

Article

Proposed Theorems on the Lifts of Kenmotsu Manifolds Admitting a Non-Symmetric Non-Metric Connection (NSNMC) in the Tangent Bundle

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Abstract: The main aim of the proposed paper is to investigate the lifts of Kenmotsu manifolds that admit NSNMC in the tangent bundle. We investigate several properties of the lifts of the curvature tensor, the conformal curvature tensor, and the conharmonic curvature tensor of Kenmotsu manifolds that admit NSNMC in the tangent bundle. We also study and discover that the lift of the Kenmotsu manifold that admit NSNMC is regular in the tangent bundle. Additionally, we find that the data provided by the lift of Ricci soliton on the lift of Ricci semi-symmetric Kenmotsu manifold that admits NSNMC in the tangent bundle are expanding.

Keywords: Kenmotsu manifolds; non-symmetric non-metric connection; vertical and complete lifts; tangent bundle; partial differential equations; curvature tensor; Ricci semi-symmetric manifolds; mathematical operators; Einstein manifolds; Ricci soliton



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1. Introduction

The geometry of tangent bundles has been an important domain in differential geometry because the theory provides many new problems in the study of modern differential geometry. Using the lift function, it is convenient to generalize to differentiable structures on any manifold M to its tangent bundle. In the field of differential geometry, numerous geometers, such as Yano and Kobayashi [1], Yano and Ishihara [2], Tani [3], Pandey and Chaturvedi [4], have explored the tangent bundle of differential geometry. The vertical, complete, and horizontal lifts of tensors as well as the connection from the manifold to its tangent bundle were developed by Yano and Ishihara [2]. The tangent bundles on different manifolds and their submanifolds with different connections and partial differential equations were studied by Khan in [5–8].

On the other hand, Kenmotsu [9] in 1971 introduced a new class of almost-contact manifold which became known as the Kenmotsu manifold. Since then, many geometers, such as Sinha and Pandey [10], Cihan and De [11], and others, have studied the properties of the Kenmotsu manifold. In the early 1930s, Friedman and Schouten [12] and Hayden [13] started the study of semi-symmetric metric and linear connections on the differential manifolds. If the torsion tensor \check{T} of the connection $\check{\nabla}$ is zero, it is called a symmetric and torsion-free connection, otherwise it is called a non-symmetric connection when the torsion tensor \check{T} is not zero. In an n -dimensional manifold M^n , if the torsion tensor \check{T} of the linear connection $\check{\nabla}$ satisfies $\check{T}(X_0, Y_0) = 2g(\Phi X_0, Y_0)$, it is called a non-symmetric connection for all vector fields X_0, Y_0 on M^n and, additionally, if the Riemannian metric g is such that

$\ddot{\nabla}g = 0$, then it is called a metric connection, and it is called non-metric if $\ddot{\nabla}g \neq 0$. Different geometers have studied and defined different types of connections, which can be seen in [14–30].

We start this paper with the introduction in Section 1. Section 2 is about the preliminaries. In Section 3, we obtain the lifts of the Kenmotsu manifolds in the tangent bundle. Section 4 deals with the lifts of NSNMC on Kenmotsu manifolds to the tangent bundle. In Sections 5 and 6, we study the lifts of the curvature tensors of NSNMC on Kenmotsu manifolds and the Ricci semi-symmetric Kenmotsu manifold in the tangent bundle, and some proposed theorems are also proved. Section 7 provides the conclusions and results that we obtained. Lastly, an example of the lifts of three-dimensional Kenmotsu manifolds in the tangent bundle is shown in Section 8.

2. Preliminaries

Let M be a differentiable manifold and $T_0M = \bigcup_{p \in M} T_{0p}M$ be the tangent bundle, where $T_{0p}M$ is the tangent space at point $p \in M$, and $\pi : T_0M \rightarrow M$ is the natural bundle structure of T_0M over M . For any coordinate system (Q, x^h) in M , where (x^h) is the local coordinate system in the neighborhood Q , then $(\pi^{-1}(Q), x^h, y^h)$ is the coordinate system in T_0M , where (x^h, y^h) is the induced coordinate system in $\pi^{-1}(Q)$ from (x^h) [2].

2.1. Vertical and Complete Lifts

Let f_0 be a function, X_0 a vector field, ω_0 a 1-form, a tensor field F_0 of type $(1, 1)$, and ∇ an affine connection in M . The vertical and complete lifts of a function f_0 , a vector field X_0 , a 1-form ω_0 , a tensor field F_0 of type $(1, 1)$, and ∇ an affine connection are given by $f_0^v, X_0^v, \omega_0^v, F_0^v, \nabla^v$ and $f_0^c, X_0^c, \omega_0^c, F_0^c, \nabla^c$, respectively. The following formulas for complete and vertical lifts are defined by [2]

$$(f_0 X_0)^v = f_0^v X_0^v, \quad (f_0 X_0)^c = f_0^c X_0^c + f_0^v X_0^c, \tag{1}$$

$$X_0^v f_0^v = 0, \quad X_0^v f_0^c = X_0^c f_0^c = (X_0 f_0)^v, \quad X_0^c f_0^c = (X_0 f_0)^c, \tag{2}$$

$$\omega_0(f_0^v) = 0, \quad \omega_0^v(X_0^c) = \omega_0^c(X_0^v) = \omega_0(X_0)^v, \quad \omega_0^c(X_0^c) = \omega_0(X_0)^c, \tag{3}$$

$$F_0^v X_0^c = (F_0 X_0)^v, \quad F_0^c X_0^c = (F_0 X_0)^c, \tag{4}$$

$$[X_0, Y_0]^v = [X_0^c, Y_0^v] = [X_0^v, Y_0^c], \quad [X_0, Y_0]^c = [X_0^c, Y_0^c], \tag{5}$$

$$\nabla^c_{X_0^c} Y_0^c = (\nabla_{X_0} Y_0)^c, \quad \nabla^c_{X_0^c} Y_0^v = (\nabla_{X_0} Y_0)^v. \tag{6}$$

2.2. Kenmotsu Manifolds

Let M be a $(2n + 1)$ -dimensional, almost-contact metric manifold with an almost-contact metric structure (Φ, A, α, g) consisting of a $(1, 1)$ tensor field Φ , a vector field A , a 1-form α , and Riemannian metric g on M satisfying

$$\alpha(A) = 1, \quad \Phi A = 0, \quad \alpha(\Phi(X_0)) = 0, \quad g(X_0, A) = \alpha(X_0), \tag{7}$$

$$\Phi^2(X_0) = -X_0 + \alpha(X_0)A, \quad g(X_0, \Phi Y_0) = -g(\Phi X_0, Y_0), \tag{8}$$

$$g(\Phi X_0, \Phi Y_0) = g(X_0, Y_0) - \alpha(X_0)\alpha(Y_0). \tag{9}$$

An almost-contact metric structure (Φ, A, α, g) is said to be a Kenmotsu manifold if and only if

$$(\ddot{\nabla}_{X_0} \Phi)Y_0 = g(\Phi X_0, Y_0)A - \alpha(Y_0)\Phi X_0. \tag{10}$$

