Article
The Separability Problem in Two Qubits Revisited

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Abstract: In this paper, we present an overview of the progress in the separability problem in bipartite systems, more specifically in a two quantum bits (qubits) system, from the separability criterion based on Bell's inequalities in 1964 to the recent separability criteria.

Keywords: qubits; separability criteria; entanglement

1. Introduction

A paper by D. Bruss, et al., published in 2001, listed four important issues of motivation to study the entanglement problem [1], which are, the essential role of entanglement in apparent "paradoxes" and counter-intuitive consequences of quantum mechanics [2–4], the characterization of entanglement which is one of the most fundamental open problems of quantum mechanics [5], the essential role of entanglement in applications of quantum mechanics to quantum information processing [6,7], the quantum cryptography [8,9], and the quantum communication [10], the direct link of entanglement to one of the most challenging open problems of linear algebra and functional analysis: the characterization and classification of positive maps on $C^\star$ algebras [11–14]. These issues are still relevant nowadays. Indeed, a recent paper by P. Horodecki, et al., titled Five Open Problems in Quantum Information Theory includes open problems in quantum metrology, quantum entanglement and its distillability [15]. This paper boosts our motivation and deepens our interest in studying quantum entanglement. The definition of entanglement is nowadays a mathematical one, that is rather simple, compared to the phenomenological description which is still difficult. The wave function describing a quantum system is entangled if it cannot be written as a product state of subsystems. The simplest example is the singlet state of two spin-$\frac{1}{2}$ particles [16]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle), \tag{1}$$

where $|0\rangle$ standing for "spin up" state and $|1\rangle$ standing for "spin down" state. It can be proven that $|\psi\rangle \neq |\varphi\rangle|\phi\rangle$ for any $|\varphi\rangle$ and $|\phi\rangle$ describing subsystems. The above definition is naturally generalized to the entanglement of multiparticle pure-state. A bipartite pure quantum state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled when it cannot be written in the form

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B, \tag{2}$$

for some $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$. A mixed state or density matrix ρ_{AB} which is semi-definite operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled when it cannot be written in the following form

$$\rho = \sum_i p_i |\psi_i\rangle_{AA}\langle\psi_i| \otimes |\psi_i\rangle_{BB}\langle\psi_i|. \tag{3}$$

Here the coefficients p_i are probabilities, that means $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. Note that in general neither $\{|\psi_i\rangle_A\}$ nor $\{|\psi_i\rangle_B\}$ have to be orthogonal.

A fundamental problem in the study of entanglement is the determination of the separability of quantum states. A separability condition can be necessary or necessary and sufficient. A necessary condition for separability has to be fulfilled by every separable state. In that case, if a state does not fulfill the condition, it has to be entangled, but if it fulfills we cannot conclude. On the other hand, a necessary and sufficient condition for separability can only be satisfied by separable states, if a state fulfills a necessary and sufficient condition, then we can be sure that the state is separable [17]. The separability problem has been solved for pure states [18,19], and for 2×2 and 2×3 systems [20]. There are still a lot of investigations into the case of higher dimensions. In addition to the fact that quantum entanglement is still an emergent problem, a motivation to study the separability of quantum states is the fact that it is directly linked to challenging open questions of modern mathematics. In this paper, we present an overview of the operational and nonoperational criteria of separability in two quantum bits (qubits) systems starting from Bell's inequalities in 1964 to the recent ones, to provide a good starting point for reading about the topic and to direct the interested reader to more in-depth resources.

The paper starts with a historical prelude in Section 2. In Section 3, we present the operational separability criteria followed by the nonoperational separability criteria in Section 4. The conclusion is given in Section 5.

2. Prelude

Quantum mechanics is a beautiful and fascinating theory, started from 1900 to 1920s, and grew into its present form in the late 1920s. A collection of views about the meaning of quantum mechanics principally attributed to Niels Bohr and Werner Heisenberg is called the Copenhagen interpretation [21]. There is no definitive historical statement of what the Copenhagen interpretation is. It is one of the oldest and numerous proposed interpretations of quantum mechanics as features of it date to the development of quantum mechanics during 1925–1927, and it remains one of the most commonly taught [22]. Albert Einstein was skeptical of quantum mechanics, particularly its Copenhagen interpretation [23]. In the 15 May 1935 issue of Physical Review, Albert Einstein co-authored a paper with Boris Podolsky and Nathan Rosen who were his two postdoctoral research associates at the Institute for Advanced Study. The article was entitled *Can quantum mechanical description of physical reality be considered complete*? [2]. In this study, the three scientists proposed a thought experiment known today as the EPR paradox that attempted to show that the quantum mechanical description of physical reality given by wave functions is not complete.

However, Einstein, Podolsky and Rosen did not coin the word entanglement. Erwin Schrödinger in his correspondence with Einstein, following the EPR paper, used the word Verschränkung (in German) translated by himself in English as entanglement, to describe the correlations between two particles that interact and then separate as in the EPR thought experiment. He shortly thereafter published a seminal paper defining and discussing the notion of entanglement [3]. In this seminal paper, Schrödinger recognized the importance of the concept, and stated: "I would not call (entanglement) one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought". Einstein was disturbed by the theoretical concept of quantum entanglement, which he called *Spooky action at distance*. Einstein did not believe two particles could remain connected to each other over great distances: doing so, he said, would require them to communicate faster than the speed of light, something he had previously shown to be impossible. Like Einstein, Schrödinger was dissatisfied with the concept of entanglement, because it seemed to violate the speed limit on the transmission of information implicit in the theory of relativity. The EPR paper generated significant interest among physicists and quickly became a centerpiece in debates over the interpretation of quantum theory, debates that continue today. The question of the expected locality of the entangled quantum sys-

tems raised by EPR allowed John Stewart Bell to discover his famous inequalities serving as a test and demonstration of strange properties of the simplest entangled wave function represented by a singlet state [24,25]. Still one had to wait long for the proposals and practical applications of quantum entanglement. Clauser and his collaborators proposed an experiment to test local hidden-variable theories [26] and that experiment is considered the first experimental verifications of Bell inequalities. Until 1975, a decisive experiment based on the violation of Bell's inequalities and verifying the veracity of quantum entanglement was missing. The experiment led by French physicist Alain Aspect at the Ecole Supérieure d'optique in Orsay between 1980 and 1982 was the first quantum mechanics experiment to demonstrate the violation of Bell's inequalities [27,28]. This experiment is called the Aspect's experiment. It confirmed the predictions of quantum mechanics and thus confirmed its incompatibilities with local theories. In 1992–1993, Lucien Hardy demonstrated Bell's theorem without using inequalities. Hardy's paradox is a thought experiment in quantum mechanics in which a particle and its antiparticle may interact without annihilating each other [29–31]. In 2015, Ronald Hanson and colleagues performed a Loophole-free Bell inequality violation using electron spins separated by 1.3 km. Their results are consistent with a violation of the inequality [32]. Subsequently, in the same year, Lynden K. Shalm et al. presented a loophole-free violation of local realism [33,34]. In addition to its fundamental importance, a loophole-free Bell test is an important building block in quantum information processing.

3. Operational Separability Criteria

An operational criterion is a recipe that can be applied to an explicit density matrix ρ, giving some immediate answer like "ρ is entangled" or "ρ is separable" or this "criterion is not strong enough to decide whether ρ is separable or entangled".

3.1. Bell-CHSH Inequalities

The Bell inequality was originally designated to test predictions of quantum mechanics against those of a local hidden variables theory [24]. Bell's inequalities were initially dealing with two qubits, that is two-level systems and provide a necessary criterion for the separability of 2-qubits states. For pure states, Bell's inequalities are also sufficient for separability. It has been proven by Gisin that any non-product state of two-particle systems violates a Bell-inequality [35]. This inequality which involves three vectors in real space $\mathbb{R}^3$ determining which component of a spin to be measured by each party or three, has been extended for the case involving four vectors by Clauser, Horne, Shimony and Holt (CHSH) in 1969 [26]. The Bell-CHSH inequality also provides a test to distinguish entangled from non-entangled states.

