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Integral Operators Applied to Classes of Convex and Close-to-Convex Meromorphic p-Valent Functions

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Abstract: We consider a newly introduced integral operator that depends on an analytic normalized function and generalizes many other previously studied operators. We find the necessary conditions that this operator has to meet in order to preserve convex meromorphic functions. We know that convexity has great impact in the industry, linear and non-linear programming problems, and optimization. Some lemmas and remarks helping us to obtain complex functions with positive real parts are also given.

Keywords: convex meromorphic functions; close-to-convex meromorphic functions; integral operators

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1. Introduction and Preliminaries

This paper belongs to the so-called "geometric functions theory", which is perhaps, the most important field of complex analysis. This theory deals with normalized univalent functions, p-valent functions, meromorphic functions, meromorphic p-valent functions, harmonic functions, fractional regular functions, etc. The geometric function theory was first originated by Riemann in 1850. In 1907, Koebe introduced the concept of univalent functions in his monograph and a lot of important properties for different new classes of univalent functions are stated. In 1957, the class of meromorphic functions started to attract attention due to results from the work of Z. Nehari and E. Netanyahu [1]. Two years later, J. Clunie came up with a simplified proof in [2]. Some twenty years later, S. S. Miller and P. T. Mocanu revealed the theory of differential subordinations with the very useful method of admissible functions, and many authors returned to the study of meromorphic functions, as we can see from works [3–6]. Nowadays, although the class of meromorphic functions is not as used as other classes of functions, there are many recent papers that have dealt with its properties (see [7–10]).

The method of admissible functions, known for simplifying many proofs, is also used by the authors to prove some lemmas necessary for the results of the paper. These lemmas help us to obtain complex functions with positive real parts.

In this work, we use the integral operator (defined for the first time in [11]) to obtain some results regarding the conservation of the class of convex meromorphic functions. We chose to study the preservation of the class of convex meromorphic functions since convexity is a fundamental concept in mathematics and plays an essential role in optimization, programming, geometry, statistics, and many other fields.

We consider $U = \{z \in \mathbb{C} : |z| < 1\}$ as the unit disc, $\dot{U} = U \setminus \{0\}$ as the punctured unit disc, $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$, we have $\Sigma_p = \left\{ g/g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots,\ z \in \dot{U},\ a_{-p} \neq 0 \right\}$ the class of meromorphic p-valent functions in U.

We also use the following notations: $\Sigma K_p = \left\{ g \in \Sigma_p : \mathrm{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] < 0,\, z \in U \right\}$, $\Sigma \mathcal{C}_p = \left\{ g \in \Sigma_p : \text{ there is } \varphi \in \Sigma K_p \text{ such that } \mathrm{Re}\left[\frac{g'(z)}{\varphi'(z)}\right] > 0,\, z \in U, \text{ and } \frac{g}{\varphi}\Big|_{z=0} = 1 \right\}$, $H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots\}$ for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, $A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and, for $n = 1$, we denote A_1 by A. This set is called *the class of analytic functions normalized at the origin*.

Since our results will use the "Open Door" function, we now give its defintion:

Definition 1 ([12], p. 46). *Let c be a complex number such that $\mathrm{Re}\, c > 0$, let n be a positive integer, and let*

$$C_n = C_n(c) = \frac{n}{\mathrm{Re}\, c}\left[|c|\sqrt{1 + \frac{2\mathrm{Re}\, c}{n}} + \mathrm{Im}\, c\right]. \tag{1}$$

If $R(z)$ is the univalent function defined in U by $R(z) = \frac{2C_n z}{1 - z^2}$, then the "Open Door" function is defined by

$$R_{c,n}(z) = R\left(\frac{z + b}{1 + \bar{b}z}\right) = 2C_n \frac{(z + b)(1 + \bar{b}z)}{(1 + \bar{b}z)^2 - (z + b)^2}, \tag{2}$$

where $b = R^{-1}(c)$.

Theorem 1 ([12]). *Let $p \in H[a,n]$ with $\mathrm{Re}\, a > 0$. If $\psi \in \Psi_n\{a\}$, then*

$$\mathrm{Re}\, \psi(p(z), zp'(z); z) > 0,\, z \in U \Rightarrow \mathrm{Re}\, p(z) > 0,\, z \in U.$$

We remember here that a function $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ belongs to the class $\Psi_n\{a\}$ (where $n \in \mathbb{N}^*$, $a \in \mathbb{C}$, $\mathrm{Re}\, a > 0$), when we have

$$\mathrm{Re}\, \psi(\rho i, \sigma; z) \leq 0, \text{ for } \rho, \sigma \in \mathbb{R}, z \in U, \text{ with } \sigma \leq -\frac{n}{2} \cdot \frac{|a - i\rho|^2}{\mathrm{Re}\, a}.$$

For the results of the present paper, we will use the operator $J_{p,\gamma,h}$, introduced for the first time in [11].