From now on, we use the notation M to represent the Kenmotsu manifolds of dimension $(2n + 1)$. Using the above relations, we have some properties as given below:

$$\ddot{\nabla}_{X_0} A = X_0 - \alpha(X_0)A, \tag{11}$$

$$(\ddot{\nabla}_{X_0} \alpha)Y_0 = g(X_0, Y_0) - \alpha(X_0)\alpha(Y_0) = g(\Phi X_0, \Phi Y_0), \tag{12}$$

$$\ddot{R}(X_0, Y_0)A = \alpha(X_0)Y_0 - \alpha(Y_0)X_0, \tag{13}$$

$$\ddot{R}(A, X_0)Y_0 = \alpha(Y_0)X_0 - g(X_0, Y_0)A, \tag{14}$$

$$\ddot{R}(A, X_0)A = X_0 - \alpha(X_0)A, \tag{15}$$

$$\alpha(\ddot{R}(X_0, Y_0)Z_0) = g(X_0, Z_0)\alpha(Y_0) - g(Y_0, Z_0)\alpha(X_0), \tag{16}$$

$$\ddot{S}(\Phi X_0, \Phi Y_0) = \ddot{S}(X_0, Y_0) + 2n\alpha(X_0)\alpha(Y_0), \tag{17}$$

$$\ddot{S}(X_0, A) = -2n\alpha(X_0), \tag{18}$$

$$\ddot{S}(X_0, Y_0) = g(\ddot{Q}_0, Y_0), \tag{19}$$

where \ddot{R} , \ddot{S} , and \ddot{Q}_0 are the curvature tensor, the Ricci tensor, and the Ricci operator associated with the Levi-Civita connection.

Definition 1. An almost-contact metric manifold M is said to be an η -Einstein manifold if there exists the real valued functions λ_1, λ_2 such that [31,32]

$$\ddot{S}(X_0, Y_0) = \lambda_1 g(X_0, Y_0) + \lambda_2 \alpha(X_0)\alpha(Y_0). \tag{20}$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

Definition 2. A Ricci soliton (g, V_0, λ_0) on a Riemannian manifold is defined by [32]

$$\ddot{L}_{V_0} g + 2\ddot{S} + 2\lambda_0 g = 0, \tag{21}$$

where $L_{V_0} g$ is a Lie derivative of Riemannian metric g associated with vector field V_0 and real constant λ_0 . It is known to be shrinking, steady, and expanding if $\lambda_0 < 0$, $\lambda_0 = 0$, and $\lambda_0 > 0$ accordingly.

3. Kenmotsu Manifolds in the Tangent Bundle

Suppose T_0M is the tangent bundle, and $X_0 = X_0^i \frac{\partial}{\partial x^i}$ is a local vector field on M ; then, its vertical and complete lifts in terms of partial differential equations are

$$X_0^v = X_0^i \frac{\partial}{\partial y^i}, \tag{22}$$

$$X_0^c = X_0^i \frac{\partial}{\partial x^i} + \frac{\partial X_0^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \tag{23}$$

Let T_0M be the tangent bundle on the Kenmotsu manifolds M ; operating the complete lifts from Equations (7)–(21), we have

$$\alpha^c(A^c) = 1, \quad (\Phi A)^c = 0, \quad \alpha^c((\Phi(X_0))^c) = 0, \quad g^c(X_0^c, A^c) = \alpha^c(X_0^c), \tag{24}$$

$$(\Phi^2(X_0))^c = -X_0^c + \alpha^c(X_0^c)A^v + \alpha^v(X_0^c)A^c, \tag{25}$$

$$g^c(X_0^c, (\Phi Y_0)^c) = -g^c((\Phi X_0)^c, Y_0^c), \tag{26}$$

$$g^c((\Phi X_0)^c, (\Phi Y_0)^c) = g^c(X_0^c, Y_0^c) - \alpha^c(X_0^c)\alpha^v(Y_0^c) - \alpha^v(X_0^c)\alpha^c(Y_0^c), \tag{27}$$

$$\begin{aligned} (\ddot{\nabla}_{X_0^c}^c \Phi^c)Y_0^c &= g^c((\Phi X_0)^c, Y_0^c)A^v + g^c((\Phi X_0)^v, Y_0^c)A^c - \alpha^c(Y_0^c)(\Phi X_0)^v \\ &\quad - \alpha^v(Y_0^c)(\Phi X_0)^c, \end{aligned} \tag{28}$$

$$\check{\nabla}_{X_0^c}^c A^c = X_0^c - \alpha^c(X_0^c)A^v - \alpha^v(X_0^c)A^c, \quad (29)$$

$$\begin{aligned} (\check{\nabla}_{X_0^c}^c \alpha^c)Y_0^c &= g^c(X_0^c, Y_0^c) - \alpha^c(X_0^c)\alpha^v(Y_0^c) - \alpha^v(X_0^c)\alpha^c(Y_0^c) \\ &= g^c((\Phi X_0)^c, (\Phi Y_0)^c), \end{aligned} \quad (30)$$

$$\begin{aligned} \check{R}^c(X_0^c, Y_0^c)A^c &= \alpha^c(X_0^c)Y_0^v + \alpha^v(X_0^c)Y_0^c - \alpha^c(Y_0^c)X_0^v \\ &\quad - \alpha^v(Y_0^c)X_0^c, \end{aligned} \quad (31)$$

$$\begin{aligned} \check{R}^c(A^c, X_0^c)Y_0^c &= \alpha^c(Y_0^c)X_0^v + \alpha^v(Y_0^c)X_0^c - g^c(X_0^c, Y_0^c)A^v \\ &\quad - g^c(X_0^v, Y_0^c)A^c, \end{aligned} \quad (32)$$

$$\check{R}^c(A^c, X_0^c)A^c = X_0^c - \alpha^c(X_0^c)A^v - \alpha^v(X_0^c)A^c, \quad (33)$$

$$\begin{aligned} \alpha^c(\check{R}^c(X_0^c, Y_0^c)Z_0^c) &= g^c(X_0^c, Z_0^c)\alpha^v(Y_0^c) + g^c(X_0^v, Z_0^c)\alpha^c(Y_0^c) \\ &\quad - g^c(Y_0^c, Z_0^c)\alpha^v(X_0^c) - g^c(Y_0^v, Z_0^c)\alpha^c(X_0^c), \end{aligned} \quad (34)$$

$$\begin{aligned} \check{S}^c((\Phi X_0)^c, (\Phi Y_0)^c) &= \check{S}^c(X_0^c, Y_0^c) + 2n\alpha^c(X_0^c)\alpha^v(Y_0^c) \\ &\quad + 2n\alpha^v(X_0^c)\alpha^c(Y_0^c), \end{aligned} \quad (35)$$

