent, and the abbreviation that we will use through will be CSOI, if $h_k^{2\bigstar} = h_k$, for $k = 1, \ldots, t$, $h_k\bigstar h_l = 0$, if $k \neq l$ and $1 \leq k, l \leq t$, and $\sum_{j=1}^{t} h_j = e$. An idempotent of $\mathcal{B}$ is primitive if it is a non-zero idempotent of $\mathcal{B}$ and cannot be written as a sum of two non-zero orthogonal idempotent. A Jordan frame of $\mathcal{B}$ is a set $S = \{h_1, h_2, \ldots, h_s\}$ of non-zero idempotent of $\mathcal{B}$ such that S is a complete system of orthogonal idempotent, along the text we will use the abbreviation JF to designate it, such that each idempotent is primitive.

Theorem 1 ([2], p. 43)**.** *Let us consider an RFEJA $\mathcal{B}$ with the unit* **e**. *Then, for h in $\mathcal{B}$, there exist unique real numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$, all distinct, and a unique CSOI $\{f_1, f_2, \ldots, f_l\}$ such that*

$$h = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_l f_l. \quad (1)$$

Decomposition (1) is called the first spectral decomposition of h.

Theorem 2 ([2], p. 44)**.** *Let us consider an RFEJA $\mathcal{B}$ with unit* **e** *and such that rank$(\mathcal{B}) = r$. Then, for each* **h** *in $\mathcal{B}$, there exists an JF $\{f_1, f_2, \cdots, f_r\}$ and real numbers $\beta_1, \cdots, \beta_{r-1}$ and β_r such that (2) is verified.*

$$h \;=\; \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_r f_r. \quad (2)$$

Decomposition (2) is called the second spectral decomposition of h.

In an RFEJA, all the Jordan frames have the same cardinally as their rank.

3. Some Properties of Symmetric Association Schemes

A symmetric associative scheme $\mathcal{F}$ with d classes is a finite set $\mathcal{X}$ provided with $d+1$ relations $\mathbb{C}_i$ such that

1. $\{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{d-1}, \mathcal{C}_d\}$ is a partition of $\mathcal{X} \times \mathcal{X}$;
2. $\mathcal{C}_0 = \{(a, a) : a \in \mathcal{X}\}$;
3. For $i \in \{0, 1, \cdots, d\}$ if $(a, b) \in \mathcal{C}_i$, then $(b, a) \in \mathcal{C}_i$;
4. For each $i, j, l \in \{0, 1, \cdots, d\}$, there exists a real number p^l_{ij} such that, for all (a, b) in $\mathcal{C}_l$, we have (3).

$$|\{c \in \mathcal{X} : (a, c) \in \mathcal{C}_i \wedge (c, b) \in \mathcal{C}_j\}| \;=\; p^l_{ij}. \quad (3)$$

5. $p^l_{ij} = p^l_{ji}, \forall i, j, l \in \{0, 1, \ldots, d\}$.

The classes $\{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_d\}$ of symmetric association schemes can be described through their adjacency matrices $\{A_0, A_1, \ldots, A_d\}$, where A_i for $i = 0, 1, \ldots, d$ is defined as $(A_i)_{rs} = 1$ if and only if $(r, s) \in \mathcal{C}_i$ and $(A_i)_{rs} = 0$ otherwise.

So, the matrices associated to a symmetric association scheme $\mathcal{F}$ satisfy the following equalities.

1. $A_0 = I_n$;
2. $\sum_{i=0}^{d} A_i = J_n$;
3. $A_i = A_i^T$;
4. $A_i A_j = \sum_{l=0}^{d} p_{ij}^l A_l, \forall i, j \in \{0, 1, \ldots, d\}$;
5. $\forall i = 1, \cdots, d,\ (A_i)_{rs} \in \{0, 1\}, \forall r, s = 1, \cdots, n.$

We must note that Property 2 implies that the matrices $A_i, i = 0, 1, \ldots, d$ form a linear independent set of matrices of the vector space formed by real symmetric matrices of order n over the field $\mathbb{R}$ with the usual operations of addition of matrices and multiplication of a matrix by a scalar.