Consider a system of two qubits. Let A and A' denote observables on the first qubit, B and B' denote observables on the second qubit, the Bell-CHSH inequality says that for non-entangled states, means for states of the form $\rho = \rho_1 \otimes \rho_2$, or mixtures of such states, the following inequality holds:

$$|\langle\, A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B' \,\rangle_\rho| \leq 2. \tag{4}$$

As an example, we consider a two qubits state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the observables

$$A = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z), \ A' = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z), \ B = \sigma_x, \ B' = \sigma_z, \tag{5}$$

where σ_x and σ_z are Pauli matrices. We have then explicitly

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{6}$$

and

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; A' = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}; B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; B' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7)$$

It is easy to check that

$$\langle \psi | \; A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B' \; |\psi\rangle = 2\sqrt{2}. \quad (8)$$

The state $|\psi\rangle$ which violates the Bell-CHSH inequality is a well-known entangled state, and is one of the Bell pairs, maximally entangled state. The maximal violation of (4), for entangled states follows from an inequality of Cirelson [36]

$$|\langle A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B' \rangle_\rho| \le 2\sqrt{2}. \quad (9)$$

The equality in Equation (9) can be attained by the singlet state. Historically, Bell-CHSH inequalities were the first tool for the recognition of entanglement; however, it has been well-known for some time that the violation of a Bell-CHSH inequality is only a sufficient condition for entanglement and not a necessary one, and that many entangled states that satisfy them [37]. Bell-CHSH inequalities were generalized to N qubits, whose violations provide a criterion to distinguish the separable states from the entangled states [38–40].

3.2. Schmidt Decomposition Criterion

For pure states there is a very simple necessary and sufficient criterion for separability, the Schmidt decomposition. It allows one to write any pure state of a bipartite system as a linear combination of bi-orthogonal product states or, equivalently, of a non-superfluous set of product states built from local bases [5,41–43]. Let us first recall the following.

Theorem 1. *Consider quantum systems A and B with dimensions d_A, d_B respectively, and let $d = \min(d_A, d_B)$. Any pure bipartite state $|\psi\rangle_{AB}$ has a Schmidt decomposition*

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} \lambda_i |u_i\rangle_A |v_i\rangle_B, \quad (10)$$

where $\lambda_i \ge 0$ and $\{|u_i\rangle_A\}_i$, $\{|u_i\rangle_B\}_i$ are orthogonal sets. The coefficients λ_i are called the Schmidt coefficients and $|u_i\rangle_A$, $|v_i\rangle_B$, the Schmidt vectors.

The Schmidt coefficients are the singular values of the matrix coefficients of development of the pure state on a product state orthonormal basis. They are uniquely defined and the number of non-zero Schmidt coefficients is called the Schmidt rank, defined as follows. For any bipartite pure state with Schmidt decomposition of the form (10), the Schmidt rank is defined as the number of non-zero coefficients λ_i. A corollary of the Schmidt decomposition theorem is the following.

Theorem 2. *A pure state in a composite system is a product state if and only if the Schmidt rank is 1, and is an entangled if and only if the Schmidt rank is greater than 1.*

Example 1. *let us consider in two qubits system, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are span by $\{|0\rangle, |1\rangle\}$, the state*

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} |0\rangle_A |1\rangle_B + \frac{1}{\sqrt{2}} |1\rangle_A |0\rangle_B. \quad (11)$$

It is easy to check that $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$, and that the state has Schmidt rank 2 and therefore entangled.

Example 2. *We consider now*

$$|\psi\rangle_{AB} = \frac{1}{2}(|0\rangle_A|1\rangle_B + |1\rangle_A|1\rangle_B + |0\rangle_A|0\rangle_B + |1\rangle_A|0\rangle_B).\tag{12}$$

It is easy to check that

$$|\psi\rangle_{AB} = \frac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes (|1\rangle_B + |0\rangle_B) = |+\rangle_A|+\rangle_B.\tag{13}$$

Here $|+\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$ *and* $|+\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B)$. *In this example, the Schmidt rank is* 1, *then* $|\psi\rangle_{AB}$ *is separable.*

3.3. Entropy of Entanglement Criterion

A good way of characterizing the degree of entanglement of pure states is to measure the entanglement entropy, S_A, that is for a state $|\psi\rangle_{AB}$, as in Equation (10),

$$S_A = -\sum_{j=1}^{d} \lambda_j^2 \log\lambda_j^2 = -\text{Tr}_A \, \rho_A \log\rho_A.\tag{14}$$

The entanglement entropy, S_B, is determined as in the Equation (14).
The entanglement entropy is zero for separable states.
We consider now a 2 qubits system. Suppose the system is in the pure state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\, |00\rangle + |11\rangle\,),\tag{15}$$

so the density operator is $\rho_{AB} = |\psi\rangle\langle\psi|$. The reduced density matrix of system A is

$$
\begin{aligned}
\rho_A &= \text{Tr}_B\rho_{AB};\\
&= \frac{1}{2}\,_B\langle 0|(\, |00\rangle + |11\rangle\,)(\, \langle 11| + \langle 00|\,)|0\rangle_B\\
&\quad + \frac{1}{2}\,_B\langle 1|(\, |00\rangle + |11\rangle\,)(\, \langle 11| + \langle 00|\,)|1\rangle_B;\\
&= \frac{1}{2}(|0\rangle_{AA}\langle 0| + |1\rangle_{AA}\langle 1|).\\
&= \frac{1}{2}\mathbb{I}_{2\times 2},
\end{aligned}
$$

where Tr_B is a map of operators known as the partial trace over the system B. The partial trace Tr_B is defined as

$$\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \, \text{Tr}_B(|b_1\rangle\langle b_2|)) = |a_1\rangle\langle a_2|\langle b_2|b_1\rangle.\tag{16}$$

More generally, assuming $\rho_{AB} = \sigma \otimes \tau$, where σ is the density operator for system A and τ the density operator for system B,

$$\text{Tr}_B(\rho_{AB}) = \text{Tr}_B(\sigma \otimes \tau) = \sigma\text{Tr}\tau = \sigma.\tag{17}$$

In the same manner, $\text{Tr}_A(\rho_{AB}) = \tau$.
The entanglement entropy of subsystem A can be calculate using the formula in the Equation (14) as follows

$$
\begin{aligned}
S_A &= -\text{tr}\rho_A \log \rho_A;\\
&= -2 \times \frac{1}{2} \log \frac{1}{2};\\
&= \log 2 \neq 0.
\end{aligned}
$$

3.4. The Positive Partial Transpose (PPT) Criterion

This criterion is also called the Peres-Horodecki criterion. It was first proposed as a necessary condition for every separable state by A. Peres in 1996 [44]. He noticed that the separable states remain positive if subjected to partial transposition and then conjectured that this is also a sufficient condition. Later, M. Horodecki and al. studied the criterion in detail and discovered that Peres's conjecture is valid for separable states of 2×2 and 2×3 dimensions [20,45]. Thus for two qubits, the PPT criterion can be used to confirm exactly whether a state is entangled or not. The criterion can be formulated as follows.

Let $\rho_{AB} = [\rho_{i\mu,j\nu}]$ the density matrix of two qubits A and B. The entries of the density matrix that is partially transposed concerning A are given by

$$\rho^{T_A} = [\rho_{j\mu,i\nu}], \tag{18}$$

where $\rho_A = [\rho_{ij}]$ and $\rho_B = [\rho_{\mu\nu}]$.

The theorem describing the criterion is as follows.

Theorem 3. *The partial transpose of a separable state ρ_{AB} with respect to any subsystem is positive.*

For 2×2 and 2×3 systems ρ_{AB} is separable if and only if ρ^{T_A} is positive. The demonstration of the theorem can be found in [20,43,45].

As an illustration, we consider a Werner state

$$\rho = p \, |\psi\rangle\langle\psi| + \frac{1-p}{4}\mathbb{I}_{4\times4}, \tag{19}$$

where $0 < p < 1$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(\, |01\rangle + |10\rangle \,)$. For $p = 0$ or $p = 1$, ρ becomes a pure state. Explicitly we have

$$\begin{aligned}
\rho \;=\; & \frac{p}{2}(\, (|0\rangle\langle0| \otimes |1\rangle\langle1|) + (|0\rangle\langle1| \otimes |1\rangle\langle0|) + (|1\rangle\langle0| \otimes |0\rangle\langle1|) + (|1\rangle\langle1| \otimes |0\rangle\langle0|) \,) \\
+ & \frac{1-p}{4}(|0\rangle\langle0| \otimes |0\rangle\langle0| + |1\rangle\langle1| \otimes |1\rangle\langle1| + |0\rangle\langle0| \otimes |1\rangle\langle1| + |1\rangle\langle1| \otimes |0\rangle\langle0|).
\end{aligned} \tag{20}$$

For the calculations, it is better to write ρ in matrix form as follows

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \langle0| = (1 \; 0); \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \langle1| = (0 \; 1). \tag{21}$$

It is easy to find

$$\rho = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & 2p & 0 \\ 0 & 2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}. \tag{22}$$

This density matrix has to be positive definite, all of its eigenvalues should be positive. The eigenvalues of ρ are given by $\frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1+3p}{4}$. For $0 < p < 1$, they are all positive.

