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Bipolar Fuzzy Multi-Criteria Decision-Making Technique Based on Probability Aggregation Operators for Selection of Optimal Artificial Intelligence Framework

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Abstract: Artificial intelligence (AI) frameworks are essential for development since they offer pre-built tools and libraries that speed up and simplify the production of AI models, leveraging symmetry to save time and effort. They guarantee effective computing by modifying code for particular hardware, facilitating quicker testing and deployment. The identification of a suitable and optimal AI framework for development is a multi-criteria decision-making (MCDM) dilemma, where the considered AI frameworks for development are evaluated by considering various criteria and these criteria may have dual aspects (positive and negative). Thus, in this manuscript, we diagnosed a technique of MCDM within the bipolar fuzzy set (BFS) for identification and selection of optimal AI framework for development. In this regard, we diagnosed probability aggregation operators (AOs) within BFS, such as probability bipolar fuzzy weighted averaging (P-BFWA), probability bipolar fuzzy ordered weighted averaging (P-BFOWA), immediate probability bipolar fuzzy ordered weighted averaging (IP-BFOWA), probability bipolar fuzzy weighted geometric (P-BFWG), probability bipolar fuzzy ordered weighted geometric (P-BFOWH), and immediate probability bipolar fuzzy ordered weighted geometric (IP-BFOWG) operators. The diagnosed technique would be based on these invented probably AOs. Afterward, in this manuscript, we took a case study and obtained the optimal AI framework for development by employing the diagnosed technique of MCDM. We also investigated the comparison of the devised theory with certain prevailing theories to reveal the dominance and significance of the devised theory.

Keywords: artificial intelligence framework for development; bipolar fuzzy set; probability averaging/geometric aggregation operators; MCDM technique

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1. Introduction

A complete collection of resources, libraries, and tools called an AI framework for development is intended to speed up the process of developing AI applications. These frameworks give programmers an organized and effective environment in which AI models are created, trained, and used. They simplify implementation details and difficult mathematical calculations so that developers may concentrate on the important parts of their projects. TensorFlow, PyTorch, Keras, and sci-kit-learn are a few well-known AI frameworks. It is impossible to exaggerate the value of AI frameworks for development. By providing pre-built components for typical activities like data pretreatment, model architecture building, and optimization algorithms, the development process is sped up. This accelerates the iterative loop of testing and improving models, making it possible for developers to create AI applications more quickly. The second benefit is that AI frameworks offer a level of uniformity that encourages cooperation and information exchange among AI practitioners. With a shared set of tools and methods, developers may more easily share concepts and fixes. This supports the replication of research and makes it possible to regularly apply innovative strategies and evaluate them against current ways.

Thirdly, these frameworks make it easier to introduce AI models into real-world settings. They provide connections with deployment systems, enabling developers to effortlessly go from testing phases to practical applications. Understanding the utility of AI in real-world applications, such as self-driving cars, medical diagnosis, and applications for natural language processing, is crucial. As a last point, AI frameworks encapsulate the inner workings of hardware optimization. Frameworks may automatically use hardware acceleration, such as GPUs and TPUs, without developers having to go into hardware-specific specifics, as AI computations frequently need significant processing capacity. The effectiveness and performance of AI applications are considerably improved by this easy access to specialized hardware.

A decision-making (DM) process known as MCDM takes into account several criteria or considerations while assessing and choosing options. It weighs and ranks these criteria to assist decision analysts in making well-informed choices. Structured procedures for managing complicated decisions are offered by MCDM techniques. MCDM is a popular tool for handling decisions involving several, frequently at odds with one another, domains such as business, engineering, and environmental planning. It aids in maximizing the results of decisions that are in line with the intended objectives and goals. More, there are various situations where the information contains uncertainty and ambiguity, which cannot be tackled by traditional set theory. The foundation of traditional set theory is the idea of sharp borders, where a member either fully belongs to a set or does not. However, many ideas in the actual world are difficult to classify as wholly in or totally out of a set. Fuzzy set (FS) diagnosed by Zadeh [1] in 1965, is more effective at simulating these circumstances. There is not a distinct line that divides tall from not tall or elderly from not old when expressing notions like “tall” or “old”, for instance. Decision-making (DM) procedures frequently use the notion of FS, particularly when dealing with complicated and ambiguous data. It offers a structure for including expert and subjective assessments in DM models. On the other hand, bipolar fuzzy set (BFS) diagnosed by Zhang [2] in 1994, enables the representation of both positive and negative aspects of the element or object. This is especially helpful when dealing with circumstances in which elements might display both desirable and negative features, or when making decisions that entail weighing both benefits and drawbacks at once. BFS can better capture this dual feeling in sentiment analysis, where an opinion may be both somewhat positive and slightly negative.

1.1. Literature Review

Wang et al. [3] investigated the development of an AI framework for photo identification. Jiang et al. [4] devised an AI framework for data-driven prediction. Yang et al. [5] diagnosed AI frameworks, applications, and case studies. Bennett and Hauser [6] devised an AI framework for simulating clinical DM. John et al. [7] AI framework for business development. The human movement recognition in the AI framework was deduced by Gupta et al. [8]. Khan et al. [9] devised the AI framework for smart cities for discussing challenges and opportunities. Haener et al. [10] investigated the research agenda, structure, and review in the setting of AI and innovation management. In the field of brain tumor segmentation, the AI framework was devised by Das et al. [11]. For online transient stability evaluation of power systems, the AI framework was originated by Wehenkel et al. [12]. Soenksen et al. [13] investigated the AI framework in the field of healthcare. Ghillani [14] utilized the AI framework to expand cyber security. For detecting crime patterns, Raja et al. [15] employed an AI framework. Parekh et al. [16] investigated the AI framework for detecting fatigue. Through the AI technique, the selection of feature extraction was deduced by Cateni et al. [17].