In the following, we will designate a symmetric association scheme by SAS. We will define an SAS by the matrices A_is, and we will say consider the SAS $\mathcal{F} = \{A_0, A_1, \ldots, A_d\}$.

Example 1. *Let us consider the SAS* $\mathcal{F} = \{A_0, A_1, A_2.A_3\}$, *where* $A_0 = I_6$ *and*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

The SAS $\mathcal{F}$ *defined by the matrices presented in (4) verifies the multiplication in Table 1.*

In Table 1 of the SAS $\mathcal{F}$, *we have that* $B = 2A_0 + A_2$ *and* $C = 2A_3 + A_1$.

Herein, we must say that the algebra $\mathcal{A}$ *spanned by the elements of the SAS* $\mathcal{F} = \{A_0, A_1, A_2, \ldots, A_d\}$ *is an RFEJA such that* $rank(\mathcal{A}) = d + 1$ *and* $\dim(\mathcal{A}) = d + 1$. *As a consequence, there exists a unique JF,* $\{E_0, E_1, \cdots, E_d\}$ *that is the basis of* $\mathcal{A}$ *with* $E_0 = \frac{J_n}{n}$. *Since the algebra* $\mathcal{A}$ *is a commutative algebra spanned by symmetric matrices, we can say that this Jordan frame can be obtained by considering a matrix* A *with* $d + 1$ *distinct eigenvalues of* $\mathcal{A}$ *and next determine the projectors on each common proper subspace of* A *using the equality (5)*

$$E_i = \prod_{l=0, l \neq i}^{d} \frac{A - \lambda_l I_n}{\lambda_i - \lambda_l} \tag{5}$$

where the λ_i*'s for* $i = 0, 1, \ldots, d$ *are the distinct eigenvalues of* A.

Table 1. Table of multiplication.

	A_0	A_1	A_2	A_3
A_0	A_0	A_1	A_2	A_3
A_1	A_1	B	C	A_2
A_2	A_2	C	B	A_1
A_3	A_3	A_2	A_1	A_0

Remark 1. *To obtain a JF, $\{E_0, E_1, E_2, E_3\}$ of the RFEJA, $\mathcal{A}$ spanned by the matrices of the SAS $\mathcal{F}$ of the Example 1 and using the notation $\lambda_0 = -2, \lambda_1 = -1, \lambda_2 = 1$ and $\lambda_3 = 2$, we can write that $E_i = \prod_{l=0, l\neq i}^{3} \frac{A_1 - \lambda_l I_6}{\lambda_i - \lambda_l}$ for $i = 0, 1, \ldots, 3$, since A_1 is a matrix with the distinct eigenvalues $\lambda_0, \lambda_1, \lambda_2$, and λ_3. So, after some calculations, we obtain:*

$$\begin{aligned} E_0 &= \frac{1}{6}A_0 - \frac{1}{6}A_1 + \frac{1}{6}A_2 - \frac{1}{6}A_3, \\ E_1 &= \frac{1}{3}A_0 - \frac{1}{6}A_1 - \frac{1}{6}A_2 + \frac{1}{3}A_3, \\ E_2 &= \frac{1}{3}A_0 + \frac{1}{6}A_1 - \frac{1}{6}A_2 - \frac{1}{3}A_3 - \frac{1}{3}A_3, \\ E_3 &= \frac{1}{6}A_0 + \frac{1}{6}A_1 + \frac{1}{6}A_2 + \frac{1}{6}A_3 = \frac{J_6}{6}. \end{aligned}$$

Example 2. *Let us consider the SAS $\mathcal{F} = \{A_0, A_1, A_2, A_3\}$, where*

$$\begin{aligned} A_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The process of multiplication of the matrices of the SAS $\mathcal{F}$ is described in Table 2.

