symmetry

MDPI

Article

On Fractional Ostrowski-Mercer-Type Inequalities and Applications

Sofia Ramzan [1], Muhammad Uzair Awan [1], Miguel Vivas-Cortez [2,*] and Hüseyin Budak [3]

1 Department of Mathematics, Government College University, Faisalabad 38000, Pakistan; sofiaramzan.202008391@gcuf.edu.pk or ssofi227@gmail.com (S.R.); muawan@gcuf.edu.pk or awan.uzair@gmail.com (M.U.A.)
2 Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076, Apartado, Quito 17-01-2184, Ecuador
3 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey; useyinbudak@duzce.edu.tr or hsyn.budak@gmail.com
* Correspondence: mjvivas@puce.edu.ec

Abstract: The objective of this research is to study in detail the fractional variants of Ostrowski–Mercer-type inequalities, specifically for the first and second order differentiable s-convex mappings of the second sense. To obtain the main outcomes of the paper, we leverage the use of conformable fractional integral operators. We also check the numerical validations of the main results. Our findings are also validated through visual representations. Furthermore, we provide a detailed discussion on applications of the obtained results related to special means, q-digamma mappings, and modified Bessel mappings.

Keywords: s-convex mappings; Jensen's inequality; Jensen-Mercer inequality; fractional calculus

MSC: 26A33; 26A51; 26D07; 26D10; 26D15; 26D20

check for updates

Citation: Ramzan, S.; Awan, M.U.; Vivas-Cortez, M.; Budak, H. On Fractional Ostrowski-Mercer-Type Inequalities and Applications. *Symmetry* **2023**, *15*, 2003. https://doi.org/10.3390/sym15112003

Academic Editor: Wei-Shih Du

Received: 27 July 2023
Revised: 17 October 2023
Accepted: 24 October 2023
Published: 31 October 2023

1. Introduction

In 1937, Alexandar Markowich Ostrowski [1] discovered an integral inequality known as the Ostrowski inequality, stated as:

Let $\psi : \mathcal{I} = [\mu, \kappa] \subseteq \mathbb{R} \to \mathbb{R}$ (Real numbers) be a differentiable mapping on the interior of $\mathcal{I}$ such that ψ is integrable on $[\mu, \kappa]$, where $\mu, \kappa \in \mathcal{I}$ with $\mu < \kappa$. If $|\psi'(\lambda)| \leq M$ for all $\lambda \in (\mu, \kappa)$ and $M > 0$, then

$$\left|\psi(\ell) - \frac{1}{\kappa - \mu}\int_{\mu}^{\kappa}\psi(\lambda)d\lambda\right| \leq M(\kappa - \mu)\left[\frac{1}{4} + \frac{\left(\ell - \frac{\mu+\kappa}{2}\right)^2}{(\kappa - \mu)^2}\right],$$

holds for all $\ell \in [\mu, \kappa]$, and $\frac{1}{4}$ is the best possible constant.

Ostrowski's inequality provides an approximation of the difference between mapping values and their integral average over a given interval. For more than 5000 years, inequalities have been seen in wide applications. The oldest was recorded in ancient Chinese mathematics, called the He Chengtian inequality [2]. By utilizing this inequality, He Chengtian calculated the approximate values of the fractional day of a moon and a year. Over the course of time, researchers have broadened the scope of convex mappings, leading to the discovery of various variants of the Hermite–Hadamard inequality, see [3–14].

The class of convex mappings is regarded as cornerstone of the theory of inequalities with a wide range of applications in many areas of mathematics such as in numerical integration, special means and special functions.

A mapping $\psi : \mathcal{I} \to \mathbb{R}$ is said to be convex, if

$$\psi(\lambda\mu + (1-\lambda)\kappa) \leq \lambda\psi(\mu) + (1-\lambda)\psi(\kappa), \quad \forall \mu, \kappa \in \mathcal{I}, \quad \lambda \in [0,1]. \tag{1}$$

One of the prolific results concerning to the convexity property of the mappings is Jensen's inequality (see [15]) interpreted as:

For a convex mapping $\psi : \mathcal{I} = [\mu, \kappa] \subseteq \mathbb{R} \to \mathbb{R}$, we have

$$\psi\left(\sum_{j=1}^{n} \wp_j \gamma_j\right) \leq \sum_{j=1}^{n} \wp_j \psi(\gamma_j),$$

where $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and $\wp = (\wp_1, \wp_2, \cdots, \wp_n)$ are non-negative weights with $\sum_{i=1}^{n} \wp_i = 1$, $\gamma_j \in [\mu, \kappa]$, $\wp_j \in [0,1]$ and $j = 1, \cdots, n$.

A new variant of Jensen's inequality known as Jensen–Mercer inequality was introduced by Mercer [16] in 2003, stated as:

For a convex mapping $\psi : \mathcal{I} = [\mu, \kappa] \subseteq \mathbb{R} \to \mathbb{R}$, we have

$$\psi\left(\mu + \kappa - \sum_{j=1}^{n} \wp_j \gamma_j\right) \leq \psi(\mu) + \psi(\kappa) - \sum_{j=1}^{n} \wp_j \psi(\gamma_j),$$

for all $\gamma_j \in [\mu, \kappa]$, $\wp_j \in [0,1]$ and $j = 1, \cdots, n$.

Kian and Moslehian [17] obtained the Hermite–Hadamard–Jensen–Mercer inequality for convex mappings, as follows:

For a convex mapping $\psi : [\mu, \kappa] \subseteq \mathbb{R} \to \mathbb{R}$ and $\phi_1, \phi_2 \in [\mu, \kappa]$, we have

$$\begin{aligned}
\psi\left(\mu + \kappa - \frac{\varphi_1 + \varphi_2}{2}\right) &\leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \psi(\mu + \kappa - \lambda) d\lambda \\
&\leq \frac{\psi(\mu + \kappa - \varphi_1) + \psi(\mu + \kappa - \varphi_2)}{2} \\
&\leq \psi(\mu) + \psi(\kappa) - \frac{\psi(\varphi_1) + \psi(\varphi_2)}{2}.
\end{aligned}$$

For more work on Jensen–Mercer-type inequalities, see [18–22].

Over time, the researchers have extended the definition of convex mappings to obtain different variants of Hermite–Hadamard inequality. The concept of s-Breckner convex mappings, or s-convex mappings in the second sense $(0 < s \leq 1)$, was introduced by Breckner [23] in 1978; this is a generalized class of convex mappings such that for $s = 1$, it reduces back to convexity.

A mapping $\psi : [0, \infty) \to \mathbb{R}$ is said to be s-convex mapping in the second sense, when

$$\psi(\lambda\mu + (1-\tau)\kappa) \leq \lambda^s \psi(\mu) + (1-\lambda)^s \psi(\kappa), \tag{2}$$

holds for all $\mu, \kappa \in [0, \infty)$, $s \in (0,1]$ and $\lambda \in [0,1]$. The geometrical aspect of s-convexity $(0 < s < 1)$ is that a curved chord joining any two points always lies above the mapping's graph. The inequality (2) is identical to the inequality (1), when $s = 1$.

Cortez and Hernández [24] have proved Jensen–Mercer inequality for s-convex mapping $\psi : \mathcal{I} = [0, \infty) \to \mathbb{R}$ with $(0 < s \leq 1)$ as follows:

$$\psi\left(\mu + \kappa - \sum_{j=1}^{n} \wp_j \gamma_j\right) \leq \psi(\mu) + \psi(\kappa) - \sum_{j=1}^{n} \wp_j^s \psi(\gamma_j),$$

for all $\gamma_j \in \mathcal{I}$, $\wp_j \in [0,1]$ and $j = 1, \cdots, n$.

For s-convex mapping, $\psi : [\mu, \kappa] \subseteq \mathbb{R} \to \mathbb{R}$, $\phi_1, \phi_2 \in [\mu, \kappa]$ and $s \in (0, 1]$, the Hermite–Hadamard–Jensen–Mercer inequality is given in [24] as follows:

$$\psi\left(\mu + \kappa - \frac{\varphi_1 + \varphi_2}{2}\right) \leq \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} \psi(\mu + \kappa - \lambda) d\lambda$$
$$\leq \psi(\mu) + \psi(\kappa) - \frac{\psi(\varphi_1) + \psi(\varphi_2)}{s + 1}.$$

Fractional calculus, dealing with integrals and derivatives of arbitrary real order, has significantly contributed to the characterization of diverse real materials, such as polymers. The fractional models are more adequate than the previous used models of integer orders, see [25–27]. In addition to classical derivatives, fractional order derivatives offer superior capabilities in describing the memory and hereditary characteristics of diverse processes. In [28], Podlubny discussed various applications of fractional derivatives. Riemann, Liouville, Grünwald and other researchers defined the fractional derivatives in several ways given in [28,29].

To investigate the characteristics of fractional differentiability and local scaling, the fractional derivatives were not suitable because of their non-local nature [30]. By renormalization of the Riemann–Liouville definition, Kolwankar and Gangal [30,31] proposed the idea of local fractional derivatives. The calculus of fractal space-time is studied with the help of local fractional derivatives. In addition, two-scale fractal theory is utilized to study problems involving porous media and unsmooth boundaries [32–34]. Moreover, the fractional derivatives are also utilized to find the approximate solutions of the fractional differential equations, see [35,36].

Probably the most frequently used definition of fractional integrals is due to B. Riemann and J. Liouville, commonly known as the Riemann–Liouville fractional integrals, defined as follows:

Let ψ be an integrable mapping on $[\mu, \kappa]$. Then, the left and right sided Reimann–Liouville fractional integrals $I^{\omega}_{\mu^+}\psi$ and $I^{\omega}_{\kappa^-}\psi$ of order $\nu > 0$ with $\mu \geq 0$ are defined by:

$$I^{\nu}_{\mu^+}\psi(y) = \frac{1}{\Gamma(\nu)} \int_{\mu}^{y} (y - \lambda)^{\nu-1} \psi(\lambda) d\lambda, \quad y > \mu, \tag{3}$$

and

$$I^{\nu}_{\kappa^-}\psi(y) = \frac{1}{\Gamma(\nu)} \int_{y}^{\kappa} (\lambda - y)^{\nu-1} \psi(\lambda) d\lambda, \quad y < \kappa, \tag{4}$$

where $\Gamma(\nu)$ is the gamma mapping defined as:

$$\Gamma(\nu) = \int_{0}^{\infty} \lambda^{\nu-1} e^{-\lambda} d\lambda, \quad Re(\nu) > 0.$$

For more details, see [37].

When the fractional operators are closely examined, various features such as singularity, locality, generalization and differences in their kernel structures become apparent. Although generalizations and inferences are the foundations of mathematical methods, the new fractional operators add new features to solutions, particularly for the time memory effect. In the literature, there are various fractional operators with local, nonlocal, singular and non-singular kernels, [38–41]. Jarad et al. [42] defined conformable fractional integrals and derivatives with two parameters and kernels, which are helpful to the better understanding of the complexity of fractional variational problems, optimal control problems and modelling of complex systems.

For an integrable mapping ψ on $[\mu,\kappa]$, the left and right-conformable fractional integrals ${}^{\nu}_{\mu}I^{\omega}\psi$ and ${}^{\nu}I^{\omega}_{\kappa}\psi$ of order $\nu \in \mathbb{C}$ (Complex numbers), $Re(\nu) > 0$, $\omega \in (0,1]$ are defined as:

$$ {}^{\nu}_{\mu}I^{\omega}\psi(y) = \frac{1}{\Gamma(\nu)}\int_{\mu}^{y}\left(\frac{(y-\mu)^{\omega}-(\lambda-\mu)^{\omega}}{\omega}\right)^{\nu-1}\frac{\psi(\lambda)}{(\lambda-\mu)^{1-\omega}}d\lambda, \quad y > \mu, \tag{5} $$

and

$$ {}^{\nu}I^{\omega}_{\kappa}\psi(y) = \frac{1}{\Gamma(\nu)}\int_{y}^{\kappa}\left(\frac{(\kappa-y)^{\omega}-(\kappa-\lambda)^{\omega}}{\omega}\right)^{\nu-1}\frac{\psi(\lambda)}{(\kappa-\lambda)^{1-\omega}}d\lambda, \quad y < \kappa. \tag{6} $$

When $\omega = 1$ in (5) and (6), then they coincides with (3) and (4), respectively. They also coincide with the Hadamard fractional integral [43] by setting $\mu = 0$ and $\omega \to 0$ in (5) and $\kappa = 0$ and $\omega \to 0$ in (6). In addition, by choosing $\mu = 0$ in (5) and $\kappa = 0$ in (6), we have the generalized fractional integrals [44].

Let us recall beta mapping or Euler integral of the first kind with two variables defined by:

$$ \mathrm{B}(\mu_1,\kappa_1) = \int_{0}^{1} t^{\mu_1-1}(1-t)^{\kappa_1-1}dt, \quad Re(\mu_1) > 0, \quad Re(\kappa_1) > 0. \tag{7} $$

In terms of gamma mapping, it is defined as:

$$ \mathrm{B}(\mu_1,\kappa_1) = \frac{\Gamma(\mu_1)\Gamma(\kappa_1)}{\Gamma(\mu_1+\kappa_1)}. $$

Some properties of beta function are:

1. The beta function is symmetric i.e., $B(\mu_1,\kappa_1) = B(\kappa_1,\mu_1)$.
2. $B(\mu_1+1,\kappa_1) = B(\mu_1,\kappa_1)\frac{\mu_1}{\mu_1+\kappa_1}$.
3. $B(\mu_1,\kappa_1+1) = B(\mu_1,\kappa_1)\frac{\kappa_1}{\mu_1+\kappa_1}$.
4. $B(\mu_1,\kappa_1) = B(\mu_1+1,\kappa_1) + B(\mu_1,\kappa_1+1)$.

