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# Three-Dimensional Semi-Symmetric Almost $\alpha$-Cosymplectic Manifolds

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**Abstract:** The main objective of this paper is to study semi-symmetric almost $\alpha$-cosymplectic three-manifolds. We present basic formulas for almost $\alpha$-cosymplectic manifolds. Using curvature properties, we obtain some necessary and sufficient conditions on semi-symmetric almost $\alpha$-cosymplectic three-manifolds. We obtain the main results under an additional condition. The paper concludes with two illustrative examples.

**Keywords:** almost $\alpha$-cosymplectic manifold; almost $\alpha$-Kenmotsu manifold; almost cosymplectic manifold; semi-symmetric space; locally symmetric space

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## 1. Introduction

Semi-symmetric spaces are a broad and exciting class of Riemannian manifolds, and they have applications in various areas of mathematics, particularly in the study of homogeneous spaces and differential geometry. They serve as an essential class of examples for understanding the interplay between curvature, symmetry, and geometry on manifolds. Researchers in Riemannian and differential geometry have studied these spaces to understand better their geometric properties and applications in physics, such as in the study of Einstein's field equations in general relativity [1]. Nomizu introduced the notion of semi-symmetric manifolds. A Riemannian manifold $M$ is called semi-symmetric if

$$R(X,Y) \times R = 0 \tag{1}$$

for all vector fields $X$ and $Y$ on $M$, $R(X,Y)$ acts as a derivation on $R$ [2].

A Riemannian manifold $M$, which is not necessarily complete, is locally symmetric if its curvature tensor is parallel, i.e., $\nabla R = 0$. In other words, $M$ is locally symmetric if and only if there exists a symmetric space $S$ such that $M$ is locally isometric to $S$. Nomizu proved that if $M^n$ is a complete, connected semi-symmetric hypersurface of Euclidean space $R^{n+1}$ $(n > 3)$, then $M^n$ is locally symmetric. Then, Sekigawa and Tanno showed that the manifold is locally symmetric if the Riemannian curvature tensor provided some conditions related to the covariant derivatives for $dimM \geq 3$ [3]. For the case of a compact Kaehler manifold, Ogawa proved that if it is semi-symmetric, it must be locally symmetric [4]. In the case of contact structures, Tanno showed no proper semi-symmetric or Ricci semi-symmetric K-contact manifold [5]. Moreover, Szabó gave a complete intrinsic classification of these spaces [6].

It is well known that semi-symmetric manifolds include the set of locally symmetric manifolds as a proper subset. Semi-symmetric spaces are the natural generalization of locally symmetric spaces. Such a space is called semi-symmetric since $R_q$ is the same as the curvature tensor of a symmetric space at a point of $q \in M$. Namely, locally symmetric spaces are semi-symmetric, but the converse is generally untrue.

Semi-symmetric contact metric manifolds have been studied by numerous authors [7–9]. In particular, Takahashi proved that the constant sectional curvature of a semi-symmetric Sasakian manifold is 1. In addition, semi-symmetric contact manifolds satisfying $(\kappa, \mu)$-nullity condition for dimensions greater than 3 were investigated by Papantoniu. Moreover, if $M$ is semi-symmetric and the tensor field $h$ is $\xi$-parallel, then $M$ is either smooth or has a constant curvature of 1. Then, Perrone proved that a semi-symmetric contact Riemannian three-manifold is flat or has constant sectional curvature of 1. On the other hand, Blair and Sharma proved that the constant curvature of a locally symmetric contact metric three-manifold is 0 or 1 [10].

Later, Calvaruso and Perrone investigated semi-symmetric contact three-manifolds [11]. Under some additional conditions, they obtained several classification results. Then, conformally flat semi-symmetric spaces were investigated by Calvaruso [12]. The author obtained that a conformally flat semi-symmetric space $M$ of dimension greater than 2 is either locally symmetric or irreducible and isometric to a semi-symmetric real cone. In [13], if $M$ is a locally symmetric contact metric manifold with dimensions 3 and 5, it is either Sasakian and has constant curvature of 1 or locally isometric to the unit tangent sphere bundle of Euclidean space.

Almost contact metric structure has a special subclass called almost cosymplectic manifold. It was first introduced to the literature by Goldberg and Yano [14]. An almost contact metric manifold is said to be an almost cosymplectic manifold if $d\eta = 0$ and $d\Phi = 0$. Here, $d$ is the exterior differential operator. An almost cosymplectic manifold with constant curvature is cosymplectic if and only if it is locally flat. A comprehensive study of almost cosymplectic manifolds has been undertaken by Olszak [15,16]. The author obtained some sufficient conditions and proved that no almost cosymplectic manifolds of non-vanishing constant curvature exist in dimensions greater than 5. In addition, Perrone classified simply connected homogeneous almost cosymplectic three-manifolds [17]. The author showed that if an almost cosymplectic three-manifold is locally symmetric, then its structure is cosymplectic and it is locally a product of a one-dimensional manifold and a Kaehler surface of constant curvature $c$. After this study, the author classified connected homogeneous dimensional almost $\alpha$-coKaehler structures [18].

Kenmotsu manifolds were first introduced by Kenmotsu [19]. A Kenmotsu manifold can be defined as a normal almost contact metric manifold. Kenmotsu showed that a locally symmetric Kenmotsu manifold has constant curvature of $-1$. Therefore, local symmetry is an essential restriction for Kenmotsu manifolds. The author obtained that if the Kenmotsu structure satisfies the semi-symmetric condition, it has constant negative curvature. Furthermore, if the Kenmotsu manifold $M$ is conformally flat, then $M$ is a space of constant negative curvature of $-1$ for $dimM > 3$. A $(2n+1)$-dimensional almost contact metric manifold is said to be an almost $\alpha$-Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\alpha(\eta \wedge \Phi)$, where $\alpha$ is a non-zero real constant [20]. The geometric properties and examples of these manifolds were studied [16,19,20]. Remark that almost $\alpha$-Kenmotsu structures are related to certain local conformal deformations of almost cosymplectic structures [16,21]. If we consider these two classes jointly, we introduce a new notion called an almost $\alpha$-cosymplectic manifold for any real constant $\alpha$, which is given by $d\eta = 0$ and $d\Phi = 2\alpha(\eta \wedge \Phi)$ [22].

On the other hand, a systematic study of semi-symmetric almost contact metric manifolds still needs to be undertaken. In [23], the authors studied certain classification results related to the nullity condition for an almost Kenmotsu manifold $M$ with the characteristic vector field $\xi$ belonging to the $(k, \mu)\prime$-nullity distribution. They showed that if $M$ is $\xi$-Riemannian semi-symmetric, then $M$ is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature of $-4$ and a flat $n$-dimensional manifold. Furthermore, if $M$ is a $\xi$-Riemannian semi-symmetric almost Kenmotsu manifold such that $\xi$ belongs to the null distribution, then $M$ has constant sectional curvature of $-1$. In [24], Öztürk studied semi-symmetric conditions for $\alpha$-Kenmotsu manifolds. In addition, many authors on these topics have studied almost Kenmotsu manifolds [25–28].