For $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}\, \gamma > p$ and $h \in A$, we have

$$J_{p,\gamma,h} : \Sigma_p \to \Sigma_p,\; J_{p,\gamma,h}(g) = \frac{\gamma - p}{h^\gamma(z)} \int_0^z g(t) h^{\gamma-1}(t) h'(t) dt. \tag{3}$$

Theorem 2 ([11]). *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}\, \gamma > p$ and $h \in A$ with $\frac{h(z)}{z} \cdot h'(z) \neq 0$. Let $g \in \Sigma_p$ with*

$$\frac{zg'(z)}{g(z)} + \frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 \prec R_{\gamma - p, p}(z),\, z \in U. \tag{4}$$

If $G = J_{p,\gamma,h}(g)$ is defined by (3), then $G \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\frac{zG'(z)}{G(z)} + \gamma \frac{zh'(z)}{h(z)}\right] > 0,\, z \in U.$$

All powers in (3) are principal ones.

2. Main Results

First, for $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}\, \gamma > p$ and $h \in A$ with $\frac{h(z)}{z} \cdot h'(z) \neq 0$, we denote by $\Sigma_{p,\gamma,h}$ the class of meromorphic functions $g \in \Sigma_p$ satisfying subordination (4).

It is clear that for $h(z) = z$, we have the class $\Sigma_{p,\gamma,h}$ made of functions $g \in \Sigma_p$ which verify the following subordination

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U.$$

Moreover, since $\operatorname{Re}\gamma > p$, we have $g(z) = \dfrac{1}{z^p} \in \Sigma_{p,\gamma,h}$.

Remark 1. *Using the fact that, from $G = J_{p,\gamma,h}(g)$, we have the equality*

$$\gamma \cdot G \cdot h' + h \cdot G' = (\gamma - p) \cdot g \cdot h',$$

it is easy to verify that for g of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots,\ z \in \dot{U},\ a_{-p} \neq 0,$$

we have G of the form

$$G(z) = \frac{a_{-p}}{z^p} + b_0 + b_1 z + \cdots,\ z \in \dot{U}.$$

This means that

$$\left.\frac{g}{G}\right|_{z=0} = 1.$$

In order to prove the main results of this paper, we need the following lemma and some of its particular cases.

Lemma 1. *Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and $n \in \mathbb{N}^*$. Let us consider the complex functions*

$$A,\ B,\ C,\ D : U \to \mathbb{C},$$

which verify the conditions:

- $\operatorname{Re} A(z) > 0,\ z \in U$;
- $n \cdot \operatorname{Re} A(z) + 2\operatorname{Re} a \cdot \operatorname{Re} B(z) > 0,\ z \in U$;
- $[n \cdot \operatorname{Im} a - \operatorname{Re} a \cdot \operatorname{Im} C(z)]^2 \leq [n\operatorname{Re} A(z) + 2\operatorname{Re} a \cdot \operatorname{Re} B(z)] \cdot [n|a|^2\operatorname{Re} A(z) - 2\operatorname{Re} a \cdot\operatorname{Re} D(z)]$.

  If $p \in H[a,n]$, then

$$\operatorname{Re}\Big[A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)\Big] > 0,\ z \in U \Rightarrow \operatorname{Re} p(z) > 0,\ z \in U.$$

Proof . To prove this result, we use the class of admissible functions. We consider the function $\psi(r,s;z) = A(z)s + B(z)r^2 + C(z)r + D(z)$.

We need to show that $\operatorname{Re}\psi(\rho i, \sigma; z) \leq 0$, when $\rho, \sigma \in \mathbb{R},\ z \in U$, with

$$\sigma \leq -\frac{n}{2} \cdot \frac{|a - i\rho|^2}{\operatorname{Re} a}.$$

This means that we have $\psi \in \Psi_n\{a\}$.

We have

$$\psi(\rho i, \sigma; z) = A(z)\sigma - B(z)\rho^2 + C(z)\rho i + D(z).$$

Therefore,

$$\operatorname{Re}\psi(\rho i, \sigma; z) = \sigma \cdot \operatorname{Re} A(z) - \rho^2 \cdot \operatorname{Re} B(z) - \rho \cdot \operatorname{Im} C(z) + \operatorname{Re} D(z). \tag{5}$$

Since $\sigma \leq -\frac{n}{2} \cdot \frac{|a-i\rho|^2}{\operatorname{Re} a} = -\frac{n}{2\operatorname{Re} a} \cdot \left(|a|^2 - 2\rho \cdot \operatorname{Im} a + \rho^2\right)$ and $\operatorname{Re} A(z) > 0$, $z \in U$, we obtain from (5) that

$$\operatorname{Re} \psi(\rho i, \sigma; z) \leq$$

$$-\frac{n}{2\operatorname{Re} a} \cdot \left(|a|^2 - 2\rho \cdot \operatorname{Im} a + \rho^2\right) \cdot \operatorname{Re} A(z) - \rho^2 \cdot \operatorname{Re} B(z) - \rho \cdot \operatorname{Im} C(z) + \operatorname{Re} D(z)$$

$$= \rho^2\left[-\frac{n}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z) - \operatorname{Re} B(z)\right] + \rho\left[\frac{n}{\operatorname{Re} a} \cdot \operatorname{Im} a - \operatorname{Im} C(z)\right] + \operatorname{Re} D(z) - \frac{n|a|^2}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z).$$