The motivation of this paper is to establish several new fractional variants of Ostrowski–Mercer-type inequalities using the first and the second order s-convex mappings of second sense. To achieve this goal, we employ conformable fractional integral operators. The main results' relevance has also been analyzed numerically and graphically. In addition, we also demonstrate some applications to means, q-digamma mappings, and modified Bessel mappings.

2. Ostrowski–Mercer-Type Inequalities for the First Order Differentiable s-Convex Mappings

In this section, we first establish a key lemma for the first differentiable mappings involving conformable fractional integrals. Then, by utilizing this result, we obtain several inequalities for the first order differentiable mappings whose absolute values are s-convex in the second sense.

Lemma 1. *Let $\psi : [\mu,\kappa] \to \mathbb{R}$ be a differentiable mapping on (φ_1,φ_2) and ψ' is integrable mapping on $[\varphi_1,\varphi_2]$, then for all $\ell \in [\varphi_1,\varphi_2]$, $\varphi_1,\varphi_2 \in [\mu,\kappa]$, $\omega \in (0,1]$ and $Re(\nu) > 0$, the following identity holds:*

$$ (\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) $$

$$+ (\varphi_2 - \ell)^\nu \psi(\ell + \kappa - \varphi_2) - \frac{\omega^\nu \Gamma(\nu + 1)}{(\varphi_2 - \ell)^{\omega\nu - \nu}} \left({}^{\nu}_{\ell+\kappa-\varphi_2} I^\omega \psi(\kappa) \right)$$

$$= (\ell - \varphi_1)^{\nu+1} \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell]) d\tau$$

$$- (\varphi_2 - \ell)^{\nu+1} \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \kappa - [\tau\varphi_2 + (1-\tau)\ell]) d\tau. \tag{8}$$

Proof. Consider

$$(\ell - \varphi_1)^{\nu+1} \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell]) d\tau$$

$$- (\varphi_2 - \ell)^{\nu+1} \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \kappa - [\tau\varphi_2 + (1-\tau)\ell]) d\tau$$

$$= (\ell - \varphi_1)^{\nu+1} Y_1 - (\varphi_2 - \ell)^{\nu+1} Y_2. \tag{9}$$

Applying integration by parts, we have

$$Y_1 = \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell]) d\tau$$

$$= \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \frac{\psi(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell])}{\ell - \varphi_1} \Bigg|_0^1$$

$$- \nu \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^{\nu-1} \frac{\psi(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell])}{\ell - \varphi_1} (1-\tau)^{\omega-1} d\tau$$

$$= \frac{1}{\omega^\nu} \frac{\psi(\ell + \mu - \varphi_1)}{\ell - \varphi_1} - \frac{\nu}{\ell - \varphi_1} \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^{\nu-1} \psi(\ell + \mu - [\tau\varphi_1 + (1-\tau)\ell]) (1-\tau)^{\omega-1} d\tau$$

$$= \frac{1}{\omega^\nu} \frac{\psi(\ell + \mu - \varphi_1)}{\ell - \varphi_1} - \frac{\nu}{(\ell - \varphi_1)^{\omega\nu+1}}$$

$$\int_\mu^{\ell+\mu-\varphi_1} \left(\frac{(\ell - \varphi_1)^\omega - (\ell + \mu - \varphi_1 - \lambda)^\omega}{\omega} \right)^{\nu-1} \psi(\lambda) (\ell + \mu - \varphi_1 - \lambda)^{\omega-1} d\lambda$$

$$= \frac{1}{\omega^\nu} \frac{\psi(\ell + \mu - \varphi_1)}{\ell - \varphi_1} - \frac{\Gamma(\nu+1)}{(\ell - \varphi_1)^{\omega\nu+1}} \left({}^{\nu} I^\omega_{\ell+\mu-\varphi_1} \psi(\mu) \right). \tag{10}$$

Similarly,

$$Y_2 = \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \psi'(\ell + \kappa - [\tau\varphi_2 + (1-\tau)\ell]) d\tau$$

$$= \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^\nu \frac{\psi(\ell + \kappa - [\tau\varphi_2 + (1-\tau)\ell])}{\ell - \varphi_2} \Bigg|_0^1$$

$$- \nu \int_0^1 \left(\frac{1 - (1-\tau)^\omega}{\omega} \right)^{\nu-1} \frac{\psi(\ell + \kappa - [\tau\varphi_2 + (1-\tau)\ell])}{\ell - \varphi_2} (1-\tau)^{\omega-1} d\tau$$

$$= -\frac{1}{\omega^{\nu}}\frac{\psi(\ell+\kappa-\varphi_2)}{\varphi_2-\ell} + \frac{\nu}{\varphi_2-\ell}\int_0^1 \left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu-1} \psi(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])(1-\tau)^{\omega-1}d\tau$$

$$= -\frac{1}{\omega^{\nu}}\frac{\psi(\ell+\kappa-\varphi_2)}{\varphi_2-\ell} + \frac{\nu}{\varphi_2-\ell}$$

$$\int_{\ell+\kappa-\varphi_2}^{\kappa} \left(\frac{(\varphi_2-\ell)^{\omega}-(\varphi_2-\ell-(\kappa-\lambda))^{\omega}}{\omega}\right)^{\nu-1} \psi(\lambda)(\varphi_2-\ell-(\kappa-\lambda))^{\omega-1}d\lambda$$

$$= -\frac{1}{\omega^{\nu}}\frac{\psi(\ell+\kappa-\varphi_2)}{\varphi_2-\ell} + \frac{\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu+1}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right). \tag{11}$$

Now using (10) and (11) in (9) and multiply with ω^{ν}, we obtain

$$\begin{aligned} &= (\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\ &+ (\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right). \end{aligned} \tag{12}$$

The proof is completed. □

Remark 1. *Setting $\varphi_1 = \mu, \varphi_2 = \kappa$ and $\omega = 1$ in Lemma 1, we obtain Lemma 2 in [1].*

Remark 2. *Setting $\varphi_1 = \mu, \varphi_2 = \kappa, \omega = 1$ and $\nu = 1$ in Lemma 1, we obtain Lemma 1 in [3].*

Theorem 1. *For a differentiable mapping $\psi : [\mu,\kappa] \to \mathbb{R}$ on (μ,κ) and if $|\psi'|$ is an s-convex mapping in the second sense on $[\mu,\kappa]$. Then, under the assumptions of Lemma 1, the following inequality holds:*

$$\begin{aligned} &\Bigg|(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\ &+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg| \\ &\leq (\ell-\varphi_1)^{\nu+1}\left\{\frac{1}{\omega^{v+1}}B\left(\nu+1,\frac{1}{\omega}\right)[\psi'|\ell|+\psi'|\mu|] \right. \\ &\left. -[\mathcal{C}_1(s,\nu,\omega)\psi'|\varphi_1|+\mathcal{C}_2(s,\nu,\omega)\psi'|\ell|]\right\} \\ &+ (\varphi_2-\ell)^{\nu+1}\left\{\frac{1}{\omega^{v+1}}B\left(\nu+1,\frac{1}{\omega}\right)[\psi'|\ell|+\psi'|\kappa|] \right. \\ &\left. -[\mathcal{C}_1(s,\nu,\omega)\psi'|\varphi_2|+\mathcal{C}_2(s,\nu,\omega)\psi'|\ell|]\right\}. \end{aligned} \tag{13}$$

where

$$\mathcal{C}_1(s,\nu,\omega) = \int_0^1 \left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\tau^s d\tau,$$

$$\mathcal{C}_2(s,\nu,\omega) = \int_0^1 \left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}(1-\tau)^s d\tau,$$

and the beta mapping $B(\cdot,\cdot)$ is defined in (7).

Proof. Using Lemma 1 and the Jensen–Mercer inequality with the s-convexity of $|\psi'|$ on $[\mu,\kappa]$, we obtain

$$\begin{aligned}
&\Bigg|(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)-\frac{\omega^{\nu}\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\\
&+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)-\frac{\omega^{\nu}\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg|\\
&=\Bigg|(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])d\tau\\
&-(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])d\tau\Bigg|\\
&\le(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\left|\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])\right|d\tau\\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\left|\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])\right|d\tau\\
&\le(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\left\{\psi'|\ell|+\psi'|\mu|-\left[\tau^{s}\psi'|\varphi_1|+(1-\tau)^{s}\psi'|\ell|\right]\right\}d\tau\\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}\left\{\psi'|\ell|+\psi'|\kappa|-\left[\tau^{s}\psi'|\varphi_2|+(1-\tau)^{s}\psi'|\ell|\right]\right\}d\tau
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu}d\tau&=\frac{1}{\omega^{\nu}}\frac{\Gamma(1+\nu)\Gamma\left(1+\frac{1}{\omega}\right)}{\Gamma\left(1+\nu+\frac{1}{\omega}\right)}\\
&=\frac{1}{\omega^{\nu}}\frac{\nu\Gamma(\nu)\frac{1}{\omega}\Gamma\left(\frac{1}{\omega}\right)}{\nu+\frac{1}{\omega}\Gamma\left(\nu+\frac{1}{\omega}\right)}=\frac{\nu}{\omega^{\nu}(\omega\nu+1)}B\left(\nu,\frac{1}{\omega}\right)\\
&=\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right).
\end{aligned}$$

Which implies

$$\begin{aligned}
&\le(\ell-\varphi_1)^{\nu+1}\left\{\frac{1}{\omega^{v+1}}B\left(\nu+1,\frac{1}{\omega}\right)[\psi'|\ell|+\psi'|\mu|]\right.\\
&\left.-\left[\mathcal{C}_1(s,\nu,\omega)\psi'|\varphi_1|+\mathcal{C}_2(s,\nu,\omega)\psi'|\ell|\right]\right\}\\
&+(\varphi_2-\ell)^{\nu+1}\left\{\frac{1}{\omega^{v+1}}B\left(\nu+1,\frac{1}{\omega}\right)[\psi'|\ell|+\psi'|\kappa|]\right.\\
&\left.-\left[\mathcal{C}_1(s,\nu,\omega)\psi'|\varphi_2|+\mathcal{C}_2(s,\nu,\omega)\psi'|\ell|\right]\right\}.
\end{aligned}\tag{14}$$

The proof is completed. □

Remark 3. *By setting $\varphi_1=\mu,\varphi_2=\kappa$ and $\omega=1$ in Theorem 1, we obtain Theorem 7 in [1].*

Remark 4. *By setting $\varphi_1=\mu,\varphi_2=\kappa$, $\omega=1$, and $s=1$ in Theorem 1, we obtain Theorem 2 in [4].*

Corollary 1. *If we set $\varphi_1 = \mu, \varphi_2 = \kappa$, $\omega = 1$ and $\nu = 1$ in Theorem 1, we obtain*

$$\begin{aligned}
&\left|(\kappa-\mu)\psi(\ell) - \int_{\mu}^{\kappa} \psi(\lambda)d\lambda\right| \\
&\leq (\ell-\mu)^2\left\{\frac{1}{2}\left(\psi'|\ell| + \psi'|\mu|\right) - \left[\frac{1}{s+2}\psi'|\mu| + \frac{1}{2+3s+s^2}\psi'|\ell|\right]\right\} \\
&+ (\kappa-\ell)^2\left\{\frac{1}{2}\left(\psi'|\ell| + \psi'|\kappa|\right) - \left[\frac{1}{s+2}\psi'|\kappa| + \frac{1}{2+3s+s^2}\psi'|\ell|\right]\right\}.
\end{aligned}$$

Corollary 2. *If we set $\omega = 1$ and $\nu = 1$ in Theorem 1, we obtain*

$$\begin{aligned}
&|(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1) + (\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2) \\
&-\left\{\int_{\mu}^{\ell+\mu-\varphi_1} \psi(\lambda)d\lambda + \int_{\ell+\kappa-\varphi_2}^{\kappa} \psi(\lambda)d\lambda\right\}\Bigg| \\
&\leq (\ell-\varphi_1)^2\left\{\frac{1}{2}\left[\psi'|\ell| + \psi'|\mu|\right] - \left[\frac{1}{s+2}\psi'|\varphi_1| + \frac{1}{2+3s+s^2}\psi'|\ell|\right]\right\} \\
&+ (\varphi_2-\ell)^2\left\{\frac{1}{2}\left[\psi'|\ell| + \psi'|\kappa|\right] - \left[\frac{1}{s+2}\psi'|\varphi_2| + \frac{1}{2+3s+s^2}\psi'|\ell|\right]\right\}.
\end{aligned}$$

Corollary 3. *If we set $\omega = 1$, $\nu = 1$ and $s = 1$ in Theorem 1, we obtain*

$$\begin{aligned}
&|(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1) + (\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2) \\
&-\left\{\int_{\mu}^{\ell+\mu-\varphi_1} \psi(\lambda)d\lambda + \int_{\ell+\kappa-\varphi_2}^{\kappa} \psi(\lambda)d\lambda\right\}\Bigg| \\
&\leq (\ell-\varphi_1)^2\left\{\frac{1}{2}\left[\psi'|\ell| + \psi'|\mu|\right] - \left[\frac{1}{3}\psi'|\varphi_1| + \frac{1}{6}\psi'|\ell|\right]\right\} \\
&+ (\varphi_2-\ell)^2\left\{\frac{1}{2}\left[\psi'|\ell| + \psi'|\kappa|\right] - \left[\frac{1}{3}\psi'|\varphi_2| + \frac{1}{6}\psi'|\ell|\right]\right\}.
\end{aligned}$$