Let's consider A the first subsystem and B the second subsystem. Let's apply the transpose to A only, that is called partial transpose to A,

$$\begin{aligned}
\rho^{T_A} \;=\; & \frac{p}{2}(\, |0\rangle\langle0| \otimes |1\rangle\langle1|) + (|1\rangle\langle0| \otimes |1\rangle\langle0|) + (|0\rangle\langle1| \otimes |0\rangle\langle1|) + (|1\rangle\langle1| \otimes |0\rangle\langle0|) \\
+ & \frac{1-p}{4} \, \mathbb{I}_{4\times4}.
\end{aligned} \tag{23}$$

In the matrix form we have

$$\rho^{T_A} = \frac{1}{4} \begin{pmatrix} 1-p & 0 & 0 & 2p \\ 0 & p+1 & 0 & 0 \\ 0 & 0 & p+1 & 0 \\ 2p & 0 & 0 & 1-p \end{pmatrix}. \tag{24}$$

According to the Peres-Horodecki criterion, ρ is separable if and only if both ρ and ρ^{T_A} are positive semi-definite, which means they are symmetric and all their eigenvalues are non-negative. In the case there exists at least one negative eigenvalue then ρ is entangled. The matrix ρ^{T_A} has 3 eigenvalues that are equal to $\frac{p+1}{4}$ and another lowest eigenvalue $\frac{1-3p}{4}$, $0 < p < 1$. In order to have both eigenvalues of ρ^{T_A} positive, p should be ranged between 0 and 1/3, that is ($0 < p < 1/3$). For instance for $p = 2/3$, we have the eigenvalues $\frac{5}{12}$; $\frac{5}{12}$; $\frac{5}{12}$; $-\frac{1}{4}$. The fact that one eigenvalue is negative violates the theorem and therefore for $p = 2/3$ the state ρ is entangled. The PPT criterion is so strong that it characterizes entanglement for 2×2 and 2×3 systems, which means necessary and sufficient for qubit-qubit and qubit-qutrit systems only.

3.5. Negativity Criterion

Negativity is a measure of quantum entanglement deriving from the PPT criterion for separability [46]. Let $\rho_{AB} = [\rho_{i\mu,j\nu}]$ be the density matrix of two qubits A and B and the entries of the density matrix that is partially transposed concerning A are given as in Equation (18). Negativity is defined as

$$\mathcal{N}(\rho) = \frac{||\rho^{T_A}||_{\mathrm{Tr}} - 1}{2} = \frac{\sum_i |\lambda_i| - 1}{2}, \tag{25}$$

where $\{\lambda_i\}$ are the eigenvalues of ρ^{T_A} and $|| \cdot ||_{\mathrm{Tr}}$ is the trace norm.

For separable states, the negativity is zero. In order to illustrate the negativity criterion, we consider the Werner's state

$$\rho = p \, |\psi\rangle\langle\psi| + \frac{1-p}{4} \mathbb{I}_{4\times 4}, \tag{26}$$

where $0 < p < 1$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The matrix ρ^{T_A} is given as in the Equations (23) and (24), and its eigenvalues are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which are respectively $\frac{p+1}{4}, \frac{p+1}{4}, \frac{p+1}{4}, \frac{1-3p}{4}$. For $0 < p < \frac{1}{3}$, we have

$$\sum_{i=1}^{4} |\lambda_i| = 1, \text{ hence } \mathcal{N}(\rho) = 0. \tag{27}$$

For $0 < p < \frac{1}{3}$, the Werner state is separable and this conclusion confirm the illustration in the PPT criterion for the separability condition of the Werner state.

3.6. The Reduction Criterion

Applying the positivity criterion to the positive map $\Lambda(\sigma) = \mathbb{I} \, \mathrm{Tr}\sigma - \sigma$ (with respect to the subsystems A and B), the following criterion was been provided [20,47].

Proposition 1. *For any separable state ρ_{AB}, $\rho_A \otimes \mathbb{I}_B - \rho_{AB} \geq 0$, $\mathbb{I}_A \otimes \rho_B - \rho_{AB} \geq 0$. We consider a density operator ρ_{AB} of a bipartite system AB. The reduced density operator for system A is defined by $\rho_A = Tr_B(\rho_{AB})$.*

The reduction criterion of separability is stated as follows.

Theorem 4. *If ρ_{AB} is separable then*

$$\rho_A \otimes \mathbb{I}_B - \rho_{AB} \geq 0, \text{ and } \mathbb{I}_A \otimes \rho_B - \rho_{AB} \geq 0, \tag{28}$$

where ρ_A is the reduced matrix for system A and ρ_B the reduced density of system B.

Some simple examples are given as follows:

Example 3. *Let's consider the state $|\psi\rangle_{AB} = |++\rangle_{AB} = |+\rangle_A|+\rangle_B$, the density matrix is then $\rho_{AB} = |++\rangle_{AB}\langle++|_{AB} = (|+\rangle_A\langle+|_A) \otimes (|+\rangle_B\langle+|_B)$. In the basis $\{|+\rangle, |-\rangle\}$. Let's determine now the reduced matrices ρ_A and ρ_B.*

$$\rho_A = Tr_B(\rho_{AB}) = |+\rangle_A\langle+|_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \rho_A \otimes \mathbb{I}_{2\times2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

$$\rho_B = Tr_A(\rho_{AB}) = |+\rangle_B\langle+|_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad \mathbb{I}_{2\times2} \otimes \rho_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Let's check the reduction criterion theorem.

$$\rho_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

and

$$\rho_A \otimes \mathbb{I}_{2\times2} - \rho_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbb{I}_{2\times2} \otimes \rho_B - \rho_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

$\rho_A \otimes \mathbb{I}_{2\times2} - \rho_{AB} \geq 0$ and $\mathbb{I}_{2\times2} \otimes \rho_B - \rho_{AB} \geq 0$ because it is easy to see that their eigenvalues are all greater or equal to zero, therefore the state is separable.

We consider now a second example that is the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$. The associated density matrix

$$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

$$\rho_A = \text{Tr}_B\rho_{AB} = \frac{1}{2}(|0\rangle_A\langle0|_A + |1\rangle_A\langle1|_A) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{34}$$

we have

$$\rho_A \otimes \mathbb{I}_{2\times2} - \rho_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

It is easy to check that $\lambda = -1/2$ is an eigenvalue of $\rho_A \otimes \mathbb{I}_{2\times2} - \rho_{AB}$, therefore the state $|\psi\rangle_{AB}$ is entangled since $\rho_A \otimes \mathbb{I}_{2\times2} - \rho_{AB}$ is not positive.

A remark is that the reduction criterion is a necessary and sufficient separability condition only for the dimensions 2×2 and 2×3 systems but it is just a necessary condition in higher dimensions.

Another remark is that there is another trick to recognize a pure entangled state related to reduced density is the following: if the reduced matrix ρ_A for a pure state ρ_{AB} is mixed then ρ_{AB} is entangled.

3.7. Concurrence Criterion

For a pure state $|\psi\rangle$ of a pair of qubits, the concurrence denoted $C(|\psi\rangle)$ is defined as

$$C(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle|, \tag{36}$$

where $|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$, with σ_y the Pauli operator $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $|\psi^*\rangle$ the complex conjugate of $|\psi\rangle$. Explicitly

$$|\tilde{\psi}\rangle = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} |\psi^*\rangle. \tag{37}$$

The state $|\tilde{\psi}\rangle$ is called the "spin flip" state of $|\psi\rangle$ [48,49]. The spin-flip operation, when applied to a pure product state, takes the state of each qubit to the orthogonal state.