$$\check{S}^c(X_0^c, A^c) = -2n\alpha^c(X_0^c), \quad (36)$$

$$\check{S}^c(X_0^c, Y_0^c) = g^c((\check{Q}_0 X_0)^c, Y_0^c), \quad (37)$$

$$\check{S}^c(X_0^c, Y_0^c) = \lambda_1 g^c(X_0^c, Y_0^c) + \lambda_2 \alpha^c(X_0^c)\alpha^v(Y_0^c) + \lambda_2 \alpha^v(X_0^c)\alpha^c(Y_0^c), \quad (38)$$

$$(\check{L}_{V_0} g)^c + 2\check{S}^c + 2\lambda_0 g^c = 0. \quad (39)$$

4. Lifts of NSNMC on Kenmotsu Manifolds in the Tangent Bundle

In a Kenmotsu manifolds M , let the linear connection be given by [32]

$$\check{\nabla}_{X_0} Y_0 = \check{\nabla}_{X_0} Y_0 + g(\Phi X_0, Y_0)A, \quad (40)$$

satisfying the torsion tensor

$$\check{T}(X_0, Y_0) = 2g(\Phi X_0, Y_0)A, \quad (41)$$

and

$$(\check{\nabla}_{X_0} g)(Y_0, Z_0) = -\alpha(Z_0)g(\Phi X_0, Y_0) - \alpha(Y_0)g(\Phi X_0, Z_0), \quad (42)$$

for arbitrary vector fields X_0, Y_0, Z_0 ; this is called an NSNMC. Now, let us take the complete lifts of Equations (40)–(42) by mathematical operators; we have

$$\check{\nabla}_{X_0^c}^c Y_0^c = \check{\nabla}_{X_0^c}^c Y_0^c + g^c((\Phi X_0)^c, Y_0^c)A^v + g^c((\Phi X_0)^v, Y_0^c)A^c, \quad (43)$$

$$\check{T}^c(X_0^c, Y_0^c) = 2g^c((\Phi X_0)^c, Y_0^c)A^v + 2g^c((\Phi X_0)^v, Y_0^c)A^c. \tag{44}$$

$$\begin{aligned} (\check{\nabla}_{X_0^c}^c g^c)(Y_0^c, Z_0^c) &= -\alpha^c(Z_0^c)g^c((\Phi X_0)^v, Y_0^c) - \alpha^v(Z_0^c)g^c((\Phi X_0)^c, Y_0^c) \\ &\quad - \alpha^c(Y_0^c)g^c((\Phi X_0)^v, Z_0^c) - \alpha^v(Y_0^c)g^c((\Phi X_0)^c, Z_0^c). \end{aligned} \tag{45}$$

We also have

$$\begin{aligned} (\check{\nabla}_{X_0^c}^c \Phi^c)(Y_0^c) &= (\check{\nabla}_{X_0^c}^c \Phi^c)(Y_0^c) + g^c((\Phi X_0)^c, (\Phi Y_0)^c)A^v \\ &\quad + g^c((\Phi X_0)^v, (\Phi Y_0)^c)A^c, \end{aligned} \tag{46}$$

$$(\check{\nabla}_{X_0^c}^c \alpha^c)(Y_0^c) = (\check{\nabla}_{X_0^c}^c \alpha^c)(Y_0^c) - g^c((\Phi X_0)^c, (Y_0)^c), \tag{47}$$

$$\begin{aligned} (\check{\nabla}_{X_0^c}^c g^c)((\Phi Y_0)^c, Z_0^c) &= -\alpha^c(Z_0^c)g^c((\Phi X_0)^v, (\Phi Y_0)^c) \\ &\quad - \alpha^v(Z_0^c)g^c((\Phi X_0)^c, (\Phi Y_0)^c). \end{aligned} \tag{48}$$

Replacing Y_0^c with A^c in Equation (43), we obtain

$$(\check{\nabla}_{X_0^c}^c A^c) = \check{\nabla}_{X_0^c}^c A^c. \tag{49}$$

Again, putting $X_0^c = A^c$ in Equation (42), we obtain

$$(\check{\nabla}_{A^c}^c g^c)(Y_0^c, Z_0^c) = 0. \tag{50}$$

Hence, we can propose the following proposition.

Proposition 1. *The complete lift of vector field A^c in the tangent bundle T_0M is invariant with respect to the lift of the Levi-Civita connection $\check{\nabla}^c$ and the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle T_0M .*

Proposition 2. *The complete lift of the co-variant differentiation of Riemannian metric g^c associated with the lift of contra-variant vector field A^c vanishes identically in a contact metric manifold admitting the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle T_0M .*

5. Lifts of Curvature Tensor of NSNMC on Kenmotsu Manifolds in the Tangent Bundle

The curvature tensor \check{R} of NSNMC $\check{\nabla}$ is given by [32]

$$\check{R}(X_0, Y_0)Z_0 = \check{\nabla}_{X_0} \check{\nabla}_{Y_0} Z_0 - \check{\nabla}_{Y_0} \check{\nabla}_{X_0} Z_0 - \check{\nabla}_{[X_0, Y_0]} Z_0. \tag{51}$$

Taking a complete lift by mathematical operators on Equation (51), we obtain

$$\check{R}^c(X_0^c, Y_0^c)Z_0^c = \check{\nabla}_{X_0^c}^c \check{\nabla}_{Y_0^c}^c Z_0^c - \check{\nabla}_{Y_0^c}^c \check{\nabla}_{X_0^c}^c Z_0^c - \check{\nabla}_{[X_0^c, Y_0^c]}^c Z_0^c \tag{52}$$

for all $X_0, Y_0, Z_0 \in T_0M$. By using Equation (43), we obtain

$$\begin{aligned} \check{R}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c + g^c\left(\left(\check{\nabla}_{X_0}\Phi\right)Y_0^c, Z_0^c\right)A^v \\ &\quad + g^c\left(\left(\check{\nabla}_{X_0}\Phi\right)Y_0^c, Z_0^c\right)A^c - g^c\left(\left(\check{\nabla}_{Y_0}\Phi\right)X_0^c, Z_0^c\right)A^v \\ &\quad - g^c\left(\left(\check{\nabla}_{Y_0}\Phi\right)X_0^c, Z_0^c\right)A^c + g^c\left(\Phi Y_0^c, Z_0^c\right)\left(\check{\nabla}_{X_0}A\right)^v \\ &\quad + g^c\left(\Phi Y_0^c, Z_0^c\right)\left(\check{\nabla}_{X_0}A\right)^c - g^c\left(\Phi X_0^c, Z_0^c\right)\left(\check{\nabla}_{Y_0}A\right)^v \\ &\quad - g^c\left(\Phi X_0^c, Z_0^c\right)\left(\check{\nabla}_{Y_0}A\right)^c, \end{aligned} \tag{53}$$

where $\check{R}^c(X_0^c, Y_0^c)Z_0^c$ is the complete lift of the Riemannian curvature tensor of the Levi-Civita connection in the tangent bundle such that

$$\check{R}^c(X_0^c, Y_0^c)Z_0^c = \check{\nabla}_{X_0^c}^c \check{\nabla}_{Y_0^c}^c Z_0^c - \check{\nabla}_{Y_0^c}^c \check{\nabla}_{X_0^c}^c Z_0^c - \check{\nabla}_{[X_0^c, Y_0^c]}^c Z_0^c. \tag{54}$$