Zhao et al. [18] devised a multi-criteria mission aborting policy for systems subjected to a two-stage degradation procedure. Aruldoss et al. [19] devised various approaches to MCDM. Shao et al. [20] investigated the applications of MCDM for the selection of sites for renewable energy. For sustainable energy, Wang et al. [21] deduced the MCDM approach. Abdullah [22] devised various applications of the fuzzy MCDM approach. Kaya et al. [23] deduced various fuzzy MCDM techniques. Yalcin et al. [24] assessed the financial performance by employing a fuzzy MCDM approach. Maiers and Sherif [25] devised numerous applications of the concept of FS. Roberts [26] deduced ordination relying on the FS. The association among some extensions of the concept of FS was originated by Deschriiver and Kerre [27]. Yager and Filey [28] investigated the dilemma of defuzzification and selection by employing FS. Dubois and Prade [29] originated fuzzy aggregation connectives. Dubois and Prade [30] analyzed FS in AI and Garibaldi [31] discussed the requirement of fuzzy AI. Pedrycz [32] devised the FS framework for development. Kandel and Schneider [33] devised FS and its application in AI. More, Yager [34] deduced AI and fuzzy logic, and Negoita and Ralescu [35] devised AI and fuzzy systems. Akram et al. [36] devised an approach of TOPSIS for bipolar FS. Alghamdi et al. [37] discussed a procedure of MCDM for bipolar. FS and Jana [38] discussed a technique of MABAC for bipolar FS. Further, The SWARA-MABAC approach and MULTIMOORA approach for bipolar FS were investigated by Liu et al. [39] and Stanujkic et al. [40], respectively. Shumaiza et al. [41] devised the ELECTRE II approach for bipolar FS. The bipolar fuzzy graphs and their applications were discussed by Akram [42,43].

1.2. Motivation

By aggregating and translating different criterion values or preferences into an overall evaluation or ranking of options, aggregation operators (AOs) play a significant role in MCDM. To help decision analysts make well-informed decisions, these operators synthesize complicated and varied information from many criteria while taking into consideration their interdependencies and relative relevance. The outcome of MCDM is substantially influenced by the choice of a suitable AO, which also affects the robustness and dependability of the decision-making process. Because of the significance of the AOs in MCDM, various scholars diagnosed various AOs in the setting of BFS, such as Jana et al. [44] devised Dombi, Wei et al. [45] devised Hamacher, Riaz et al. [46] diagnosed sine trigonometric, and Jana et al. [47] interpreted logarithmic AOs within bipolar FS. There are various genuine-life dilemmas within BFS, where probabilistic information is required but all abovementioned AOs cannot consider the probabilistic information while aggregating the bipolar fuzzy information. Thus, in this manuscript, we investigated the averaging and geometric AOs within BFS that consider the probabilistic information. These AOs are P-BFWA, P-BFOWA, IP-BFOWA, P-BFWG, P-BFOWG, and IP-BFOWG. Further, the identification of a suitable and optimal AI framework for development is an MCDM dilemma, where the considered AI frameworks for development are evaluated by considering various criteria such as performance and scalability, ease of integration and adoption, community and support, and feature set and flexibility, and these criteria may have dual aspects (positive and negative). To handle this dilemma, no technique can cope with the negative and positive aspects and the probabilistic information of the dilemma. Thus, in this manuscript, we also invented an MCDM technique for tackling such real-world problems. Moreover, we discussed a case study "Prioritization and Selection of AI framework for development".

The rest of the manuscript is developed as follows: In Section 2, we recall the notion of bipolar FS and related properties. In Section 3, we investigate probable averaging and geometric AOs that are P-BFWA, P-BFOWA, IP-BFOWA, P-BFWG, P-BFOWG, and IP-BFOWG operators. We also diagnose the linked properties of the invented operators in Section 3. In Section 4 of this manuscript, we demonstrate a technique of MCDM by employing the invented operator and then tackle a decision-making problem "Selection of optimal AI framework for development" by employing the invented technique of MCDM. In Section 5, we compare the devised theory with current work, and in Section 6, we demonstrate the conclusion.

2. Preliminaries

In this segment of the article, we recall the notion of bipolar FS and related properties.

Definition 1 ([2]). *The model of BFS is expressed below*

$$\mathfrak{B}_{BFS} = \left\{ \left(\text{y}, \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}}(\text{y}),\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}(\text{y}) \right) \,\middle|\, \text{y} \in \grave{\text{Y}} \right\} \tag{1}$$

Note that $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}}(\text{y})$ *is utilized as a positive truth degree that is placed in* $[0,\ 1]$ *and* $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}(\text{y})$ *is utilized as a negative truth degree that is placed in* $[-1,\ 0]$*. The bipolar fuzzy number (BFN) will be revealed as* $\mathfrak{B}_{BFS} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}} \right)$.

Definition 2 ([48]). *Let two BFNs be* $\mathfrak{B}_{BFS-1} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} \right)$ *and* $\mathfrak{B}_{BFS-2} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}} \right)$*, and* $\eth \geq 0$*. Then*

$$\mathfrak{B}_{BFS-1} \oplus \mathfrak{B}_{BFS-2} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}} + \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}} - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}} \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}, -\left(\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}} \right) \right) \tag{2}$$

$$\mathfrak{B}_{BFS-1} \otimes \mathfrak{B}_{BFS-2} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}} \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}} + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}} \right) \tag{3}$$

$$\eth \mathfrak{B}_{BFS-1} = \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}} \right)^{\eth}, -\left(\left| \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} \right|^{\eth} \right) \right) \tag{4}$$

$$\mathfrak{B}^{\eth}_{BFS-1} = \left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}} \right)^{\eth}, -1 + \left(1 + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}} \right)^{\eth} \right) \tag{5}$$

Definition 3 ([45]). *The score value of a BFN* $\mathfrak{B}_{BFS} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}} \right)$ *would be found as*

$$\dot{\text{S}}(\mathfrak{B}_{BFS}) = \frac{1}{2}\left(1 + \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}} + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}} \right) \quad \dot{\text{S}}_{\mathcal{B}}(\mathfrak{B}_{BFS}) \in [0,\ 1] \tag{6}$$

and the accuracy value of a BFN $\mathfrak{B}_{BFS} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}} \right)$ *would be found as*

$$\ddot{\text{H}}(\mathfrak{B}_{BFS}) = \frac{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}} - \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}}{2},\ \ \ddot{\text{H}}(\mathfrak{B}_{BFS}) \in [0,\ 1] \tag{7}$$