Corollary 4. *By considering $|\psi'(\ell)| \leq \mathcal{M}$ in Theorem 1, we obtain*

$$\begin{aligned}
&\Bigg|(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\
&+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2) - \frac{\omega^{\nu}\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg| \\
&\leq M\left[\frac{2}{\omega^{\nu+1}}B\left(\nu+1, \frac{1}{\omega}\right) - \{\mathcal{C}_1(s,\nu,\omega) + \mathcal{C}_2(s,\nu,\omega)\}\right]\left\{(\ell-\varphi_1)^{\nu+1} + (\varphi_2-\ell)^{\nu+1}\right\}.
\end{aligned}$$

Corollary 5. *Taking $\varphi_1 = \mu, \varphi_2 = \kappa$ in Corollary 4, we obtain*

$$\begin{aligned}
&\left|\left\{(\ell-\mu)^{\nu} + (\kappa-\ell)^{\nu}\right\}\psi(\ell) - \omega^{\nu}\Gamma(\nu+1)\right. \\
&\left.\left\{\frac{1}{(\ell-\mu)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell}\psi(\mu)\right) + \frac{1}{(\kappa-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell}I^{\omega}\psi(\kappa)\right)\right\}\right| \\
&\leq M\left[\frac{2}{\omega^{\nu+1}}B\left(\nu+1, \frac{1}{\omega}\right) - \{\mathcal{C}_1(s,\nu,\omega) + \mathcal{C}_2(s,\nu,\omega)\}\right]\left\{(\ell-\mu)^{\nu+1} + (\kappa-\ell)^{\nu+1}\right\}.
\end{aligned}$$

Remark 5. *If we set $\omega = 1$ and $\nu = 1$ in Corollary 5, we obtain Theorem 2 in [5].*

Remark 6. *Taking $\varphi_1 = \mu, \varphi_2 = \kappa, \omega = 1$ and $s = 1$ in Corollary 4, we obtain Corollary 1 in [1].*

Theorem 2. *For a differentiable mapping $\psi : [\mu, \kappa] \to \mathbb{R}$ on (μ, κ) and if $|\psi'|^q$ is an s-convex mapping in the second sense on $[\mu, \kappa]$ and $p, q > 1$. Then, under the assumptions of Lemma 1, the following inequality holds:*

$$\begin{aligned}
&\bigg|(\ell-\varphi_1)^\nu \psi(\ell+\mu-\varphi_1) - \frac{\omega^\nu \Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\
&+(\varphi_2-\ell)^\nu \psi(\ell+\kappa-\varphi_2) - \frac{\omega^\nu \Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\bigg| \\
&\leq (\ell-\varphi_1)^{\nu+1}\frac{1}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}} \\
&\left(|\psi'(\ell)|^q + |\psi'(\mu)|^q - \frac{1}{s+1}\left[|\psi'(\varphi_1)|^q + |\psi'(\ell)|^q\right]\right)^{\frac{1}{q}} \\
&+(\varphi_2-\ell)^{\nu+1}\frac{1}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}} \\
&\left(|\psi'(\ell)|^q + |\psi'(\kappa)|^q - \frac{1}{s+1}\left[|\psi'(\varphi_2)|^q + |\psi'(\ell)|^q\right]\right)^{\frac{1}{q}},
\end{aligned} \tag{15}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$ and $B(\cdot,\cdot)$ is the beta mapping defined in (7).

Proof. Using Lemma 1 and the Hölder inequality for integrals, we have

$$\begin{aligned}
&\bigg|(\ell-\varphi_1)^\nu \psi(\ell+\mu-\varphi_1) - \frac{\omega^\nu \Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\
&+(\varphi_2-\ell)^\nu \psi(\ell+\kappa-\varphi_2) - \frac{\omega^\nu \Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\bigg| \\
&\leq (\ell-\varphi_1)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu |\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|d\tau \\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu |\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|d\tau \\
&\leq (\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|^q d\tau\right)^{\frac{1}{q}} \\
&+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned} \tag{16}$$

Now, by applying the Jensen–Mercer inequality with the s-convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\leq (\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p}d\tau\right)^{\frac{1}{p}} \\
&\left(\int_0^1\left\{|\psi'(\ell)|^q + |\psi'(\mu)|^q - \left[\tau^s|\psi'(\varphi_1)|^q + (1-\tau)^s|\psi'(\ell)|^q\right]\right\}d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

$$+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^{\omega}}{\omega}\right)^{\nu p}d\tau\right)^{\frac{1}{p}}$$

$$\left(\int_0^1\left\{|\psi'(\ell)|^q+|\psi'(\kappa)|^q-\left[\tau^s|\psi'(\varphi_2)|^q+(1-\tau)^s|\psi'(\ell)|^q\right]\right\}d\tau\right)^{\frac{1}{q}}$$

$$\leq(\ell-\varphi_1)^{\nu+1}\frac{1}{\omega^{\nu}}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}}$$

$$\left(|\psi'(\ell)|^q+|\psi'(\mu)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_1)|^q+|\psi'(\ell)|^q\right]\right)^{\frac{1}{q}}$$

$$+(\varphi_2-\ell)^{\nu+1}\frac{1}{\omega^{\nu}}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}}$$

$$\left(|\psi'(\ell)|^q+|\psi'(\kappa)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_2)|^q+|\psi'(\ell)|^q\right]\right)^{\frac{1}{q}}. \quad (17)$$

The proof is completed. □

Remark 7. *By setting $\varphi_1=\mu,\varphi_2=\kappa$ and $\omega=1$ in Theorem 2, we obtain Theorem 8 in [1].*

Remark 8. *By setting $\varphi_1=\mu,\varphi_2=\kappa$, $\omega=1$, and $s=1$ in Theorem 2, we obtain Theorem 3 in [4].*

Corollary 6. *If we set $\omega=1$ and $\nu=1$ in Theorem 2, we obtain*

$$\left|(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)-\left\{\int_{\mu}^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^{\kappa}\psi(\lambda)d\lambda\right\}\right|$$

$$\leq(\ell-\varphi_1)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\left(|\psi'(\ell)|^q+|\psi'(\mu)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_1)|^q+|\psi'(\ell)|^q\right]\right)^{\frac{1}{q}}$$

$$+(\varphi_2-\ell)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\left(|\psi'(\ell)|^q+|\psi'(\kappa)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_2)|^q+|\psi'(\ell)|^q\right]\right)^{\frac{1}{q}}. \quad (18)$$

Corollary 7. *If we set $\omega=1$, $\nu=1$ and $s=1$ in Theorem 2, we obtain*

$$\left|(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)-\left\{\int_{\mu}^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^{\kappa}\psi(\lambda)d\lambda\right\}\right|$$

$$\leq(\ell-\varphi_1)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\left(|\psi'(\ell)|^q+|\psi'(\mu)|^q-\frac{1}{2}\left[|\psi'(\varphi_1)|^q+|\psi'(\ell)|^q\right]\right)^{\frac{1}{q}}$$

$$+(\varphi_2-\ell)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\left(\left|\psi'(\ell)\right|^q+\left|\psi'(\kappa)\right|^q-\frac{1}{2}\left[\left|\psi'(\varphi_2)\right|^q+\left|\psi'(\ell)\right|^q\right]\right)^{\frac{1}{q}}.$$

Corollary 8. *If we set* $\varphi_1=\mu,\varphi_2=\kappa$*,* $\omega=1$ *and* $\nu=1$ *in Theorem 2, we obtain*

$$\begin{aligned}&\left|(\kappa-\mu)\psi(\ell)-\int_\mu^\kappa\psi(\lambda)d\lambda\right|\\&\leq(\ell-\mu)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{s}{s+1}\left[\left|\psi'(\mu)\right|^q+\left|\psi'(\ell)\right|^q\right]\right)^{\frac{1}{q}}\\&+(\kappa-\ell)^2\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{s}{s+1}\left[\left|\psi'(\kappa)\right|^q+\left|\psi'(\ell)\right|^q\right]\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 9. *By considering* $|\psi'(\ell)|\leq\mathcal{M}$ *in Theorem 2, we obtain*

$$\begin{aligned}&\Bigg|(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)-\frac{\omega^\nu\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\\&+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)-\frac{\omega^\nu\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg|\\&\leq\frac{\mathcal{M}}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}}\left(\frac{2s}{s+1}\right)^{\frac{1}{q}}\left[(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\right].\end{aligned}$$

Corollary 10. *Taking* $\varphi_1=\mu,\varphi_2=\kappa$ *in Corollary 9, we obtain*

$$\begin{aligned}&\left|\{(\ell-\mu)^\nu+(\kappa-\ell)^\nu\}\psi(\ell)-\omega^\nu\Gamma(\nu+1)\left\{\frac{1}{(\ell-\mu)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell}\psi(\mu)\right)+\frac{1}{(\kappa-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell}I^{\omega}\psi(\kappa)\right)\right\}\right|\\&\leq\frac{\mathcal{M}}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}}\left(\frac{2s}{s+1}\right)^{\frac{1}{q}}\left[(\ell-\mu)^{\nu+1}+(\kappa-\ell)^{\nu+1}\right].\end{aligned}$$

Remark 9. *If we set* $\omega=1$ *and* $\nu=1$ *in Corollary 10, we obtain Theorem 3 in [5].*

Remark 10. *Taking* $\varphi_1=\mu,\varphi_2=\kappa$*,* $\omega=1$ *and* $s=1$ *in Corollary 9, we obtain Corollary 2 in [1].*

Theorem 3. *For a differentiable mapping* $\psi:[\mu,\kappa]\to\mathbb{R}$ *on* (μ,κ) *and if* $|\psi'|^q$ *is an s-convex mapping in the second sense on* $[\mu,\kappa]$ *and* $q>1$*. Then, under the assumptions of Lemma 1, the following inequality holds:*

$$\begin{aligned}&\Bigg|(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)-\frac{\omega^\nu\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\\&+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)-\frac{\omega^\nu\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg|\\&\leq(\ell-\varphi_1)^{\nu+1}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\&\left(\left\{\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\left(\left|\psi'(\ell)\right|^q+\left|\psi'(\mu)\right|^q\right)-\left[\mathcal{C}_1(s,\nu,\omega)\left|\psi'(\varphi_1)\right|^q+\mathcal{C}_2(s,\nu,\omega)\left|\psi'(\ell)\right|^q\right]\right\}\right)^{\frac{1}{q}}\\&+(\varphi_2-\ell)^{\nu+1}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\end{aligned}$$

$$\left(\left\{\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\left(\left|\psi'(\ell)\right|^q+\left|\psi'(\kappa)\right|^q\right)-\left[\mathcal{C}_1(s,\nu,\omega)\left|\psi'(\varphi_2)\right|^q+\mathcal{C}_2(s,\nu,\omega)\left|\psi'(\ell)\right|^q\right]\right\}\right)^{\frac{1}{q}}, \tag{19}$$

where $B(\cdot,\cdot)$ is the beta mapping defined by (7) and $\mathcal{C}_1(s,\nu,\omega)$ and $\mathcal{C}_2(s,\nu,\omega)$ are defined in Theorem 1.