We have that the projectors on the proper sub-spaces of the matrix A_1 that has the eigenvalues -1 and 1 are $P_0 = \frac{1}{2}A_0 - \frac{1}{2}A_1, P_1 = \frac{1}{2}A_0 + \frac{1}{2}A_1$ and the projectors associated with the proper sub-spaces of A_2 are: $P_2 = \frac{1}{2}A_0 - \frac{1}{2}A_2, P_3 = \frac{1}{2}A_0 + \frac{1}{2}A_2$. Hence, the JF of the EJA spanned by the SAS $\mathcal{F}$ is the set $\{E_0, E_1, E_2, E_3\}$ such that $E_0 = P_0P_2 = \frac{1}{4}A_4 - \frac{1}{4}A_1 - \frac{1}{4}A_2 + \frac{1}{4}A_3$, $E_1 = P_1P_2 = \frac{1}{4}A_4 + \frac{1}{4}A_1 - \frac{1}{4}A_2 - \frac{1}{4}A_3$, $E_2 = P_0P_3 = \frac{1}{4}A_4 - \frac{1}{4}A_1 + \frac{1}{4}A_2 - \frac{1}{4}A_3$, $E_3 = P_1P_3 = \frac{1}{4}A_4 + \frac{1}{4}A_1 + \frac{1}{4}A_2 + \frac{1}{4}A_3 = \frac{J_4}{4}$.

Table 2. Table of multiplication 2.

	$\boldsymbol{A_0}$	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	$\boldsymbol{A_3}$
A_0	A_0	A_1	A_2	A_3
A_1	A_1	A_0	A_3	A_2
A_2	A_2	A_3	A_0	A_1
A_3	A_3	A_2	A_1	A_0

4. Krein Parameters of a Particular Euclidean Jordan Algebra

Let us consider an SAS with d classes, $\mathcal{F} = \{A_0, A_1, \ldots, A_d\}$ and the real vector space $\mathcal{A} = \{\alpha_0 A_0 + \alpha_1 A_1 + \cdots + \alpha_d A_d : \alpha_0 \in \mathbb{R}, \alpha_i \in \mathbb{R}, i = 1, \cdots, d\}$. Then, $\mathcal{A}$ is a real algebra, closed for the Schur product of two of its matrices, when it is equipped with the usual product of real matrices of order n. Now, one proves that $\mathcal{A}$ is a commutative algebra.

Firstly, we will show that $A_i A_j = A_j A_i$. Indeed, we have $A_i A_j = A_i^T A_j^T = (A_j A_i)^T = \left(\sum_{k=0}^{d} p_{ji}^k A_k\right)^T = \sum_{k=0}^{d} p_{ji}^k A_k^T = \sum_{k=0}^{d} p_{ji}^k A_k = A_j A_i$. Let u and v be two elements of $\mathcal{A}$. Let us consider the following notation: $u = \sum_{i=0}^{d} \alpha_i A_i$ and $v = \sum_{j=0}^{d} \beta_j A_j$. Then, we have the following calculations:

$$
\begin{aligned}
uv &= \left(\sum_{i=0}^{d} \alpha_i A_i\right)\left(\sum_{j=0}^{d} \beta_j A_j\right) \\
&= \left(\sum_{i=0}^{d} \alpha_i A_i^T\right)\left(\sum_{j=0}^{d} \beta_j A_j^T\right) \\
&= \left(\sum_{i=0}^{d} \alpha_i A_i\right)^T \left(\sum_{j=0}^{d} \beta_j A_j\right)^T \\
&= \left(\left(\sum_{j=0}^{d} \beta_j A_j\right)\left(\sum_{i=0}^{d} \alpha_i A_i\right)\right)^T \\
&= \left(\sum_{j=0}^{d}\sum_{i=0}^{d} \beta_j \alpha_i A_j A_i\right)^T \\
&= \left(\sum_{j=0}^{d}\sum_{i=0}^{d} \beta_j \alpha_i (A_j A_i)^T\right) \\
&= \left(\sum_{j=0}^{d}\sum_{i=0}^{d} \beta_j \alpha_i (A_i^T A_j^T)\right) \\
&= \left(\sum_{j=0}^{d}\sum_{i=0}^{d} \beta_j \alpha_i (A_i A_j)\right) \\
&= \left(\sum_{j=0}^{d}\sum_{i=0}^{d} \beta_j \alpha_i (A_j A_i)\right) \\
&= \left(\sum_{j=0}^{d} (\beta_j A_j)\right)\left(\sum_{i=0}^{d} \alpha_i A_i\right) \\
&= vu.
\end{aligned}
$$