Utilizing Equations (6) and (7), we obtain

1. *If* $\dot{\text{S}}(\mathfrak{B}_{BFS-1}) < \dot{\text{S}}(\mathfrak{B}_{BFS-2})$ *then* $\mathfrak{B}_{BFS-1} < \mathfrak{B}_{BFS-2}$
2. *If* $\dot{\text{S}}(\mathfrak{B}_{BFS-1}) > \dot{\text{S}}(\mathfrak{B}_{BFS-2})$ *then* $\mathfrak{B}_{BFS-1} > \mathfrak{B}_{BFS-2}$
3. *If* $\dot{\text{S}}(\mathfrak{B}_{BFS-1}) = \dot{\text{S}}(\mathfrak{B}_{BFS-2})$ *then we have*
    i. *If* $\ddot{\text{H}}(\mathfrak{B}_{BFS-1}) < \ddot{\text{H}}(\mathfrak{B}_{BFS-2})$ *then* $\mathfrak{B}_{BFS-1} < \mathfrak{B}_{BFS-2}$
    ii. *If* $\ddot{\text{H}}(\mathfrak{B}_{BFS-1}) > \ddot{\text{H}}(\mathfrak{B}_{BFS-2})$ *then* $\mathfrak{B}_{BFS-1} > \mathfrak{B}_{BFS-2}$
    iii. *If* $\ddot{\text{H}}(\mathfrak{B}_{BFS-1}) = \ddot{\text{H}}(\mathfrak{B}_{BFS-2})$ *then* $\mathfrak{B}_{BFS-1} = \mathfrak{B}_{BFS-2}$

3. Main Results (Probabilistic AOs for BFNs)

This part of the article contains the averaging and geometric AOs, such as P-BFWA, P-BFOWA, IP-BFOWA, P-BFWG, P-BFOWG, and IP-BFOWG operators.

Definition 4. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\ddot{x}}) = \underset{\tilde{\sigma}=1}{\overset{\ddot{x}}{\oplus}} \Theta_{\mathbb{PW}\sim-\tilde{\sigma}'}\mathfrak{B}_{BFS-\tilde{\sigma}'} \tag{8}$$

demonstrates a P-BFWA operator. Notice that $\Theta_{\mathbb{PW}-\tilde{\sigma}} = \mathfrak{f}\mathfrak{B}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}$ *is a weight vector that fuses the weight* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \leq \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \leq 1$, $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *and* $\mathfrak{f} \in [0, 1]$.

Theorem 1. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$. *Then the aggregated result of this class obtained by employing the P-BFWA operator is a BFN and*

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(1 - \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\ddot{x}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\ddot{x}}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \tag{9}$$

Proof. We will employ mathematical induction to prove this theorem. Consider $\ddot{x} = 2$, then using Equation (9), we obtain

$$\Theta_{\mathbb{PW}-1}\mathfrak{B}_{BFS-1} = \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}}\right)^{\Theta_{\mathbb{PW}-1}}, -\left(\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}}\right|^{\Theta_{\mathbb{PW}-1}}\right)\right)$$

$$\Theta_{\mathbb{PW}-2}\mathfrak{B}_{BFS-2} = \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}, -\left(\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}}\right|^{\Theta_{\mathbb{PW}-2}}\right)\right)$$

then,

$$\begin{gathered} P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}) = \Theta_{\mathbb{PW}-1}\mathfrak{B}_{BFS-1} \oplus \Theta_{\mathbb{PW}-2}\mathfrak{B}_{BFS-2} \\ = \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}, -\left(\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}}\right|^{\Theta_{\mathbb{PW}-2}}\right)\right) \oplus \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}, -\left(\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}}\right|^{\Theta_{\mathbb{PW}-2}}\right)\right) \\ = \left(1 - \prod_{\tilde{\sigma}=1}^{2}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{2}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \end{gathered}$$

$\Rightarrow$ Equation (9) holds for $\ddot{x} = 2$. Now, consider the Equation (9) holds for $\ddot{x} = \dot{E}$, then

$$P - BFWA\left(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\dot{E}}\right) = \left(1 - \prod_{\tilde{\sigma}=1}^{\dot{E}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\dot{E}}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)$$

Below, we will demonstrate that the Equation (9) holds for $\ddot{x} = \dot{E} + 1$, hence

$$\begin{gathered} P - BFWA\left(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\dot{E}+1}\right) = P - BFWA\left(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\dot{E}}\right) \oplus \mathfrak{B}_{BFS-\dot{E}+1} \\ = \left(1 - \prod_{\tilde{\sigma}=1}^{\dot{E}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\dot{E}}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \oplus \left(1 - \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\dot{E}+1}}\right)^{\Theta_{\mathbb{PW}-\dot{E}+1}}, -\left(\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\dot{E}+1}}\right|^{\Theta_{\mathbb{PW}-\dot{E}+1}}\right)\right) \\ = \left(1 - \prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \\ = P - BFWA\left(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\dot{E}}, \mathfrak{B}_{BFS-\dot{E}+1}\right) \end{gathered}$$

This reveals that the Equation (9) holds for $\ddot{x} = \dot{E} + 1$ and hence, holds $\forall\, \ddot{x}$. □

Properties: the P-BFWA operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \mathfrak{B}_{BFS-\tilde{\sigma}}$$

Proof. As we have that

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(1 - \prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)$$

and $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$, then we obtain

$$\begin{aligned} P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) &= \left(1 - \prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\dot{E}+1}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \\ &= \left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}}\right)^{\sum_{\tilde{\sigma}=1}^{\ddot{x}}\Theta_{\mathbb{PW}-\tilde{\sigma}}}, \left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}\right|^{\sum_{\tilde{\sigma}=1}^{\ddot{x}}\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS}}\right) \end{aligned}$$