Proof. Using Lemma 1, power mean inequality and the Jensen–Mercer inequality with the s-convexity of $|\psi'|^q$, we have

$$\begin{aligned}
&\Bigg|(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)-\frac{\omega^\nu\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\\
&+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)-\frac{\omega^\nu\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg|\\
&\le(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left|\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])\right|d\tau\\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left|\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])\right|d\tau\\
&\le(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left|\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])\right|^q d\tau\right)^{\frac{1}{q}}\\
&+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left|\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])\right|^q d\tau\right)^{\frac{1}{q}}\\
&\le(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left\{\left|\psi'(\ell)\right|^q+\left|\psi'(\mu)\right|^q-\left[\tau^s\left|\psi'(\varphi_1)\right|^q+(1-\tau)^s\left|\psi'(\ell)\right|^q\right]\right\}d\tau\right)^{\frac{1}{q}}\\
&+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau\right)^{1-\frac{1}{q}}\\
&\left(\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\left\{\left|\psi'(\ell)\right|^q+\left|\psi'(\kappa)\right|^q-\left[\tau^s\left|\psi'(\varphi_2)\right|^q+(1-\tau)^s\left|\psi'(\ell)\right|^q\right]\right\}d\tau\right)^{\frac{1}{q}}\\
&\le(\ell-\varphi_1)^{\nu+1}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\
&\left(\left\{\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\left(\left|\psi'(\ell)\right|^q+\left|\psi'(\mu)\right|^q\right)-\left[\mathcal{C}_1(s,\nu,\omega)\left|\psi'(\varphi_1)\right|^q+\mathcal{C}_2(s,\nu,\omega)\left|\psi'(\ell)\right|^q\right]\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

$$+(\varphi_2-\ell)^{\nu+1}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}$$
$$\left(\left\{\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\left(|\psi'(\ell)|^q+|\psi'(\kappa)|^q\right)-\left[\mathcal{C}_1(s,\nu,\omega)|\psi'(\varphi_2)|^q+\mathcal{C}_2(s,\nu,\omega)|\psi'(\ell)|^q\right]\right\}\right)^{\frac{1}{q}}. \quad (20)$$

The proof is completed. □

Remark 11. *By setting $\varphi_1=\mu,\varphi_2=\kappa$ and $\omega=1$ in Theorem 3, we obtain Theorem 9 in [1].*

Remark 12. *By setting $\varphi_1=\mu,\varphi_2=\kappa$, $\omega=1$, and $s=1$ in Theorem 3, we obtain Theorem 4 in [4].*

Corollary 11. *If we set $\varphi_1=\mu,\varphi_2=\kappa$, $\omega=1$ and $\nu=1$ in Theorem 3, we obtain*

$$\left|(\kappa-\mu)\psi(\ell)-\int_\mu^\kappa\psi(\lambda)d\lambda\right|$$
$$\le(\ell-\mu)^2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s}{2s+4}|\psi'(\mu)|^q+\frac{3s+s^2}{4+6s+2s^2}|\psi'(\ell)|^q\right)^{\frac{1}{q}}$$
$$+(\kappa-\ell)^2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s}{2s+4}|\psi'(\kappa)|^q+\frac{3s+s^2}{4+6s+2s^2}|\psi'(\ell)|^q\right)^{\frac{1}{q}}.$$

Corollary 12. *By considering $|\psi'(\ell)|\le\mathcal{M}$ in Theorem 3, we obtain*

$$\left|(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)-\frac{\omega^\nu\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\right.$$
$$\left.+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)-\frac{\omega^\nu\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right|$$
$$\le\mathcal{M}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\left[\frac{2}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)-\{\mathcal{C}_1(s,\nu,\omega)+\mathcal{C}_2(s,\nu,\omega)\}\right]^{\frac{1}{q}}$$
$$\left[(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\right].$$

Corollary 13. *Taking $\varphi_1=\mu,\varphi_2=\kappa$ in Corollary 12, we obtain*

$$\left|\{(\ell-\mu)^\nu+(\kappa-\ell)^\nu\}\psi(\ell)-\omega^\nu\Gamma(\nu+1)\left\{\frac{1}{(\ell-\mu)^{\omega\nu-\nu}}({}^{\nu}I^{\omega}_{\ell}\psi(\mu))+\frac{1}{(\kappa-\ell)^{\omega\nu-\nu}}({}^{\nu}_{\ell}I^{\omega}\psi(\kappa))\right\}\right|$$
$$\le\mathcal{M}\left(\frac{1}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\left[\frac{2}{\omega^{\nu+1}}B\left(\nu+1,\frac{1}{\omega}\right)-\{\mathcal{C}_1(s,\nu,\omega)+\mathcal{C}_2(s,\nu,\omega)\}\right]^{\frac{1}{q}}$$
$$\left[(\ell-\mu)^{\nu+1}+(\kappa-\ell)^{\nu+1}\right].$$

Remark 13. *If we set $\omega=1$ and $\nu=1$ in Corollary 13, we obtain Theorem 4 in [5].*

Remark 14. *Taking $\varphi_1=\mu,\varphi_2=\kappa$, $\omega=1$ and $s=1$ in Corollary 12, we obtain Corollary 3 in [1].*

Theorem 4. *For a differentiable mapping $\psi : [\mu, \kappa] \to \mathbb{R}$ on (μ, κ) and if $|\psi'|^q$ is an s-convex mapping in the second sense on $[\mu, \kappa]$ with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, under the assumptions of Lemma 1, the following inequality holds:*

$$\begin{aligned} &\Bigg|(\ell-\varphi_1)^\nu \psi(\ell+\mu-\varphi_1) - \frac{\omega^\nu \Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\ &+(\varphi_2-\ell)^\nu \psi(\ell+\kappa-\varphi_2) - \frac{\omega^\nu \Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg| \\ &\leq (\ell-\varphi_1)^{\nu+1}\Bigg[\frac{1}{p\omega^{\nu p+1}}B\left(\nu p+1, \frac{1}{\omega}\right) \\ &+\frac{1}{q}\left\{|\psi'(\ell)|^q + |\psi'(\mu)|^q - \frac{1}{s+1}\Big[|\psi'(\varphi_1)|^q + |\psi'(\ell)|^q\Big]\right\}\Bigg] \\ &+(\varphi_2-\ell)^{\nu+1}\Bigg[\frac{1}{p\omega^{\nu p+1}}B\left(\nu p+1, \frac{1}{\omega}\right) \\ &+\frac{1}{q}\left\{|\psi'(\ell)|^q + |\psi'(\kappa)|^q - \frac{1}{s+1}\Big[|\psi'(\varphi_2)|^q + |\psi'(\ell)|^q\Big]\right\}\Bigg]. \end{aligned} \tag{21}$$

Proof. Taking modulus of Lemma 1 and using Young's inequality, i.e., $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ (equality holds when $x^p = y^q$), we have

$$\begin{aligned} &\Bigg|(\ell-\varphi_1)^\nu \psi(\ell+\mu-\varphi_1) - \frac{\omega^\nu \Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right) \\ &+(\varphi_2-\ell)^\nu \psi(\ell+\kappa-\varphi_2) - \frac{\omega^\nu \Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\Bigg| \\ &\leq (\ell-\varphi_1)^{\nu+1}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu |\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|d\tau \\ &+(\varphi_2-\ell)^{\nu+1}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu |\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|d\tau \\ &\leq (\ell-\varphi_1)^{\nu+1}\left[\frac{1}{p}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p} d\tau + \frac{1}{q}\int_0^1 |\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|^q d\tau\right] \\ &+(\varphi_2-\ell)^{\nu+1}\left[\frac{1}{p}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu} p d\tau + \frac{1}{q}\int_0^1 |\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|^q d\tau\right] \end{aligned}$$

Now, applying the Jensen–Mercer inequality with the s-convexity of $|\psi'|^q$, we obtain

$$\begin{aligned} &\leq (\ell-\varphi_1)^{\nu+1}\Bigg[\frac{1}{p}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p} d\tau \\ &+\frac{1}{q}\int_0^1 \left\{|\psi'(\ell)|^q + |\psi'(\mu)|^q - \Big[\tau^s|\psi'(\varphi_1)|^q + (1-\tau)^s|\psi'(\ell)|^q\Big]\right\}d\tau\Bigg] \\ &+(\varphi_2-\ell)^{\nu+1}\Bigg[\frac{1}{p}\int_0^1 \left(\frac{1-(1-\tau)^\omega}{\omega}\right)^{\nu p} d\tau \end{aligned}$$

$$+\frac{1}{q}\int_0^1 \left\{|\psi'(\ell)|^q+|\psi'(\kappa)|^q-\left[\tau^s|\psi'(\varphi_2)|^q+(1-\tau)^s|\psi'(\ell)|^q\right]\right\}d\tau\Bigg]$$
$$\leq (\ell-\varphi_1)^{\nu+1}\left[\frac{1}{p\omega^{\nu p+1}}B\left(\nu p+1,\frac{1}{\omega}\right)\right.$$
$$\left.+\frac{1}{q}\left\{|\psi'(\ell)|^q+|\psi'(\mu)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_1)|^q+|\psi'(\ell)|^q\right]\right\}\right]$$
$$+(\varphi_2-\ell)^{\nu+1}\left[\frac{1}{p\omega^{\nu p+1}}B\left(\nu p+1,\frac{1}{\omega}\right)\right.$$
$$\left.+\frac{1}{q}\left\{|\psi'(\ell)|^q+|\psi'(\kappa)|^q-\frac{1}{s+1}\left[|\psi'(\varphi_2)|^q+|\psi'(\ell)|^q\right]\right\}\right]. \quad (22)$$

The proof is completed. □

Corollary 14. *By considering* $|\psi'(\ell)|\leq\mathcal{M}$ *in Theorem 4, we obtain*

$$\left|(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)-\frac{\omega^\nu\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu-\nu}}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)\right.$$
$$\left.+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)-\frac{\omega^\nu\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu-\nu}}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right|$$
$$\leq\left\{\frac{1}{p\omega^{\nu p+1}}B\left(\nu p+1,\frac{1}{\omega}\right)+\frac{2\mathcal{M}^q}{q}\frac{s}{s+1}\right\}\left[(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\right].$$

Remark 15. *If we set* $\omega=1$ *and* $s=1$ *in Theorem 4, we obtain Theorem 5 in [4].*

3. Ostrowski–Mercer-Type Inequalities for the Twice Differentiable *s*-Convex Mappings

In this section, we first establish a key result for the twice differentiable mappings involving conformable fractional integrals. Then, by utilizing this result, we obtain several inequalities for the twice differentiable mappings whose absolute values are *s*-convex in the second sense.

Lemma 2. *Let* $\psi:[\mu,\kappa]\to\mathbb{R}$ *be twice differentiable mapping on* (φ_1,φ_2) *and* ψ'' *is integrable mapping on* $[\varphi_1,\varphi_2]$*, then for all* $\ell\in[\varphi_1,\varphi_2]$*,* $\varphi_1,\varphi_2\in[\mu,\kappa]$*,* $\omega\in(0,1]$ *and* $Re(\nu)>0$*, the following identity holds:*

$$\left\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\right\}$$
$$+\int_0^1\left(1-(1-\tau)^\omega\right)^\nu d\tau\left\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\right\}$$
$$-\omega^\nu\Gamma(\nu+1)\left\{(\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)+(\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right\}$$
$$=(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])d\tau$$
$$-(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])d\tau. \quad (23)$$

Proof. Consider

$$\begin{aligned}
&(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])d\tau\\
&-(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])d\tau\\
&=(\ell-\varphi_1)^{\nu+1}Y_1-(\varphi_2-\ell)^{\nu+1}Y_2.
\end{aligned}\tag{24}$$

Applying integration by parts, we have

$$\begin{aligned}
Y_1&=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])d\tau\\
&=\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\frac{\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])}{\ell-\varphi_1}\Bigg|_0^1\\
&+\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\frac{\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])}{\ell-\varphi_1}d\tau.
\end{aligned}$$

Since we have proved in Lemma 1:

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\psi'(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])d\tau\\
&=\frac{1}{\omega^\nu}\frac{\psi(\ell+\mu-\varphi_1)}{\ell-\varphi_1}-\frac{\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu+1}}\left({}^\nu I^\omega_{\ell+\mu-\varphi_1}\psi(\mu)\right).
\end{aligned}$$

Which implies

$$\begin{aligned}
&=-\frac{\psi'(\mu)}{\ell-\varphi_1}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau\\
&+\frac{1}{\omega^\nu}\frac{\psi(\ell+\mu-\varphi_1)}{\ell-\varphi_1}-\frac{\Gamma(\nu+1)}{(\ell-\varphi_1)^{\omega\nu+1}}\left({}^\nu I^\omega_{\ell+\mu-\varphi_1}\psi(\mu)\right).
\end{aligned}\tag{25}$$

Similarly,

$$\begin{aligned}
Y_2&=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])d\tau\\
&=\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\frac{\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])}{\ell-\varphi_2}\Bigg|_0^1\\
&+\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu\frac{\psi'(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])}{\varphi_2-\ell}d\tau\\
&=-\frac{\psi'(\kappa)}{\varphi_2-\ell}\int_0^1\left(\frac{1-(1-\tau)^\omega}{\omega}\right)^\nu d\tau-\frac{1}{\omega^\nu}\frac{\psi(\ell+\kappa-\varphi_2)}{\varphi_2-\ell}\\
&+\frac{\Gamma(\nu+1)}{(\varphi_2-\ell)^{\omega\nu+1}}\left({}^\nu_{\ell+\kappa-\varphi_2}I^\omega\psi(\kappa)\right).
\end{aligned}\tag{26}$$

Now using (25) and (26) in (24) and multiplying by ω^{ν}, we obtain

$$\begin{aligned}
&\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\} \\
&+\int_0^1 (1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\} \\
&-\omega^{\nu}\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}.
\end{aligned} \tag{27}$$

The proof is completed. □

Theorem 5. *For a twice differentiable mapping $\psi:[\mu,\kappa]\to\mathbb{R}$ on (μ,κ) and if $|\psi''|$ is an s-convex mapping in the second sense on $[\mu,\kappa]$. Then, under the assumptions of Lemma 2, the following inequality holds:*

$$\begin{aligned}
&\Big|\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\} \\
&+\int_0^1 (1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\} \\
&-\omega^{\nu}\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big| \\
&\le(\ell-\varphi_1)^{\nu+1}\big[\mathcal{C}_3(\nu,\omega)\{\psi''|\ell|+\psi''|\mu|\}-\{\mathcal{C}_4(s,\nu,\omega)\psi''|\varphi_1|+\mathcal{C}_5(s,\nu,\omega)\psi''|\ell|\}\big] \\
&+(\varphi_2-\ell)^{\nu+1}\big[\mathcal{C}_3(\nu,\omega)\{\psi''|\ell|+\psi''|\kappa|\}-\{\mathcal{C}_4(s,\nu,\omega)\psi''|\varphi_2|+\mathcal{C}_5(s,\nu,\omega)\psi''|\ell|\}\big],
\end{aligned} \tag{28}$$

where

$$\mathcal{C}_3(\nu,\omega)=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau,$$
$$\mathcal{C}_4(s,\nu,\omega)=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)\tau^s d\tau$$

and

$$\mathcal{C}_5(s,\nu,\omega)=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)(1-\tau)^s d\tau.$$