Proposition 3. *The relation between the complete lifts of \check{R}^c associated with the lift of NSNMC $\check{\nabla}^c$ and \check{R}^c associated with the lift of the Levi-Civita connection $\check{\nabla}$ is given by Equation (53).*

Using Equations (25), (26), (28) and (29) in Equation (53), we obtain

$$\begin{aligned} \check{R}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c + 2\alpha^c(Z_0^c)g^c\left(\left(\Phi X_0\right)^c, Y_0^c\right)A^v \\ &\quad + 2\alpha^c(Z_0^c)g^c\left(\left(\Phi X_0\right)^v, Y_0^c\right)A^c + 2\alpha^v(Z_0^c)g^c\left(\left(\Phi X_0\right)^c, Y_0^c\right)A^c \\ &\quad - g^c\left(\left(\Phi X_0\right)^c, Z_0^c\right)Y_0^v - g^c\left(\left(\Phi X_0\right)^v, Z_0^c\right)Y_0^c \\ &\quad + g^c\left(\left(\Phi Y_0\right)^c, Z_0^c\right)X_0^v + g^c\left(\left(\Phi Y_0\right)^v, Z_0^c\right)X_0^c. \end{aligned} \tag{55}$$

Taking the contraction of Equation (55) with respect to X_0 , we obtain

$$\check{S}^c(Y_0^c, Z_0^c) = \check{S}^c(Y_0^c, Z_0^c) + 2ng^c\left(\left(\Phi Y_0\right)^c, Z_0^c\right). \tag{56}$$

Putting Equation (37) into Equation (56), we obtain

$$\check{Q}_0^c(Y_0^c) = \check{Q}_0^c(Y_0^c) + 2n(\Phi Y_0)^c. \tag{57}$$

Again, from Equation (56), by taking the contraction, it follows that

$$\check{r}^c = \check{r}^c, \tag{58}$$

where $\check{S}^c, \check{S}^c, \check{Q}_0^c, \check{Q}_0^c, \check{r}^c$, and \check{r}^c are the complete lifts of the Ricci tensors, Ricci operators, and scalar curvatures of the lifts of NSNMC $\check{\nabla}^c$ and Levi-Civita connection $\check{\nabla}^c$, respectively.

By interchanging Z_0^c with A^c in Equation (55), and inputting Equations (25) and (26), we obtain

$$\check{R}^c(A^c, Y_0^c)Z_0^c = R^c(A^c, Y_0^c)Z_0^c + g^c\left(\left(\Phi Y_0\right)^c, Z_0^c\right)A^v + g^c\left(\left(\Phi Y_0\right)^v, Z_0^c\right)A^c. \tag{59}$$

From Equations (32) and (59), we obtain

$$\begin{aligned} \check{R}^c(A^c, Y_0^c)Z_0^c &= \alpha^c(Z_0^c)Y_0^v + \alpha^v(Z_0^c)Y_0^c \\ &\quad - g^c(Y_0^c, Z_0^c)A^v - g^c(Y_0^v, Z_0^c)A^c \\ &\quad + g^c\left(\left(\Phi Y_0\right)^c, Z_0^c\right)A^v + g^c\left(\left(\Phi Y_0\right)^v, Z_0^c\right)A^c. \end{aligned} \tag{60}$$

Again, interchanging Z_0^c with A^c in Equation (55) and inputting Equations (25), (27) and (31), we obtain

$$\begin{aligned}
 \check{R}^c(X_0^c, Y_0^c)A^c &= \check{R}^c(X_0^c, Y_0^c)A^c + 2g^c((\Phi Y_0)^c, Z_0^c)A^v \\
 &\quad + 2g^c((\Phi Y_0)^v, Z_0^c)A^c \\
 &= \alpha^c(X_0^c)Y_0^v + \alpha^v(X_0^c)Y_0^c \\
 &\quad - \alpha^c(Y_0^c)X_0^v - \alpha^v(Y_0^c)X_0^c \\
 &\quad + 2g^c((\Phi Y_0)^c, Z_0^c)A^v + 2g^c((\Phi Y_0)^v, Z_0^c)A^c \\
 &\neq 0.
 \end{aligned}
 \tag{61}$$

Thus, we state the following theorem.

Theorem 1. Every $(2n + 1)$ -dimensional Kenmotsu manifold associated with the lift of the NSNMC $\check{\nabla}^c$ in the tangent bundle is regular with respect to $\check{\nabla}^c$.

Let us operate α on both sides of Equation (55) and, inputting Equation (25), we obtain

$$\begin{aligned}
 \alpha^c(\check{R}^c(X_0^c, Y_0^c)Z_0^c) &= 2g^c((\Phi X_0)^c, Y_0^c)\alpha^v(Z_0^c) + 2g^c((\Phi X_0)^v, Y_0^c)\alpha^c(Z_0^c) \\
 &\quad + \alpha^c(Y_0^c)g^c(X_0^v, Z_0^c) + \alpha^v(Y_0^c)g^c(X_0^c, Z_0^c) \\
 &\quad - \alpha^c(X_0^c)g^c(Y_0^v, Z_0^c) - \alpha^v(X_0^c)g^c(Y_0^c, Z_0^c) \\
 &\quad + g^c((\Phi Y_0)^c, Z_0^c)\alpha^v(X_0^c) + g^c((\Phi Y_0)^v, Z_0^c)\alpha^c(X_0^c) \\
 &\quad - g^c((\Phi X_0)^c, Z_0^c)\alpha^v(Y_0^c) - g^c((\Phi X_0)^v, Z_0^c)\alpha^c(Y_0^c).
 \end{aligned}
 \tag{62}$$

Now, taking the contraction of Equation (61) with respect to X_0^c , we obtain

$$\check{S}^c(Y_0^c, A^c) = -2n\alpha^c(Y_0^c).
 \tag{63}$$

Putting $\check{R}^c(X_0^c, Y_0^c)Z_0^c = 0$ into Equation (55), we obtain

$$\begin{aligned}
 \check{R}^c(X_0^c, Y_0^c)Z_0^c &= g^c((\Phi X_0)^c, Z_0^c)Y_0^v + g^c((\Phi X_0)^v, Z_0^c)Y_0^c \\
 &\quad - g^c((\Phi Y_0)^c, Z_0^c)X_0^v - g^c((\Phi Y_0)^v, Z_0^c)X_0^c \\
 &\quad - 2g^c((\Phi X_0)^c, Y_0^c)\alpha^c(Z_0^c)A^v - 2g^c((\Phi X_0)^c, Y_0^c)\alpha^v(Z_0^c)A^c \\
 &\quad - 2g^c((\Phi X_0)^v, Y_0^c)\alpha^c(Z_0^c)A^c.
 \end{aligned}
 \tag{64}$$