Notice that $\sum_{\tilde{\sigma}=1}^{\ddot{x}}\Theta_{\mathbb{PW}-\tilde{\sigma}} = \sum_{\tilde{\sigma}=1}^{\ddot{x}}\left(\mathfrak{f}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}\right) = \mathfrak{f}\sum_{\tilde{\sigma}=1}^{\ddot{x}}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\sum_{\tilde{\sigma}=1}^{\ddot{x}}\mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = \mathfrak{f} + 1 - \mathfrak{f} = 1$. □

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq P - BFWA\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

Proof. We have

$$\begin{aligned} &\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}} \\ &\Rightarrow 1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \geq 1 - \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}} \\ &\Rightarrow \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \geq \left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \\ &\Rightarrow \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \geq \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \\ &\Rightarrow 1 - \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \leq 1 - \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \end{aligned}$$

and

$$\begin{aligned} &\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}} \\ &\Rightarrow \left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right| \geq \left|\eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right| \\ &\Rightarrow \left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \geq \left|\eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \\ &\Rightarrow \prod_{\tilde{\sigma}=1}^{\ddot{x}} \left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \geq \prod_{\tilde{\sigma}=1}^{\ddot{x}} \left|\eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \\ &\Rightarrow -\prod_{\tilde{\sigma}=1}^{\ddot{x}} \left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \leq -\prod_{\tilde{\sigma}=1}^{\ddot{x}} \left|\eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \end{aligned}.$$

Thus,

$$P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq P - BFWA\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right).$$

□

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}^{-}_{BFS} = \left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS} = \left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then

$$\mathfrak{B}^{-}_{BFS} \leq P - BFWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq \mathfrak{B}^{+}_{BFS}$$

Proof. This proof can be obtained by employing idempotency and monotonicity. □

Definition 5. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\ddot{x}}) = \bigoplus_{\tilde{\sigma}=1}^{\ddot{x}} \Theta_{\mathbb{PW}-\tilde{\sigma}} \mathfrak{B}_{BFS-\Pi(\tilde{\sigma})} \tag{10}$$

demonstrates a P-BFOWA operator. Notice that $\Theta_{\mathbb{PW}-\tilde{\sigma}} = \mathfrak{f}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}$ *is a weight vector that fuses the weight* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \leq \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \leq 1$, $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *and* $\mathfrak{f} \in [0, 1]$. $(\Pi(1), \Pi(2), \ldots, \Pi(\ddot{x}))$ *is a permutation of* $(1, 2, \ldots, \ddot{x})$*, such that* $\Pi(\tilde{\sigma}-1) \geq \Pi(\tilde{\sigma})$, for $\tilde{\sigma} = 2, 3, \ldots, \ddot{x}$.

Theorem 2. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$*. Then the aggregated result of this class obtained by employing the P-BFOWA operator is a BFN and*

$$P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(1 - \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\ddot{x}}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right|^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \tag{11}$$

Proof. Similar to the Theorem 1. □

Properties: the P-BFOWA operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then

$$P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \mathfrak{B}_{BFS-\tilde{\sigma}}$$

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then

$$P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq P - BFOWA\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}^{-}_{BFS} = \left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS} = \left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then

$$\mathfrak{B}^{-}_{BFS} \leq P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq \mathfrak{B}^{+}_{BFS}$$

Definition 6. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, *and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$IP - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\ddot{x}}) = \bigoplus_{\tilde{\sigma}=1}^{\ddot{x}} \mathcal{M}_{\mathbb{IP}-\tilde{\sigma}} \mathfrak{B}_{BFS-\Pi(\tilde{\sigma})} \tag{12}$$

demonstrates an IP-BFOWA operator. Notice that $\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}} = \frac{\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}}}{\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}}}$ *is an IP given to BFN, where* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \leq \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \leq 1$, $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *are associated weight and probability, respectively.* $(\Pi(1), \Pi(2), \ldots, \Pi(\ddot{x}))$ *is a permutation of* $(1, 2, \ldots, \ddot{x})$, *such that* $\Pi(\tilde{\sigma}-1) \geq \Pi(\tilde{\sigma})$, *for* $\tilde{\sigma} = 2, 3, \ldots, \ddot{x}$.

Theorem 3. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, *and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$. *Then the aggregated result of this class obtained by employing the P-BFOWA operator is a BFN and*

$$P - BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(1 - \prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1 - \eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}}}, -\prod_{\tilde{\sigma}=1}^{\ddot{x}}\left|\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right|^{\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}}}\right) \tag{13}$$

Proof. Similar to the Theorem 1. □

Properties: the IP-BFOWA operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then

$$IP-BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \mathfrak{B}_{BFS-\tilde{\sigma}}$$

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then

$$IP-BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq IP-BFOWA\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}^{-}_{BFS} = \left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS} = \left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then

$$\mathfrak{B}^{-}_{BFS} \leq IP-BFOWA(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq \mathfrak{B}^{+}_{BFS}$$

Definition 7. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$P-BFWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\ddot{x}}) = \underset{\tilde{\sigma}=1}{\overset{\ddot{x}}{\otimes}} \left(\mathfrak{B}_{BFS-\tilde{\sigma}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \tag{14}$$

demonstrates a P-BFWG operator. Notice that $\Theta_{\mathbb{PW}-\tilde{\sigma}} = \mathfrak{f}\mathfrak{B}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}$ *is a weight vector that fuses the weight* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \leq \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \leq 1, \sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *and* $\mathfrak{f} \in [0, 1]$.