So, since $\mathcal{A}$ is a commutative algebra, the algebra $\mathcal{A}$ provided with the inner product $\bullet|\bullet$ defined by $u|v = \text{trace}(uv)$ for any u and v of $\mathcal{A}$ becomes an RFEJA.

Indeed, since $\mathcal{A}$ is a commutative, associative algebra, we have the following calculations. $A^2(AB) = (A^2A)B = (AA^2)B = A(A^2B)$, and finally, we conclude that $(AB)|C = \text{trace}((AB)C) = \text{trace}((BA)C) = \text{trace}(B(AC)) = B|(AC)$. Next, noting that $\dim(\mathcal{A}) = d + 1$, we will show that $\text{rank}(\mathcal{A}) = d + 1$. Now, for each matrix A of

$\mathcal{A}$, we can say that the set $\{I, A, A^2, \ldots, A^d, A^{d+1}\}$ is a linearly dependent set of $\mathcal{A}$, then $\text{rank}(\mathcal{A}) \leq d+1$. Next, we will deduce that $\text{rank}(\mathcal{A}) = d+1$.

But, since $\{I_n, A_1, A_2, \ldots, A_d\}$ is the bais of $\mathcal{A}$ formed by commuting symmetric matrices, then they are simultaneously diagonalizable, and therefore, there exists a basis of projectors $\mathcal{B} = \{E_0 = \frac{I}{n}, E_1, \cdots, E_d\}$ on the proper sub-spaces common of all the matrices of the SAS $\mathcal{F}$, of $\mathcal{A}$. Now, let us consider the element

$$X = \alpha_0 E_0 + \alpha_1 E_1 + \alpha_2 E_2 + \cdots + \alpha_d E_d \tag{6}$$

with all the α_i's distinct. Then, the decomposition (6) is the first spectral decomposition of X, and $\mathcal{B}$ is the unique CSOI associated with X. Now, we will deduce that $\mathcal{B}$ is a JF of $\mathcal{A}$. But, firstly, we will show that $\text{rank}(\mathcal{A}) = d+1$. For that, we show that $\text{rank}(X) = d+1$.

We have

$$\begin{aligned}
I_n &= E_0 + E_1 + E_2 + \cdots + E_d, \\
X &= \alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_d E_d \\
X^2 &= \alpha_0^2 E_0 + \alpha_1^2 E_1 + \cdots + \alpha_d^2 E_d \\
\vdots &= \vdots \\
X^d &= \alpha_0^d E_0 + \alpha_1^d E_1 + \cdots + \alpha_d^d E_d
\end{aligned}$$

Since the set $\{I_n, X, X^2, \ldots, X^d\}$ is linearly independent if and only if the determinant (7) is a non-null determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \alpha_0 & \alpha_1 & \ldots & \alpha_d \\ \alpha_0^2 & \alpha_1^2 & \ldots & \alpha_d^2 \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_0^d & \alpha_1^d & \ldots & \alpha_d^d \end{vmatrix} = \prod_{1 \leq i < j \leq n}^{d} (\alpha_j - \alpha_i), \tag{7}$$