The paper is organized in the following way: In Section 2, we recall the concept of almost $\alpha$-cosymplectic manifolds. In Section 3, we give some basic formulas on almost $\alpha$-cosymplectic manifolds. In Section 4, we obtain several results for three-dimensional almost $\alpha$-cosymplectic manifolds. Section 5 obtains the results of the semi-symmetric almost $\alpha$-cosymplectic three-manifolds. In Section 6, we give illustrative examples of almost $\alpha$-Kenmotsu manifolds. The last section of the paper is devoted to the discussion.

## 2. Preliminaries

Let $M$ be a $(2n+1)$-dimensional smooth manifold. Then, $M$ is said to be an almost contact manifold if its structure group is reducible to $U(n)x1$. This corresponds to an almost contact structure defined by a triple $(\phi, \xi, \eta)$ satisfying the following conditions

$$\eta(\xi) = 1, \ \varphi^2 = -I + \eta \otimes \xi, \tag{2}$$

which yield

$$\varphi\xi = 0, \eta \circ \varphi = 0, \ rank(\varphi) = 2n \tag{3}$$

Here, the $\xi$ is called the Reeb vector field or characteristic vector field. Then, we have a compatible Riemannian metric $g$ on $M$ defined by [29]

$$g(\varphi X, Y) = -g(X, \varphi Y), \eta(X) = g(X, \xi) \tag{4}$$

for arbitrary vector fields on $M$. Such $(M, \varphi, \xi, \eta, g)$ is said to be an almost contact metric manifold [30]. The fundamental two-form $\Phi$ of $M$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. Additionally, if $M$ holds the condition $d\eta = \Phi$, then $M$ is said to be a contact metric manifold. It is well known that Tanno classified the structures into three classes using their automorphism groups [5]. Blair analyzed the contact metric structure, which also includes the Sasakian structure for class (1). Cosymplectic structures characterize the geometrical relations of class (2). The first simple example that comes to mind for class (2) is local products of a real line or a circle and a Kaehler manifold. Class (3) was extended by Kenmotsu, which is expressed locally by a warped product of an open interval and a Kaehler manifold [31]. This type of manifold is called Kenmotsu and has normal structure. We have noted that every orientable surface admits a Kaehler metric. If we take a warped product metric on the product space $IR \times N$, then we have a cosymplectic or a Kenmotsu three-manifold, respectively. A cosymplectic or a Kenmotsu structure satisfies the normallity and CR-integrability [32]. An almost complex structure $J$ on $M \times IR$ is defined by [29].

$$J\left(Y, f\frac{d}{dt}\right) = (\varphi - f\xi, \eta(Y)\frac{d}{dt}) \tag{5}$$

Here, $Y$ is a vector field tangent to $M$, $t$ is the standart coordinate of $IR$, and $f$ is a function on $M \times IR$. If (5) is integrable, then $M$ is called normal. In addition, it is well known that $M$ is normal if and only if $M$ satisfies

$$[\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] + 2d\eta(X, Y)\xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion tensor field of $\varphi$. We recall that we have much broader classes without normality. Note that a normal almost $\alpha$-cosymplectic manifold is said to be an $\alpha$-cosymplectic manifold. An $\alpha$-cosymplectic manifold is either cosymplectic (when $\alpha = 0$) or $\alpha$-Kenmotsu (when $\alpha \neq 0$) [22].

We denote by $\nabla$ the Levi Civita connection of $M$, by $R$ the corresponding Riemannian curvature tensor for a Riemannian manifold $M$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

by $S$ the Ricci tensor, and by $Q$ the Ricci operator given by $S(X,Y) = g(QX,Y)$. For an almost contact manifold, the (1,1)-tensor field $h$ is defined by

$$h = (1/2)\big(L_\xi \varphi\big),$$

where $L_X$ denotes the Lie derivative in the direction of $X$ [30].

**Lemma 1.** *Ref. [33]. Let $M$ be a $(2n+1)$-dimensional almost contact metric manifold. Then, $M$ is normal if and only if the tensor field $h$ identically vanishes.*

Throughout the paper, we shall denote by $\Gamma(TM)$ and $\nabla$ the Lie algebra of all tangent vector fields on $M$ and the Levi Civita connection of Riemannian metric $g$, respectively.

## 3. Basic Properties

This section recalls the below basic formulas on almost $\alpha$-cosymplectic manifolds.

**Proposition 1.** *Let $M$ be a $(2n+1)$-dimensional almost contact metric manifold and $\nabla$ be the Riemannian connection. Then, we have:*

$$(\nabla_X \Phi)(Y,Z) = g(Y,(\nabla_X \varphi)Z), \tag{6}$$

$$(\nabla_X \Phi)(Y,Z) + (\nabla_X \Phi)(\varphi Y, \varphi Z) = \eta(Z)(\nabla_X \eta)\varphi Y - \eta(Y)(\nabla_X \eta)\varphi Z, \tag{7}$$

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi) = (\nabla_X \Phi)(\xi, \varphi Y), \tag{8}$$

$$2d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X, \tag{9}$$

$$3d\Phi(X,Y,Z) = \oplus_{X,Y,Z}(\nabla_X \Phi)(Y,Z). \tag{10}$$

Here, $\oplus_{X,Y,Z}$ denotes the cyclic sum over the vector fields $X, Y$, and $Z$ [34].

**Lemma 2.** *Let $M$ be a $(2n+1)$-dimensional almost contact metric manifold. Then, we have that:*

$$\begin{aligned} 2g((\nabla_X \varphi)Y,Z) &= 3d\Phi(X,\varphi Y,\varphi Z) - 3d\Phi(X,Y,Z) + g(N^{(0)}(Y,Z),\varphi X) \\ &+ N^{(1)}(Y,Z)\eta(X) + 2d\eta(\varphi Y,X)\eta(Z) - 2d\eta(\varphi Z,X)\eta(Y), \end{aligned} \tag{11}$$

*for any $X,Y,Z \in \Gamma(TM)$ where $N^{(0)}, N^{(1)}$ are defined by*

$$N^{(0)}(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi \tag{12}$$

*and*

$$N^{(1)}(X,Y) = \big(L_{\varphi X}\eta\big)Y - \big(L_{\varphi Y}\eta\big)X, \tag{13}$$

*respectively. Here, $L_X$ denotes the Lie derivative in the direction of $X$ [30].*

**Proposition 2.** *Let $M$ be a $(2n+1)$-dimensional almost $\alpha$-cosymplectic manifold. Then, we have:*

$$trh = 0, \; h(\xi) = 0, \tag{14}$$

$$\nabla_X \xi = -\alpha\varphi^2 X - \varphi h X, \tag{15}$$

$$\nabla_\xi \xi = 0, \ \nabla_\xi \varphi = 0, \tag{16}$$

$$(\varphi \circ h)X + (h \circ \varphi)X = 0, \tag{17}$$

$$(\nabla_X \eta)Y = \alpha[\varepsilon g(X,Y) - \eta(X)\eta(Y)] + \varepsilon g(\varphi Y, hX), \tag{18}$$