By using the notations:

- $-\frac{n}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z) - \operatorname{Re} B(z) = \alpha(z)$;
- $\frac{n}{\operatorname{Re} a} \cdot \operatorname{Im} a - \operatorname{Im} C(z) = \beta(z)$;
- $\operatorname{Re} D(z) - \frac{n|a|^2}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z) = \gamma(z)$;

we have

$$\operatorname{Re} \psi(\rho i, \sigma; z) \leq \rho^2\alpha(z) + \rho\beta(z) + \gamma(z),\ \rho \in \mathbb{R},\ z \in U.$$

If we consider

$$\Delta(z) = \beta^2(z) - 4\alpha(z)\gamma(z) = \left[\frac{n}{\operatorname{Re} a} \cdot \operatorname{Im} a - -IMC(z)\right]^2 -$$

$$4\left[-\frac{n}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z) - \operatorname{Re} B(z)\right]\left[\operatorname{Re} D(z) - \frac{n|a|^2}{2\operatorname{Re} a} \cdot \operatorname{Re} A(z)\right]$$

$$= \frac{1}{(\operatorname{Re} a)^2}\Big\{[n \cdot \operatorname{Im} a - \operatorname{Re} a \cdot \operatorname{Im} C(z)]^2$$

$$-[n\operatorname{Re} A(z) + 2\operatorname{Re} a \cdot \operatorname{Re} B(z)] \cdot [n|a|^2\operatorname{Re} A(z) - 2\operatorname{Re} a \cdot \operatorname{Re} D(z)]\Big\},$$

by using the last condition from the hypothesis, we obtain

$$\Delta(z) \leq 0,\ z \in U.$$

We remark that, from the second condition of the hypothesis, we have $\alpha(z) < 0$, $z \in U$. Therefore, the sign of the equation (in ρ)

$$\rho^2\alpha(z) + \rho\beta(z) + \gamma(z)$$

is less than or equal to zero.

Thus, $\operatorname{Re} \psi(\rho i, \sigma; z) \leq 0$, when $\rho, \sigma \in \mathbb{R}$, $z \in U$, with

$$\sigma \leq -\frac{n}{2} \cdot \frac{|a-i\rho|^2}{\operatorname{Re} a}.$$

This means that we have $\psi \in \Psi_n\{a\}$.

From Theorem 1, since $\psi \in \Psi_n\{a\}$ and $\operatorname{Re} \psi(p(z), zp'(z); z) > 0$, for $z \in U$, we obtain $\operatorname{Re} p(z) > 0$. □

Particular cases of Lemma 1 may be found in different papers that have studied the admissible functions.

We will consider the following particular cases:

Remark 2. *($a > 0$, $A = B$) Let $a > 0$ and $n \in \mathbb{N}^*$. Let us consider the complex functions*

$$A,\ C,\ D : U \to \mathbb{C},$$

which verify the conditions:

- $\operatorname{Re} A(z) > 0$, $z \in U$;
- $[a \cdot \operatorname{Im} C(z)]^2 \leq (n + 2a)\operatorname{Re} A(z) \cdot [n \cdot a^2 \operatorname{Re} A(z) - 2a \cdot \operatorname{Re} D(z)]$.

If $p \in H[a, n]$, *then*

$$\operatorname{Re}\left[A(z)(zp'(z) + p^2(z)) + C(z)p(z) + D(z)\right] > 0,\ z \in U \Rightarrow \operatorname{Re} p(z) > 0,\ z \in U.$$

We will need Remark 2 to prove Theorem 3.

Remark 3. *($a > 0$, $D = 0$) Let $a > 0$ and $n \in \mathbb{N}^*$. Let us consider the complex functions*

$$A,\ B,\ C : U \to \mathbb{C},$$

which verify the conditions:

- $\operatorname{Re} A(z) > 0$, $z \in U$;
- $\operatorname{Re} B(z) > 0$, $z \in U$;
- $[\operatorname{Im} C(z)]^2 \leq n \cdot \operatorname{Re} A(z) \cdot [n\operatorname{Re} A(z) + 2a \cdot \operatorname{Re} B(z)]$.

If $p \in H[a, n]$, *then*

$$\operatorname{Re}\left[A(z)zp'(z) + B(z)p^2(z) + C(z)p(z)\right] > 0,\ z \in U \Rightarrow \operatorname{Re} p(z) > 0,\ z \in U.$$

Remark 4. *($a > 0$, $D = B = 0$) Let $a > 0$ and $n \in \mathbb{N}^*$. Let us consider the complex functions $A,\ C : U \to \mathbb{C}$, which verify the conditions:*

- $\operatorname{Re} A(z) > 0$, $z \in U$;
- $|\operatorname{Im} C(z)| \leq n \cdot \operatorname{Re} A(z)$.