Proof. Using Lemma 2 and the Jensen–Mercer inequality with the s-convexity of $|\psi''|$, we obtain

$$\begin{aligned}
&\Big|\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\} \\
&+\int_0^1 (1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\} \\
&-\omega^{\nu}\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big| \\
&\le(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|d\tau \\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|d\tau
\end{aligned}$$

$$\leq (\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\{\psi''|\ell|+\psi''|\mu|-[\tau^s\psi''|\varphi_1|+(1-\tau)^s\psi''|\ell|]\}d\tau$$
$$+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)\{\psi''|\ell|+\psi''|\kappa|-[\tau^s\psi''|\varphi_2|+(1-\tau)^s\psi''|\ell|]\}d\tau$$
$$\leq (\ell-\varphi_1)^{\nu+1}[\mathcal{C}_3(\nu,\omega)\{\psi''|\ell|+\psi''|\mu|\}-\{\mathcal{C}_4(s,\nu,\omega)\psi''|\varphi_1|+\mathcal{C}_5(s,\nu,\omega)\psi''|\ell|\}]$$
$$+(\varphi_2-\ell)^{\nu+1}[\mathcal{C}_3(\nu,\omega)\{\psi''|\ell|+\psi''|\kappa|\}-\{\mathcal{C}_4(s,\nu,\omega)\psi''|\varphi_2|+\mathcal{C}_5(s,\nu,\omega)\psi''|\ell|\}]. \quad (29)$$

The proof is completed. □

Corollary 15. *Setting $\varphi_1=\mu,\varphi_2=\kappa$ with $\omega=1$ and $\nu=1$ in Theorem 5*

$$\left|\{(\kappa-\mu)\psi(\ell)\}+\frac{1}{2}\{(\kappa-\ell)\psi'(\kappa)-(\ell-\mu)\psi'(\mu)\}-\int_\mu^\kappa\psi(\lambda)d\lambda\right|$$
$$\leq (\ell-\mu)^2\left[\left\{\left(\frac{1}{3}-\frac{1}{(s+1)(s+3)}\right)\psi''|\mu|+\left(\frac{1}{3}-\frac{1}{2(s+1)}+\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\psi''|\ell|\right\}\right]$$
$$+(\kappa-\ell)^2\left[\left\{\left(\frac{1}{3}-\frac{1}{(s+1)(s+3)}\right)\psi''|\kappa|+\left(\frac{1}{3}-\frac{1}{2(s+1)}+\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\psi''|\ell|\right\}\right].$$

Corollary 16. *If we set $\omega=1$ and $\nu=1$ in Theorem 5, we obtain*

$$\Bigg|\{(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)\}$$
$$+\frac{1}{2}\{(\varphi_2-\ell)\psi'(\kappa)-(\ell-\varphi_1)\psi'(\mu)\}-\left\{\int_\mu^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^\kappa\psi(\lambda)d\lambda\right\}\Bigg|$$
$$\leq (\ell-\varphi_1)^2\left[\frac{1}{3}\{\psi''|\ell|+\psi''|\mu|\}-\left\{\frac{1}{(s+1)(s+3)}\psi''|\varphi_1|+\left(\frac{1}{2(s+1)}-\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\psi''|\ell|\right\}\right]$$
$$+(\varphi_2-\ell)^2\left[\frac{1}{3}\{\psi''|\ell|+\psi''|\kappa|\}-\left\{\frac{1}{(s+1)(s+3)}\psi''|\varphi_2|+\left(\frac{1}{2(s+1)}-\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\psi''|\ell|\right\}\right].$$

Corollary 17. *By considering $|\psi''(\ell)|\leq\mathcal{M}_1$ in Theorem 5, we obtain*

$$\Big|\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\}$$
$$+\int_0^1(1-(1-\tau)^\omega)^\nu d\tau\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\}$$
$$-\omega^\nu\Gamma(\nu+1)\left\{(\ell-\varphi_1)^{\nu-\omega\nu}\left({}^\nu I^\omega_{\ell+\mu-\varphi_1}\psi(\mu)\right)+(\varphi_2-\ell)^{\nu-\omega\nu}\left({}^\nu_{\ell+\kappa-\varphi_2}I^\omega\psi(\kappa)\right)\right\}\Big|$$
$$\leq (\ell-\varphi_1)^{\nu+1}\mathcal{M}_1[2\mathcal{C}_3(\nu,\omega)-\{\mathcal{C}_4(s,\nu,\omega)+\mathcal{C}_5(s,\nu,\omega)\}]$$
$$+(\varphi_2-\ell)^{\nu+1}\mathcal{M}_1[2\mathcal{C}_3(\nu,\omega)-\{\mathcal{C}_4(s,\nu,\omega)+\mathcal{C}_5(s,\nu,\omega)\}].$$

Theorem 6. *For a twice differentiable mapping $\psi:[\mu,\kappa]\to\mathbb{R}$ on (μ,κ) and if $|\psi''|^{q_1}$ is an s-convex mapping in the second sense on $[\mu,\kappa]$ and $p_1,q_1>1$. Then, under the assumptions of Lemma 2, the following inequality holds:*

$$\Big|\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\}$$
$$+\int_0^1(1-(1-\tau)^\omega)^\nu d\tau\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\}$$

$$-\omega^{\nu}\Gamma(\nu+1)\left\{(\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)+(\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right\}\Big|$$
$$\leq (\mathcal{C}_6(\nu,\omega))^{\frac{1}{p_1}}\Bigg[(\ell-\varphi_1)^{\nu+1}\left(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\frac{1}{s+1}\left[|\psi''(\varphi_1)|^{q_1}+|\psi''(\ell)|^{q_1}\right]\right)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\left(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\frac{1}{s+1}\left[|\psi''(\varphi_2)|^{q_1}+|\psi''(\ell)|^{q_1}\right]\right)^{\frac{1}{q_1}}\Bigg], \tag{30}$$

where

$$\mathcal{C}_6(\nu,\omega)=\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau,$$

and $\frac{1}{p_1}=1-\frac{1}{q_1}$.

Proof. Using Lemma 2 and the Hölder inequality for integrals, we have

$$\big|\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\}$$
$$+\int_0^1(1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\}$$
$$-\omega^{\nu}\Gamma(\nu+1)\left\{(\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)+(\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right\}\Big|$$
$$\leq(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|d\tau$$
$$+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|d\tau$$
$$\leq(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{\frac{1}{p_1}}\left(\int_0^1|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|^{q_1}d\tau\right)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{\frac{1}{p_1}}\left(\int_0^1|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|^{q_1}d\tau\right)^{\frac{1}{q_1}}.$$

Now, by applying the Jensen–Mercer inequality with the s-convexity of $|\psi''|^{q_1}$, we have

$$\leq(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{\frac{1}{p_1}}$$
$$\left(\int_0^1\left\{|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\left[\tau^s|\psi''(\varphi_1)|^{q_1}+(1-\tau)^s|\psi''(\ell)|^{q_1}\right]\right\}d\tau\right)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{\frac{1}{p_1}}$$
$$\left(\int_0^1\left\{|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\left[\tau^s|\psi''(\varphi_2)|^{q_1}+(1-\tau)^s|\psi''(\ell)|^{q_1}\right]\right\}d\tau\right)^{\frac{1}{q_1}}$$

$$\leq (\mathcal{C}_6(\nu,\omega))^{\frac{1}{p_1}}\Bigg[(\ell-\varphi_1)^{\nu+1}\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_1)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\Big)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\Big(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_2)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\Big)^{\frac{1}{q_1}}\Bigg]. \quad (31)$$

The proof is completed. □

Corollary 18. *Setting $\varphi_1=\mu,\varphi_2=\kappa$ with $\omega=1$ and $\nu=1$ in Theorem 30*

$$\Bigg|\{(\kappa-\mu)\psi(\ell)\}+\frac{1}{2}\{(\kappa-\ell)\psi'(\kappa)-(\ell-\mu)\psi'(\mu)\}-\int_\mu^\kappa\psi(\lambda)d\lambda\Bigg|$$
$$\leq\left(\frac{1}{3}\right)^{\frac{1}{p_1}}\Bigg[(\ell-\mu)^2\Big(\frac{s}{s+1}\Big\{|\psi''(\mu)|^{q_1}+|\psi''(\ell)|^{q_1}\Big\}\Big)^{\frac{1}{q_1}}$$
$$+(\kappa-\ell)^2\Big(\frac{s}{s+1}\Big\{|\psi''(\mu)|^{q_1}+|\psi''(\ell)|^{q_1}\Big\}\Big)^{\frac{1}{q_1}}\Bigg].$$

Corollary 19. *If we set $\omega=1$ and $\nu=1$ in Theorem 30, we obtain*

$$|\{(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)\}$$
$$+\frac{1}{2}\{(\varphi_2-\ell)\psi'(\kappa)-(\ell-\varphi_1)\psi'(\mu)\}$$
$$-\left\{\int_\mu^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^{\kappa}\psi(\lambda)d\lambda\right\}\Bigg|$$
$$\leq\left(\frac{1}{3}\right)^{\frac{1}{p_1}}\Bigg[(\ell-\varphi_1)^2\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_1)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\Big)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^2\Big(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_2)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\Big)^{\frac{1}{q_1}}\Bigg].$$

Corollary 20. *By considering $|\psi''(\ell)|\leq\mathcal{M}_1$ in Theorem 30, we obtain*

$$\big|\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\}$$
$$+\int_0^1(1-(1-\tau)^\omega)^\nu d\tau\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\}$$
$$-\omega^\nu\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big|$$
$$\leq(\mathcal{C}_6(\nu,\omega))^{\frac{1}{p_1}}\mathcal{M}_1\left(\frac{2s}{s+1}\right)^{\frac{1}{q_1}}\Big[(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\Big].$$

Theorem 7. *For a twice differentiable mapping $\psi:[\mu,\kappa]\to\mathbb{R}$ on (μ,κ) and if $|\psi''|^{q_1}$ is an s-convex mapping in the second sense on $[\mu,\kappa]$ and $q_1>1$. Then, under the assumptions of Lemma 2, the following inequality holds:*

$$\big|\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\}$$
$$+\int_0^1(1-(1-\tau)^\omega)^\nu d\tau\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\}$$
$$-\omega^\nu\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big|$$

$$\leq (\mathcal{C}_3(\nu,\omega))^{1-\frac{1}{q_1}}$$
$$\Big[(\ell-\varphi_1)^{\nu+1}\Big(\Big\{\mathcal{C}_3(\nu,\omega)\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}\Big)-\Big[\mathcal{C}_4(s,\nu,\omega)|\psi''(\varphi_1)|^{q_1}+\mathcal{C}_5(s,\nu,\omega)|\psi''(\ell)|^{q_1}\Big]\Big\}\Big)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\Big(\Big\{\mathcal{C}_3(\nu,\omega)\Big(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}\Big)-\Big[\mathcal{C}_4(s,\nu,\omega)|\psi''(\varphi_2)|^{q_1}+\mathcal{C}_5(s,\nu,\omega)|\psi''(\ell)|^{q_1}\Big]\Big\}\Big)^{\frac{1}{q_1}}\Big]. \tag{32}$$

where $\mathcal{C}_3(\nu,\omega)$, $\mathcal{C}_4(s,\nu,\omega)$ and $\mathcal{C}_5(s,\nu,\omega)$ are defined in Theorem 5.

Proof. Using Lemma 2, power mean inequality and the Jensen–Mercer inequality with the s-convexity of $|\psi''|^{q_1}$, we have

$$\Big|\Big\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\Big\}$$
$$+\int_0^1(1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\}$$
$$-\omega^{\nu}\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big|$$
$$\leq(\ell-\varphi_1)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|d\tau$$
$$+(\varphi_2-\ell)^{\nu+1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|d\tau$$
$$\leq(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{1-\frac{1}{q_1}}$$
$$\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])|^{q_1}d\tau\right)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{1-\frac{1}{q_1}}$$
$$\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])|^{q_1}d\tau\right)^{\frac{1}{q_1}}$$
$$\leq(\ell-\varphi_1)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{1-\frac{1}{q_1}}$$
$$\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)\Big\{|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\Big[\tau^s|\psi''(\varphi_1)|^{q_1}+(1-\tau)^s|\psi''(\ell)|^{q_1}\Big]\Big\}d\tau\right)^{\frac{1}{q_1}}$$
$$+(\varphi_2-\ell)^{\nu+1}\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)d\tau\right)^{1-\frac{1}{q_1}}$$
$$\left(\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^{\omega}}{\omega}\right)^{\nu}dx\right)\Big\{|\psi'(\ell)|^{q_1}+|\psi'(\kappa)|^{q_1}-\Big[\tau^s|\psi'(\varphi_2)|^{q_1}+(1-\tau)^s|\psi'(\ell)|^{q_1}\Big]\Big\}d\tau\right)^{\frac{1}{q_1}}$$
$$\leq(\mathcal{C}_3(\nu,\omega))^{1-\frac{1}{q_1}}$$

$$\Big[(\ell-\varphi_1)^{\nu+1}\Big(\Big\{\mathcal{C}_3(\nu,\omega)\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}\Big)-\Big[\mathcal{C}_4(s,\nu,\omega)|\psi''(\varphi_1)|^{q_1}+\mathcal{C}_5(s,\nu,\omega)|\psi''(\ell)|^{q_1}\Big]\Big\}\Big)^{\frac{1}{q_1}}$$

$$+(\varphi_2-\ell)^{\nu+1}\Big(\Big\{\mathcal{C}_3(\nu,\omega)\Big(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}\Big)-\Big[\mathcal{C}_4(s,\nu,\omega)|\psi''(\varphi_2)|^{q_1}+\mathcal{C}_5(s,\nu,\omega)|\psi''(\ell)|^{q_1}\Big]\Big\}\Big)^{\frac{1}{q_1}}\Big]. \quad (33)$$