In view of $'\ddot{R}^c(X_0^c, Y_0^c, Z_0^c, W_0) = g^c(\ddot{R}^c(X_0^c, Y_0^c)Z_0^c, W_0^c)$ and Equation (64), we get

$$\begin{aligned} '\ddot{R}^c(X_0^c, Y_0^c, Z_0^c, W_0) &= g^c((\Phi X_0)^c, Z_0^c)g^c(Y_0^v, W_0^c) \\ &+ g^c((\Phi X_0)^v, Z_0^c)g^c(Y_0^c, W_0^c) \\ &- g^c((\Phi Y_0)^c, Z_0^c)g^c(X_0^v, W_0^c) \\ &- g^c((\Phi Y_0)^v, Z_0^c)g^c(X_0^c, W_0^c) \\ &- 2g^c((\Phi X_0)^c, Y_0^c)\alpha^c(Z_0^c)\alpha^v(W_0^c) \\ &- 2g^c((\Phi X_0)^c, Y_0^c)\alpha^v(Z_0^c)\alpha^c(W_0^c) \\ &- 2g^c((\Phi X_0)^v, Y_0^c)\alpha^c(Z_0^c)\alpha^c(W_0^c). \end{aligned} \tag{65}$$

Taking the contraction of Equation (65) with respect to X_0^c , we obtain

$$\ddot{S}^c(Y_0^c, Z_0^c) = -2ng^c((\Phi Y_0)^c, Z_0^c). \tag{66}$$

Putting Equation (37) into Equation (66), we obtain

$$\ddot{Q}_0^c(Y_0^c) = -2n(\Phi Y_0)^c. \tag{67}$$

From Equation (66), it follows that

$$\check{r}^c = 0. \tag{68}$$

The conformal curvature tensor of $\check{\nabla}$ is defined by [32]

$$\begin{aligned} \check{C}(X_0, Y_0)Z_0 &= \check{R}(X_0, Y_0)Z_0 - \frac{1}{2n-1} [\check{S}(Y_0, Z_0) - \check{S}(X_0, Z_0)Y_0 \\ &+ g(Y_0, Z_0)\check{Q}X_0 - g(X_0, Z_0)\check{Q}Y_0] \\ &+ \frac{\check{r}}{2n(2n-1)} [g(Y_0, Z_0)X_0 - g(X_0, Z_0)Y_0]. \end{aligned} \tag{69}$$

Taking a complete lift by mathematical operators on Equation (69), we obtain

$$\begin{aligned} \check{C}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{2n-1} [\check{S}^c(Y_0^c, Z_0^c)X_0^v + \check{S}^v(Y_0^c, Z_0^c)X_0^c \\ &- \check{S}^c(X_0, Z_0)Y_0^v - \check{S}^v(X_0, Z_0)Y_0^c \\ &+ g^c(Y_0^c, Z_0^c)(\check{Q}X_0)^v + g^c(Y_0^v, Z_0^c)(\check{Q}X_0)^c \\ &- g^c(X_0^c, Z_0^c)(\check{Q}Y_0)^v - g^c(X_0^v, Z_0^c)(\check{Q}Y_0)^c] \\ &+ \frac{\check{r}}{2n(2n-1)} [g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c \\ &- g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c]. \end{aligned} \tag{70}$$

Putting Equations (55)–(57) into Equation (70), we obtain

$$\begin{aligned}
\check{C}^c(X_0^c, Y_0^c)Z_0^c - \check{C}^c(X_0^c, Y_0^c)Z_0^c &= 2g^c\left((\Phi X_0)^c, Y_0^c\right)\alpha^c(Z_0^c)A^v \\
&+ 2g^c\left((\Phi X_0)^c, Y_0^c\right)\alpha^v(Z_0^c)A^c \\
&+ 2g^c\left((\Phi X_0)^v, Y_0^c\right)\alpha^c(Z_0^c)A^c \\
&+ g^c\left((\Phi Y_0)^c, Z_0^c\right)X_0^v + g^c\left((\Phi Y_0)^v, Z_0^c\right)X_0^c \\
&- g^c\left((\Phi X_0)^c, Z_0^c\right)Y_0^v - g^c\left((\Phi X_0)^v, Z_0^c\right)Y_0^c \\
&- \frac{2n}{(2n-1)}\left[g^c\left((\Phi Y_0)^c, Z_0^c\right)X_0^v \right. \\
&+ g^c\left((\Phi Y_0)^v, Z_0^c\right)X_0^c - g^c\left((\Phi X_0)^c, Z_0^c\right)Y_0^v \\
&- g^c\left((\Phi X_0)^v, Z_0^c\right)Y_0^c + g^c(Y_0^c, Z_0^c)(\Phi X_0)^v \\
&+ g^c(Y_0^v, Z_0^c)(\Phi X_0)^c - g^c(X_0^c, Z_0^c)(\Phi Y_0)^v \\
&\left. - g^c(X_0^v, Z_0^c)(\Phi Y_0)^c\right], \tag{71}
\end{aligned}$$

where \check{C}^c is the complete lift of the conformal curvature tensor associated with the lift of the Levi-Civita connection $\check{\nabla}^c$ in the tangent bundle as

$$\begin{aligned}
\check{C}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{2n-1}\left[\check{S}^c(Y_0^c, Z_0^c)X_0^v + \check{S}^v(Y_0^c, Z_0^c)X_0^c \right. \\
&- \check{S}^c(X_0, Z_0)Y_0^v - \check{S}^v(X_0, Z_0)Y_0^c \\
&+ g^c(Y_0^c, Z_0^c)(\check{Q}X_0)^v + g^c(Y_0^v, Z_0^c)(\check{Q}X_0)^c \\
&- g^c(X_0^c, Z_0^c)(\check{Q}Y_0)^v - g^c(X_0^v, Z_0^c)(\check{Q}Y_0)^c\left. \right] \\
&+ \frac{\check{r}}{2n(2n-1)}\left[g^c(Y_0^c, Z_0^c)X_0^v + g^c(Y_0^v, Z_0^c)X_0^c \right. \\
&- g^c(X_0^c, Z_0^c)Y_0^v - g^c(X_0^v, Z_0^c)Y_0^c\left. \right]. \tag{72}
\end{aligned}$$

The conharmonic curvature tensor of NSNMC $\check{\nabla}$ is given by [32]

$$\begin{aligned}
\check{L}(X_0, Y_0)Z_0 &= \check{R}(X_0, Y_0)Z_0 - \frac{1}{(2n-1)}\left[\check{S}(Y_0, Z_0)X_0 \right. \\
&- \check{S}(X_0, Z_0)Y_0 + g(Y_0, Z_0)\check{Q}X_0 \\
&- g(X_0, Z_0)\check{Q}Y_0\left. \right]. \tag{73}
\end{aligned}$$