If $p \in H[a, n]$, *then*

$$\operatorname{Re}\left[A(z)zp'(z) + C(z)p(z)\right] > 0,\ z \in U \Rightarrow \operatorname{Re} p(z) > 0,\ z \in U.$$

Considering in Remark 4 that $C(z) = 1$, we have the following result, which is necessary to prove one of our theorems:

Remark 5. *($a > 0$, $D = B = 0$, $C = 1$) Let $a > 0$ and $n \in \mathbb{N}^*$. Let us consider the complex function $A : U \to \mathbb{C}$, with $\operatorname{Re} A(z) > 0$, $z \in U$. If $p \in H[a, n]$, then*

$$\operatorname{Re}\left[A(z)zp'(z) + p(z)\right] > 0,\ z \in U \Rightarrow \operatorname{Re} p(z) > 0,\ z \in U.$$

Remark 6. *For $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, $h \in A$ with $\dfrac{h(z)}{z} \cdot h'(z) \neq 0$, $z \in U$, we will consider some new functions defined as:*

$H(z) = \dfrac{h(z)}{h'(z)}$,

$P(z) = -1 - \dfrac{zG''(z)}{G'(z)}$, *where* $G = J_{p,\gamma,h}(g)$, $g \in \Sigma_{p,\gamma,h}$,

$Q(z) = z\gamma + zH' - H(P + 1)$.

For Q with $Q(z) \neq 0$, $z \in U$, let be:

$A(z) = \dfrac{H(z)}{Q(z)}$,

$C(z) = \dfrac{z\gamma + 2zH'(z) - 2H(z)}{Q(z)}$,

$D(z) = \dfrac{zH'(z) - z^2H''(z) - H(z)}{Q(z)}$.

Theorem 3. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $h \in A$ with $\dfrac{h(z)}{z} \cdot h'(z) \neq 0$.*

Additionally, let $g \in \Sigma_{p,\gamma,h}$ and $G = J_{p,\gamma,h}(g)$, with $z^{p+1}G'(z) \neq 0$, $z \in U$.
We consider the functions $A, C, D : U \to \mathbb{C}$, defined as above, satisfying the conditions:

$$\begin{cases} \operatorname{Re} A(z) > 0,\ z \in U \\ p[\operatorname{Im} C(z)]^2 \leq (3p+1)\operatorname{Re} A(z) \cdot [p(p+1)\operatorname{Re} A(z) - 2 \cdot \operatorname{Re} D(z)]. \end{cases}$$

If $g \in \Sigma K_p$, then $G \in \Sigma K_p$ with $z^p G(z) \neq 0$, $z \in U$ and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

Proof . Since the hypothesis of Theorem 2 is fulfilled, we have $G \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

Since $G \in \Sigma_p$ and $z^{p+1}G'(z) \neq 0$, $z \in U$, we have $P \in H[p, p+1]$.

We want to prove now that we have $\operatorname{Re}\left(-1 - \frac{zG''}{G'}\right) > 0$, $z \in U$, meaning that $\operatorname{Re} P(z) > 0$, $z \in U$.

From $P = -1 - \frac{zG''}{G'}$ we obtain

$$zG'' = -G'(P+1),\ z^2G''' = G'[(P+1)(P+2) - zP']. \tag{6}$$

On the other hand, from $G = J_{p,\gamma,h}(g) = \frac{\gamma - p}{h^\gamma(z)} \int_0^z g(t)h^{\gamma-1}(t)h'(t)dt$, we have

$$\gamma G + HG' = (\gamma - p)g.$$

Therefore,

$$\gamma G' + H'G' + HG'' = (\gamma - p)g' \Rightarrow z\gamma G' + zH'G' + zHG'' = z(\gamma - p)g'$$

and, from (6), we obtain

$$z\gamma G' + zH'G' - HG'(P+1) = z(\gamma - p)g'. \tag{7}$$

From $\gamma G' + H'G' + HG'' = (\gamma - p)g'$ we have

$$\gamma G'' + H''G' + 2H'G'' + HG''' = (\gamma - p)g'' \Rightarrow$$

$$z^2\gamma G'' + z^2H''G' + 2z^2H'G'' + z^2HG''' = z^2(\gamma - p)g''$$

and, from (6), we obtain

$$-z\gamma G'(P+1) + z^2H''G' - 2zH'G'(P+1) + HG'[(P+1)(P+2) - zP'] = z^2(\gamma - p)g''. \tag{8}$$