and since the α_i's are all distinct, then the set $\{I, X, X^2, \ldots, X^d\}$ is a free set of $\mathcal{A}$. Now, since the set $\{I, X, X^2, \ldots, X^d\}$ is linearly independent and the set $\{I, X, X^2, \ldots, X^d, X^{d+1}\}$ is linearly dependent set (note that $\dim(\mathcal{A}) = d+1$,), then we conclude that $\text{rank}(X) = d+1$, and therefore, $\text{rank}(\mathcal{A}) = d+1$. So, we can say that $\mathcal{B} = \{E_0, E_1, \ldots, E_d\}$ is a JF of $\mathcal{A}$ since $\mathcal{B}$ has cardinality equal to $d+1 = \text{rank}(\mathcal{A})$. And, we also must say that since $\dim(\mathcal{A}) = d+1$, then $\mathcal{B}$ is the bais of $\mathcal{A}$. Next, we will show that $\mathcal{A}$ has a unique JF, this is we will prove that the unique JF of $\mathcal{A}$ is $\mathcal{B}$. Indeed, if $\mathcal{C} = \{F_0, F_1, \ldots, F_r\}$ is another JF of $\mathcal{A}$, then we would obtain

$$X = \beta_0 F_0 + \beta_1 F_1 + \cdots + \beta_r F_r. \tag{8}$$

But, since $\mathcal{C}$ is a CSOI, then X would have two first spectral spectral decomposition's if $\mathcal{C} \neq \mathcal{B}$. Therefore $\mathcal{B} = \mathcal{C}$.

In what follows, we will define the Krein parameters of $\mathcal{A}$ as being the coordinates of $E_i \circ E_j$ relatively to the basis $\mathcal{B} = \{E_1 = \frac{I_n}{n}, E_2, \ldots, E_{d+1}\}$ of $\mathcal{A}$, where $A \circ B$ represents the Schur product of the real square matrices A and B with same order, that is, as being the real numbers q_{ijl} for $i, j, l = 1, \ldots, d+1$ such that

$$E_i \circ E_j = \sum_{l=1}^{d+1} q_{ijl} E_l \tag{9}$$

Next, we will present some inequalities involving the Krein parameters of $\mathcal{A}$.

Theorem 3. *Let us consider the RFEJA $\mathcal{A}$ spanned by an SAS $\mathcal{F}$ with d classes with the unique Jordan frame $\mathcal{B} = \{P_1 = \frac{J_n}{n}, P_2, \ldots, P_{d+1}\}$. Then, we have for $l = 1, \cdots, d+1$ that*

$$-1 \quad \leq \quad \sum_{i=1}^{d+1} q_{ii\,l} - 2\sum_{i=2}^{d+1} q_{1jl} + 2 \sum_{2=i<j\leq d+1}^{d+1} q_{ij\,l} \leq 1 \tag{10}$$

for $l = 1, \cdots, d+1$.

Proof. We have that $((P_1 - P_2 - \cdots - P_{d+1}) \otimes (P_1 - F_2 - \cdots - P_{d+1}))^3 = P_1 - P_2 - \cdots - P_{d+1}$, where $A \otimes B$ represents the Kronecker product of the matrices A and B. Since the matrix $Z = (P_1 - P_2 - \cdots - P_{d+1}) \otimes (P_1 - P_2 - \cdots - P_{d+1})$ verifies the equality $Z^3 - Z = O$, where O is the null matrix, then we have that if λ is an eigenvalue of Z, and we must have $-1 \leq \lambda \leq 1$. Since $Y = (P_1 - P_2 - \cdots - P_{d+1}) \circ (P_1 - P_2 - \cdots - P_{d+1})$, where $A \circ B$ represent the Schur product of the matrices A and B, is a principal sub-matrix of Z, then we conclude that if λ is an eigenvalue of Y, and we must have $-1 \leq \lambda \leq 1$. Since each eigenvalue of Y, is $\lambda_l = \sum_{i=1}^{d+1} q_{iil} - 2\sum_{i=2}^{d+1} q_{1jl} + 2\sum_{2=i<j\leq d+1}^{d+1} q_{ijl}$, then Inequality (10) follows. □