A more clear definition of concurrence establishing a connection with entanglement is given in [49–51] as follows. A pure state $|\psi\rangle$ in a two-qubit system can be expressed in the standard basis as

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \eta|11\rangle, \tag{38}$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\eta|^2 = 1$. It can be shown that $|\psi\rangle$ is factorizable only under the case $\alpha\eta = \beta\gamma$. Therefore, the difference between $\alpha\eta$ and $\beta\gamma$ can be taken as a measurement of entanglement. In this way, a definition of the concurrence is given by

$$C(|\psi\rangle) = 2|\alpha\eta - \beta\gamma|. \tag{39}$$

From this definition, the concurrence is zero for separable state, and that is a criterion of separability. For example for a two qubit state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$, the concurrence is given by $C(|\psi\rangle) = 2|\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - 0 \times 0| = 1$.

The concurrence of a mixed state ρ of two qubits can be defined as the average concurrence of an ensemble of pure states representing ρ, minimized overall decomposition of ρ:

$$C(\rho) = \inf_{\{p_j, |\psi_j\rangle\}}\left\{\sum_j p_j C(\psi_j)\right\}, \tag{40}$$

where $\rho = \sum_j p_j|\psi_j\rangle\langle\psi_j|$ and the states $|\psi_j\rangle$ are distinct normalized pure states of the bipartite system, not necessarily orthogonal. As a consequence, a state ρ is separable if and only if $C(\rho) = 0$.

An explicit formula for concurrence is given by [50] as

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{41}$$

where $\lambda_i(i = 1, 2, 3, 4)$ is the non-negative eigenvalue of the Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, in decreasing order. Here $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where ρ^* is the complex conjugate of ρ, and as a mixed state $\rho = \sum_j p_j|\psi_j\rangle\langle\psi_j|$.

3.8. The Majoration Criterion

We start by defining what we mean by majoration. Consider two d-dimensional real vectors

$$x = (x_1, x_2, \ldots, x_d) \quad \text{and} \quad y = (y_1, y_2, \ldots y_d). \tag{42}$$

It is often assumed in addition that x and y are probability distributions, that is, the components are non-negative and sum to one as follows.

$$x_i \geq 0, \quad 1 \leq i \leq d \quad \text{and} \quad \sum_{i=1}^{d} x_i = 1; \tag{43}$$

$$y_i \geq 0, \quad 1 \leq i \leq d \quad \text{and} \quad \sum_{i=1}^{d} y_i = 1. \tag{44}$$

Let's introduce now the notation $\downarrow$ to denote the components of a vector rearranged into decreasing order. So $x^{\downarrow} = \left(x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_d^{\downarrow} \right)$ in the sense that $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_d^{\downarrow}$. The vector $x^{\downarrow}$ is majorized by the vector $y^{\downarrow}$, that is denoted $x^{\downarrow} \prec y^{\downarrow}$ when

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}, \quad 1 \leq k \leq d - 1, \tag{45}$$

and the equality holds for $k = d$, with d being the dimension of the vectors. More details about the majoration can be found in the references [52,53]. We consider now the density matrix ρ_{AB} of a pair of qubits. Let ρ_A be the reduced density matrix of the system of the qubit A and ρ_B be the reduced density matrix of qubit B. If $\lambda_{\rho_{AB}}$ consist of the eigenvalues of ρ_{AB} and λ_{ρ_A}, λ_{ρ_B} the eigenvalues of ρ_A, ρ_B respectively,

The majoration criterion of separability says that if the state ρ_{AB} is separable then

$$\lambda_{\rho_{AB}}^{\downarrow} \prec \lambda_{\rho_A}^{\downarrow}, \quad \text{and} \quad \lambda_{\rho_{AB}}^{\downarrow} \prec \lambda_{\rho_B}^{\downarrow}. \tag{46}$$

Zeros are appended to the vectors $\lambda_{\rho_A}^{\downarrow}$ and $\lambda_{\rho_B}^{\downarrow}$ in the Equation (46) to make their dimensions equal to the one of $\lambda_{\rho_{AB}}$. For a separable state the ordered vector of eigenvalues of the whole density operator is majorized by the ones of the reduced density matrices [53]. An important remark is that the spectra of a density matrix and its reduced density matrices do not allow for distinguishing separable and entangled states. The majoration criterion is only a necessary and not a sufficient condition of separability. For the system of two qubits, the operational separability criteria follow this logical ordering:

ρ_{AB} separable $\Leftrightarrow$ ρ_{AB} satisfies the reduction criterion $\Rightarrow$ ρ_{AB} satisfies the majoration criterion.

As illustration, we consider the Werner state

$$\rho_{AB} = p|\psi\rangle\langle\psi| + \frac{1-p}{4}\mathbb{I}_{4\times4}, \tag{47}$$

where $0 \leq p \leq 1$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state has already been studied and its matrix matrix form is

$$\rho_{AB} = \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & 2p & 0 \\ 0 & 2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}. \tag{48}$$

Its eigenvalues are

$$\lambda_{\rho_{AB}}^{\downarrow} = \left\{ \frac{3p+1}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4} \right\}, 0 \leq p \leq 1. \tag{49}$$

The reduced density matrix ρ_A and the reduced density matrix ρ_B are respectively given in matrix form as

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_B. \tag{50}$$

The eigenvalues of ρ_A and ρ_B respectively appended by zeros are given respectively by

$$\lambda_{\rho_A}^{\downarrow} = \left\{ \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}; \quad \lambda_{\rho_B}^{\downarrow} = \left\{ \frac{1}{2}, \frac{1}{2}, 0, 0 \right\}, \quad 0 \le p \le 1. \tag{51}$$

All conditions are fulfilled to check the separability condition using the majoration criterion. We know that $0 \le p \le 1/3$, is separable according to the PPT criterion and this is fulfilled by

$$\sum_{j=1}^{k} \lambda_{\rho_{AB},j}^{\downarrow} \le \sum_{j=1}^{k} \lambda_{\rho_A,j}^{\downarrow}, \quad \text{and} \quad \sum_{j=1}^{k} \lambda_{\rho_{AB},j}^{\downarrow} \le \sum_{j=1}^{k} \lambda_{\rho_B,j}^{\downarrow} \tag{52}$$

so

$$\lambda_{\rho_{AB}}^{\downarrow} \prec \lambda_{\rho_A}^{\downarrow} \quad \text{and} \quad \lambda_{\rho_{AB}}^{\downarrow} \prec \lambda_{\rho_B}^{\downarrow} \tag{53}$$

One can realize that for $p > 1/3$ the majoration condition is not satisfied.

3.9. The Computable Cross Norm or Realignment (CCNR) Criterion

For finite-dimensional systems, a criterion for separability is the so-called computable cross norm or realignment [CCNR] criterion proposed by Rudoph [54,55], Chen and Wu [56].

The CCNR criteria have been found in two different forms, that are, the computable cross norm criterion by Rudolph and the the realignment criterion by Chen and Wu. The separability criterion is given either by defining a new norm (a cross norm) or by realigning the density operator and then taking the usual trace norm of the realigned matrix.

1. The computable cross norm (CCN) criterion is an analytical and computable separability criterion for bipartite quantum states developed by O. Rudolph [54], known to systematically detect bound entanglement and complements in certain aspects of the well-known Peres positive partial transpose (PPT) criterion. It can be formulated in different equivalent ways. A very useful and instructive way is the following procedure [57]. Consider a quantum state ρ_{AB} defined on a tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We denote the canonical real basis $(|i\rangle)_i$ and expand ρ_{AB} in terms of the operators $E_{ij} = |i\rangle\langle j|$, we write

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl} E_{ij} \otimes E_{kl}. \tag{54}$$

Next, we define an operator $\mathcal{U}(\rho_{AB})$ that acts on $T(\mathcal{H}_A \otimes \mathcal{H}_B)$ by

$$\mathcal{U}(\rho_{AB}) = \sum_{ijkl} \rho_{ijkl} |E_{ij}\rangle\langle E_{kl}|, \tag{55}$$

where $T(\mathcal{H}_A \otimes \mathcal{H}_B)$ denotes the trace class operators on $\mathcal{H}_A \otimes \mathcal{H}_B$. Here, the operator $|E_{ij}\rangle$ denotes the ket vector with respect to Hilbert Schmidt inner product $\langle A, B \rangle \equiv \text{Tr}(A^\dagger B)$ in $T(\mathcal{H}_A \otimes \mathcal{H}_B)$. We also write $||A||_2 \equiv \langle A, A \rangle^{1/2}$. The norm $||A||_2$ is often called the Hilbert-Schmidt norm or the Frobenius norm of A and is equal to the sum of the squares of the singular values of A. The sum of the absolute values of the singular values of A is called the trace class norm, or simply trace norm, and is denoted by $||A||_1$. The CCN criterion is then formulated as follows. *The CCN criterion asserts that if ρ is separable, then the trace norm of $\mathcal{U}(\rho)$ is less than or equal to one. Whenever a quantum state ρ satisfies $||\mathcal{U}(\rho)||_1 > 1$, this signals that ρ is entangled.*