Theorem 4. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$. *Then, the aggregated result of this class obtained by employing the P-BFWG operator is a BFN and*

$$P-BFWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(\prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -1+\prod_{\tilde{\sigma}=1}^{\ddot{x}}\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \tag{15}$$

Proof. We will employ mathematical induction to prove this theorem. Consider $\ddot{x} = 2$, then using Equation (15), we obtain

$$(\mathfrak{B}_{BFS-1})^{\Theta_{\mathbb{PW}-1}} = \left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}}\right)^{\Theta_{\mathbb{PW}-1}}, -1+\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}}\right)^{\Theta_{\mathbb{PW}-1}}\right)$$

$$(\mathfrak{B}_{BFS-2})^{\Theta_{\mathbb{PW}-2}} = \left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}, -1+\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}\right)$$

then,

$$P-BFWG(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2})=(\mathfrak{B}_{BFS-1})^{\Theta_{\mathbb{PW}-1}}\otimes(\mathfrak{B}_{BFS-2})^{\Theta_{\mathbb{PW}-2}}$$
$$=\left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-1}}\right)^{\Theta_{\mathbb{PW}-1}},-1+\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-1}}\right)^{\Theta_{\mathbb{PW}-1}}\right)\otimes\left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}},-1+\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-2}}\right)^{\Theta_{\mathbb{PW}-2}}\right)$$
$$=\left(\prod_{\tilde{\sigma}=1}^{2}\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}},-1+\prod_{\tilde{\sigma}=1}^{2}\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)$$

$\Rightarrow$ Equation (15) holds for $\ddot{x}=2$. Now, consider the Equation (15) holds for $\ddot{x}=\dot{\mathrm{E}}$, then

$$P-BFWG\left(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\dot{\mathrm{E}}}\right)=\left(\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}}\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}},-1+\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}}\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)$$

Below, we will demonstrate that the Equation (15) holds for $\ddot{x}=\dot{\mathrm{E}}+1$, hence

$$P-BFWG\left(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\dot{\mathrm{E}}+1}\right)=B-BFWG\left(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\dot{\mathrm{E}}}\right)\otimes\mathfrak{B}_{BFS-\dot{\mathrm{E}}+1}$$
$$=\left(\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}}\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}},-1+\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}}\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)\otimes\left(\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\dot{\mathrm{E}}+1}}\right)^{\Theta_{\mathbb{PW}-\dot{\mathrm{E}}+1}},-1+\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\dot{\mathrm{E}}+1}}\right)^{\Theta_{\mathbb{PW}-\dot{\mathrm{E}}+1}}\right)$$
$$=\left(\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}+1}\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}},-1+\prod_{\tilde{\sigma}=1}^{\dot{\mathrm{E}}+1}\left(1+\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right)$$
$$=P-BFWG\left(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\dot{\mathrm{E}}},\ \mathfrak{B}_{BFS-\dot{\mathrm{E}}+1}\right)$$

This reveals that the Equation (15) holds for $\ddot{x}=\dot{\mathrm{E}}+1$ and hence, holds $\forall\ \ddot{x}$. □

Properties: the P-BFWG operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}}=\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma}=1,\ 2,\ \ldots,\ \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}}=\mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then
$$P-BFWG(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\ddot{x}})=\mathfrak{B}_{BFS-\tilde{\sigma}}$$

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}}=\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}=\left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}},\ \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma}=1,\ 2,\ \ldots,\ \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\leq\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\leq\eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then
$$P-BFWG(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\ddot{x}})\leq P-BFWG\left(\mathfrak{B}^{\#}_{BFS-1},\ \mathfrak{B}^{\#}_{BFS-2},\ \ldots\mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}}=\left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}},\ \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma}=1,\ 2,\ \ldots,\ \ddot{x}$, if $\mathfrak{B}^{-}_{BFS}=\left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\},\ \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS}=\left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\},\ \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then
$$\mathfrak{B}^{-}_{BFS}\leq P-BFWG(\mathfrak{B}_{BFS-1},\ \mathfrak{B}_{BFS-2},\ \ldots\mathfrak{B}_{BFS-\ddot{x}})\leq\mathfrak{B}^{+}_{BFS}$$

Definition 8. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$P - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\ddot{x}}) = \underset{\tilde{\sigma}=1}{\overset{\ddot{x}}{\otimes}} \left(\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}} \tag{16}$$

demonstrates a P-BFOWA operator. Notice that $\Theta_{\mathbb{PW}-\tilde{\sigma}} = \mathfrak{f}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}$ *is a weight vector that fuses the weight* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \leq \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \leq 1, \sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *and* $\mathfrak{f} \in [0, 1]$. $(\Pi(1), \Pi(2), \ldots, \Pi(\ddot{x}))$ *is a permutation of* $(1, 2, \ldots, \ddot{x})$*, such that* $\Pi(\tilde{\sigma}-1) \geq \Pi(\tilde{\sigma})$, for $\tilde{\sigma} = 2, 3, \ldots, \ddot{x}$.

Theorem 5. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$*. Then, the aggregated result of this class obtained by employing the P-BFWG operator is a BFN and*

$$P - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(\prod_{\tilde{\sigma}=1}^{\ddot{x}} \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}, -1 + \prod_{\tilde{\sigma}=1}^{\ddot{x}} \left(1 + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\Theta_{\mathbb{PW}-\tilde{\sigma}}}\right) \tag{17}$$

Proof. Similar to Theorem 4. □

Properties: the P-BFOWG operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then

$$P - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \mathfrak{B}_{BFS-\tilde{\sigma}}$$

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \leq \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then

$$P - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq P - BFOWG\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}^{-}_{BFS} = \left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS} = \left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then

$$\mathfrak{B}^{-}_{BFS} \leq P - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \leq \mathfrak{B}^{+}_{BFS}$$

Definition 9. *Under the incidence of a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$,

$$IP - BFOWG\left(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots, \mathfrak{B}_{BFS-\check{\Upsilon}}\right) = \underset{\tilde{\sigma}=1}{\overset{\ddot{x}}{\otimes}} \left(\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}\right)^{\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}}} \tag{18}$$

Demonstrates an IP-BFOWG operator. Notice that $\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}} = \frac{\mathbb{W}_{\mathbb{V}-\tilde{\sigma}}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}}}{\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}}\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}}}$ *is an IP given to BFN, where* $\mathbb{W}_{\mathbb{V}} = (\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\ddot{x}})$ *with* $0 \le \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} \le 1$, $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{W}_{\mathbb{V}-\tilde{\sigma}} = 1$ *and probabilities* $\mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} > 0$ *with* $\sum_{\tilde{\sigma}=1}^{\ddot{x}} \mathbb{E}_{\mathbb{PB}-\tilde{\sigma}} = 1$ *are associated weight and probability, respectively.* $(\Pi(1), \Pi(2), \ldots, \Pi(\ddot{x}))$ *is a permutation of* $(1, 2, \ldots, \ddot{x})$*, such that* $\Pi(\tilde{\sigma} - 1) \ge \Pi(\tilde{\sigma})$*, for* $\tilde{\sigma} = 2, 3, \ldots, \ddot{x}$.