Next, we divide (8) by (7), and we obtain

$$\frac{zg''}{g'} = \frac{-z\gamma(P+1) + z^2H'' - 2zH'(P+1) + H[(P+1)(P+2) - zP']}{z\gamma + zH' - H(P+1)},$$

so

$$-1 - \frac{zg''}{g'} = -1 - \frac{-z\gamma(P+1) + z^2H'' - 2zH'(P+1) + H[(P+1)(P+2) - zP']}{z\gamma + zH' - H(P+1)} \tag{9}$$

$$= \frac{zP'H + P^2H + (\gamma z + 2zH' - 2H)P + zH' - z^2H'' - H}{Q(z)}$$
$$= A(z)(zP' + P^2) + C(z)P + D(z).$$

Therefore, we have

$$-1 - \frac{zg''}{g'} = A(z)zP' + A(z)P^2 + C(z)P + D(z),\ z \in U. \tag{10}$$

By using the fact that $g \in \Sigma K_p$, which is equivalent to $\mathrm{Re}\left(-1 - \frac{zg''}{g'}\right) > 0$, we obtain from (10) that

$$\mathrm{Re}\left(A(z)zP' + A(z)P^2 + C(z)P + D(z)\right) > 0,\ z \in U.$$

Since we have $P \in H[p, p = 1]$, we see that the conditions from the hypothesis of Remark 2 are verified for $a = p$ and $n = p + 1$.

Using now Remark 2, we obtain from

$$\mathrm{Re}\left(A(z)zP' + A(z)P^2 + C(z)P + D(z)\right) > 0,\ z \in U,$$

that $\mathrm{Re}\, P(z) > 0,\ z \in U$.

Since $P = -1 - \frac{zG''}{G'}$ we obtain $G \in \Sigma K_p$ with $z^pG(z) \neq 0,\ z \in U$ and

$$\mathrm{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U,$$

which means that the proof of the theorem is complete. □

Before continuing with some corollaries of Theorem 3, we will show that the conditions given in the hypothesis of the theorem are met for some particular cases. Taking $h(z) = z$, $\gamma \in \mathbb{R}$ with $\gamma > p$ and $g(z) = \frac{1}{z^p}$ we have

$$H(z) = z,\ Q(z) = z(\gamma - p),\ A(z) = \frac{1}{\gamma - p},\ C(z) = \frac{\gamma}{\gamma - p},\ D(z) = 0.$$

This means that:

$$\mathrm{Re}\, A(z) > 0 \Leftrightarrow \gamma > p\ (true),$$
$$p[\mathrm{Im}\, C(z)]^2 \leq (3p+1)\mathrm{Re}\, A(z) \cdot [p(p+1)\mathrm{Re}\, A(z) - 2 \cdot \mathrm{Re}\, D(z)]$$
$$\Leftrightarrow 0 \leq \frac{(3p+1)p(p+1)}{(\gamma - p)^2}\ (true).$$

Moreover, if we consider in Theorem 3 only that $h(z) = z$, we have $J_{p,\gamma,h} = J_{p,\gamma}$, (introduced in [13]),

$$\frac{zh'(z)}{h(z)} = 1,\ H(z) = z,\ Q(z) = z\gamma - zP,$$

$$A(z) = \frac{H(z)}{Q(z)} = \frac{1}{\gamma - P},\ C(z) = \frac{\gamma}{\gamma - P} = \gamma \cdot A(z),\ D(z) = 0$$

and

$$\frac{zh''(z)}{h'(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 = \gamma.$$

It is obvious that we have $\mathrm{Re}\,A(z) > 0 \Leftrightarrow \mathrm{Re}\,P(z) < \mathrm{Re}\,\gamma$ and

$$[\mathrm{Im}\,C(z)]^2 \le (3p+1)(p+1)[\mathrm{Re}\,A(z)]^2 \Leftrightarrow |\mathrm{Im}\,\gamma P| \le \sqrt{(3p+1)(p+1)}\mathrm{Re}\,(\gamma - P).$$

Thus, we obtain:

Corollary 1. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\,\gamma > p$, $g \in \Sigma_p$ *and* $G = J_{p,\gamma}(g)$ *with*

$$z^{p+1}G'(z) \neq 0,\ z \in U.$$

We denote by P the function $P(z) = -1 - \dfrac{zG''(z)}{G'(z)}$, $z \in U$.

Suppose that $\mathrm{Re}\,P(z) < \mathrm{Re}\,\gamma$, $z \in U$, *and* $|\mathrm{Im}\,\gamma P| \le \mathrm{Re}\,(\gamma - P)$.