*for any $X, Y, Z \in \Gamma(TM)$ [35].*

**Proposition 3.** *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost $\alpha$-cosymplectic manifold. Then, we have:*

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX] + (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y, \tag{19}$$

*for any $X, Y \in \Gamma(TM)$ [36].*

**Proposition 4.** *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost $\alpha$-cosymplectic manifold. Then, the following relations are satisfied:*

$$R(X,\xi)\xi = \alpha^2\varphi^2 X + 2\alpha\varphi hX - h^2 X + \varphi(\nabla_\xi h)X, \tag{20}$$

$$(\nabla_\xi h)X = -\varphi R(X,\xi)\xi - \alpha^2\varphi X - 2\alpha hX - \varphi h^2 X, \tag{21}$$

$$R(X,\xi)\xi - \varphi R(\varphi X,\xi)\xi = 2[\alpha^2\varphi^2 X - h^2 X], \tag{22}$$

$$S(X,\xi) = -2n\alpha^2\eta(X) - (div(\varphi h))X \tag{23}$$

$$S(\xi,\xi) = -[2n\alpha^2 + tr(h^2)] \tag{24}$$

*for any $X, Y \in \Gamma(TM)$ [37].*

## 4. Almost $\alpha$-Cosymplectic Three-Manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic three-manifold. Let us consider the the open subsets

$$V = \{q \in M : \ h \neq 0 \text{ in a neighborhood of } q\}$$

$$W = \{q \in M : \ h = 0 \ \text{ in a neighborhood of } q\}.$$

Then, the union set $V \cup W$ is an open dense subset of $M$. There exists a local orthonormal basis $\{E, \varphi E, \xi\}$ of smooth eigenvectors of $h$ in a neighborhood of $q$ for any point $q \in V \cup W$. This basis is called the $\varphi$-basis of $M$. Let $hE = \mu E$ on $V$, where $\mu$ is a positive non-vanishing smooth function. Next, using (14) and (15), we have $h\varphi E = -\mu\varphi E$. Thus, we can state the following lemma:

**Lemma 3.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic three-manifold. Then, we have on $V$:*

$$\begin{aligned} &\nabla_\xi E = -a\varphi E, && \nabla_E E = b\varphi E - \alpha\xi, && \nabla_{\varphi E}\varphi E = cE - \alpha\xi, \\ &\nabla_\xi \varphi E = aE, && \nabla_E \varphi E = -bE + \mu\xi, && \nabla_{\varphi E} E = -c\varphi E + \mu\xi. \\ &\nabla_E \xi = \alpha E - \mu\varphi E, && \nabla_{\varphi E}\xi = -\mu E + \alpha\varphi E, \end{aligned} \tag{25}$$

Here, $a$ is a smooth function and $b, c$ are defined by

$$b = (1/2\mu)[(\varphi E)(\mu) + A], \; A = \sigma(E) = S(\xi, E) = g(Q\xi, E),$$

and

$$c = (1/2\mu)[E(\mu) + B], \; B = \sigma(\varphi E) = S(\xi, \varphi E) = g(Q\xi, \varphi E),$$

respectively.

**Proof.** For any $X \in \Gamma(TM)$, using the definition of covariant derivation, it follows that

$$\begin{aligned} \nabla_E \xi &= -\alpha\varphi^2 E - \varphi h E, & \nabla_{\varphi E}\xi &= -\alpha\varphi^3 E - \varphi h \varphi E, \\ &= \alpha E - \alpha\eta(E)\xi + h\varphi E, & &= \alpha\varphi E - hE, \\ &= \alpha E - \mu\varphi E, & &= \alpha\varphi E - \mu E. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \nabla_\xi E &= g\big(\nabla_\xi E, E\big)E + g\big(\nabla_\xi E, \varphi E\big)\varphi E + g\big(\nabla_\xi E, \xi\big)\xi \\ &= 0 + g\big(\nabla_\xi E, \varphi E\big)\varphi E - g\big(E, \nabla_\xi \xi\big)\xi = g\big(\nabla_\xi E, \varphi E\big)\varphi E. \end{aligned}$$

Here, if we set $a = g\big(E, \nabla_\xi \varphi E\big)$, we obtain $\nabla_\xi E = -a\varphi E$. In a similar way, we assume that $b = g\big(\nabla_\xi E, \varphi E\big)$ and $c = g\big(\nabla_{\varphi E}\varphi E, E\big)$, then the other covariant derivatives can be obtained.

It is well known that Weyl conformal curvature tensor vanishes in dimension 3. That is to say, we have:

$$\begin{aligned} R(X,Y)Z &= S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &\quad - g(Y,Z)QX - \left(\tfrac{r}{2}\right)[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{26}$$

Replacing $X = E, \; Y = \varphi E$, and $Z = \xi$ in (26), we find

$$R(E, \varphi E)\xi = -g(QE, \xi)\varphi E + g(Q\varphi E, \xi)E. \tag{27}$$

Since $\sigma(X) = g(Q\xi, X)$, we obtain

$$R(E, \varphi E)\xi = -\sigma(E)\varphi E + \sigma(\varphi E)E. \tag{28}$$

Then, using (28), it follows that

$$\begin{aligned} R(E, \varphi E)\xi &= \big(\nabla_{\varphi E}\varphi h\big)E - \big(\nabla_E \varphi h\big)\varphi E \\ &= (\varphi E)(\mu)\varphi E + \mu(cE - \mu\xi) + c\mu E - E(\mu)E - \mu(b\varphi E - \alpha\xi) - \mu b\varphi E \\ &= (2\mu c - E(\mu))E + (-2\mu b + (\varphi E)(\mu))\varphi E. \end{aligned} \tag{29}$$

From (28) and (29), we have

$$\sigma(E) = 2\mu b - (\varphi E)(\mu), \; \sigma(\varphi E) = 2\mu c - E(\mu). \tag{30}$$

Hence, the smooth functions $b$ and $c$ take the form

$$b = (1/2\mu)[(\varphi E)(\mu) + \sigma(E)], \; c = (1/2\mu)[E(\mu) + \sigma(\varphi E)].$$

Thus, it completes the proof. □

**Proposition 5.** *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional locally symmetric almost $\alpha$-cosymplectic manifold. Then, we have $\nabla_\xi h = 0$ [37].*

**Proposition 6.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost α-cosymplectic three-manifold. On $V$, we have:*

$$\nabla_{\xi} h = 2ah\varphi + \xi(\mu)p, \tag{31}$$

*where $p$ is a (1,1)-tensor field such that $p\xi = 0,\ pE = E$, and $p\varphi E = -\varphi E$.*

**Proof.** Taking the covariant derivative of $h$ with respect to $\xi$, we have

$$\begin{aligned} (\nabla_{\xi} h)E &= \nabla_{\xi} hE - h(\nabla_{\xi} E), & (\nabla_{\xi} h)\varphi E &= \nabla_{\xi} h\varphi E - h(\nabla_{\xi}\varphi E), \\ &= \nabla_{\xi}\mu E + \mu\nabla_{\xi} E + ah\varphi E, & &= -\nabla_{\xi}\mu\varphi E - haE, \\ &= \xi(\mu)E - 2a\mu\varphi E, & &= -\xi(\mu)\varphi E - 2a\mu E, \end{aligned} \tag{32}$$