We must note on a symmetric association scheme $\mathcal{F}$ with $2d - 1$ classes considering the unique Jordan frame $S = \{E_1 = \frac{J_n}{n}, E_2, \cdots, E_{2d}\}$ of the Euclidean Jordan algebra spanned by $\mathcal{A}$, we have that the matrix $D = (E_1 + E_2 + \cdots + E_d) \otimes (E_{d+1} + \cdots + E_{2d})$ is an idempotent. Next since $G = (E_1 + E_2 + \cdots + E_d) \circ (E_{d+1} + \cdots + E_{2d}) + (E_{d+1} + \cdots + E_{2d}) \circ (E_1 + E_2 + \cdots + E_d)$ is a principal matrix of D, by the analysis of the Krein parameters of G and using a similar proof like the one made in Theorem 3, we obtain Theorem 4.

Theorem 4. *Let $\mathcal{A}$ be the RFEJA spanned by an SAS $\mathcal{F}$ with $2d - 1$ classes, and considering the unique JF of $\mathcal{A}$, $S = \{P_1 = \frac{J}{n}, P_2, \cdots, P_{2d}\}$, the Krein parameters $q_{i\,j\,t}$ with $1 \leq i \leq d, d+1 \leq j \leq 2d$ with $t = 1, \cdots, 2d$ verify Inequality (11).*

$$\sum_{i=1}^{d} \sum_{j=d+1}^{2d} q_{ijt} \quad \leq \quad \frac{1}{2} \tag{11}$$

We conclude Theorem 5 by generalizing Theorem 4.

Theorem 5. *Let $\mathcal{A}$ be the RFEJA spanned by an SAS $\mathcal{F}$ with $2d - 1$ classes, and considering the unique JF of $\mathcal{A}$, $S = \{P_1 = \frac{J}{n}, P_2, \cdots, P_{2d}\}$, u be a natural number such that $1 < u < 2d$,, the Krein parameters $q_{i\,j\,t}$ with $1 \leq i \leq u, u+1 \leq j \leq 2d$ with $t = 1, \cdots, 2d$ verify the inequality (12).*

$$\sum_{i=1}^{u} \sum_{j=u+1}^{2d} q_{ijt} \quad \leq \quad \frac{1}{2} \tag{12}$$

5. Some Concepts and Properties about Strongly Regular Graphs

R. C, Bose introduced strongly regular graphs in [13]. In the following, we will present some relevant properties. For a very perceptible text about concepts and the algebraic properties of strongly regular graphs and algebraic properties of the strongly regular graphs, see Algebraic Graph Theory [14].

The order of a graph is the number of vertexes it has. If a graph has neither parallel edges nor loops, it is called a simple graph.

One defines the eigenvalues of a graph G as the eigenvalues of its adjacency matrix.

If all pairs of distinct vertices of a simple graph are adjacent, then this graph is called a complete graph.

One defines the complement of a simple graph G, which one denotes by $\overline{G}$, as being a simple graph with the same set of vertexes of G and such that two any of its vertexes are adjacent if and only if they are not adjacent vertexes of G. In the following text, we only treat non-empty, simple, and non-complete graphs.

The degree of a vertex x of a graph G is the number of incident edges on x. A graph G is called $l-$regular if all its vertexes have the same degree l.

One says that a graph G, is a $(m, l; c, d)-$strongly regular graph if G is $l-$regular graph with order m, and if any pair of adjacent vertexes have c common neighbor vertexes and any pair of non-adjacent vertexes have d common neighbor vertexes.

In the following, the abbreviation srg will be used to designate a strongly regular graph.

If G is a $(m, l; c, d)-$srg ,then the complement graph of $G, \overline{G}$ is a $(m, m-l-1; m-2l+d-2, m-2l+c)$ srg.

A $(m, l; c, d)-$srg G is primitive if and only if G and $\overline{G}$ are connected. A $(m, l; c, d)-$srg is a non-primitive srg if and only if $d = l$ or $d = 0$.