The proof is completed. □

Corollary 21. *Setting $\varphi_1=\mu,\varphi_2=\kappa$ with $\omega=1$ and $\nu=1$ in Theorem 7*

$$\left|\{(\kappa-\mu)\psi(\ell)\}+\frac{1}{2}\{(\kappa-\ell)\psi'(\kappa)-(\ell-\mu)\psi'(\mu)\}-\int_{\mu}^{\kappa}\psi(\lambda)d\lambda\right|$$

$$\leq\left(\frac{1}{3}\right)^{1-\frac{1}{q_1}}\Bigg[(\ell-\mu)^2\Bigg(\Bigg\{\left(\frac{1}{3}-\frac{1}{s^2+4s+3}\right)|\psi''(\mu)|^{q_1}+\left(\frac{1}{3}-\frac{1}{1+s}+\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)|\psi''(\ell)|^{q_1}\Bigg\}\Bigg)^{\frac{1}{q_1}}$$

$$+(\kappa-\ell)^2\Bigg(\Bigg\{\left(\frac{1}{3}-\frac{1}{s^2+4s+3}\right)|\psi''(\kappa)|^{q_1}+\left(\frac{1}{3}-\frac{1}{1+s}+\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)|\psi''(\ell)|^{q_1}\Bigg\}\Bigg)^{\frac{1}{q_1}}\Bigg].$$

Corollary 22. *If we set $\omega=1$ and $\nu=1$ in Theorem 7, we obtain*

$$|\{(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)\}$$

$$+\frac{1}{2}\{(\varphi_2-\ell)\psi'(\kappa)-(\ell-\varphi_1)\psi'(\mu)\}-\left\{\int_{\mu}^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^{\kappa}\psi(\lambda)d\lambda\right\}\Bigg|$$

$$\leq\left(\frac{1}{3}\right)^{1-\frac{1}{q_1}}\Big[(\ell-\varphi_1)^2$$

$$\Bigg(\Bigg\{\frac{1}{3}\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}\Big)-\Bigg[\frac{1}{s^2+4s+3}|\psi''(\varphi_1)|^{q_1}+\left(\frac{1}{1+s}-\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)|\psi''(\ell)|^{q_1}\Bigg]\Bigg\}\Bigg)^{\frac{1}{q_1}}$$

$$+(\varphi_2-\ell)^2$$

$$\Bigg(\Bigg\{\frac{1}{3}\Big(|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}\Big)-\Bigg[\frac{1}{s^2+4s+3}|\psi''(\varphi_2)|^{q_1}+\left(\frac{1}{1+s}-\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)|\psi''(\ell)|^{q_1}\Bigg]\Bigg\}\Bigg)^{\frac{1}{q_1}}\Bigg].$$

Corollary 23. *By considering $|\psi''(\ell)|\leq\mathcal{M}_1$ in Theorem 7, we obtain*

$$\big|\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\}$$

$$+\int_0^1(1-(1-\tau)^{\omega})^{\nu}d\tau\{(\varphi_2-\ell)^{\nu}\psi'(\kappa)-(\ell-\varphi_1)^{\nu}\psi'(\mu)\}$$

$$-\omega^{\nu}\Gamma(\nu+1)\Big\{(\ell-\varphi_1)^{\nu-\omega\nu}\Big({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\Big)+(\varphi_2-\ell)^{\nu-\omega\nu}\Big({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\Big)\Big\}\Big|$$

$$\leq(\mathcal{C}_3(\nu,\omega))^{1-\frac{1}{q_1}}\mathcal{M}_1\{2\mathcal{C}_3(\nu,\omega)-\mathcal{C}_4(s,\nu,\omega)-\mathcal{C}_5(s,\nu,\omega)\}^{\frac{1}{q_1}}\Big[(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\Big].$$

Theorem 8. *For a twice differentiable mapping $\psi:[\mu,\kappa]\to\mathbb{R}$ on (μ,κ) and if $|\psi''|^{q_1}$ is an s-convex mapping in the second sense on $[\mu,\kappa]$ with $p_1,q_1>1$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Then, under the assumptions of Lemma 2, the following inequality holds:*

$$\big|\{(\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2)\}$$

$$
\begin{aligned}
&+\int_0^1 \left(1-(1-\tau)^{\omega}\right)^{\nu} d\tau \left\{ (\varphi_2-\ell)^{\nu}\psi'(\kappa) - (\ell-\varphi_1)^{\nu}\psi'(\mu) \right\} \\
&-\omega^{\nu}\Gamma(\nu+1)\left\{ (\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu) \right) + (\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa) \right) \right\}\Big| \\
&\leq (\ell-\varphi_1)^{\nu+1}\Bigg[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau \\
&+\frac{1}{q_1}\left\{ |\psi''(\ell)|^{q_1} + |\psi''(\mu)|^{q_1} - \frac{1}{s+1}\left[|\psi''(\varphi_1)|^{q_1} + |\psi''(\ell)|^{q_1} \right] \right\} \Bigg] \\
&+(\varphi_2-\ell)^{\nu+1}\Bigg[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau \\
&+\frac{1}{q_1}\left\{ |\psi''(\ell)|^{q_1} + |\psi''(\kappa)|^{q_1} - \frac{1}{s+1}\left[|\psi''(\varphi_2)|^{q_1} + |\psi''(\ell)|^{q_1} \right] \right\} \Bigg].
\end{aligned} \tag{34}
$$

Proof. Taking modulus of Lemma 2 and using Young's inequality, i.e., $xy \leq \frac{1}{p_1}x^{p_1} + \frac{1}{q_1}y^{q_1}$ (equality holds when $x^{p_1} = y^{q_1}$), we have

$$
\begin{aligned}
&\Big| \left\{ (\ell-\varphi_1)^{\nu}\psi(\ell+\mu-\varphi_1) + (\varphi_2-\ell)^{\nu}\psi(\ell+\kappa-\varphi_2) \right\} \\
&+\int_0^1 \left(1-(1-\tau)^{\omega}\right)^{\nu} d\tau \left\{ (\varphi_2-\ell)^{\nu}\psi'(\kappa) - (\ell-\varphi_1)^{\nu}\psi'(\mu) \right\} \\
&-\omega^{\nu}\Gamma(\nu+1)\left\{ (\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu) \right) + (\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa) \right) \right\}\Big| \\
&\leq (\ell-\varphi_1)^{\nu+1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right) \left|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])\right| d\tau \\
&+(\varphi_2-\ell)^{\nu+1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right) \left|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])\right| d\tau \\
&\leq (\ell-\varphi_1)^{\nu+1}\left[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau + \frac{1}{q_1}\int_0^1 \left|\psi''(\ell+\mu-[\tau\varphi_1+(1-\tau)\ell])\right|^{q_1} d\tau \right] \\
&+(\varphi_2-\ell)^{\nu+1}\left[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau + \frac{1}{q_1}\int_0^1 \left|\psi''(\ell+\kappa-[\tau\varphi_2+(1-\tau)\ell])\right|^{q_1} d\tau \right]
\end{aligned}
$$

Now, applying the Jensen–Mercer inequality with the s-convexity of $|\psi''|^{q_1}$, we obtain

$$
\begin{aligned}
&\leq (\ell-\varphi_1)^{\nu+1}\Bigg[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau \\
&+\frac{1}{q_1}\int_0^1 \left\{ |\psi''(\ell)|^{q_1} + |\psi''(\mu)|^{q_1} - \left[\tau^{s}|\psi''(\varphi_1)|^{q_1} + (1-\tau)^{s}|\psi''(\ell)|^{q_1} \right] \right\} d\tau \Bigg] \\
&+(\varphi_2-\ell)^{\nu+1}\Bigg[\frac{1}{p_1}\int_0^1 \left(\int_\tau^1 \left(\frac{1-(1-x)^{\omega}}{\omega} \right)^{\nu} dx \right)^{p_1} d\tau
\end{aligned}
$$

$$+\frac{1}{q_1}\int_0^1\Big\{|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\Big[\tau^s|\psi''(\varphi_2)|^{q_1}+(1-\tau)^s|\psi''(\ell)|^{q_1}\Big]\Big\}d\tau\Bigg]$$

$$\leq(\ell-\varphi_1)^{\nu+1}\left[\frac{1}{p_1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)^{p_1}d\tau\right.$$

$$\left.+\frac{1}{q_1}\left\{|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_1)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\right\}\right]$$

$$+(\varphi_2-\ell)^{\nu+1}\left[\frac{1}{p_1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)^{p_1}d\tau\right.$$

$$\left.+\frac{1}{q_1}\left\{|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_2)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\right\}\right]. \tag{35}$$

The proof is completed. □

Corollary 24. *Setting $\varphi_1=\mu,\varphi_2=\kappa$ with $\omega=1$ and $\nu=1$ in Theorem 8*

$$\left|\{(\kappa-\mu)\psi(\ell)\}+\frac{1}{2}\{(\kappa-\ell)\psi'(\kappa)-(\ell-\mu)\psi'(\mu)\}-\int_\mu^\kappa\psi(\lambda)d\lambda\right|$$

$$\leq\left\{(\ell-\mu)^2+(\kappa-\ell)^2\right\}\left[\frac{\sqrt{\pi}\Gamma(1+p_1)}{2^{p_1+1}p_1\Gamma(\frac{3}{2}+p_1)}+\frac{s}{q_1(s+1)}\Big(|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}\Big)\right]$$

Corollary 25. *If we set $\omega=1$ and $\nu=1$ in Theorem 8, we obtain*

$$|\{(\ell-\varphi_1)\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\psi(\ell+\kappa-\varphi_2)\}$$

$$+\frac{1}{2}\{(\varphi_2-\ell)\psi'(\kappa)-(\ell-\varphi_1)\psi'(\mu)\}$$

$$-\left\{\int_\mu^{\ell+\mu-\varphi_1}\psi(\lambda)d\lambda+\int_{\ell+\kappa-\varphi_2}^{\kappa}\psi(\lambda)d\lambda\right\}\Bigg|$$

$$\leq(\ell-\varphi_1)^2\left[\frac{\sqrt{\pi}\Gamma(1+p_1)}{2^{p_1+1}p_1\Gamma(\frac{3}{2}+p_1)}+\frac{1}{q_1}\left\{|\psi''(\ell)|^{q_1}+|\psi''(\mu)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_1)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\right\}\right]$$

$$+(\varphi_2-\ell)^2\left[\frac{\sqrt{\pi}\Gamma(1+p_1)}{2^{p_1+1}p_1\Gamma(\frac{3}{2}+p_1)}+\frac{1}{q_1}\left\{|\psi''(\ell)|^{q_1}+|\psi''(\kappa)|^{q_1}-\frac{1}{s+1}\Big[|\psi''(\varphi_2)|^{q_1}+|\psi''(\ell)|^{q_1}\Big]\right\}\right].$$

Corollary 26. *By considering $|\psi''(\ell)|\leq\mathcal{M}_1$ in Theorem 8, we obtain*

$$\big|\{(\ell-\varphi_1)^\nu\psi(\ell+\mu-\varphi_1)+(\varphi_2-\ell)^\nu\psi(\ell+\kappa-\varphi_2)\}$$

$$+\int_0^1(1-(1-\tau)^\omega)^\nu d\tau\{(\varphi_2-\ell)^\nu\psi'(\kappa)-(\ell-\varphi_1)^\nu\psi'(\mu)\}$$

$$-\omega^\nu\Gamma(\nu+1)\left\{(\ell-\varphi_1)^{\nu-\omega\nu}\left({}^{\nu}I^{\omega}_{\ell+\mu-\varphi_1}\psi(\mu)\right)+(\varphi_2-\ell)^{\nu-\omega\nu}\left({}^{\nu}_{\ell+\kappa-\varphi_2}I^{\omega}\psi(\kappa)\right)\right\}\Big|$$

$$\leq\left\{(\ell-\varphi_1)^{\nu+1}+(\varphi_2-\ell)^{\nu+1}\right\}\left[\frac{1}{p_1}\int_0^1\left(\int_\tau^1\left(\frac{1-(1-x)^\omega}{\omega}\right)^\nu dx\right)^{p_1}d\tau+\frac{2\mathcal{M}_1{}^{q_1}}{q_1}\frac{s}{s+1}\right].$$

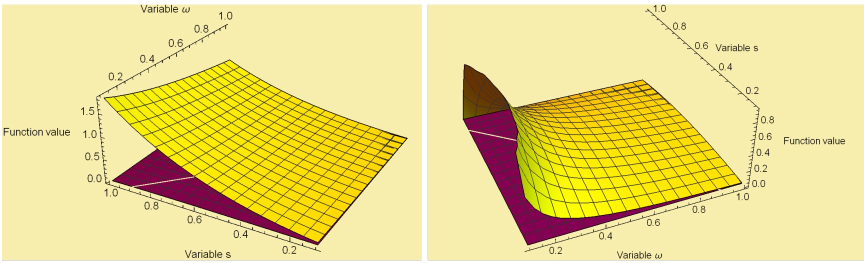
4. Numerical Examples and Visual Analysis

Throughout this section, for the numerical verification, the following assumptions will be considered:

Suppose $\psi(\tau) = \tau^s$ with $s = 0.5, \omega = 0.5$, $[\mu, \kappa] = [1, 5]$, $[\varphi_1, \varphi_2] = [2, 4]$, $\ell = 3$, $\nu = 1$ and $(p, q, p_1, q_1 = 2)$.