Theorem 6. *Let a class of BFNs* $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$*, and* $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$*. Then the aggregated result of this class obtained by employing the IP-BFWG operator is a BFN and*

$$IP - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \left(\prod_{\tilde{\sigma}=1}^{\ddot{x}} \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}}}, -1 + \prod_{\tilde{\sigma}=1}^{\ddot{x}} \left(1 + \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\Pi(\tilde{\sigma})}}\right)^{\mathcal{M}_{\mathbb{IP}-\tilde{\sigma}}}\right) \tag{19}$$

Proof. Similar to the Theorem 4. □

Properties: the IP-BFOWG operator has the below properties.

1. Idempotency: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}_{BFS-\tilde{\sigma}} = \mathfrak{B}_{BFS}$ for all $\tilde{\sigma}$, then

$$IP - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) = \mathfrak{B}_{BFS-\tilde{\sigma}}$$

2. Monotonicity: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$ and $\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \le \eta^{\mathcal{P}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}} \le \eta^{\mathcal{N}}_{\mathfrak{B}^{\#}_{BFS-\tilde{\sigma}}}$, then

$$IP - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \le IP - BFOWG\left(\mathfrak{B}^{\#}_{BFS-1}, \mathfrak{B}^{\#}_{BFS-2}, \ldots \mathfrak{B}^{\#}_{BFS-\ddot{x}}\right)$$

3. Boundedness: under the incidence of a class of BFNs $\mathfrak{B}_{BFS-\tilde{\sigma}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right)$, and $\tilde{\sigma} = 1, 2, \ldots, \ddot{x}$, if $\mathfrak{B}^{-}_{BFS} = \left(\min_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \max_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$ and $\mathfrak{B}^{+}_{BFS} = \left(\max_{\tilde{\sigma}}\left\{\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}, \min_{\tilde{\sigma}}\left\{\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\sigma}}}\right\}\right)$, then

$$\mathfrak{B}^{-}_{BFS} \le IP - BFOWG(\mathfrak{B}_{BFS-1}, \mathfrak{B}_{BFS-2}, \ldots \mathfrak{B}_{BFS-\ddot{x}}) \le \mathfrak{B}^{+}_{BFS}$$

4. MCDM Technique under BFNs

Let us suppose an MCDM dilemma where $\ddot{x}$ alternatives $\mathfrak{W}_{at} = \{\mathfrak{W}_{at-1}, \mathfrak{W}_{at-2}, \ldots, \mathfrak{W}_{at-\ddot{x}}\}$ are under consideration and these alternatives would be assessed by considering $\underline{z}$ criteria $\mathfrak{Y}_{ct} = \{\mathfrak{Y}_{ct-1}, \mathfrak{Y}_{ct-2}, \ldots, \mathfrak{Y}_{ct-\underline{z}}\}$. Since each criterion has its significance, the decision analyst would give weight $\mathbb{W}_{\mathbb{V}} = \left(\mathbb{W}_{\mathbb{V}-1}, \mathbb{W}_{\mathbb{V}-2}, \ldots, \mathbb{W}_{\mathbb{V}-\underline{z}}\right)$ with $0 \leq \mathbb{W}_{\mathbb{V}-\mathfrak{k}} \leq 1, \sum_{\mathfrak{k}=1}^{z} \mathbb{W}_{\mathbb{V}-\mathfrak{k}} = 1$ to each criterion by his/her preference. After assessing the alternatives by keeping in mind the criteria, the decision analyst would interpret the evaluation values within BFNs $\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}} = \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}\right)$, where $\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}} \in [0, 1]$ and $\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}} \in [-1, 0]$ form a bipolar fuzzy decision matrix $\mathcal{D}_{BFDM}$. To tackle this, we demonstrate the underlying stages.

Stage 1: There is a possibility in the problems, that the criteria can be of cost or benefit type. If any criteria are of the cost type, the bipolar fuzzy decision matrix needs to be normalized as follows:

$$(\mathcal{D}_{BFDM})^{\mathbb{N}} = \begin{cases} \left(\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}, \eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}\right) & for\ benefit\ type \\ \left(1-\eta^{\mathcal{P}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}, -1-\eta^{\mathcal{N}}_{\mathfrak{B}_{BFS-\tilde{\mathfrak{O}}\mathfrak{k}}}\right) & for\ cost\ type \end{cases}$$

Stage 2: By employing the investigated weight and probabilistic information, determine $\Theta_{\mathbb{PW}-\underset{\cdot}{\mathrm{V}}} = \mathfrak{f}\mathfrak{B}_{\mathbb{PB}-\tilde{\mathfrak{O}}} + (1-\mathfrak{f})\mathbb{W}_{\mathbb{V}-\underset{\cdot}{\mathrm{V}}}$ and $\mathcal{M}_{\mathbb{IP}-\underset{\cdot}{\mathrm{V}}} = \frac{\mathbb{W}_{\mathbb{V}-\underset{\cdot}{\mathrm{V}}}\mathfrak{B}_{\mathbb{PB}-\underset{\cdot}{\mathrm{V}}}}{\sum_{\underset{\cdot}{\mathrm{V}}=1}^{\dot{z}} \mathbb{W}_{\mathbb{V}-\underset{\cdot}{\mathrm{V}}}\mathfrak{B}_{\mathbb{PB}-\underset{\cdot}{\mathrm{V}}}}$.

Stage 3: To achieve the aggregated values of alternatives, employ any of the invented operators that is P-BFWA, P-BFOWA, IP-BFOWA, P-BFWA, P-BFOWA, and IP-BFOWA.