Let's consider the following two qubit example. We consider two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ span by $\{|0\rangle, |1\rangle\}$, respectively. Next we consider the family of states on $\mathcal{H}_A \otimes \mathcal{H}_B$

$$\rho_p = p|00\rangle\langle00| + (1-p)|\psi\rangle\langle\psi|, \tag{56}$$

where $0 \leq p \leq 1$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The matrix operator in Equation (56) can be expanded as follows

$$\begin{aligned}
\rho_p &= p|0\rangle\langle0| \otimes |0\rangle\langle0| + \frac{1-p}{2}|0\rangle\langle0| \otimes |1\rangle\langle1| + \frac{1-p}{2}|0\rangle\langle1| \otimes |1\rangle\langle0| \\
&+ \frac{1-p}{2}|1\rangle\langle0| \otimes |0\rangle\langle1| + \frac{1-p}{2}|1\rangle\langle1| \otimes |0\rangle\langle0|.
\end{aligned} \tag{57}$$

Defining now $E_{ij} = |i\rangle\langle j|$, we have

$$\rho_p = pE_{00} \otimes E_{00} + \frac{1-p}{2}(E_{00} \otimes E_{11} + E_{01} \otimes E_{10} + E_{10} \otimes E_{01} + E_{11} \otimes E_{00}). \tag{58}$$

Next, we define an operator $\mathcal{U}(\rho_p)$ that acts on $T(\mathcal{H}_A \otimes \mathcal{H}_B)$ by

$$\mathcal{U}(\rho_p) = p|E_{00}\rangle\langle E_{00}| + \frac{1-p}{2}(|E_{00}\rangle\langle E_{11}| + |E_{11}\rangle\langle E_{00}| + |E_{01}\rangle\langle E_{10}| + |E_{10}\rangle\langle E_{01}|), \tag{59}$$

where $|E_{ij}\rangle$ denotes the ket vector with respect to Hilbert-Schmidt inner product. The trace class norm of $\mathcal{U}(\rho_p)$ can be computed as follows [55].

$$\begin{aligned}
||\mathcal{U}(\rho_p)||_1 &= 1 - p + \sqrt{\frac{p^2}{2} + \frac{(1-p)^2}{4} + \frac{p}{2}\sqrt{p^2 + (1-p)^2}} \\
&+ \sqrt{\frac{p^2}{2} + \frac{(1-p)^2}{4} - \frac{p}{2}\sqrt{p^2 + (1-p)^2}} \\
&\geq 1 - p + p\sqrt{1 + \frac{(1-p)^2}{2p^2}} \geq 1.
\end{aligned} \tag{60}$$

We have egality if and only if $p = 1$, so ρ_p is separable if and only if $p = 1$.

The CCN criterion is in general not a sufficient criterion for separability in dimension 2×2. For two qubit states with maximally disordered subsystems the CCN criterion is necessary and sufficient. It is shown that for all pure states, for Bell diagonal states, for Werner states in dimension $d = 2$, and for isotropic states in arbitrary dimensions, the CCN criterion is necessary and sufficient.

2. A matrix realignment criterion

Motivated by the Kronecker product approximation technique, a method to assess the inseparability of bipartite quantum systems, based on a realigned matrix constructed from the density matrix has been developed by Chen and Wu [56]. Let's define the realigned matrix as follows.

We consider a bipartite system A, B represented by two Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimension d_A and d_B respectively. Let's consider now a density matrix ρ_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, the realigned matrix $\mathcal{R}(\rho_{AB})$ is such that the matrix elements are

$$\langle a_i, b_l| \mathcal{R}(\rho_{AB}) |b_k, a_j\rangle = \langle a_i, b_k| \rho_{AB} |a_j, b_l\rangle, \tag{61}$$

where $\{|a_i, b_k\rangle\}$, $i = 1, \dots, d_A; k = 1, \dots, d_B$, is a basis of $\mathcal{H}_A \otimes \mathcal{H}_B$. The realigned matrix is of dimension $d_A^2 \times d_B^2$.

Consider a bipartite state ρ_{AB} acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ as

$$\rho_{AB} = \sum_{i,j} \sum_{k,l} c_{ijkl} |a_i, b_k\rangle\langle a_j, b_l|, \tag{62}$$

and its realigned matrix

$$\mathcal{R}(\rho_{AB}) = \sum_{i,j} \sum_{k,l} c_{ijkl} |a_i, b_l\rangle\langle b_k, a_j|, \tag{63}$$

The realignment criterion based on the matrix $\mathcal{R}(\rho_{AB})$ states that if the state ρ_{AB} is separable, then $|| \, \mathcal{R}(\rho_{AB}) \, ||_1 \leq 1$ must hold [20].

The CCN criterion and the realignment criterion are equivalent and known under computable cross norm or realignment criterion (CCNR).

The CCNR criterion states that if ρ_{AB} is a separable state on $\mathcal{H}_A \otimes \mathcal{H}_B$ with $dim(\mathcal{H}_A \otimes \mathcal{H}_B) < +\infty$, then the trace norm $||\mathcal{R}(\rho_{AB})||_1$ of the realignment matrix $\mathcal{R}(\rho_{AB})$ of ρ_{AB} is not greater than 1.

3.10. The Correlation Matrix (Or De Vicente) Criterion (2008)

The correlation matrix (or de Vicente) criterion involves the Bloch representation of density operators. About Bloch operators (density matrices), one may read in [58,59]. The criterion, developed in 2007 by de Vicente, is a necessary condition and can detect PPT entangled states [60]. The correlation matrix criterion is formulated as follows.

In the case of $d_A \times d_B$ bipartite quantum systems $d_A \leq d_B$, the Bloch representation, also known as Fano form [61] can be written as follows.

$$\rho_{AB} = \frac{1}{d_A d_B}\mathbb{I}_{d_A} \otimes \mathbb{I}_{d_B} + \frac{1}{2d_B}\vec{r} \cdot \vec{\lambda} \otimes \mathbb{I}_{d_B} + \frac{1}{2d_A}\mathbb{I}_{d_A} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} \tau_{ij}\lambda_i \otimes \sigma_j, \quad (64)$$

where $\mathbb{I}_{d_A}$ and $\mathbb{I}_{d_B}$ are the identity matrices, $\vec{r}$ and $\vec{\sigma}$ have the components $r_i = \text{Tr}[\rho_{AB}(\lambda_i \otimes \mathbb{I}_{d_B})]$, $i = 1, 2, \ldots d_A^2 - 1$ and $s_j = \text{Tr}[\rho_{AB}(\mathbb{I}_{d_A} \otimes \sigma_j)]$, $j = 1, 2, \ldots d_B^2 - 1$, and the correlation matrix $\tau_{ij} = \text{Tr}[\rho_{AB}(\lambda_i \otimes \sigma_j)]$. The vector $\vec{\lambda}$ is defined to be $\lambda \equiv (\lambda_1, \lambda_2 \ldots \lambda_{d_A^2-1})^T$, with λ_i, being the generators of $SU(d_A)$. The vector $\vec{\sigma}$ is defined similarly with σ_j, being the generators of $SU(d_B)$. Here $\vec{r}$ and $\vec{s}$ are called the Bloch vectors of the density matrices.

If the state is separable, then the inequality [60],

$$||\tau||_1 \leq \sqrt{\frac{4(d_A - 1)(d_B - 1)}{d_A d_B}}. \quad (65)$$

must hold, with τ the matrix corresponding to the matrix elements τ_{ij}.

The matrix τ accounts for the possible correlations between the subsystems and is therefore called the correlation matrix. Note that if $\tau = 0$, then the state ρ is separable, but the converse is not true. Physically, this necessary condition means that there is an upper bound to the amount of correlation contained in a separable state, which means that the correlations in separable states cannot be "too large". For the case of two qubits system with a maximally mixed reduced density matrix, the criterion is necessary and sufficient in reference to example 2 in [60].