If $g \in \Sigma K_p$, *such that*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U, \tag{11}$$

then $G \in \Sigma K_p$ *with* $z^pG(z) \neq 0$, $z \in U$ *and*

$$\mathrm{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U.$$

If we consider in Theorem 3 that $h \in A$ satisfies the equality $z\gamma + 2zH' - 2H = 0$, where $H = \dfrac{h}{h'}$, we obtain:

$$C(z) = 0,\ D(z) = 0,\ Q(z) = \frac{z\gamma}{2} - HP,$$

so we may consider the next corollary:

Corollary 2. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\,\gamma > p$ *and* $h \in A$ *with*

$$\frac{h(z)}{z} \cdot h'(z) \neq 0,\ z \in U,\ \text{and}\ z\gamma + 2zH' - 2H = 0,\ \text{where}\ H = \frac{h}{h'}.$$

Let $g \in \Sigma_{p,\gamma,h}$ *and* $G = J_{p,\gamma,h}(g)$, *with* $z^{p+1}G'(z) \neq 0$, $z \in U$. *We denote by P the function* $P(z) = -1 - \dfrac{zG''(z)}{G'(z)}$. *Suppose that* $\mathrm{Re}\,P(z) < \mathrm{Re}\,\dfrac{z\gamma h'}{2h}$, $z \in U$.

If $g \in \Sigma K_p$, *then* $G \in \Sigma K_p$ *with* $z^pG(z) \neq 0$, $z \in U$ *and*

$$\mathrm{Re}\left[\frac{zG'(z)}{G(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

In order to prove the next theorem, we need the following lemma:

Lemma 2. *Let* $p \in H[a,n]$, *where* $n \in \mathbb{N}^*$, $a \in \mathbb{C}$ *with* $\mathrm{Re}\,a > 0$.

We have

$$\mathrm{Re}\left[\frac{zp'(z)}{p(z)} - \frac{1}{p(z)}\right] > 0,\ z \in U \Rightarrow \mathrm{Re}\,p(z) > 0,\ z \in U.$$

Proof . To prove this result, we use the class of admissible functions. We consider the function $\psi(r,s,t;z) = \dfrac{s-1}{r}$.

We need to show that $\operatorname{Re}\psi(\rho i,\sigma,\mu+i\nu;z)\leq 0$, when $\rho,\sigma,\mu,\nu\in\mathbb{R},\ z\in U$, with

$$\sigma\leq-\frac{n}{2}\cdot\frac{|a-i\rho|^2}{\operatorname{Re}a},\ \sigma+\mu\leq 0.$$

We have

$$\psi(\rho i,\sigma,\mu+i\nu;z)=\frac{\sigma-1}{i\rho}=\frac{(1-\sigma)i}{\rho}.$$

Therefore,

$$\operatorname{Re}\psi(\rho i,\sigma,\mu+i\nu;z)=\operatorname{Re}\left[\frac{(1-\sigma)i}{\rho}\right]=0,$$

for $\rho\in\mathbb{R}^*,\ \sigma\leq-\frac{n}{2}\cdot\frac{|a-i\rho|^2}{\operatorname{Re}a}$.

Thus, $\operatorname{Re}\psi(\rho i,\sigma,\mu+i\nu;z)\leq 0$, when $\rho,\sigma,\mu,\nu\in\mathbb{R},\ z\in U$, with

$$\sigma\leq-\frac{n}{2}\cdot\frac{|a-i\rho|^2}{\operatorname{Re}a},\ \sigma+\mu\leq 0.$$

This means that we have $\psi\in\Psi_n\{a\}$.

From Theorem 1, since $\psi\in\Psi_n\{a\}$ and $\operatorname{Re}\psi(p(z),zp'(z),z^2p''(z);z)>0$, for $z\in U$, we obtain $\operatorname{Re}p(z)>0$. □

For $p\in\mathbb{N}^*$, $\gamma\in\mathbb{C}$ with $\operatorname{Re}\gamma>p$ and $h\in A$ with $\frac{h(z)}{z}\cdot h'(z)\neq 0$, let us define the classes:

$\Sigma K_{p,\gamma,h}=\Sigma K_p\cap\Sigma_{p,\gamma,h}$,

$\Sigma\mathcal{C}_{p,\gamma,h}=\left\{g\in\Sigma_{p,\gamma,h}/(\exists)\varphi\in\Sigma K_{p,\gamma,h}\ \text{such that}\ \operatorname{Re}\frac{g'}{\varphi'}>0,\ z\in U,\ \text{and}\ \left.\frac{g}{\varphi}\right|_{z=0}=1\right\}$.

Remark 7. *Taking Remark 1 into account, it is not difficult to see that if we have $g\in\Sigma\mathcal{C}_{p,\gamma,h}$, then*

$$\left.\frac{J_{p,\gamma,h}(g)}{J_{p,\gamma,h}(\varphi)}\right|_{z=0}=1.$$

Theorem 4. *Let $p\in\mathbb{N}^*$, $\gamma\in\mathbb{C}$ with $\operatorname{Re}\gamma>p$ and $h\in A$ with $\frac{h(z)}{z}\cdot h'(z)\neq 0$, such that*

$$\operatorname{Re}\frac{\gamma z\cdot h'(z)}{h(z)}<0,\ z\in U.$$

If $J_{p,\gamma,h}\left(\Sigma K_{p,\gamma,h}\right)\subset\Sigma K_p$, then $J_{p,\gamma,h}\left(\Sigma\mathcal{C}_{p,\gamma,h}\right)\subset\Sigma\mathcal{C}_p$.