Now from Theorem 1, we have $0.04834 < 0.40370$ and from Theorem 5, we have $0.04377 < 0.10530$. This proves the numerical validation of these results.

Next in Figure 1, we present the graphical visualization of Theorems 1 and 5. For this we consider the above mentioned assumptions and $s \in (0, 1]$ and $\omega \in (0, 1]$.

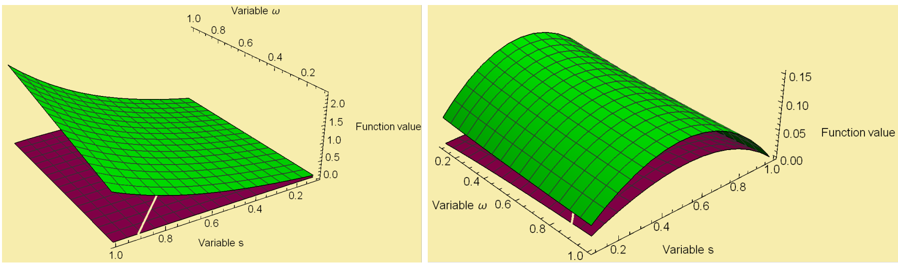


(**a**) (**b**)

Figure 1. In the above figures, the yellow and purple surfaces show the right and left sides of inequalities (**a**) (13) and (**b**) (28), respectively. Clearly one can see that the inequalities (13) and (28) hold good by varying both the parameters s and ω.

Now from Theorem 2, we have $0.04834 < 0.27498$ and from Theorem 6, we have $0.04377 < 0.13139$. This proves the numerical validation of these results.

In Figure 2, we present the graphical visualization of Theorems 2 and 6. For this we consider the above mentioned assumptions and $s \in (0, 1]$ and $\omega \in (0, 1]$.

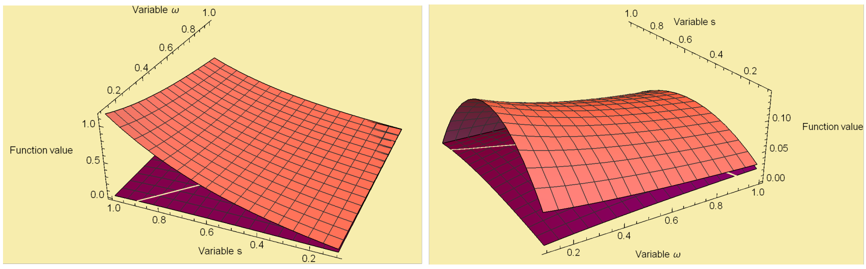


(**a**) (**b**)

Figure 2. In the above figures, the green and purple surfaces show the right and left sides of inequalities (**a**) (15) and (**b**) (30), respectively. Clearly one can see that the inequalities (15) and (30) hold good by varying both the parameters s and ω.

Now from Theorem 3, we have $0.04834 < 0.12172$ and from Theorem 7, we have $0.04377 < 0.09022$. This proves the numerical validation of these results.

Next in Figure 3, we present the graphical visualization of Theorems 3 and 7. For this we consider the above mentioned assumptions and $s \in (0, 1]$ and $\omega \in (0, 1]$.

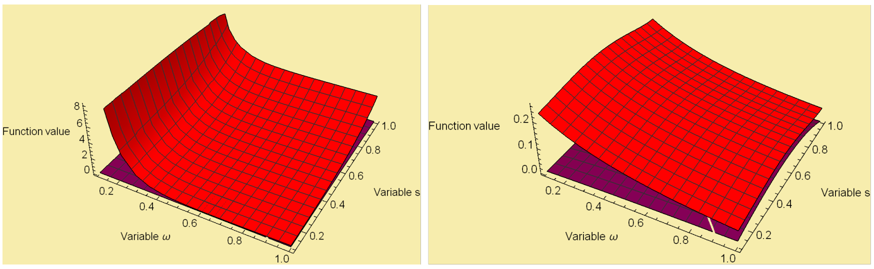


(**a**) (**b**)

Figure 3. In the above figures, the pink and purple surfaces show the right and left sides of inequalities (**a**) (19) and (**b**) (32), respectively. Clearly one can see that the inequalities (19) and (32) hold good by varying both the parameters s and ω.

Now from Theorem 4, we have $0.04834 < 0.78191$ and from Theorem 8, we have $0.04377 < 0.28330$. This proves the numerical validation of these results.

Now in Figure 4, we present the graphical visualization of Theorems 4 and 8. For this we consider the above mentioned assumptions and $s \in (0,1]$ and $\omega \in (0,1]$.



(**a**) (**b**)

Figure 4. In the above figures, the red and purple surfaces show the right and left sides of inequalities (**a**) (21) and (**b**) (34), respectively. Clearly one can see that the inequalities (21) and (34) hold good by varying both the parameters s and ω.

5. Applications

In this section, we will discuss some applications of our results.

5.1. Special Means

For positive real numbers $\varphi_1, \varphi_2, \varphi_1 \neq \varphi_2$, the following means are well known in the literature:

1. The arithmetic mean

$$A(\varphi_1, \varphi_2) = \frac{\varphi_1 + \varphi_2}{2}, \quad \varphi_1, \varphi_2 \in \mathcal{R}.$$

2. The generalized log mean

$$L_n(\varphi_1, \varphi_2) = \left(\frac{{\varphi_2}^{n+1} - {\varphi_1}^{n+1}}{(n+1)(\varphi_2 - \varphi_1)} \right)^{\frac{1}{n}}, \quad n \in \mathcal{R} \setminus \{-1, 0\}, \quad \varphi_1, \varphi_2 > 0$$

Proposition 1. *If $\mu, \kappa \in \mathbb{R}, \mu < \kappa$ and $s \in (0,1]$. Then, for all $\ell \in [\varphi_1, \varphi_2]$ and $\varphi_1, \varphi_2 \in [\mu, \kappa]$, the following inequality holds:*

$$\begin{aligned} &\big|(\ell - \varphi_1)(2A(\ell, \mu) - \varphi_1)^s + (\varphi_2 - \ell)(2A(\ell, \kappa) - \varphi_2)^s \\ &- \{(\ell - \varphi_1)L_s^s(\ell + \mu - \varphi_1, \mu) + (\varphi_2 - \ell)L_s^s(\ell + \kappa - \varphi_2, \kappa)\}\big| \end{aligned}$$

$$\leq (\ell-\varphi_1)^2 s\left\{A\left(\ell^{s-1},\mu^{s-1}\right)-\frac{2}{s+2}A\left({\varphi_1}^{s-1},\frac{\ell^{s-1}}{s+1}\right)\right\}$$
$$+(\varphi_2-\ell)^2 s\left\{A\left(\ell^{s-1},\kappa^{s-1}\right)-\frac{2}{s+2}A\left({\varphi_2}^{s-1},\frac{\ell^{s-1}}{s+1}\right)\right\}.$$

Proof. Setting $\psi(\lambda)=\lambda^s$ in Corollary 2, we obtain the desired inequality. □

Proposition 2. *If $\mu,\kappa\in\mathbb{R},\mu<\kappa$ and $s\in(0,1]$. Then, for all $\ell\in[\varphi_1,\varphi_2]$ and $\varphi_1,\varphi_2\in[\mu,\kappa]$, the following inequality holds:*

$$\begin{aligned}
&\big|(\ell-\varphi_1)(2A(\ell,\mu)-\varphi_1)^s+(\varphi_2-\ell)(2A(\ell,\kappa)-\varphi_2)^s\\
&+\frac{s}{2}\left\{(\varphi_2-\ell)\kappa^{s-1}-(\ell-\varphi_1)\mu^{s-1}\right\}\\
&-\{(\ell-\varphi_1)L_s^s(\ell+\mu-\varphi_1,\mu)+(\varphi_2-\ell)L_s^s(\ell+\kappa-\varphi_2,\kappa)\}|\\
&\leq(\ell-\varphi_1)^2 s(s-1)\left\{\frac{2}{3}A\left(\ell^{s-2},\mu^{s-2}\right)-\left(\frac{1}{(s+1)(s+3)}\varphi_1^{s-2}+\left(\frac{1}{2(s+1)}-\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\ell^{s-2}\right)\right\}\\
&+(\varphi_2-\ell)^2 s(s-1)\left\{\frac{2}{3}A\left(\ell^{s-2},\kappa^{s-2}\right)-\left(\frac{1}{(s+1)(s+3)}\varphi_2^{s-2}+\left(\frac{1}{2(s+1)}-\frac{\Gamma(s+1)}{\Gamma(s+4)}\right)\ell^{s-2}\right)\right\}.
\end{aligned}$$

Proof. Setting $\psi(\lambda)=\lambda^s$ in Corollary 16, we obtain the desired inequality. □

5.2. $\mathfrak{q}$-*Digamma Mapping*

For $0<\mathfrak{q}<1$, the $\mathfrak{q}$-digamma mapping $\zeta_\mathfrak{q}$ is given in [45,46] as follows:

$$\begin{aligned}
\zeta_\mathfrak{q}(\gamma)&=-\ln(1-\mathfrak{q})+\ln\mathfrak{q}\sum_{j=0}^{\infty}\frac{\mathfrak{q}^{k+\gamma}}{1-\mathfrak{q}^{k+\gamma}}\\
&=-\ln(1-\mathfrak{q})+\ln\mathfrak{q}\sum_{j=1}^{\infty}\frac{\mathfrak{q}^{k\gamma}}{1-\mathfrak{q}^{k}}.
\end{aligned}$$

For $\mathfrak{q}>1$ and $\gamma>0$, the $\mathfrak{q}$-digamma mapping $\zeta_\mathfrak{q}$ can be defined as follows:

$$\begin{aligned}
\zeta_\mathfrak{q}(\gamma)&=-\ln(\mathfrak{q}-1)+\ln\mathfrak{q}\left[\gamma-\frac{1}{2}-\sum_{j=0}^{\infty}\frac{\mathfrak{q}^{-(k+\gamma)}}{1-\mathfrak{q}^{-(k+\gamma)}}\right]\\
&=-\ln(\mathfrak{q}-1)+\ln\mathfrak{q}\left[\gamma-\frac{1}{2}-\sum_{j=1}^{\infty}\frac{\mathfrak{q}^{-(k\gamma)}}{1-\mathfrak{q}^{-(k\gamma)}}\right].
\end{aligned}$$

Proposition 3. *Let $0<\mu<\kappa$, $\mathfrak{p},\mathfrak{q}>1$, $0<\mathfrak{q}<1$ and $\mathfrak{q}^{-1}=1-\mathfrak{p}^{-1}$. Then, for all $\ell\in[\varphi_1,\varphi_2]$ and $\varphi_1,\varphi_1\in[\mu,\kappa]$, we have*

$$\begin{aligned}
&|(\ell-\varphi_1)\zeta_\mathfrak{q}(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\zeta_\mathfrak{q}(\ell+\kappa-\varphi_2)\\
&-\left\{\int_{\mu}^{\ell+\mu-\varphi_1}\zeta_\mathfrak{q}(\gamma)d\gamma+\int_{\ell+\kappa-\varphi_2}^{\kappa}\zeta_\mathfrak{q}(\gamma)d\gamma\right\}\Bigg|\\
&\leq(\ell-\varphi_1)^2\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}\\
&\left(\left|{\zeta_\mathfrak{q}}'(\ell)\right|^\mathfrak{q}+\left|{\zeta_\mathfrak{q}}'(\mu)\right|^\mathfrak{q}-\frac{1}{2}\left[\left|{\zeta_\mathfrak{q}}'(\varphi_1)\right|^\mathfrak{q}+\left|{\zeta_\mathfrak{q}}'(\ell)\right|^\mathfrak{q}\right]\right)^{\frac{1}{\mathfrak{q}}}\\
&+(\varphi_2-\ell)^2\left(\frac{1}{\mathfrak{p}+1}\right)^{\frac{1}{\mathfrak{p}}}
\end{aligned}$$

$$\left(\left|\zeta_{\mathrm{q}}{}'(\ell)\right|^{\mathrm{q}}+\left|\zeta_{\mathrm{q}}{}'(\kappa)\right|^{\mathrm{q}}-\frac{1}{2}\left[\left|\zeta_{\mathrm{q}}{}'(\varphi_2)\right|^{\mathrm{q}}+\left|\zeta_{\mathrm{q}}{}'(\ell)\right|^{\mathrm{q}}\right]\right)^{\frac{1}{\mathrm{q}}}.$$

Proof. The assertion can be obtained immediately by using Corollary 7 with the $\psi:\gamma\to\zeta_{\mathrm{q}}(\gamma)$ is a completely monotone mapping on $(0,\infty)$ for all $\gamma>0$ and consequently, $\psi'(\gamma):=\zeta_{\mathrm{q}}{}'(\gamma)$ is convex. □

5.3. Modified Bessel Function

In [46], the modified Bessel mapping of the first kind $\xi_\omega(\gamma)$ is given as follows:

$$\xi_\omega(\gamma)=\sum_{n=0}^{\infty}\frac{\left(\frac{\gamma}{2}\right)^{\omega+2n}}{n!\Gamma(\omega+n+1)},$$

where $\gamma\in\mathbb{R}$ and $\omega<-1$.