We have $d_A = d_B = 2$, so

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{I}_2 \otimes \mathbb{I}_2 + \sum_{i=1}^{3} r_i\lambda_i \otimes \mathbb{I}_2 + \sum_{j=1}^{3} s_j\mathbb{I}_2 \otimes \sigma_j + \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\lambda_i \otimes \sigma_j\right), \quad (66)$$

where $\{\lambda_i\}_{i=1}^{3}$, $\{\sigma_i\}_{i=1}^{3}$ denotes the generators of $SU(2)$, and $r_i = \text{Tr}(\lambda_i \otimes \mathbb{I}_2 \, \rho_{AB})$, $s_i = \text{Tr}(\mathbb{I}_2 \otimes \sigma_i \, \rho_{AB})$, $i = 1, 2, 3$. The matrix τ is given by $\tau_{ij} = \text{Tr}(\lambda_i \otimes \sigma_j)\rho_{AB}$, $i, j = 1, 2, 3$. The three generators of $SU(2)$ are Pauli matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1; \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2; \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \quad (67)$$

The separability criterion (65) for the case of two qubits becomes

$$||\tau||_1 \leq 1. \quad (68)$$

3.11. Enhanced Entanglement Criterion via SIC POVMs

This criterion is based on a special measurement called the symmetric informationally complete positive operator valued measures (SIC POVMs) proposed in 2018 by Shang, Asadian, Zhu and Gühne [62]. Let's recall first the definition of positive-operator valued measure (POVM) as follows.

A set of operator $\{O_i, \ i \in M\}$ is called a POVM if each operator O_i satisfies the following properties: (1) Hermitian: $O_i = O_i^\dagger$; (2) O_i is positive semi-definite: $O_i \geq 0$; (3) $\{O_i, \ i \in M\}$ forms a resolution of the identity on M: $\sum_{i \in M} O_i = \mathbb{I}_M$. The probability of obtaining outcome i for a given state specified by density matrix ρ is given by

$$p_i = Tr(O_i \rho). \tag{69}$$

A POVM in a *d*-dimensional Hilbert space is called informationally complete if the probabilities p_i uniquely determine the density operator, which means that it must contain at least d^2 elements to span the Hilbert Schmidt space $\mathcal{HS}(\mathcal{H})$, with d the dimension of $\mathcal{H}$ [63].

The POVM is said to be symmetric informationally complete (SIC) known as SIC POVMs if it is composed of d^2 elements

$$\Pi_k = \frac{1}{d} |\psi_k\rangle\langle\psi_k|, \tag{70}$$

which are subnormalized rank 1 projectors onto pure states with equal pairwise fidelity, that is

$$|\langle\psi_k|\psi_l\rangle|^2 = \frac{d \, \delta_{kl} + 1}{d + 1}, \quad k, l = 1, \dots d^2, \tag{71}$$

and satisfying the completeness condition $\sum_{k=1}^{d^2} \Pi_k = \mathbb{I}$. It is still a conjecture that SIC POVMs exist in all finite dimensions.

Given a SIC POVM $\mathcal{M}_s = \{\Pi_k\}_{k=1}^{d^2}$ and a quantum state ρ, the probability of obtaining outcome k is given by the Born rule, $p_k = \langle \pi_k \rangle = Tr(\rho \Pi_k)$.

Consider now a bipartite state ρ acting on the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimension $d_{AB} = d_A \times d_B$, and let $\{E_k^A\}_{k=1}^{d_A^2}$ and $\{E_k^B\}_{k=1}^{d_B^2}$ be normalized SIC POVMs for the two respective subsystems: the linear correlations between the two SIC POWMs E^A and E^B are

$$[P_s]_{kl} = Tr(E_k^A \otimes E_l^B \, \rho) \tag{72}$$

from which the entanglement criterion based on the SIC POVMs, called ESIC criterion is as follows.

If a state ρ is separable, then $||P_s||_1 \leq 1$ has to hold, otherwise, it is entangled.

While the existence of SIC-POVMs for all dimensions is yet to be proven, there are analytical and/or numerical solutions for dimensions up to 151.

4. Non Operational Separability Criteria

Non operational separability criteria do not provide us with a simple procedure to check the separability properties of a given state. So they are not easy to use. However, they are necessary and sufficient criteria for any bipartite system.

4.1. Positive Maps Criterion

A standard approach to deciding on the separability of a given mixed state relies on positive maps. Quantum entanglement theory is linked with the theory of positive maps [11,12,14,64,65]. A map $\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}))$ is called positive if it maps positive operators on positive ones, means

$$\Lambda(\rho) \geq 0, \quad \text{for all} \quad \rho \geq 0, \tag{73}$$

where the positivity of an operator ρ is to state that ρ is positive semi-definite that means ρ has only non-negative eigenvalue. Here $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$. A crucial property of positive maps is that a trivial extension $\Lambda \otimes \mathbb{I}$ is not necessarily positive and this fact can be used to conclude the separability of a mixed state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. The positivity of Λ means that $\Lambda(X) \geq 0$ for any $X \geq 0$. Recall that X is positive (which is denoted by $X \geq 0$) if and only if $\langle \psi | X | \psi \rangle \geq 0$ for any vector $|\psi\rangle$. This is equivalent to the requirement that X is a Hermitian operator with nonnegative spectrum [20]. The positive maps criterion of separability is as follows.

A state ρ_{AB} defined on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if for all possible linear maps

$$\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A) \tag{74}$$

one has $(\mathbb{I} \otimes \Lambda)\rho_{AB} \geq 0$.

While the separability of pure states can easily be checked, it turns out to be much more difficult to decide whether a given mixed state is entangled or separable. This statement does not allow us to derive a sufficient separability criterion for the very general case, since the classification of positive maps is still an unsolved problem.

4.2. The Entanglement Witnesses

There is a natural characterization of separable states in terms of mean values of Hermitian operators [20,66]. The scalar separability criteria are represented as some bounds on values of scalar functions of the density matrix ρ [66,67]. The term entanglement witness was first used by Terhal in 2000 [66], and refers to operators that can detect entangled states. The definition of entanglement witness is as follows.

A Hermitian operator W acting on a bipartite system $\mathcal{H}_{AB}$ is called an entanglement witness if it satisfies the following properties:

$$\langle \phi \otimes \psi | \, W \, | \phi \otimes \psi \rangle \geq 0, \quad \forall \quad |\phi\rangle \otimes |\psi\rangle \in \mathcal{H}_{AB}; \tag{75}$$

$$\exists \, |\chi\rangle \in \mathcal{H}_{AB}, \, \langle \chi \, | \, W \, | \chi \rangle < 0. \tag{76}$$

The criterion based on entanglement witness is as follows.

Proposition 2. *The state ρ is separable if and only if $Tr(\rho W) \geq 0$ for all Hermitian operators W called entanglement witnesses (EW) such that*

- *$Tr\,(\sigma W) \geq 0$ for all separable σ;*
- *there exists some entangled ρ such that $Tr\,(W\rho) < 0$.*

Any fixed entanglement witness W provides a necessary condition for separability $\mathrm{Tr}(W\rho) \geq 0$. The first explicit example of operators satisfying the properties in the proposition was provided in [37] as follows.

This was "flip operator" V defined for $d \otimes d$ systems as $V|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle$ for all $|\phi\rangle, \psi\rangle \in C^d$. It reveals entanglement for $|\psi_-\rangle$ as $\mathrm{Tr}(V|\psi_-\rangle\langle\psi_-|) = -1 < 0$, with $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

The separability criterion based on entanglement witnesses is a necessary and sufficient criterion but not practical. Some widely encountered entanglement witnesses and their properties are given in [66,68–70]. The geometric entanglement witnesses are discussed in [71], and are of great use.

4.3. Local Uncertainty Relations (LURs) Criterion

The local uncertainties relations (LURs) by Hofmann and Takeuchi [72] are based on the reformulation of the uncertainty principle, adapting it to arbitrary properties of N-level systems and providing unconditional limitations for the predictabilities of measurement outcomes for any selection of non-commuting physical properties. Local uncertainty limits valid for all non-entangled states can be then derived. Since no separable quantum state can

overcome these limits, any violation of such uncertainty relations is an unambiguous proof of entanglement. The local uncertainty relations (LURs) criterion is formulated as follows.