Proof . Let $g\in\Sigma\mathcal{C}_{p,\gamma,h}\subset\Sigma_{p,\gamma,h}$ and $G=J_{p,\gamma,h}(g)$. Since the hypothesis of Theorem 2 is fulfilled, we have $G\in\Sigma_p$ with $z^pG(z)\neq 0$, $z\in U$, and

$$\operatorname{Re}\left[\frac{zG'(z)}{G(z)}+\gamma\frac{zh'(z)}{h(z)}\right]>0,\ z\in U.$$

From $G=J_{p,\gamma,h}(g)$, we have $\gamma G+HG'=(\gamma-p)g$, where $H(z)=\frac{h(z)}{h'(z)}$. Therefore,

$$\gamma G'+H'G'+HG''=(\gamma-p)g'. \tag{12}$$

Because $g \in \Sigma\mathcal{C}_{p,\gamma,h}$, we have from the definition of the class that there is a function $\varphi \in\in \Sigma K_{p,\gamma,h}$ such that

$$\operatorname{Re}\frac{g'}{\varphi'} > 0,\ z \in U,\ \text{ and } \left.\frac{g}{\varphi}\right|_{z=0} = 1. \tag{13}$$

Since $\varphi \in\in \Sigma K_{p,\gamma,h} \subset \Sigma_{p,\gamma,h}$, the hypothesis of Theorem 2 is fulfilled, so we have $\Phi = J_{p,\gamma,h}(\varphi) \in \Sigma_p$ with $z^p\Phi(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{z\Phi'(z)}{\Phi(z)} + \gamma\frac{zh'(z)}{h(z)}\right] > 0,\ z \in U.$$

From $\Phi = J_{p,\gamma,h}(\varphi)$ we have $\gamma\Phi + H\Phi' = (\gamma - p)\varphi$. Therefore,

$$\gamma\Phi' + H'\Phi' + H\Phi'' = (\gamma - p)\varphi'. \tag{14}$$

Because we have in the hypothesis of our theorem $J_{p,\gamma,h}\left(\Sigma K_{p,\gamma,h}\right) \subset \Sigma K_p$, we obtain that $\Phi = J_{p,\gamma,h}(\varphi) \in \Sigma K_p$.

Let us denote $P(z) = \dfrac{G'(z)}{\Phi'(z)}$, $z \in U$. First of all, we show that we have $P \in H[1, p+1]$. It is not difficult to see, due to Remark 1, that

$$\left.\frac{g}{G}\right|_{z=0} = 1,\ \left.\frac{\varphi}{\Phi}\right|_{z=0} = 1$$

and, since $\left.\dfrac{g}{\varphi}\right|_{z=0} = 1$, we obtain that $\left.\dfrac{G}{\Phi}\right|_{z=0} = 1$.

Using now the fact that $G, \Phi \in \Sigma_p$, after a little computation, we obtain $\dfrac{G'(z)}{\Phi'(z)} \in H[1, p+1]$.

From $P(z) = \dfrac{G'(z)}{\Phi'(z)}$, $z \in U$, we have

$$G'(z) = \Phi'(z)\cdot P(z) \text{ and } G''(z) = \Phi''(z)\cdot P(z) + \Phi'(z)\cdot P'(z),\ z \in U. \tag{15}$$

By replacing G' and G'' from (12) with the forms from (15), we obtain:

$$\gamma\Phi' P + H'\Phi' P + H\Phi'' P + H\Phi' P' = (\gamma - p)g'. \tag{16}$$

Since from (14) we have $\gamma\Phi' + H'\Phi' + H\Phi'' = (\gamma - p)\varphi'$, by replacing it in (16), we obtain

$$(\gamma - p)\varphi'\cdot P + H\Phi'\cdot P' = (\gamma - p)g'.$$

Therefore,

$$P + \frac{H\Phi'\cdot P'}{(\gamma - p)\varphi'} = \frac{g'}{\varphi'},$$

which is equivalent to

$$P(z) + A(z)\cdot zP'(z) = \frac{g'(z)}{\varphi'(z)},\ z \in U,\ \text{ where } A(z) = \frac{H\Phi'}{(\gamma - p)z\varphi'}.$$

From (13), we have $\operatorname{Re}\dfrac{g'(z)}{\varphi'(z)} > 0$, $z \in U$. Thus, we obtain

$$\operatorname{Re}\left(P(z) + A(z)\cdot zP'(z)\right) > 0,\ z \in U. \tag{17}$$

Next, we prove that we have $\operatorname{Re} A(z) > 0$, $z \in U$.