Now, taking a complete lift by mathematical operators on Equation (73), we obtain

$$\begin{aligned}
\check{L}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{(2n-1)}\left[\check{S}^c(Y_0^c, Z_0^c)X_0^v \right. \\
&+ \check{S}^v(Y_0^c, Z_0^c)X_0^c - \check{S}^c(X_0^c, Z_0^c)Y_0^v \\
&- \check{S}^v(X_0^c, Z_0^c)Y_0^c + g^c(Y_0^c, Z_0^c)(\check{Q}_0X_0)^v \\
&+ g^c(Y_0^v, Z_0^c)(\check{Q}_0X_0)^c - g^c(X_0^c, Z_0^c)(\check{Q}_0Y_0)^v \\
&\left. - g^c(X_0^v, Z_0^c)(\check{Q}_0Y_0)^c\right]. \tag{74}
\end{aligned}$$

Putting Equations (55)–(58) into Equation (74), we obtain

$$\begin{aligned}
 \check{L}^c(X_0^c, Y_0^c)Z_0^c - \check{L}^c(X_0^c, Y_0^c)Z_0^c &= 2g^c\left((\Phi X_0)^c, Y_0^c\right)\alpha^c(Z_0^c)A^v \\
 &+ 2g^c\left((\Phi X_0)^c, Y_0^c\right)\alpha^v(Z_0^c)A^c \\
 &+ 2g^c\left((\Phi X_0)^v, Y_0^c\right)\alpha^c(Z_0^c)A^c \\
 &+ g^c\left((\Phi Y_0)^c, Z_0^c\right)X_0^v + g^c\left((\Phi Y_0)^v, Z_0^c\right)X_0^c \\
 &- g^c\left((\Phi X_0)^c, Z_0^c\right)Y_0^v - g^c\left((\Phi X_0)^v, Z_0^c\right)Y_0^c \\
 &- \frac{2n}{(2n-1)}\left[g^c\left((\Phi Y_0)^c, Z_0^c\right)X_0^v \right. \\
 &+ g^c\left((\Phi Y_0)^v, Z_0^c\right)X_0^c - g^c\left((\Phi X_0)^c, Z_0^c\right)Y_0^v \\
 &- g^c\left((\Phi X_0)^v, Z_0^c\right)Y_0^c + g^c(Y_0^c, Z_0^c)(\Phi X_0)^v \\
 &+ g^c(Y_0^v, Z_0^c)(\Phi X_0)^c - g^c(X_0^c, Z_0^c)(\Phi Y_0)^v \\
 &\left. - g^c(X_0^v, Z_0^c)(\Phi Y_0)^c\right], \tag{75}
 \end{aligned}$$

where \check{L}^c is the complete lift of the conharmonic curvature tensor associated with the lift of the Levi-Civita connection $\check{\nabla}^c$ in the tangent bundle as

$$\begin{aligned}
 \check{L}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c - \frac{1}{(2n-1)}\left[\check{S}^c(Y_0^c, Z_0^c)X_0^v \right. \\
 &+ \check{S}^v(Y_0^c, Z_0^c)X_0^c - \check{S}^c(X_0^c, Z_0^c)Y_0^v \\
 &- \check{S}^v(X_0^c, Z_0^c)Y_0^c + g^c(Y_0^c, Z_0^c)(\check{Q}_0 X_0)^v \\
 &+ g^c(Y_0^v, Z_0^c)(\check{Q}_0 X_0)^c - g^c(X_0^c, Z_0^c)(\check{Q}_0 Y_0)^v \\
 &\left. - g^c(X_0^v, Z_0^c)(\check{Q}_0 Y_0)^c\right]. \tag{76}
 \end{aligned}$$

Hence, we can proposed the following.

Proposition 4. *The relationship between the complete lift of the conformal curvature tensor \check{C}^c associated with the lift of NSNMC $\check{\nabla}^c$ and the complete lift of the conformal curvature tensor \check{C}^c associated with the lift of the Levi-Civita connection $\check{\nabla}^c$ is given by Equation (71).*

Proposition 5. *The relationship between the complete lift of the conharmonic curvature tensor \check{L}^c associated with the lift of NSNMC $\check{\nabla}^c$ and the complete lift of the conharmonic curvature tensor \check{L}^c associated with the lift of the Levi-Civita connection $\check{\nabla}^c$ is given by Equation (75).*

Theorem 2. *For a Kenmotsu manifold admitting an NSNMC $\check{\nabla}^c$, the necessary and sufficient condition for a lift of the conformal curvature tensor associated with $\check{\nabla}^c$ that coincides with $\check{\nabla}^c$ is that the lift of the conharmonic curvature tensor associated with $\check{\nabla}^c$ is equal to that of $\check{\nabla}^c$ in the tangent bundle.*

The concircular curvature tensor of a Riemannian manifold is given by [32]

$$\begin{aligned}
 \check{V}(X_0, Y_0)Z_0 &= \check{R}(X_0, Y_0)Z_0 + \frac{\check{r}}{2n(2n-1)}\left[g(Y_0, Z_0)X_0 \right. \\
 &\left. - g(X_0, Z_0)Y_0\right]. \tag{77}
 \end{aligned}$$

Taking the complete lift by mathematical operators of Equation (77), we obtain

$$\begin{aligned} \check{V}^c(X_0^c, Y_0^c)Z_0^c &= \check{R}^c(X_0^c, Y_0^c)Z_0^c + \frac{j^c}{2n(2n-1)} \left[g^c(Y_0^c, Z_0^c)X_0^v \right. \\ &\quad + g^c(Y_0^v, Z_0^c)X_0^c - g^c(X_0^c, Z_0^c)Y_0^v \\ &\quad \left. - g^c(X_0^v, Z_0^c)Y_0^c \right]. \end{aligned} \tag{78}$$

Using Equations (58), (64), (68) and (78), we can state the following theorem.

Theorem 3. For a Kenmotsu manifold admitting an NSNMC $\check{\nabla}^c$, the necessary and sufficient condition for a lift of the concircular curvature tensor that coincides with the lift of the curvature tensor is the lift of a scalar curvature of $\check{\nabla}^c$ that needs to be zero in the tangent bundle.