Here, we remark that $(\nabla_{\xi} h)\xi = 0$. In view of (31) and (32), we deduce

$$\xi(\mu)p = \nabla_{\xi} h - 2a\varphi h,$$

where $tr(p) = 0$. In addition, since $h = 0$, we obtain

$$\xi(\mu)p = \nabla_{\xi} h = 0$$

on $W$. □

**Proposition 7.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost α-cosymplectic three-manifold. Then, we have:*

$$h^2 - \alpha^2\varphi^2 = \left[\frac{tr(l)}{2}\right]\varphi^2, \tag{33}$$

*where $l = R(.,\xi)\xi$ is the Jacobi operator.*

**Proof.** Following from (24), we have

$$tr(l) = -[2\alpha^2 + tr(h^2)] = -2[\alpha^2 + \mu^2].$$

To complete the proof, let us calculate their values according to the components of the basis. In fact, we have:

$$h^2E - \alpha^2\varphi^2E = \mu^2E + \alpha^2E,\ h^2\varphi E - \alpha^2\varphi^3E = (\alpha^2 + \mu^2)\varphi E,\ h^2\xi - \alpha^2\varphi^2\xi = 0. \tag{34}$$

From (34), the proof is clear.
Since $dimM = 3$, (26) turns out to be

$$lX = tr(l)X - S(X,\xi)\xi + QX - \eta(X)Q\xi - (r/2)(X - \eta(X)\xi). \tag{35}$$

Therefore, the last formula gives

$$\begin{gathered} QX = \alpha^2\varphi^2X + 2\alpha\varphi hX - h^2X + \varphi(\nabla_{\xi} h)X - tr(l)X \\ -S(X,\xi)\xi + \eta(X)Q\xi + (r/2)(X - \eta(X)\xi). \end{gathered} \tag{36}$$

On the other hand, we have

$$S(X,\xi) = -S(\varphi^2X,\xi) + \eta(X)tr(l).$$

Then, (36) yields

$$\begin{gathered} QX = -((tr(l))/2)\varphi^2X + 2\alpha\varphi hX + \varphi(\nabla_{\xi} h)X - tr(l)X \\ -S(\varphi^2X,\xi)\xi + \eta(X)tr(l)\xi + \eta(X)Q\xi - (r/2)\varphi^2X. \end{gathered} \tag{37}$$

Moreover, we obtain

$$Q\xi = g(Q\xi, E)E + g(Q\xi, \varphi E)\varphi E + g(Q\xi, \xi)\xi = \sigma(E)E + \sigma(\varphi E)\varphi E + tr(l)\xi, \tag{38}$$

with respect to the $\varphi$-basis. Taking into account (37) and (38), we obtain

$$\begin{aligned} QX = -(1/2)(r + tr(l))\varphi^2 X + 2\alpha\varphi hX + \varphi\left(\nabla_\xi h\right)X + 2\left(\alpha^2 + \mu^2\right)X - S\left(\varphi^2 X, \xi\right)\xi \\ -2\left(\alpha^2 + \mu^2\right)\eta(X)\xi + \eta(X)[\sigma(E)E + \sigma(\varphi E)\varphi E] + \eta(X)tr(l)\xi. \end{aligned} \tag{39}$$

Next, arranging the above equation, we have

$$\begin{aligned} QX = \left[(1/2)r + \alpha^2 + \mu^2\right]X + \left[-(1/2)r - 3\alpha^2 - 3\mu^2\right]\eta(X)\xi \\ +2\alpha\varphi hX + \varphi\left(\nabla_\xi h\right)X - S\left(\varphi^2 X, \xi\right)\xi + \eta(X)\sigma(E)E + \eta(X)\sigma(\varphi E)\varphi E. \end{aligned} \tag{40}$$

Hence, if we set

$$\overline{a} = (1/2)r + \alpha^2 + \mu^2 \text{ and } \overline{b} = -(1/2)r - 3\alpha^2 - 3\mu^2,$$

then (40) turns into

$$\begin{aligned} QX = \overline{a}X + \overline{b}\eta(X)\xi + 2\alpha\varphi hX + \varphi\left(\nabla_\xi h\right)X - \sigma\left(\varphi^2 X\right)\xi \\ +\eta(X)\sigma(E)E + \eta(X)\sigma(\varphi E)\varphi E. \end{aligned} \tag{41}$$

Thus, we state the following result: □

**Lemma 4.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic three-manifold. Then, the Ricci operator satisfies the following:*

$$\begin{aligned} Q = \overline{a} + \overline{b}\eta \otimes \xi + 2\alpha\varphi h + \varphi\left(\nabla_\xi h\right) - \sigma\left(\varphi^2\right) \otimes \xi \\ +\sigma(E)\eta \otimes E + \sigma(\varphi E)\eta \otimes \varphi E. \end{aligned} \tag{42}$$

Here, the functions $\overline{a}$ and $\overline{b}$ are defined by

$$\overline{a} = (1/2)r + \alpha^2 + \mu^2 \text{ and } \overline{b} = -(1/2)r - 3\alpha^2 - 3\mu^2,$$

respectively [37].

**Proposition 8.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic three-manifold. Then, the components of the Ricci operator $Q$ with respect to the $\varphi$-basis $\{E, \varphi E, \xi\}$ are given as follows:*

$$\begin{aligned} Q\xi &= -2\left(\alpha^2 + \mu^2\right)\xi + AE + B\varphi E, \\ QE &= A\xi + \left(r/2 + \alpha^2 + \mu^2 + 2a\mu\right)E + (\xi(\mu) + 2\alpha\mu)\varphi E, \\ Q\varphi E &= B\xi + (\xi(\mu) + 2\alpha\mu)E + \left(r/2 + \alpha^2 + \mu^2 - 2a\mu\right)\varphi E, \end{aligned} \tag{43}$$

where $A = \sigma(E) = S(\xi, E)$ and $B = \sigma(\varphi E) = S(\xi, \varphi E)$.