We know that $A = \dfrac{H\Phi'}{(\gamma - p)z\varphi'}$. Thus, $(\gamma - p)z\varphi' A = H\Phi'$, and using now the logarithmic differential and then multiplying the result by z, we obtain

$$1 + \frac{z\varphi''}{\varphi'} + \frac{zA'}{A} = \frac{zH'}{H} + \frac{z\Phi''}{\Phi'}. \tag{18}$$

On the other hand, from $\gamma\Phi' + H'\Phi' + H\Phi'' = (\gamma - p)\varphi'$ (see (14)) and $A = \dfrac{H\Phi'}{(\gamma - p)z\varphi'}$, we obtain that

$$\frac{1}{A} = \frac{(\gamma - p)z\varphi'}{H\Phi'} = \frac{z\gamma\Phi' + zH'\Phi' + zH\Phi''}{H\Phi'} = \frac{z\gamma}{H} + \frac{zH'}{H} + \frac{z\Phi''}{\Phi'}.$$

This means that we have

$$\frac{zH'}{H} + \frac{z\Phi''}{\Phi'} = \frac{1}{A} - \frac{z\gamma}{H}. \tag{19}$$

Using now (19) in (18), we find that

$$1 + \frac{z\varphi''}{\varphi'} + \frac{zA'}{A} = \frac{1}{A} - \frac{z\gamma}{H} \Leftrightarrow$$

$$1 + \frac{z\varphi''}{\varphi'} + \frac{z\gamma}{H} = \frac{1}{A} - \frac{zA'}{A}. \tag{20}$$

Now, on one hand, since $\varphi \in \Sigma K_p$, we have $\operatorname{Re}\left(1 + \dfrac{z\varphi''}{\varphi'}\right) < 0$.

On the other hand, from the hypothesis of our theorem, we have $\operatorname{Re}\dfrac{z\gamma}{H} = \operatorname{Re}\dfrac{z\gamma h'}{h} < 0$. Therefore, we obtain from (20) that

$$\operatorname{Re}\left(\frac{1}{A} - \frac{zA'}{A}\right) < 0 \Leftrightarrow \operatorname{Re}\left(\frac{zA'}{A} - \frac{1}{A}\right) > 0,\ z \in U.$$

It is not difficult to remark, from the definition of A, that we have $A \in H\left[\dfrac{1}{\gamma - p}, 1\right]$.

Since we have $\operatorname{Re}\dfrac{1}{\gamma - p} > 0$, $A \in H\left[\dfrac{1}{\gamma - p}, 1\right]$ and $\operatorname{Re}\left(\dfrac{zA'}{A} - \dfrac{1}{A}\right) > 0$, $z \in U$, we may apply Lemma 2, and we obtain $\operatorname{Re} A(z) > 0$, $z \in U$.

We use now Remark 5 since we have $A : U \to \mathbb{C}$, with $\operatorname{Re} A(z) > 0$, $z \in U$, and $P \in H[1, p+1]$, such that $\operatorname{Re}\left[A(z)zP'(z) + P(z)\right] > 0$, $z \in U$ (see (17)), and we obtain that $\operatorname{Re} P(z) > 0$, $z \in U$. This means that we have $\operatorname{Re}\dfrac{G'}{\Phi'} > 0$, $z \in U$, so, $G = J_{p,\gamma,h}(g) \in \Sigma\mathcal{C}_p$.

Finally, we proved that $J_{p,\gamma,h}\left(\Sigma\mathcal{C}_{p,\gamma,h}\right) \subset \Sigma\mathcal{C}_p$. $\square$

3. Discussion

The new integral operator on meromorphic functions, denoted by $J_{p,\gamma,h}$, is used to study the conditions that allow this operator to preserve the class of convex meromorphic multivalent functions.

In addition, the integral operator used in our work depends on an analytic normalized function h. In certain particular cases of the function h, we obtain operators that have been used to study either properties related to subordination or conservation of special classes of functions. We mention here the fact that the subordination relationship between two functions can also be seen as an inclusion relationship between two domains.

The first result of the present paper is a lemma that helps us to obtain complex functions with positive real parts. Of course, we need this lemma to prove the first theorem.

This lemma is a generalization of other previous results, and some particular cases are grouped under remarks. They also may be useful to prove some new theorems.

Examples were given as corollaries for particular cases of the function h. We mention here that a result similar to Corollary 1 was proved in [13] (see Corollary 2 for $\beta = 1$ combined with Theorem 14 for $\alpha = 0$); this is a result that has in the hypothesis fewer conditions. This means that Theorem 1 can be improved.

In the last theorem, we will find out in which situation the conservation of the class of convex meromorphic functions will attract the conservation of the class of close-to-convex meromorphic functions. A useful lemma, dealing with complex functions with positive real parts, is also stated to help with the proof of the theorem.

Of course, this new integral operator can be used to introduce other subclasses of meromorphic functions, and, also, new properties of it can be investigated.

We could have presented our results using the class of meromorphic p-valent functions normalized to one (which may be found in previous papers and is denoted by $\Sigma_{p,0}$), without loss of generality, but we preferred to use the class Σ_p instead of $\Sigma_{p,0}$ because the notation was simpler.

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