6. Lifts of the Ricci Semi-Symmetric Kenmotsu Manifold in the Tangent Bundle

A $(2n + 1)$ -dimensional contact metric manifold with an NSNMC is said to be Ricci semi-symmetric if [32]

$$\left(\check{R}(X_0, Y_0) \cdot \check{S} \right) (Z_0, U_0) = 0. \tag{79}$$

Taking a complete lift of Equation (79), we obtain

$$\left(\check{R}^c(X_0^c, Y_0^c) \cdot \check{S}^c \right) (Z_0^c, U_0^c) = 0; \tag{80}$$

that is,

$$\check{S}^c \left(\check{R}^c(X_0^c, Y_0^c)Z_0^c, U_0^c \right) + \check{S}^c \left(Z_0^c, \check{R}^c(X_0^c, Y_0^c)U_0^c \right) = 0. \tag{81}$$

Interchanging X_0^c with A^c , and using Equation (60) in Equation (81), we obtain

$$\check{S}^c \left(\check{R}^c(A^c, Y_0^c)Z_0^c, U_0^c \right) + \check{S}^c \left(Z_0^c, \check{R}^c(A^c, Y_0^c)U_0^c \right) = 0, \tag{82}$$

$$\begin{aligned} &\alpha^c(Z_0^c)\check{S}^v(Y_0^c, U_0^c) + \alpha^v(Z_0^c)\check{S}^c(Y_0^c, U_0^c) - g^c(Y_0^c, Z_0^c)\check{S}^v(A^c, U_0^c) \\ &- g^c(Y_0^v, Z_0^c)\check{S}^c(A^c, U_0^c) + g^c((\Phi Y_0)^c, Z_0^c)\check{S}^v(A^c, U_0^c) \\ &+ g^c((\Phi Y_0)^v, Z_0^c)\check{S}^c(A^c, U_0^c) + \alpha^c(U_0^c)\check{S}^v(Z_0^c, Y_0^c) \\ &+ \alpha^v(U_0^c)\check{S}^c(Z_0^c, Y_0^c) - g^c(Y_0^c, U_0^c)\check{S}^v(Z_0^c, A^c) \\ &- g^c(Y_0^v, U_0^c)\check{S}^c(Z_0^c, A^c) + g^c((\Phi Y_0)^c, U_0^c)\check{S}^v(Z_0^c, A^c) \\ &+ g^c((\Phi Y_0)^v, U_0^c)\check{S}^c(Z_0^c, A^c) \\ &= 0. \end{aligned} \tag{83}$$

Using Equation (63), the above Equation (83) reduces to

$$\begin{aligned}
 & \alpha^c(Z_0^c)\check{S}^v(Y_0^c, U_0^c) + \alpha^v(Z_0^c)\check{S}^c(Y_0^c, U_0^c) + 2ng^c(Y_0^c, Z_0^c)\alpha^v(U_0^c) \\
 & + 2ng^c(Y_0^v, Z_0^c)\alpha^c(U_0^c) - 2ng^c((\Phi Y_0)^c, Z_0^c)\alpha^v(U_0^c) \\
 & - 2ng^c((\Phi Y_0)^v, Z_0^c)\alpha^c(U_0^c) + \alpha^c(U_0^c)\check{S}^v(Z_0^c, Y_0^c) \\
 & + \alpha^v(U_0^c)\check{S}^c(Z_0^c, Y_0^c) + 2ng^c(Y_0^c, U_0^c)\alpha^v(Z_0^c) \\
 & + 2ng^c(Y_0^v, U_0^c)\alpha^c(Z_0^c) - 2ng^c((\Phi Y_0)^c, U_0^c)\alpha^v(Z_0^c) \\
 & - 2ng^c((\Phi Y_0)^v, U_0^c)\alpha^c(Z_0^c) \\
 & = 0.
 \end{aligned}
 \tag{84}$$

Interchanging $U_0^c = A^c$ in Equation (84) and using Equation (63), we obtain

$$\check{S}^c(Y_0^c, Z_0^c) = 2ng^c((\Phi Y_0)^c, Z_0^c) - 2ng^c(Y_0^c, Z_0^c).
 \tag{85}$$

Putting Equation (56) into Equation (85), we obtain

$$\check{S}^c(Y_0^c, Z_0^c) = -2ng^c(Y_0^c, Z_0^c).
 \tag{86}$$

Taking the contraction of Equation (86), we obtain

$$\check{r}^c = -2n(2n + 1).
 \tag{87}$$

Using Equation (87) in Equation (58), we obtain

$$\check{r}^c = -2n(2n + 1).
 \tag{88}$$

By using Equation (86), we can propose the following theorem.

Theorem 4. *A Ricci semi-symmetric Kenmotsu manifold associated with the lift of an NSNMC $\check{\nabla}^c$ is an Einstein manifold in the tangent bundle.*

Equation (39) defined the data of the lift of the Ricci soliton (g^c, V_0^c, λ_0) in the tangent bundle, where g^c is the lift of the Riemannian metric, V_0^c is the lift of the vector field, and λ_0 is a real constant. Here we obtain two conditions with respect to $V_0^c : V_0^c \in span(A^c)$ and $V_0^c \perp span(A^c)$. Let us concentrate on the first condition, which is $V_0^c : V_0^c \in span(A^c)$. The data of the lift of the Ricci soliton (g^c, V_0^c, λ_0) of the Kenmotsu manifold associated with the lift of NSNMC $\check{\nabla}^c$ in the tangent bundle are defined as

$$(\check{L}_A g)^c(X_0^c, Y_0^c) + 2\check{S}^c(X_0^c, Y_0^c) + 2\lambda g^c(X_0^c, Y_0^c) = 0
 \tag{89}$$

for all $X_0^c, Y_0^c \in T_0M$, where $(\check{L}_A g)^c$ is the lift of the Lie-derivative of Riemannian metric g^c with respect to A^c containing a lift of NSNMC $\check{\nabla}^c$ in the tangent bundle and is defined by

$$\begin{aligned}
 (\check{L}_A g)^c(X_0^c, Y_0^c) &= A^c g^c(X_0^v, Y_0^c) + A^v g^c(X_0^c, Y_0^c) \\
 &+ g^c((\check{L}_A X_0)^c, Y_0^c) + g^c(X_0^c, (\check{L}_A Y_0)^c) \\
 &= A^c g^c(X_0^v, Y_0^c) + g^c((\check{\nabla}_A X_0)^c, Y_0^c) \\
 &+ g^c(X_0^c, (\check{\nabla}_A Y_0)^c) - g^c((\check{\nabla}_{X_0} A)^c, Y_0^c) \\
 &- g^c(X_0^c, (\check{\nabla}_{Y_0} A)^c).
 \end{aligned}
 \tag{90}$$

Hence,

$$\begin{aligned}
 (\check{L}_A g)^c(X_0^c, Y_0^c) &= (\check{\nabla}_A g)^c(X_0^c, Y_0^c) - g^c\left((\check{\nabla}_{X_0} A)^c, Y_0^c\right) \\
 &\quad - g^c\left(X_0^c, (\check{\nabla}_{X_0} A)^c\right).
 \end{aligned}
 \tag{91}$$

Using Equations (25), (27), (29), (45) and (49) in Equation (91), we obtain

$$(\check{L}_A g)^c(X_0^c, Y_0^c) = -2g^c\left((\Phi X_0)^c, (\Phi Y_0)^c\right).
 \tag{92}$$

Putting Equations (86) and (92) into Equation (89), we obtain

$$\begin{aligned}
 -g^c\left((\Phi X_0)^c, (\Phi Y_0)^c\right) - 2ng^c(X_0^c, Y_0^c) + 2ng^c\left((\Phi X_0)^c, Y_0^c\right) \\
 + \lambda_0 g^c(X_0^c, Y_0^c) \\
 = 0.
 \end{aligned}
 \tag{93}$$

Putting $X_0^c = Y_0^c = A^c$ in Equation (93) and using Equation (25), we obtain

$$\lambda_0 = 2n > 0.
 \tag{94}$$

Hence, we can state the following theorem.