Stage 4: The score value of each alternative will be demonstrated by employing the score function. In any case, similar score values for various alternatives demonstrate accuracy values.

Stage 5: Rank the interpreted alternatives with the assistance of score or accuracy values and obtain the optimal alternative.

4.1. Case Study

The choice of the optimal AI framework for a technology company's particular requirements is currently a difficult issue when starting a new project that calls for the creation of an AI-based application. With so many different AI frameworks available, each with unique advantages and disadvantages, the company wants to use an MCDM approach to rank and objectively assess these alternatives. This is a crucial decision since the final AI framework selection will have a big impact on the project's success and efficacy as a whole. The AI frameworks considered by a technology company are $\mathfrak{W}_{a\mathfrak{t}-1}$ (TensorFlow), $\mathfrak{W}_{a\mathfrak{t}-2}$(PyTorch), $\mathfrak{W}_{a\mathfrak{t}-3}$(Scikit-learn with Keras), and $\mathfrak{W}_{a\mathfrak{t}-4}$ (Microsoft Cognitive Toolkit), and the assessment criteria are as follows:

$\mathfrak{Y}_{\mathfrak{ct}-1}$: **Performance and Scalability:** This criterion evaluates the AI framework's capacity to deal with huge datasets, and intricate models, and provide fast processing. The chosen framework must be able to scale effectively as our application's user base and data requirements increase.

$\mathfrak{Y}_{\mathfrak{ct}-2}$: **Ease of Integration and Adoption:** The chosen framework should work with our current technology stack without any issues and offer our development team a relatively easy learning curve. The framework must be simple to integrate into our workflow to reduce disturbance and speed up development.

$\mathfrak{Y}_{\mathfrak{ct}-3}$: **Community and Support:** Faster issue resolution, resource access, and a wealth of shared knowledge are all benefits of a strong community and active developer support. The framework's popularity and capacity to change with emerging AI developments are seen in the community's involvement.

$\mathfrak{Y}_{\mathfrak{ct}-4}$: **Feature Set and Flexibility:** Pre-built models, optimization strategies, and particular neural network architectures are just a few of the features and capabilities that different frameworks offer. The framework we choose should be compatible with the particular demands of our project and provide flexibility for customization and extension as required.

The decision analyst of a technology company will evaluate the considered AI frameworks for development and construct a bipolar fuzzy decision matrix that is demonstrated in Table 1.

Table 1. The assessment arguments of the considered AI frameworks (hypothetical data).

	$\mathfrak{Y}_{ct-1}$	$\mathfrak{Y}_{ct-2}$	$\mathfrak{Y}_{ct-3}$	$\mathfrak{Y}_{ct-4}$
$\mathfrak{W}_{at-1}$	(0.843, −0.346)	(0.775, −0.866)	(0.765, −0.474)	(0.676, −0.546)
$\mathfrak{W}_{at-2}$	(0.245, −0.346)	(0.533, −0.336)	(0.356, −0.584)	(0.216 , −0.357)
$\mathfrak{W}_{at-3}$	(0.667, −0.674)	(0.475, −0.13)	(0.555, −0.356)	(0.468, −0.573)
$\mathfrak{W}_{at-4}$	(0.683, −0.34)	(0.778, −0.453)	(0.252, −0.675)	(0.347, −0.454)

To tackle this, the below stage will be followed.

Stage 1: The considered criteria in this case study are of the benefits type, so there is no need for normalization.

Stage 2: We obtain

$$\Theta_{\mathbb{PW}-1} = 0.37,\ \Theta_{\mathbb{PW}-2} = 0.285, \Theta_{\mathbb{PW}-3} = 0.2, \Theta_{\mathbb{PW}-4} = 0.145$$

and

$$\mathcal{M}_{\mathbb{IP}-1} = 0.45615,\ \mathcal{M}_{\mathbb{IP}-2} = 0.288462,\ \mathcal{M}_{\mathbb{IP}-3} = 0.1538,\ \mathcal{M}_{\mathbb{IP}-4} = 0.096154$$

where, $(0.3,\ 0.25,\ 0.2,\ 0.25)$ is the weight, interpreted by the decision expert, $(0.4,\ 0.3,\ 0.2,\ 0.1)$ is the probabilistic information, and $\mathfrak{f} = 0.7$.

Stage 3: The aggregated value of each AI framework for the development is demonstrated in Table 2.

Table 2. The aggregated values of the considered AI frameworks.

Operators	$\mathfrak{W}_{at-1}$	$\mathfrak{W}_{at-2}$	$\mathfrak{W}_{at-3}$	$\mathfrak{W}_{at-4}$
P-BFWA	(0.791, −0.511)	(0.359, −0.383)	(0.57, −0.362)	(0.622, −0.441)
P-BFOWA	(0.786, −0.474)	(0.378, −0.372)	(0.542, −0.299)	(0.625, −0.432)
IP-BFOWA	(0.796, −0.444)	(0.401, −0.361)	(0.533, −0.258)	(0.653, −0.412)
P-BFWG	(0.782, −0.622)	(0.324, −0.401)	(0.554, −0.486)	(0.526, −0.472)
P-BFOWG	(0.775, −0.546)	(0.336, −0.386)	(0.53, −0.408)	(0.536, −0.456)
IP-BFOWG	(0.786, −0.501)	(0.357, −0.371)	(0.523, −0.359)	(0.581, −0.433)

Stage 4: The score value of each alternative is interpreted in Table 3.

Table 3. The score values of the considered AI frameworks.

Operators	$\dot{\check{S}}_{BFS}(\mathfrak{W}_{at-1})$	$\dot{\check{S}}_{BFS}(\mathfrak{W}_{at-2})$	$\dot{\check{S}}_{BFS}(\mathfrak{W}_{at-3})$	$\dot{\check{S}}_{BFS}(\mathfrak{W}_{at-4})$
P-BFWA	0.64	0.488	0.604	0.591
P-BFOWA	0.656	0.502	0.622	0.597
IP-BFOWA	0.676	0.52	0.637	0.62
P-BFWG	0.58	0.461	0.534	0.527
P-BFOWG	0.615	0.475	0.561	0.54
IP-BFOWG	0.642	0.493	0.582	0.574

Stage 5: The ranking of the interpreted alternatives with the assistance of score values is $\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$, after employing P-BFWA, P-BFOWA, IP-BFOWA,

P-BFWG, P-BFOWG, and IP-BFOWG operators. This demonstrates that the optimal AI framework for development is $\mathfrak{W}_{at-1}$, that is TensorFlow. Thus for this new project, the technology company would prefer the TensorFlow AI framework.