**Proof.** Taking $X = \xi$ in (41), it yields that

$$Q\xi = (\overline{a} + \overline{b})\xi + \sigma(E)E + \sigma(\varphi E)\varphi E.$$

Using $\overline{a}$ and $\overline{b}$, the above equation becomes

$$Q\xi = (-2\alpha^2 - 2\mu^2)\xi + AE + B\varphi E. \tag{44}$$

Now, putting $X = E$ in (41), it follows that

$$QE = \bar{a}E + 2\alpha\varphi hE + \varphi(2ah\varphi E + \xi(\mu)pE) - \sigma(-E + \eta(E)\xi)\xi. \tag{45}$$

Then, the last equation reduces to

$$QE = \bar{a}E + 2\alpha\varphi hE + 2ahE + \xi(\mu)\varphi E + \sigma(E)\xi,$$

where $pE = E$ ve $A = \sigma(E)$. Analogously, putting $X = \varphi E$ in (41), then we obtain

$$Q\varphi E = \bar{a}\varphi E + 2\alpha hE + \varphi(2a\varphi^2 E + \xi(\mu)p\varphi E) + \sigma(\varphi E)\xi.$$

This completes the proof. Note that $p\varphi E = -\varphi E$ and $B = \sigma(\varphi E)$. □

**Proposition 9.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost $\alpha$-cosymplectic three-manifold. Then, considering (43), we have:*

$$\begin{aligned}
\big(\nabla_\xi Q\big)\xi &= -4\mu\xi(\mu)\xi + \{\xi(A) + aB\}E + \{\xi(B) - aA\}\varphi E,\\
(\nabla_E Q)E &= \big\{-3\alpha^3 - \alpha\mu^2 - (\alpha r/2) - 2a\alpha\mu + \mu\xi(\mu) + E(A) - (B/2\mu)((\varphi E)(\mu) + A)\big\}\xi\\
&+\{2\alpha A + 1/2E(r) + 2\mu E(\mu) + 2aE(\mu) + 2\mu E(a) - (1/\mu)((\varphi E)(\mu) + A)(\xi(\mu) + 2\alpha\mu)\}E\\
&+\{-\mu A + \alpha B + 2a((\varphi e)(\mu) + A + 2\alpha e(\mu) + E(\xi(\mu))\}\varphi E,\\
\big(\nabla_{\varphi E} Q\big)\varphi E &= \big\{(\varphi E)(B) - 3\alpha^3 - \alpha r/2\big) - \alpha\mu^2 + 2\alpha a\mu + \xi(\mu)\mu - A/2\lambda(E(\mu) + B)\}\xi\\
&+\{2\alpha(\varphi E)(\mu) + (\varphi E)(\xi(\mu)) - \mu B + \alpha A - 2a(E(\mu) + B)\}E + \{1/2(\varphi E)(r)\\
&+2\mu(\varphi E)(\mu) - 2a(\varphi E)(\mu) - 2\mu(\varphi E)(a) + 2\alpha B - 1/\mu(E(\mu) + B)(\xi(\mu) + 2\alpha\mu)\}\varphi E.
\end{aligned} \tag{46}$$

**Proof.** From the first equation of (43), we have

$$\big(\nabla_\xi Q\big)\xi = -2\,\nabla_\xi(\alpha^2 + \mu^2)\xi + \nabla_\xi AE + \nabla_\xi B\varphi E, \tag{47}$$

where $\nabla_\xi\xi = 0$ and $Q(0) = 0$. Then, the first equation clears from (47).
Similarly, from (25) and (43), we obtain

$$\begin{aligned}
(\nabla_E Q)E = {} & \nabla_E\big(A\xi + \big(r/2 + \alpha^2 + \mu^2 + 2a\mu\big)E\\
& +(\xi(\mu) + 2\alpha\mu)\,\varphi E\big) - bQ\varphi E + \alpha Q\xi.
\end{aligned} \tag{48}$$

Taking into account (43) and (48), we compute

$$\begin{aligned}
(\nabla_E Q)E = {} & \big\{-3\alpha^3 - \alpha\mu^2 - \alpha r/2 - 2a\alpha\mu + \mu\xi(\mu) + E(A) - bB\big\}\xi\\
& +\{2\alpha A + 1/2E(r) + 2\mu e(\mu) + 2aE(\mu) + 2\mu E(a) - 2b\xi(\mu) - 4\alpha\mu b\}E\\
& +\{-\mu A + \alpha B + 4ab\mu + 2\alpha E(\mu) + E(\xi(\mu))\}\varphi E,
\end{aligned} \tag{49}$$

which gives the second equation of (46).
Finally, considering (43) with respect to $\varphi E$, we obtain

$$\begin{aligned}
\big(\nabla_{\varphi E} Q\big)\varphi E = {} & \nabla_{\varphi E}(B\xi + (\xi(\mu) + 2\alpha\mu)E\\
& +\big(r/2 + \alpha^2 + \mu^2 - 2a\mu\big)\,\varphi E) - cQE + \alpha Q\xi.
\end{aligned} \tag{50}$$

Therefore, if we proceed similarly, we complete the last part of the proof. □

**Proposition 10.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost α-cosymplectic three-manifold. Then, the components of R with respect to the φ-basis are as follows:*

$$
\begin{aligned}
R(\xi, E)\xi &= -\big(\alpha^2 + \mu^2 - 2a\mu\big)E + (\xi(\mu) + 2\alpha\mu)\varphi E,\\
R(\xi, \varphi E)\xi &= (\xi(\mu) + 2\alpha\mu)E - \big(\alpha^2 + \mu^2 + 2a\mu\big)\varphi E,\\
R(E, \varphi E)\xi &= -BE + A\varphi E,\\
R(\xi, E)E &= \big(\alpha^2 + \mu^2 - 2a\mu\big)\xi - B\varphi E,\\
R(\xi, \varphi E)E &= -(\xi(\mu) + 2\alpha\mu)\xi + A\varphi E,\\
R(E, \varphi E)E &= B\xi + 2\big(\alpha^2 + \mu^2 + r/4\big)\varphi E,\\
R(\xi, E)\varphi E &= -(\xi(\mu) + 2\alpha\mu)\xi + BE,\\
R(\xi, \varphi E)\varphi E &= \big(\alpha^2 + \mu^2 + 2a\mu\big)\xi - AE,\\
R(E, \varphi E)\varphi E &= -A\xi - 2\big(\alpha^2 + \mu^2 + r/4\big)E.
\end{aligned}
\tag{51}
$$

**Proof.** Let us consider (26) and (43). Putting $X = Z = \xi$ and $Y = E$ in (26), we have

$$R(\xi, E)\xi = S(\xi, \xi)E - S(E, \xi)\xi + QE - (r/2)E. \tag{52}$$

Putting again $X = Z = \xi$ and $Y = \varphi E$ in (26), we obtain

$$R(\xi, \varphi E)\xi = S(\xi, \xi)\varphi E - S(\varphi E, \xi)\xi + Q\varphi E - (r/2)\varphi E. \tag{53}$$

Using (52) and (53), the first two equations can be seen. Usage of the same methodology (51) is clear. Here, we recall that

$$S(\xi, E) = A,\ S(\xi, \varphi E) = B,\ S(E, E) = \bar{a} + 2a\mu,$$

$$S(\varphi E, E) = \xi(\mu) + 2\alpha\mu,\ S(\varphi E, \varphi E) = \bar{a} - 2a\mu,\ tr(h^2) = 2\mu^2.$$

In addition, the equations given in (51) are all the possible non-zero components of the Riemannian curvature $R$. They depend on the changes in the order of the vector fields. □

## 5. Semi-Symmetric Almost α-Cosymplectic Three-Manifolds

In this section, we study semi-symmetric almost α-cosymplectic three-manifolds. Then, we prove the following:

**Theorem 1.** *Let $(M, \varphi, \xi, \eta, g)$ be an almost α-cosymplectic three-manifold. Then, M is semi-symmetric if and only if*

$$A\xi(\mu) = -2\alpha A\mu + (\alpha^2 + \mu^2 + 2a\mu)B, \tag{54}$$

$$B\xi(\mu) = -2\alpha B\mu + (\alpha^2 + \mu^2 - 2a\mu)A, \tag{55}$$

$$AB = -2(\xi(\mu) + 2\alpha\mu)(\alpha^2 + \mu^2 + r/4), \tag{56}$$

$$B^2 = -(\alpha^2 + \mu^2 - 2a\mu)(3(\alpha^2 + \mu^2) + 2a\mu + r/2) + (\xi(\mu) + 2\alpha\mu) \tag{57}$$

$$A^2 = -(\alpha^2 + \mu^2 + 2a\mu)(3(\alpha^2 + \mu^2) - 2a\mu + r/2) + (\xi(\mu) + 2\alpha\mu)^2. \tag{58}$$

**Proof.** According to the hypothesis, $M$ is an almost $\alpha$-cosymplectic three-manifold. We note that (1) is equivalent to $R(X,\xi).R = 0$, for all $X \in \Gamma(TM)$ on $M$. In other words, we have

$$\begin{aligned} 0 &= R(X,\xi)R(Y,Z)W - R(R(X,\xi)Y,Z)W \\ &-R(Y,R(X,\xi)Z)W - R(Y,Z)R(X,\xi)W, \end{aligned} \tag{59}$$

for all $X,Y,Z,W \in \Gamma(TM)$ on $M$.

Putting $X = E,\ Y = \xi,\ Z = \varphi E$, and $W = \xi$ in (59), then we have

$$\begin{aligned} 0 &= R(E,\xi)R(\xi,\varphi E)\xi - R(R(E,\xi)\xi,\varphi E)\xi \\ &-R(\xi,R(E,\xi)\varphi E)\xi - R(\xi,\varphi E)R(E,\xi)\xi. \end{aligned} \tag{60}$$

From (51) and (60), we obtain

$$\begin{aligned} 0 &= 2B(\xi(\mu) + 2\alpha\mu)\varphi E + B\big(\alpha^2 + \mu^2 + 2a\mu\big)E \\ &-2A\big(\alpha^2 + \mu^2 - 2a\mu\big)\varphi E - A(\xi(\mu) + 2\alpha\mu)E, \end{aligned} \tag{61}$$

where $R(\varphi E,\varphi E)\xi = 0$ and $R(\xi,\xi)\xi = 0$. Therefore, (61) turns into

$$\begin{aligned} 0 &= \big(-2\alpha\mu A + 2Ba\mu - \xi(\mu)A + \alpha^2 B + \lambda^2 B\big)E \\ &+\big(4\alpha\mu B + 4aA\mu - 2\alpha^2 A - 2A\mu^2 + 2B\xi(\mu)\big)\varphi E. \end{aligned} \tag{62}$$

Hence, this ends the proof of (54) and (55).

Using a similar methodology, putting $X = E$, $Y = E$, $Z = \varphi E$ ve $W = \xi$ in (59) and (51), then we obtain

$$\begin{aligned} 0 &= \big(-AB - 2(\xi(\mu) + 2\alpha\mu)\big(\alpha^2 + \mu^2 + r/4\big)\big)E \\ &+\big(-B^2 - \big(\alpha^2 + \mu^2 - 2a\mu\big)\big(\alpha^2 + \mu^2 + 2a\mu\big)\big)\varphi E \\ &+\big((\xi(\mu) + 2\alpha\mu)^2 - 2\big(\alpha^2 + \mu^2 - 2a\mu\big)\big(\alpha^2 + \mu^2 + r/4\big)\big)\varphi E, \end{aligned} \tag{63}$$

where $R(E,E)\xi = 0$ and so (56) and (57) satisfy (63). Finally, we take $X = \varphi E$, $Y = \varphi E$, $Z = E$, and $W = \xi$ in (59) and, taking account of (51), we deduce

$$\begin{aligned} 0 &= \big(-AB - 2(\xi(\mu) + 2\alpha\mu)\big(\alpha^2 + \mu^2 + r/4\big)\big)\varphi E \\ &+\big(-A^2 - \big(\alpha^2 + \mu^2 + 2a\mu\big)\big(\alpha^2 + \mu^2 - 2a\mu\big)\big)E \\ &+\big((\xi(\mu) + 2\alpha\mu)^2 - 2\big(\alpha^2 + \mu^2 + 2a\mu\big)\big(\alpha^2 + \mu^2 + r/4\big)\big)E. \end{aligned} \tag{64}$$

Thus, the proof of (58) is clear. We also note that all the other possible choices of the vector fields in the $\varphi$-basis are given again (54)–(58). Therefore, if (54)–(58) is satisfied, then (59) is also satisfied, which means $M$ is semi-symmetric. □

**Theorem 2.** *Let $(M,\varphi,\xi,\eta,g)$ be a semi-symmetric almost cosymplectic three-manifold. If the structure $(\varphi,\xi,\eta,g)$ is cosymplectic and the Ricci curvature $S(\xi,\xi)$ is constant along the characteristic vector field $\xi$, then M is locally symmetric. Otherwise, M is not locally symmetric if the structure $(\varphi,\xi,\eta,g)$ is almost cosymplectic under the same condition.*

**Proof.** Suppose that $M$ is a semi-symmetric almost cosymplectic three-manifold. Therefore, (54)–(58) satisfy $\alpha = 0$. Now, we shall classify our arguments under the following two conditions:

Case 1. If $h = 0$, then the structure is cosymplectic [38,39]. According to the hypothesis, because of (43), the Ricci curvature $S(\xi,\xi)$ constant along the characteristic vector field $\xi$ means exactly $\xi(\mu) = 0$. Hence, from (31), we have $\nabla_\xi h = 0$. In this case, whether the smooth function $a$ is different from zero will be independent. Thus, the conclusion follows from Proposition 5.

Case 2. In this case, let us consider $h \neq 0$ and $\xi(\mu) = 0$. Note that (54)–(58) satisfy on $M$. Then, as it follows from (31), if $a = 0$, we obtain $\nabla_{\xi} h = 0$. The result can be seen in Proposition 5. To end the proof, we shall obtain that the case of $a \neq 0$ cannot take place. If so, we suppose that $a \neq 0$ and consider a point $q$ at $M$, where $a(q) \neq 0$. Thus, there exists a neighbourhood $V$ of a point $q$ such that $a \neq 0$ on $V$. First, we multiply (54) by $B$ and (55) by $A$. Then, we have:

$$AB\xi(\mu) = -2\alpha AB\mu + (\alpha^2 + \mu^2 + 2a\mu)B^2 \tag{65}$$

and

$$AB\xi(\mu) = -2\alpha AB\mu + (\alpha^2 + \mu^2 - 2a\mu)A^2. \tag{66}$$

Then, we subtract (65) from (66) and take into account (57) and (58) for expressing $B^2$ and $A^2$, respectively. It follows that

$$(\alpha^2 + \mu^2 + 2a\mu)B^2 - (\alpha^2 + \mu^2 - 2a\mu)A^2 = 0, \tag{67}$$

where $\xi(\mu) = 0$. Since $a\mu \neq 0$, (67) can be written as

$$\mu^2 \pm 2a\mu = 0. \tag{68}$$

Now, we assume that

$$\mu^2 + 2a\mu = 0. \tag{69}$$

It is noted that if the other equation $\mu^2 - 2a\mu = 0$ holds, we proceed in the same manner, and since $a \neq 0$, the two equations cannot satisfy simultaneously. However, (69) shows that the function $a$ cannot vanish. In this case, we are unlikely to find a contradiction in our assumption.