Theorem 5. *On a Ricci semi-symmetric Kenmotsu manifold associated with the lift of an NSNMC $\check{\nabla}^c$, the data of the Ricci soliton (g^c, A^c, λ_0) are expanding in the tangent bundle.*

7. Example

Let M be a three-dimensional manifold defined as

$$M = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3; y_1 \neq 0 \right\},
 \tag{95}$$

where \mathbb{R} is the set of real numbers, and let y_1, y_2, y_3 be given as

$$a_1 = y_1 \frac{\partial}{\partial y_3}, \quad a_2 = y_1 \frac{\partial}{\partial y_2}, \quad a_3 = -y_1 \frac{\partial}{\partial y_1},$$

where $\{a_1, a_2, a_3\}$ constitute a linearly independent global frame on M . Let the 1-form α be given by

$$\alpha(X_0) = g(X_0, A).$$

The Riemannian metric g is defined by

$$g(a_i, a_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

Let Φ be the $(1, 1)$ tensor field defined by

$$\Phi a_i = \begin{cases} a_2, & i = 1 \\ a_1, & i = 2 \\ 0, & i = 3. \end{cases}$$

Using the linearity of Φ and g , we acquire $\alpha(a_3) = 1$, $\phi^2 Y_0 = -Y_0 + \alpha(Y_0)a_3$, and $g(\Phi X_0, \Phi Y_0) = g(X_0, Y_0) - \alpha(X_0)\alpha(Y_0)$. Thus, for $a_3 = A$, the structure (Φ, A, α, g) is an almost-contact metric structure on M , and M is called an almost-contact metric manifold. In addition, M satisfies

$$(\nabla_{X_0}^c \Phi)Y_0 = g(\Phi X_0, Y_0)a_3 - \alpha(Y_0)\Phi X_0.$$

Here, for $a_3 = A$, M is a Kenmotsu manifold. Let the complete and vertical lifts of a_1, a_2, a_3 be a_1^c, a_2^c, a_3^c and a_1^v, a_2^v, a_3^v , respectively, in the tangent bundle T_0M of manifold M , and let the complete lift of the Riemannian metric g be g^c on T_0M such that

$$g^c(X_0^v, a_3^c) = (g^c(X_0, a_3))^v = (\alpha(X_0))^v \tag{96}$$

$$g^c(X_0^c, a_3^c) = (g^c(X_0, a_3))^c = (\alpha(X_0))^c \tag{97}$$

$$g^c(a_3^c, a_3^c) = 1, \quad g^v(X_0^v, a_3^c) = 0, \quad g^v(a_3^v, a_3^c) = 0, \tag{98}$$

and so on. Let the complete and vertical lifts of the $(1, 1)$ tensor field ϕ_0 be ϕ_0^c and ϕ_0^v , respectively, and defined by

$$\Phi^v(a_3^v) = \Phi^c(a_3^c) = 0, \tag{99}$$

$$\Phi^v(a_1^v) = a_2^v, \quad \Phi^c(a_1^c) = a_2^c, \tag{100}$$

$$\Phi^v(a_2^v) = -a_1^v, \quad \Phi^c(a_2^c) = -a_1^c. \tag{101}$$

Using the linearity of Φ and g , we infer that

$$(\Phi^2 X_0)^c = -X_0^c + \alpha^c(X_0^c)a_3^v + \alpha^v(X_0^v)a_3^c, \tag{102}$$

$$g^c((\Phi a_3)^c, (\Phi a_2)^c) = g^c(a_3^c, a_2^c) - \alpha^c(a_3^c)\alpha^v(a_2^c) - \alpha^v(a_3^v)\alpha^c(a_2^c). \tag{103}$$

Thus, for $a_3 = A$ in Equations (96)–(98) and (102), the structure $(\Phi^c, A^c, \alpha^c, g^c)$ is an almost-contact metric structure on T_0M and satisfies the relation

$$\begin{aligned} (\nabla_{a_3^c}^c \Phi^c)a_2^c &= g^c((\Phi a_3)^c, a_2^c)A^v + g^c((\Phi a_3)^v, a_2^c)A^c \\ &\quad - \alpha^c(a_2^c)(\Phi a_3)^v - \alpha^v(a_2^v)(\Phi a_3)^c. \end{aligned}$$

Thus, $(\Phi^c, A^c, \alpha^c, g^c, T_0M)$ is a Kenmotsu manifold.

8. Conclusions

In the present paper, we study the lifts of Kenmotsu manifolds admitting an NSNMC in the tangent bundle. Firstly, the relationship between the lifts of the Levi-Civita connection ∇^c and the NSNMC ∇^v from a Kenmotsu manifold to the tangent bundle is established. We find that the complete lifts of vector field A^c are invariant with respect to the Levi-Civita connection in the tangent bundle and that the complete lift of the co-variant differentiation of Riemannian metric g^c associated with the lift of contra-variant vector field A^c vanishes identically in a contact metric manifold admitting NSNMC in the tangent bundle. Next, we study the lifts of the curvature tensor of Kenmotsu manifolds associated with NSNMC in the tangent bundle, and we derive the relationship between the complete lifts of curvature tensors (conformal, conharmonic, concircular), the Ricci tensor, the scalar curvature associated with the lifts of NSNMC, and the Levi-Civita connection in the tangent bundle. We find that every $(2n + 1)$ -dimensional Kenmotsu manifold associated with the lifts of the NSNMC is regular with respect to NSNMC in the tangent bundle and, if the lift of the conharmonic curvature tensor associated with NSNMC is equal to that of the Levi-Civita connection, then the lift of the conformal curvature tensor associated with NSNMC coincides with the Levi-Civita connection in the tangent bundle. We also provide the necessary and sufficient condition where the lift of the concircular curvature tensor coincides with the lift of the curvature tensor in the tangent bundle.

Lastly, the lifts of the Ricci semi-symmetric Kenmotsu manifold in the tangent bundle are investigated, and we observe that a Ricci semi-symmetric Kenmotsu manifold associ-

ated with the lift of an NSNMC is an Einstein manifold in the tangent bundle and that, in a Ricci semi-symmetric Kenmotsu manifold associated with the lift of an NSNMC \check{V}^c , the data of the Ricci soliton (g^c, A^c, λ_0) are expanding in the tangent bundle. An example of the lifts of Kenmotsu manifolds in the tangent bundle is also provided.

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