The adjacency matrix A of a $(m, l; c, d)-$strongly regular graph G satisfies Equation (13).

$$A^2 = lI_m + cA + d(J_m - A - I_m) \tag{13}$$

The real numbers l, λ_1 and λ_2 [14], where

$$\begin{aligned}\lambda_1 &= (c - d + \sqrt{(c-d)^2 + 4(l-d)})/2, \\ \lambda_2 &= (c - d - \sqrt{(c-d)^2 + 4(l-d)})/2\end{aligned}$$

are the eigenvalues of G. The real numbers f_{λ_1} and f_{λ_2} defined, respectively, by Inequalities (14) and (15).

$$f_{\lambda_1} = \frac{1}{2}\left(m - 1 + \frac{2l + (m-1)(c-d)}{\lambda_2 - \lambda_1}\right), \tag{14}$$

$$f_{\lambda_2} = \frac{1}{2}\left(m - 1 - \frac{2l + (m-1)(c-d)}{\lambda_2 - \lambda_1}\right), \tag{15}$$

are the multiplicities of the eigenvalues λ_1 and λ_2. Next, we present the admissibility conditions (16–21) over the multiplicities of the eigenvalues and over the eigenvalues and the parameters of a $(m, l; c, d)-$ primitive srg G.

$$\frac{1}{2}\left(m - 1 + \frac{2l + (m-1)(c-d)}{\lambda_1 - \lambda_2}\right) \in \mathbb{N}, \tag{16}$$

$$\frac{1}{2}\left(m - 1 - \frac{2l + (m-1)(c-d)}{\lambda_1 - \lambda_2}\right) \in \mathbb{N}, \tag{17}$$

$$(\lambda_2 + 1)(l + \lambda_2 + 2\lambda_2\lambda_1) \leq (l + \lambda_2)(\lambda_1 + 1)^2, \tag{18}$$

$$(\lambda 1 + 1)(k + \lambda_1 + 2\lambda_2\lambda_1) \leq (l + \lambda_1)(\lambda_2 + 1)^2, \tag{19}$$

$$m \leq \frac{1}{2} f_{\lambda_1}(f_{\lambda_1} + 3), \tag{20}$$

$$m \leq \frac{1}{2} f_{\lambda_2}(f_{\lambda_2} + 3). \tag{21}$$

The admissibility conditions (16) and (17) are known as the integrability conditions of a strongly regular graph, and Inequalities (18) and (19) are known as the Krein conditions of a strongly regular graph [15]. And, finally, Inequalities (20) and (21) are known as the absolute bounds conditions of a strongly regular graph [16].

6. Modified Krein Parameters of a Strongly Regular Graph

Let us consider the primitive strongly regular graph $(m, l; c, d) - G$ such that $0 < d < l - 1$ and with the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{c - d + \sqrt{(c-d)^2 + 4(l-d)}}{2}, \\ \lambda_2 &= \frac{c - d - \sqrt{(c-d)^2 + 4(l-d)}}{2}, \end{aligned}$$

and l. Next, let us consider the Euclidean Jordan sub-algebra $\mathcal{B}$ of the Euclidean Jordan algebra $\mathcal{M} = \mathrm{Sym}(m, \mathbb{R})$, equipped with the product of two matrices $\bigstar$ as being the usual product of matrices and the inner product of two matrices being the trace of these two matrices, spanned by the identity matrix of order m and the natural powers of the matrix of adjacency A_G of G. Now, we consider the Jordan frame $\{U_1, U_2, U_3\}$ of $\mathcal{B}$ that is the bais of $\mathcal{B}$, where we have

$$\begin{aligned} U_1 &= \frac{1}{m} I_m + \frac{1}{m} A_G + \frac{1}{m}(J_m - A_G - I_m) \\ U_2 &= \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_2 - \lambda_1)} I_m + \frac{-m + l - \lambda_1}{m(\lambda_2 - \lambda_1)} A_G \\ &+ \frac{l - \lambda_1}{m(\lambda_2 - \lambda_1)}(J_m - A_G - I_m) \\ U_3 &= \frac{|\lambda_2| m + \lambda_2 - l}{m(\lambda_2 - \lambda_1)} I_m + \frac{m + \lambda_2 - l}{m(\lambda_2 - \lambda_1)} A_G \\ &+ \frac{\lambda_2 - l}{m(\lambda_2 - \lambda_1)}(J_m - A_G - I_m). \end{aligned}$$