Let us continue the calculation with the thought that our assumption is true. (56) holds $AB = 0$ since $\xi(\mu) = 0$ and $\alpha = 0$. Namely, we have locally either $A = 0$ or $B = 0$. Let us suppose $A = 0$ and we shall prove that $\mu$ is constant and $B = 0$. If we suppose the other case ($B = 0$), we proceed in the same way.

Differentiating (69) with respect to $\xi$, we have $\xi(a) = 0$, where $\xi(\mu) = 0$. Then, again differentiating with respect to $E$, we obtain

$$0 = [\mu E(\mu) + aE(\mu) + \mu E(a)]. \tag{70}$$

To obtain whether $\mu$ is constant or not, let us remember the well-known formula

$$W(r) = 2\textstyle\sum_{j=1}^{n} g((\nabla_{E_j} Q)E_j, W), \tag{71}$$

for any $W \in \Gamma(TM)$, where $\{E_j\}$ is an arbitrary orthonormal basis. Applying (46) and (71) to calculate $E(r)/2$ and $(\varphi E)(r)/2$, then making use of (25), (46), (69), and (71), we observe that $\mu$ is not necessarily constant and $B$ does not have to vanish. In fact, using (25), we calculate $R(E, \varphi E)E$ as follows:

$$R(E, \varphi E)E = [-2c\mu]\xi + [-E(c) - \mu^2 - (\varphi E)(b) + b^2 + c^2]\phi E. \tag{72}$$

Then, comparing with (51), we obtain

$$B = -2c\mu, -3\mu^2 + r/4 = 0. \tag{73}$$

From (69), by a direct calculation, we deduce

$$r = 12a\mu. \tag{74}$$

Then, taking account of (57), (69), and (74), we obtain $B^2 = 32a^4$, and so we have $a \neq 0$. Moreover, a result of Olszak verifies our proof by (74) [38]. □

**Theorem 3.** *Let $(M, \varphi, \xi, \eta, g)$ be a semi-symmetric almost $\alpha$-Kenmotsu three-manifold with the Ricci curvature $S(\xi, \xi)$ constant along the characteristic vector field $\xi$. If the structure $(\varphi, \xi, \eta, g)$ is normal, then $M$ is locally symmetric only when $\alpha = 1$. Moreover, $M$ is locally symmetric if $M$ is given by a constant scalar curvature $r = -4(\alpha^2 + \mu^2)$.*

**Proof.** The geometry of almost $\alpha$-Kenmotsu manifolds differs in two cases, with the tensor field $h$ being zero or non-zero.

Case 1. Assume that $h = 0$. Then, an almost $\alpha$-Kenmotsu three-manifold is an $\alpha$-Kenmotsu manifold. A result of Dileo is that if an almost $\alpha$-Kenmotsu three-manifold has a constant curvature, then the structure is normal, and the constant curvature is $-\alpha^2$ when it is locally symmetric [25]. Furthermore, Öztürk showed that a semi-symmetric $\alpha$-Kenmotsu manifold is not of constant curvature. From Corollary 4.3 in [24], semi-symmetry implies local symmetry for $\alpha = 1$. In fact, using the hypothesis and (54)–(58), Theorem 1 is not verified except in the case $\alpha = 1$. This completes the proof.

Case 2. Suppose that $h \neq 0$. Then, applying the same technique as in Theorem 2 for $\alpha \neq 0$, $a\mu \neq 0$, and $\xi(\mu) = 0$, it follows that

$$\mu^2 \mp 2\mu\sqrt{a^2 + \alpha^2} + \alpha^2 = 0,$$

and we take

$$\mu^2 + 2\mu\sqrt{a^2 + \alpha^2} + \alpha^2 = 0. \tag{75}$$

In addition, by virtue of (56), we have

$$AB = -4\alpha\mu(\alpha^2 + \mu^2 + r/4),$$

where $\xi(\mu) = 0$. For the last equation vanishes, that is, locally $A = 0$ or $B = 0$ if and only if $r = -4(\alpha^2 + \mu^2)$.

Next, we assume $A = 0$ and we prove that $\mu$ is constant and $B = 0$ (the other case proceeds in the same way). Differentiating (75) with respect to $\xi$, we obtain $\xi\left(\sqrt{a^2 + \alpha^2}\right) = 0$, and then again differentiating with respect to $E$, we obtain

$$0 = \mu\left[E(\mu) + E\left(\sqrt{a^2 + \alpha^2}\right)\right] + \sqrt{a^2 + \alpha^2}E(\mu). \tag{76}$$

Making use of (46) and (76), we have

$$0 = -aB - \mu B + 2\mu[E(\mu) + E(a)]. \tag{77}$$

Using (76), (77) gives

$$\mu E\left(\sqrt{a^2 + \alpha^2}\right) + E(\mu)\left[\sqrt{a^2 + \alpha^2} - \mu\right] = 2\mu E(a) - B[a + \mu] \tag{78}$$

Then, differentiating (78) by $\xi$, since $\xi\left(\sqrt{a^2 + \alpha^2}\right) = \xi(\mu) = 0$, we obtain

$$\left[\sqrt{a^2 + \alpha^2} - \mu\right]\xi(E(\mu)) + \mu\xi\left(E\left(\sqrt{a^2 + \alpha^2}\right)\right) = 2[\mu\xi(E(a))] - \xi(B)[a + \mu] - B\xi(a). \tag{79}$$

Now, taking account of (46) and (71) to calculate $(\varphi E)(r)/2$, we obtain

$$\xi(B) = \mu a - B + 2\mu[(\varphi E)(a) - (\varphi E)(\mu)]. \tag{80}$$

On the other hand, using (25) and $\xi(\mu) = 0$, we also have

$$0 = [\alpha + \xi]E(\mu) + [a - \mu](\varphi E)(\mu). \tag{81}$$

Differentiating (75) by $\varphi E$, we find

$$0 = \mu(\varphi E)(\mu) + \mu(\varphi E)\left(\sqrt{a^2 + \alpha^2}\right) + \sqrt{a^2 + \alpha^2}(\varphi E)(\mu). \tag{82}$$

Moreover, from (75), we obtain

$$a = \sqrt{\left(\frac{\mu^2 + \alpha^2}{2\mu} - \alpha\right)\left(\frac{\mu^2 + \alpha^2}{2\mu} + \alpha\right)} \tag{83}$$

Then, using (78)–(83), $(\varphi E)(\mu)$ and $E(\mu)$ have to vanish. Thus, $\mu$ is constant. In addition, $B$ is identically zero. Hence, we find a contradiction in our assumption. We conclude that $a = 0$. This ends the proof. □