The modified Bessel mapping of the second kind $\varrho_\omega(\gamma)$ (see[46]) is given as follows:

$$\varrho_\omega(\gamma)=\frac{\pi}{2}\frac{\gamma_{-\omega}(\gamma)-\gamma_\omega(\gamma)}{\sin\omega\pi}.$$

The mapping $\mathcal{B}_\omega(\gamma):\mathbb{R}\to[1,\infty)$ can be defined as follows:

$$\mathcal{B}_\omega(\gamma)=2^\omega\Gamma(\omega+1)\gamma^{-\omega}\varrho_\omega(\gamma),$$

where Γ is the gamma mapping.

In [46], the following derivative formulas of $B_\omega(\gamma)$ are given as follows:

$$\mathcal{B}_\omega{}'(\gamma)=\frac{\gamma}{2(\omega+1)}\mathcal{B}_{\omega+1}(\gamma), \tag{36}$$

and

$$\mathcal{B}_\omega{}''(\gamma)=\frac{\gamma^2\mathcal{B}_{\omega+2}(\gamma)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\gamma)}{2(\omega+1)}. \tag{37}$$

Proposition 4. *Let* $0<\mu<\kappa$ *and* $\omega>-1$*. Then, for all* $\ell\in[\varphi_1,\varphi_2]$ *and* $\varphi_1,\varphi_1\in[\mu,\kappa]$*, we have*

$$\begin{aligned}
&\Bigg|(\ell-\varphi_1)\frac{(\ell+\mu-\varphi_1)}{2(\omega+1)}\mathcal{B}_{\omega+1}(\ell+\mu-\varphi_1)+(\varphi_2-\ell)\frac{(\ell+\kappa-\varphi_2)}{2(\omega+1)}\mathcal{B}_{\omega+1}(\ell+\kappa-\varphi_2)\\
&-\{(\mathcal{B}_\omega(\ell+\mu-\varphi_1)-\mathcal{B}_\omega(\mu))+(\mathcal{B}_\omega(\kappa)-\mathcal{B}_\omega(\ell+\kappa-\varphi_2))\}|\\
&\leq(\ell-\varphi_1)^2\left\{\frac{1}{2}\left[\frac{\ell^2\mathcal{B}_{\omega+2}(\ell)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\ell)}{2(\omega+1)}+\frac{\mu^2\mathcal{B}_{\omega+2}(\mu)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\mu)}{2(\omega+1)}\right]\right.\\
&\left.-\left[\frac{1}{3}\left(\frac{(\varphi_1)^2\mathcal{B}_{\omega+2}(\varphi_1)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\varphi_1)}{2(\omega+1)}\right)+\frac{1}{6}\left(\frac{\ell^2\mathcal{B}_{\omega+2}(\ell)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\ell)}{2(\omega+1)}\right)\right]\right\}\\
&+(\varphi_2-\ell)^2\left\{\frac{1}{2}\left[\frac{\ell^2\mathcal{B}_{\omega+2}(\ell)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\ell)}{2(\omega+1)}+\frac{\ell^2\mathcal{B}_{\omega+2}(\kappa)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\kappa)}{2(\omega+1)}\right]\right.\\
&\left.-\left[\frac{1}{3}\left(\frac{(\varphi_2)^2\mathcal{B}_{\omega+2}(\varphi_2)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\varphi_2)}{2(\omega+1)}\right)+\frac{1}{6}\left(\frac{\ell^2\mathcal{B}_{\omega+2}(\ell)}{4(\omega+1)(\omega+2)}+\frac{\mathcal{B}_{\omega+1}(\ell)}{2(\omega+1)}\right)\right]\right\}.
\end{aligned}$$

Proof. Using Corollary 3 to the mapping $\psi:\gamma=\mathcal{B}_\omega{}'(\gamma),\gamma>0$ (Note that all assumptions are satisfied) and the identities (36) and (37). □

6. Conclusions

To summarize, this research study introduces new fractional versions of Ostrowski–Mercer-type inequalities by using the first and the second order differentiable s-convex mappings, achieved by using the conformable fractional integral operators. The significance and applicability of our main results have been discussed thoroughly by numerical examples and graphical analysis. We have also discussed the applications of our outcomes pertaining to special means, q-digamma functions, and modified Bessel functions. We hope that this study will inspire interested readers working in this field.

Author Contributions: Conceptualization, S.R. and M.U.A.; methodology, S.R., M.U.A., M.V.-C. and H.B.; software, S.R. and M.U.A.; validation, S.R., M.U.A., M.V.-C. and H.B.; formal analysis, S.R., M.U.A., M.V.-C. and H.B.; investigation, S.R., M.U.A., M.V.-C. and H.B.; writing—original draft preparation, S.R., M.U.A., M.V.-C. and H.B.; writing—review and editing, S.R., M.U.A., M.V.-C. and H.B.; visualization, S.R., M.U.A., M.V.-C. and H.B.; supervision, M.U.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the editor and the anonymous reviewers for their valuable comments and suggestions. Miguel Vivas-Cortez thanks the Pontificia Universidad Católica del Ecuador for the support through the project: “Algunos resultados cualitativos sobre ecuaciones diferenciales fraccionales y desigualdades integrales”.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ostrowski, A. Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. *Comment. Math. Helv.* **1937**, *10*, 226–227. [CrossRef]
2. He, J.H. He Chengtian's inequality and its applications. *Appl. Math. Comput.* **2004**, *151*, 887–891. [CrossRef]
3. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
4. Cerone, P.; Dragomir, S.S. Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstr. Math.* **2004**, *37*, 299–308. [CrossRef]
5. Sahoo, S.K.; Kashuri, A.; Aljuaid, M.; Mishra, S.; De La Sen, M. On Ostrowski-Mercer's Type Fractional Inequalities for Convex Functions and Applications. *Fractal Fract.* **2023**, *7*, 215. [CrossRef]
6. Alomari, M.; Darus, M.; Dragomir, S.S.; Cerone, P. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. *Appl. Math. Lett.* **2010**, *23*, 1071–1076. [CrossRef]
7. Set, E.; Sarikaya, M.Z.; Ozdemir, M.E. Some Ostrowski's type inequalities for functions whose second derivatives are s-convex in the second sense. *Demonstr. Math.* **2014**, *47*, 37–47. [CrossRef]
8. Alomari, M.; Darus, M. Some Ostrowski type inequalities for quasi-convex functions with applications to special means. *Res. Group Math. Inequal. Appl.* **2010**, *13*, 6.
9. Dragomir, S.S.; Wang, S. An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. *Comput. Math. Appl.* **1997**, *33*, 15–20. [CrossRef]
10. Shaikh, M.A.; Khan, A.R.; Irshad, N. Generalized Ostrowski inequality with applications in numerical integration and special means. *Adv. Inequal. Appl.* **2018**, *2018*, 1–21.
11. Alomari, M.W. A companion of Dragomir's generalization of Ostrowski's inequality and applications in numerical integration. *Ukr. Mat. Zhurnal* **2012**, *64*, 435–450.
12. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Budak, H. Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions. *AIMS Math.* **2021**, *6*, 9397–9421. [CrossRef]
13. Mohammed, P.O.; Vivas-Cortez, M.; Abdeljawad, T.; Rangel-Oliveros, Y. Integral inequalities of Hermite-Hadamard type for quasi-convex functions with applications. *AIMS Math.* **2020**, *5*, 7316–7331. [CrossRef]
14. Kalsoom, H.; Latif, M.A.; Khan, Z.A.; Vivas-Cortez, M. Some New Hermite-Hadamard-Fejér fractional type inequalities for h-convex and harmonically h-Convex interval-valued Functions. *Mathematics* **2021**, *10*, 74. [CrossRef]
15. Mitrinovic, D.S.; Pecaric, J.; Fink, A.M. *Classical and New Inequalities in Analysis*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013; Volume 61.
16. Mercer, A.M. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73.
17. Kian, M.; Moslehian, M. Refinements of the operator Jensen-Mercer inequality. *Electron. J. Linear Algebra* **2013**, *26*, 742–753. [CrossRef]
18. Matković, A.; Pečarić, J.; Perić, I. A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Its Appl.* **2006**, *418*, 551–564. [CrossRef]

19. Bin-Mohsin, B.; Javed, M.Z.; Awan, M.U.; Mihai, M.V.; Budak, H.; Khan, A.G.; Noor, M.A. Jensen-Mercer Type Inequalities in the Setting of Fractional Calculus with Applications. *Symmetry* **2022**, *14*, 2187. [CrossRef]
20. Vivas-Cortez, M.; Saleem, M.S.; Sajid, S.; Zahoor, M.S.; Kashuri, A. Hermite-Jensen-Mercer-Type Inequalities via Caputo-Fabrizio Fractional Integral for h-Convex Function. *Fractal Fract.* **2021**, *5*, 269. [CrossRef]
21. Vivas-Cortez, M.; Awan, M.U.; Javed, M.Z.; Kashuri, A.; Noor, M.A.; Noor, K.I.; Vlora, A. Some new generalized k-fractional Hermite-Hadamard-Mercer type integral inequalities and their applications. *AIMS Math.* **2022**, *7*, 3203–3220. [CrossRef]
22. Zhao, J.; Butt, S.I.; Nasir, J.; Wang, Z.; Tlili, I. Hermite-Jensen-Mercer type inequalities for Caputo fractional derivatives. *J. Funct. Spaces* **2020**, *2020*, 1–11. [CrossRef]
23. Breckner, W.W. Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publ. Inst. Math.* **1978**, *23*, 13–20.
24. Cortez, M.J.V.; Hernández, J.E.H. A variant of Jensen-Mercer Inequality for h-convex functions and Operator h-convex functions. *Rev. Mat. Univ. Atl.* **2017**, *4*, 63–76.
25. Wang, W.; Khan, M.A.; Kumam, P.; Thounthong, P. A comparison study of bank data in fractional calculus. *Chaos Solitons Fractals* **2019**, *126*, 369–384. [CrossRef]
26. Xu, C.; Cui, Q.; Liu, Z.; Panc, Y.; Cui, X.; Ou, W.; Rahman, M.U.; Farman, M.; Ahmad, S.; Zeb, A. Extended Hybrid Controller Design of Bifurcation in a Delayed Chemostat Model. *MATCH Commun. Math. Comput. Chem.* **2023**, *90*, 609–648. [CrossRef]
27. Ou, W.; Xu, C.; Cui, Q.; Liu, Z.; Pang, Y.; Farman, M.; Ahmad, S.; Zeb, A. Mathematical study on bifurcation dynamics and control mechanism of tri-neuron bidirectional associative memory neural networks including delay. *Math. Methods Appl. Sci.* **2023**, 1–25. [CrossRef]
28. Podlubny, I. *Fractional Differential Equations*; Academic Press: London, UK, 1998.
29. Oldham, K.; Spanier, J. *The Fractional Calculus*; Academic Press: London, UK, 1970.
30. Kolwankar, K.M.; Gangal, A.D. Fractional differentiability of nowhere differentiable functions and dimensions. *Chaos Interdiscip. J. Nonlinear Sci.* **1996**, *6*, 505–513. [CrossRef]
31. Kolwankar, K.M.; Gangal, A.D. Hölder exponents of irregular signals and local fractional derivatives. *Pramana J. Phys.* **1997**, *48*, 49–68. [CrossRef]
32. Li, X.; Wang, D. Effects of a cavity's fractal boundary on the free front interface of the polymer filling stage. *Fractals* **2021**, *29*, 2150225. [CrossRef]
33. Zuo, Y.T.; Liu, H.J. Fractal approach to mechanical and electrical properties of graphene/sic composites. *Facta Univ. Ser. Mech. Eng.* **2021**, *19*, 271–284. [CrossRef]
34. Wang, K.J.; Zhang, P.L. Investigation of the periodic solution of the time-space fractional Sasa-Satsuma equation arising in the monomode optical fibers. *Europhys. Lett.* **2022**, *137*, 62001. [CrossRef]
35. Drăgănescu, G.E. Application of a variational iteration method to linear and nonlinear viscoelastic models with fractional derivatives. *J. Math. Phys.* **2006**, *47*, 082902. [CrossRef]
36. Wu, G.C.; Baleanu, D. Variational iteration method for the Burger's flow with fractional derivatives-new Lagrange multipliers. *Appl. Math. Model.* **2013**, *37*, 6183–6190. [CrossRef]
37. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Publishers: Amsterdam, The Netherlands, 2006.
38. Abdeljawad, T.; Baleanu, D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *J. Nonlinear Sci. Appl.* **2016**, *10*, 1098–1107. [CrossRef]
39. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci. 2016*, *20*, 763–769. [CrossRef]
40. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
41. Guzman, P.M.; Lugo, L.M.; Nápoles Valdés, J.E.; Vivas-Cortez, M. On a new generalized integral operator and certain operating properties. *Axioms* **2020**, *9*, 69. [CrossRef]
42. Jarad, F.; Ugurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 1–16. [CrossRef]
43. Anatoly, A.K. Hadamard-type fractional calculus. *J. Korean Math. Soc.* **2001**, *38*, 1191–1204.
44. Katugampola, U.N. New approach to a generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
45. Jain, S.; Mehrez, K.; Baleanu, D.; Agarwal, P. Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications. *Mathematics* **2019**, *7*, 163. [CrossRef]
46. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1995.