Now, we define the modified Krein parameters of G as being the real numbers $q_{123;l}$ with $l \in \{1, 2, 3\}$ such that

$$U_1 \circ U_2 \circ U_3 = \sum_{l=1}^{3} q_{123;l} U_l \tag{22}$$

where $A \circ B$ represents the Schur product of the real square matrices A and B of order m. Herein, we must say that $U_1 \otimes U_2 \otimes U_3$ is an idempotent matrices, and their eigenvalues are 0 or 1, and therefore, since $U_1 \circ U_2 \circ U_3$ is a principal sub-matrix of $U_1 \otimes U_2 \otimes U_3$, then we conclude that the eigenvalues of this matrix is greater than 0 and lower than 1. But, we must say that $U = U_1 \otimes U_2 \otimes U_3 + U_1 \otimes U_3 \otimes U_2 + U_2 \otimes U_1 \otimes U_3 + U_2 \otimes U_3 \otimes U_1 + U_2 \otimes U_1 \otimes U_3 + U_2 \otimes U_3 \otimes U_1$ is an idempotent. Then, we conclude that the eigenvalues of the matrix $6U_1 \circ U_2 \circ U_3$ are greater than 0 and lower than 1. And, therefore, we conclude that the modified Krein parameters $q_{123;l}$ verify the inequalities $0 \leq q_{123;l} \leq 1$ for $l = 1, 2$ and $l = 3$. Hence, we have established Theorem (6)

Theorem 6. *Let us consider a primitive* $(m, l; c, d) - G$ *strongly regular graph* $0 < d < l - 1$. *Then, the modified Krein parameters* $q_{123;l}$ *of* G *verify the inequalities:*

$$0 \leq q_{123;l} \leq \frac{1}{6}.$$

for $l = 1, 2$ *and* $l = 3$.

Now, we must remember that, in our notation, the Krein parameters of a strongly regular graph are the real numbers such that

$$U_i \circ U_j = \sum_{l=1}^{3} q_{ij;l} U_l \tag{23}$$

Next, we present Theorem 7, which is an application of the Theorem 3 to strongly regular graphs.

Theorem 7. *Let G be a* $(n, l; c, d)$ *primitive strongly regular graph such that* $0 < d < l - 1$*. Then, the Krein parameters of G,* $q_{ij;l}$ *for* $l = 1, 2$ *and* $l = 3$ *verify the inequality (24).*

$$-1 \quad \leq \quad \sum_{i=1}^{3} q_{ii;l} - 2q_{12;l} - 2q_{13;l} + 2q_{23;l} \leq 1. \tag{24}$$

7. Conclusions

Inequalities (10) and (11) presented in Theorems 3 and 4, respectively, over the Krein parameters of the finite-dimensional real Euclidean Jordan algebra spanned by a symmetric association scheme are distinct from those that were established for any symmetric association schemes (see [4]). We also have introduced modified Krein parameters of a strongly regular graph and have established some inequalities over these modified Krein parameters. Finally, we have established some new admissibility conditions over the Krein parameters of a strongly regular graph. In future work, we will recur to other spectral analysis methods of discrete structures to establish new inequalities over the spectrum of a symmetric association scheme and of a strongly regular graph.

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Abbreviations

The following abbreviations are used in this manuscript:

RFEJA	Real finite-dimensional Euclidean Jordan algebra
EJA	Euclidean Jordan algebra
JF	Jordan frame
CSOI	Complete system of orthogonal idempotent
SAS	Symmetrc Association Scheme
srg	Strongly regular graphs

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