**Corollary 1.** *Let $(M, \varphi, \xi, \eta, g)$ be a semi-symmetric $\alpha$-cosymplectic manifold with the Ricci curvature $S(\xi, \xi)$ constant along the characteristic vector field $\xi$. It is locally symmetric if it is cosymplectic when $\alpha = 0$ or Kenmotsu when $\alpha = 1$.*

## 6. Examples

**Example 1.** *Let us consider the $M \subset IR^3$ manifold such that $M = \{(x, y, z) \in IR^3\}$. Here, $(x, y, z)$ are the standart coordinates in $IR^3$. The vector fields are as follows:*

$$e_1 = \rho_2 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \rho_1 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right),\ e_2 = -\rho_1 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \rho_2 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right),\ e_3 = \left(\frac{\partial}{\partial z}\right).$$

Let $g$ be the metric tensor product given by

$$g = ({t_1}^2 + {t_2}^2)^{-1}(dx^2 + dy^2) + dz^2$$

where $t_1, t_2$ are defined by $t_1(z) = \rho_2 e^{-\alpha z},\ t_2(z) = \rho_1 e^{-\alpha z}$ with ${\rho_1}^2 + {\rho_2}^2 \neq 0,\ \alpha \neq 0$ for constants $\rho_1, \rho_2$, and $\alpha$. It is obvious that $\{e_1, e_2, e_3\}$ are linearly independent at each point of $M$. Therefore, we have

$$\phi(e_3) = 0,\ \phi(e_1) = e_2,\ \phi(e_2) = -e_1$$

$$\phi^2 X = -X + \eta(X)e_3,\ \eta(X) = g(e_3, X),\ \eta(e_3) = g(e_3, e_3) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$.

From the above relations, there exists an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$. Now, we check if the structure is almost $\alpha$-Kenmotsu metric or not. Hence, we obtain

$$\Phi\left(\left(\frac{\partial}{\partial x}\right), \left(\frac{\partial}{\partial y}\right)\right) = -({t_1}^2 + {t_2}^2)^{-1} = -({\rho_1}^2 + {\rho_2}^2)^{-1} e^{2\alpha z}.$$

Since $\eta = dz$, we deduce $d\Phi = 2\alpha(\eta \wedge \Phi)$ on $M$. Moreover, we notice that $N_\phi = 0$. Thus, $M$ is an $\alpha$-Kenmotsu manifold, and $h = 0$ with constant curvature $-\alpha^2$. Consequently, Theorem 1 and Theorem 3 are verified.

**Example 2.** *Consider the three-dimensional manifold* $M = \{(u, v, w) \in R^3,\ w \neq 0\}$, *where* $(u, v, w)$ *are the standart coordinates in* $R^3$. *The vector fields are*

$$e_1 = g_1(w)\left(\frac{\partial}{\partial u}\right) + g_2(w)\left(\frac{\partial}{\partial v}\right)$$

$$e_2 = -g_2(w)\left(\frac{\partial}{\partial u}\right) + g_1(w)\left(\frac{\partial}{\partial v}\right)$$

$$e_3 = \left(\frac{\partial}{\partial w}\right).$$

Here, $g_1$ and $g_2$ are given by

$$g_1(w) = c_2 e^{-\alpha z} cos\mu w - c_1 e^{-\alpha z} sin\mu w,$$

$$g_2(w) = c_1 e^{-\alpha z} cos\mu w + c_2 e^{-\alpha z} sin\mu w,$$

with ${c_1}^2 + {c_2}^2 \neq 0$ for constants $c_1, c_2, \mu$, and $\alpha$. It is sufficient to check that the only non-zero components of the second fundamental form $\Phi$ are

$$\Phi\left(\left(\frac{\partial}{\partial u}\right), \left(\frac{\partial}{\partial v}\right)\right) = -1/({g_1}^2 + {g_2}^2)) = -(e^{-\alpha z}/{c_1}^2 + {c_2}^2)$$

The above equation gives that

$$\Phi = -2e^{-\alpha z}/({c_1}^2 + {c_2}^2)(du \wedge dv).$$

We notice that the structure is not normal; the given structure is almost $\alpha$-cosymplectic. In addition, by simple calculation, the Riemannian curvature tensor components are as follows:

$$R(e_1, e_2)e_1 = (\alpha^2 - \mu^2)e_2,\ R(e_1, e_2)e_2 = (\mu^2 - \alpha^2)e_1,$$

$$R(e_1, e_2)e_3 = 0, R(e_1, e_3)e_1 = (\mu^2 + \alpha^2)e_3,$$

$$R(e_1, e_3)e_2 = 0, R(e_1, e_3)e_3 = -(\mu^2 + \alpha^2)e_1 - 2\mu\alpha e_2,$$

$$R(e_2, e_3)e_1 = 2\mu\alpha e_3, R(e_2, e_3)e_2 = -(\mu^2 + \alpha^2)e_3,$$

$$R(e_2, e_3)e_3 = -2\mu\alpha e_1 - (\mu^2 + \alpha^2)e_2.$$

Thus, Theorem 2 and Theorem 3 hold.

## 7. Discussion

Riemannian symmetric spaces are one of the essential Riemannian manifolds. These spaces contain many important examples for various branches of mathematics, such as compact Lie groups and bounded symmetric domains. Any symmetrical space has its unique geometry. For instance, Euclidean, elliptic, and hyperbolic geometries are the first to come to mind. On the other hand, these spaces have many common points and a wealthy theory. Symmetric spaces can be considered from many different perspectives. These spaces can be regarded as Riemannian manifolds with point reflections or parallel curvature tensors, special holonomy as a homogeneous space with special isotropy or particular Killing vector fields, or Lie triple systems [1,29,30].

Local symmetry refers to a property of a mathematical object, such as a manifold or a space, where symmetry exists at each point locally. Namely, a transformation or symmetry operation presents for every point in the object that leaves the object invariant and acts transitively on a small neighborhood around that point. Local symmetry can be declared in different ways depending on the type of object under consideration. For instance, in a locally symmetric space, such as a locally symmetric Riemannian or a pseudo-Riemannian manifold, the isometries move transitively on the entire space, not just locally around each point. Local symmetry has essential applications in various fields of mathematics and physics. It provides insights into the geometric properties of manifolds, helps classify and understand different types of spaces, and plays a crucial role in formulating physical theories [29,30].

Almost Kenmotsu manifolds have been studied extensively in Riemannian geometry and have applications in various fields, including theoretical physics and mathematical biology, which provide a geometric framework for exploring the interplay between contact geometry, Riemannian geometry, and symmetries on manifolds. While almost Kenmotsu manifolds and local symmetry are essential in the theory of manifolds, there is no inherent connection between almost Kenmotsu manifolds and local symmetry. An almost Kenmotsu manifold may or may not possess local symmetry, depending on its specific geometric properties. Moreover, it is well known that the existence of the characteristic vector field in a Kenmotsu manifold establishes the connection between Kenmotsu manifolds and local symmetry [19,22,35–37].

This study investigates the relations between semi-symmetry and local symmetry conditions on almost $\alpha$-cosymplectic three-manifolds. Our future studies on this topic will be on soliton theory.

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