5. Comparative Analysis

To reveal the significance and dominance of the newly proposed work, a comparison with certain prevailing theories is essential. Thus, in this part, we perform the comparative analysis of the diagnosed theory with certain prevailing theories to demonstrate the dominance and significance of the proposed theory. The considered prevailing theories for comparison are explained below.

- ❖ The theory of immediate probability AOs diagnosed by Wei and Merigo [49] within intuitionistic fuzzy information.
- ❖ The theory of Dombi AOs and related techniques of multi-attribute decision-making (MADM) was diagnosed by Jana et al. [44] within bipolar fuzzy information.
- ❖ The theory of Hamacher AOs and related MADM techniques was interpreted by Wei et al. [45] under the structure of BFS.
- ❖ The notion of sine trigonometric AOs invented by Riaz et al. [46] within bipolar fuzzy information.

Now reconsider the case study discussed in Section 4.1 and try to tackle that using the invented and considered prevailing theories. The result is demonstrated in Tables 4 and 5.

Table 4. The comparison between devised and current theories.

Reference	$\dot{S}_{BFS}(\mathfrak{W}_{at-1})$	$\dot{S}_{BFS}(\mathfrak{W}_{at-2})$	$\dot{S}_{BFS}(\mathfrak{W}_{at-3})$	$\dot{S}_{BFS}(\mathfrak{W}_{at-4})$
Wei and Merigo [49]	Failed	Failed	Failed	Failed
Jana et al. [44] **(BFDWA)**	0.647	0.492	0.634	0.597
Jana et al. [44] **(BFDWG)**	0.232	0.428	0.311	0.3
Wei et al. [45] **(BFHWA)**	0.629	0.48	0.589	0.569
Wei et al. [45] **(BFHWG)**	0.436	0.51	0.456	0.466
Riaz et al. [46]	0.814	0.671	0.799	0.781
Diagnosed operator (P-BFWA)	0.64	0.488	0.604	0.591
Diagnosed operator (P-BFOWA)	0.656	0.502	0.622	0.597
Diagnosed operator (IP-BFOWA)	0.676	0.52	0.637	0.62
Diagnosed operator (P-BFWG)	0.58	0.461	0.534	0.527
Diagnosed operator (P-BFOWG)	0.615	0.475	0.561	0.54
Diagnosed operator (IP-BFOWG)	0.642	0.493	0.582	0.574

The theory of immediate probabilistic AOs for intuitionistic fuzzy information is a valid and practicable theory for coping with intuitionistic fuzzy information, that is, the information contains a truth grade and falsity grade and their sum must belong to $[0, 1]$. However, when we apply this theory to cope with information that contains a positive truth grade and negative truth grade (positive and negative aspects) instead of truth grade and falsity, then this theory does not apply to this information. Thus, in abovementioned Tables, we noticed that after employing immediate probabilistic AOs for intuitionistic fuzzy information, we did not obtain any sort of score value and ranking. Further, other considered prevailing theories solved the dilemma and provided us with the optimal AI framework for development, but as we can observe, BFDWG and BFHWG interpret that $\mathfrak{W}_{\mathfrak{d}t-2}$ is the optimal AI framework, while the rest of the prevailing and even invented theories interpret that $\mathfrak{W}_{\mathfrak{d}t-1}$ is the optimal AI framework for development. In the considered prevailing theories, the probability information is missing and there is no AO in the literature that can tackle dual aspects and consider the probability information. Thus, the invented theory is more dominant and significant.

Table 5. The ranking of the comparative analysis.

Operators	Ranking
Wei and Merigo [49]	Failed
Jana et al. [44] **(BFDWA)**	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Jana et al. [44] **(BFDWG)**	$\mathfrak{W}_{at-2} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-1}$
Wei et al. [45] **(BFHWA)**	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Wei et al. [45] **(BFHWG)**	$\mathfrak{W}_{at-2} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-1}$
Riaz et al. [46]	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (P-BFWA)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (P-BFOWA)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (IP-BFOWA)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (P-BFWG)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (P-BFOWG)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$
Diagnosed operator (IP-BFOWG)	$\mathfrak{W}_{at-1} > \mathfrak{W}_{at-3} > \mathfrak{W}_{at-4} > \mathfrak{W}_{at-2}$

6. Conclusions

The selection and identification of the optimal AI framework for development is an MCDM dilemma, where various criteria are involved. The assessment of the AI framework depends on the considered criteria and these criteria can have both positive and negative aspects. To tackle such sort of dilemmas, in this manuscript, we devised a technique of MCDM within bipolar FS, and then investigated a case study and achieved the optimal AI framework for development. However, in this problem, the probability information was also necessary, and there was no such AO within BFS that could consider the probability information. Thus, firstly, we investigated probability averaging and geometric AOs under bipolar FS that are P-BFWA, P-BFOWA, IP-BFOWA, P-BFWG, P-BFOWG, and IP-BFOWG operators. We also diagnosed the linked properties of the devised AOs. By tackling a case study, we showed the practicability of the proposed theory. To demonstrate the superiority of the diagnosed work, we compared it with current theories.

Our future aims to promote probability averaging/geometric AOs in various generalizations of FS, such as complex hesitant FS [50], bipolar complex FS [51], bipolar complex fuzzy soft set [52,53], complex intuitionistic fuzzy [54], and complex picture fuzzy [55] information and other studies like the behavioral model of rational choice [56], simple heuristic [57] (was used to predict the data of the first million cases of COVID), social heuristic [58] (uncertainty leads to the evolution of social heuristics), and decision making under more intense time pressure [59].

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