We consider a bipartite system AB represented by the Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$. Given some non-commuting observables A_k on $\mathcal{H}_A$ and B_k on $\mathcal{H}_B$, one may compute strictly positive numbers C_A and C_B such that

$$\sum_{k=1}^{n} \Delta^2(A_k) \geq C_A , \quad \sum_{k=1}^{n} \Delta^2(B_k) \geq C_B \tag{77}$$

holds for all states for system A, respectively system B. Here,

$$\Delta^2(A) = < A^2 > - < A >^2 \tag{78}$$

denotes the variance of an observable A. It can then be proved that for separables states

$$\sum_{k=1}^{n} \Delta^2(A_k \otimes \mathbb{I} + \mathbb{I} \otimes B_k) \geq C_A + C_B \tag{79}$$

has to hold. Any quantum state that violates the Equation (79) is entangled. The physical interpretation of (79) may be stated as follows: separable states always inherit the uncertainty relations that hold for their reduced states [73]. The LURs criterion is strong and can be implemented with local measurements. Nevertheless, they have some disadvantages in the sense it is not clear which operators A_k and B_l one should choose to detect a given entangled state. Also, LURs can by construction characterize separable states only and they do not apply to other convex sets. More about LURs criterion and its comparison with other criteria, for instance, the CCN criterion and the criterion based on witness can be found in [72,74,75].

4.4. The Li-Qiao Criterion

A recent criterion for the separability of an arbitrary bipartite mixed state is proposed by Li and Qiao [76] by virtue of the multiplicative Horn's problem [77–79]. In Li-Qiao's criterion, a complete and finite set of inequalities to determine the separability of the compound system is obtained, which may be viewed as trade-off relations between the quantumness of subsystems. Li-Qiao's work follows the work initiated by Horodecki and al. [20] and uses the Bloch vector representation introduced to the separability problem by J. De Vicente [60]. The Li-Qiao's procedure provides the possibility to solve some problems beyond limited PPT technique.

Let's recall that a mixed bipartite state of particles *A* and *B* is separable if and only if it can be expressed as

$$\rho_{AB} = \sum_{i=1}^{L} p_i \rho_{A,i} \otimes \rho_{B,i}, \tag{80}$$

where $p_i > 0$ with $\sum_{i=1}^{L} p_i = 1$, and $\rho_{A,i}$, $\rho_{B,i}$ are local density matrices of the particles *A* and *B*. In the De Vicente criterion, we have seen that an arbitrary $d_A \times d_B$ dimensional bipartite state in the Bloch representation is

$$\rho_{AB} = \frac{1}{d_A d_B} \mathbb{I}_{d_A} \otimes \mathbb{I}_{d_B} + \frac{1}{2d_B} \vec{r} \cdot \vec{\lambda} \otimes \mathbb{I}_{d_B} + \frac{1}{2d_A} \mathbb{I}_{d_A} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{i=1}^{d_A^2-1} \sum_{j=1}^{d_B^2-1} \tau_{ij} \lambda_i \otimes \sigma_j. \tag{81}$$

Since a bipartite state is separable if it can be decomposed as the sum of direct products of local density matrices as shown in Equation (80). The Li-Qiao necessary and sufficient condition for the separability of ρ_{AB} in Equation (81) is given by

$$\sum_{i=1}^{L} p_i \vec{a}_i = \vec{r}; \quad \sum_{j=1}^{L} p_j \vec{b}_j = \vec{s}, \quad \sum_{k=1}^{L} p_k \vec{a}_k \vec{b}_k^T = \tau, \tag{82}$$

where L stands for the number of local states needed in the separable decomposition, $p_i > 0$, $\sum_{i=1}^{L} p_i = 1$ and $\rho_i^A = \frac{1}{d_A}\mathbb{I} + \frac{1}{2}\vec{a}_i \cdot \vec{\lambda}$ and $\rho_i^B = \frac{1}{d_A}\mathbb{I} + \frac{1}{2}\vec{b}_i \cdot \vec{\sigma}$, with $\vec{a}_i$, $\vec{b}_j$ being the Block vectors of the decomposed local quantum states. The Equation (82) can be set in matrix form as follows.

$$M_a \vec{p} = \vec{r}; \quad M_b \vec{p} = \vec{s}; \quad M_{ap} M_{bp}^T = \tau. \tag{83}$$

Here $M_a = (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_L)$, and $M_b = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_L)$, with $\vec{a}_i$, $\vec{b}_j$ being $d_A^2 - 1$, $d_B^2 - 1$ dimensional real vectors respectively; $\vec{p} = (p_1, p_2, \ldots p_L)^T$, and $M_{ap} = M_a D_p^{\frac{1}{2}}$, $M_{bp} = M_b D_p^{\frac{1}{2}}$ with $D_p = \text{diag}\{p_1, p_2, \ldots, p_L\}$.

Though in principle the Li-Qiao criterion is necessary and sufficient, its physical significance and practical applications need to be exemplified.

4.5. Simultaneous Hollowisability Matrix Criterion

This criterion is a generalization of the concurrence approach to investigate the general multipartite separability problem [19]. While the concurrence criterion is operational, the criterion based on a simultaneous hollowisation matrix analysis is nonoperational. Neven and Bastin extended the preconcurrence matrix formalism to arbitrary multipartite systems and showed that the separability problem can be formulated equivalently as a pure matrix analysis problem that consists of determining whether a given set of symmetric matrices is simultaneously unitarily congruent to hollow matrices, i.e., to matrices whose main diagonal is only composed of zeroes.

5. Conclusions

In this work, we have revisited the following criteria in bipartite systems focusing on the case of a system of two quantum bits: (1) the Bell inequalities criterion, (2) the Schmidt decomposition criterion, (3) the entropy of entanglement criterion, (4) the PPT or Horodecki-Peres criterion, (5) the negativity criterion, (6) the reduction criterion, (7) the concurrence criterion, (8) the majoration criterion, (9) the computable cross norm or realignment (CCNR) criterion, (10) the correlation matrix or de Vicente criterion, (11) the positive maps criterion, (12) the criterion based on entanglement witnesses, (13) the local uncertainty relations (LURs) criterion, (14) the Li-Qiao criterion, (15) the SIC POVMs criterion, (16) the simultaneous hollowisability matrix criterion has also been briefly mentioned.

While we have listed positive maps and entanglement witness criteria in the "non-operational" criteria, it is good to draw attention to the fact that these two criteria are the basis for many of the operational criteria and are nothing but examples of positive but not completely positive maps. As for entanglement witnesses, it is common in experiments to know the prepared state beforehand and to use an entanglement witness tailored to the experiment. From this point of view, entanglement witnesses are more operational than any other criterion.

All the criteria of separability are mathematically based, for instance the characterization and classification of positive maps on $C^\star$ algebras [11–14], the theory of positive maps [11,12,14,64], the multiplicative Horn's problem [77–79], the theory of majoration [80]. The Bell's inequality is only a mathematical theorem and the relation between Bell's inequalities and convex geometry is also well-known [81]. The use of uncertainty arguments to study entanglement is well known from continuous variables system [2,82,83].

In this work, We do not provide demonstrations of the theorems, and propositions, nor discuss the comparison between the various criteria since these elements are in the references cited. We do not pay attention to the higher dimensional case. For instance, we do not include the criterion for entanglement detection based on covariance matrices for an arbitrary set of observable which is said to be suitable for higher dimensions [84].

This review provides a good starting point for reading about the topic and directs the interested reader to more in-depth resources. It is certainly constructive and an inspiration

to investigate in the following research directions. (1) The separability of quantum states is directly linked to some unsolved challenges of mathematics concerning linear algebra and geometry, functional analysis and, in particular, the theory of $C^\star$-algebra. For instance the Gel'fand, Naimark and Segal (GNS) construction of a representation of $C^\star$- algebra of observable allows obtaining a representation space for the sub-algebra such that its decomposition into irreducible sub-spaces can be used to study quantum correlations [85]. (2) The distillability problem, that is the question of when the state of a composite quantum system can be transformed to an entangled pure state using local operations, is another problem that is related to challenging open questions of modern mathematics. (3) Since the entanglement measures can be used to identify and characterize quantum phase transitions, the present work is inspiring to investigate the role of entanglement in quantum phase transitions [86]. (4) The non-local two-qubit operations from a geometric perspective are also one of the interesting direction to investigate [87].

Author Contributions: Conceptualization, H.G. and L.G.; investigation, H.G.; writing original draft, H.G.; writing, review and editing, L.G.; supervision, L.G. All authors have read and agreed to the published version of the manuscript.

References

1. Bruss, D.; Cirac, J.I.; Horodecki, P.; Hulpke, F.; Kraus, B.; Lewenstein, M.; Sanpera, A. Reflections upon separability and distillability. *J. Mod. Opt.* **2001**, *49*, 8.
2. Einstein, A.; Podolsky, B.; Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **1935**, *47*, 777. [CrossRef]
3. Schrödinger, E. Discussion of probability relations between separated systems. *Proc. Camb. Phil. Soc.* **1935**, *31*, 555–563. [CrossRef]
4. Wheeler, J.A.; Zurek, W.H. (Eds.) *Quantum Theory and Measurement*; Princeton University Press: Princeton, NJ, USA, 1983.
5. Peres, A. *Quantum Theory: Concepts and Methods*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1995.
6. Bouwmeester, D.; Ekert, A.; Zeilinger, A. (Eds.) *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation*; Springer: Berlin/Heidelberg, Germany, 2000.
7. Nielsen, M.; Chuang, I. *Quantum Computation and Quantum Information*; Cambridge University Press: Cambridge, UK, 2000.
8. Bruss, D.; Lütkenhaus, N. Quantum key distribution: From principles to practicalities. *Appl. Algebra Eng. Commun. Comput.* **2000**, *10*, 383.
9. Gisin, N.; Ribordy, G.; Tittel, W.; Zbinden, H. Quantum cryptography. *Rev. Mod. Phys.* **2002**, *74*, 145. [CrossRef]
10. Bennett, C.H.; Brassard, G.; Popescu, S.; Schumacher, B.; Smolin, J.A.; Wootters, W.K. Purification of noisy entanglement and faithful teledeportation via noisy channels. *Phys. Rev. Lett.* **1996**, *76*, 722. [CrossRef] [PubMed]
11. St⌀rmer, E. Positive linear maps of operator algebras. *Acta Math.* **1963**, *110*, 233.
12. Jamio lkowski, A. Linear transformations which preserve trace and positive semidefiniteness of operators. *Rep. Math. Phys.* **1972**, *3*, 275. [CrossRef]
13. Choi, M.D. Positive semidefinite biquadratic forms. *Linear Algebra Appl.* **1975**, *12*, 95. [CrossRef]
14. Woronowicz, S.L. Positive maps of low dimensional matrix algebras. *Rep. Math. Phys.* **1976**, *10*, 165. [CrossRef]
15. Horodecki, P.; Rudnicki, L.; Zyczkowski, K. Five open problems in quantum information theory. *PRX Quantum* **2022**, *3*, 010101. [CrossRef]
16. Bohm, D. *Quantum Theory*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1951.
17. Horodecki, R.; Horodecki, P.; Horodecki, M.; Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **2009**, *81*, 865–942. [CrossRef]
18. Yu, C.S.; Song, H.S. Global entanglement for multipartite quantum states. *Phys. Rev. A* **2006**, *73*, 022325. [CrossRef]
19. Neven, A.; Bastin, T. The quantum separability problem is a simultaneous hollowisation matrix analysis problem. *J. Phys. A Math. Theor.* **2018**, *51*, 315305. [CrossRef]
20. Horodecki, M.; Horodecki, P.; Horodecki, R. Separability of mixed states: Necessity and sufficient condition. *Phys. Lett. A* **1996**, *223*, 1–8. [CrossRef]
21. Heisenberg, W. *The Principles of the Quantum Theory*; S. Hirzel Verlag: Stuttgart, Germany; University of Chicago Press: Chicago, IL, USA; Dover Publications: Mineola, NY, USA, 1930.
22. Heisenberg's 1958 Collection of Essays, Physics and Philosophy. Available online: https://riskservices.com/physics-and-philosophy-by-heisenberg-1958/ (accessed on 18 August 2023).

23. Einstein Attacks Quantum Theory. *New York Times*, 4 May 1935.
24. Bell, J.S. On the Einstein-Podolsky-Rosen paradox. *Physics* **1964**, *1*, 195–200. [CrossRef]
25. Bell, J.S. On the problem of hidden variables in quantum mechanics. *Rev. Mod. Phys.* **1966**, *38*, 447–452. [CrossRef]
26. Clauser, J.F.; Horne, M.A.; Shimony, A.; Holt, R.A. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.* **1969**, *23*, 880–884. [CrossRef]
27. Aspect, A.; Grangier, P.; Roger, G. Experimental tests of realistic local theories via Bell's theorem. *Phys. Rev. Lett.* **1981**, *47*, 460. [CrossRef]
28. Aspect, A.; Grangier, P.; Roger, G. Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of Bell's inequalities. *Phys. Rev. Lett.* **1982**, *49*, 91 [CrossRef]
29. Hardy, L. Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories. *Phys. Rev. Lett.* **1992**, *68*, 2981. [CrossRef] [PubMed]
30. Hardy, L. Nonlocality for two particles without inequalities for almost all entangled states. *Phys. Rev. Lett.* **1993**, *71*, 1665. [CrossRef] [PubMed]
31. Arraut, I. The solution to the Hardy's paradox. *arXiv* **2021**, arXiv:2106.06397.
32. Hensen, B.; Bernien, H.; Dréau, A.E.; Reiserer, A.; Kalb, N.; Blok, M.S.; Ruitenberg, J.; Vermeulen, R.F.; Schouten, R.N.; Abellán, C.; et al. Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature* **2015**, *526*, 682–686. [CrossRef]
33. Shalm, L.K.; Meyer-Scott, E.; Christensen, B.G.; Bierhorst, P.; Wayne, M.A.; Stevens, M.J.; Gerrits, T.; Glancy, S.; Hamel, D.R.; Allman, M.S.; et al. Strong Loophole-Free Test of Local Realism. *Phys. Rev. Lett.* **2015**, *115*, 250402. [CrossRef] [PubMed]
34. Giustina, M.; Versteegh, M.A.; Wengerowsky, S.; Handsteiner, J.; Hochrainer, A.; Phelan, K.; Steinlechner, F.; Kofler, J.; Larsson, J.Å; Abellán, C.; et al. Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons. *Phys. Rev. Lett.* **2015**, *115*, 250401. [CrossRef]
35. Gisin, N. Bell's inequality holds for all non-product states. *Phys. Lett. A* **1991**, *154*, 201. [CrossRef]
36. Cirelson, B.S. Quantum generalizations of Bell's inequality. *Lett. Math. Phys.* **1980**, *4*, 93. [CrossRef]
37. Werner, R.F. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden variable model. *Phys. Rev. A* **1989**, *40*, 4277. [CrossRef]
38. Mermin, N.D. Extreme quantum entanglement in a superposition of macroscopic distinct states. *Phys. Rev. Lett.* **1990**, *65*, 1838. [CrossRef]
39. Gisin, N.; Bechmann-Pasquinucci, H. Bell inequality, Bell states and maximally entangled states for N qubits. *Phys. Lett. A* **1998**, *246*, 1. [CrossRef]
40. Zukowski, M.; Brukner, Č. Bell's theorem for general N qubit states. *Phys. Rev. Lett.* **2002**, *88*, 210401. [CrossRef] [PubMed]
41. Schmidt, E. Zur Theorie der linearen und nichtlinearen Integralgleichungen. *Math. Ann.* **1907**, *63*, 433. [CrossRef]
42. Ekert, A.; Knight, P.L. Entangled quantum systems and the Schmidt decomposition. *Am. J. Phys.* **1995**, *63*, 415. [CrossRef]
43. Peres, A. Higher order Schmidt decomposition. *Phys. Lett. A* **1995**, *202*, 16. [CrossRef]
44. Peres, A. Separability criterion for density matrices. *Phys. Rev. Lett.* **1996**, *77*, 1413–1415. [CrossRef]
45. Horodecki, P. Separability criterion and inseparable mixed states with positive partial transposition. *Phys. Lett. A* **1997**, *232* 333–339. [CrossRef]
46. Vidal, G.; Werner, R.F. Computable measure of entanglement. *Phys. Rev. A* **2002**, *65*, 032314. [CrossRef]
47. Cerf, N.; Adami, C.; Gingrich, R.M. Reduction criterion for separability. *Phys. Rev.* **1999**, *60*, 898. [CrossRef]
48. Hill, S.; Wootters, W.K. Entanglement of a pair of quantum bits. *Phys. Rev. Lett.* **1997**, *78*, 5022–5025. [CrossRef]
49. Wootters, W.K. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **1998**, *80*, 2245–2248. [CrossRef]
50. Wootters, W.K. Entanglement of formation of concurrence. *Quantum Inf. Comput.* **2001**, *1*, 27–44. [CrossRef]
51. Fei, S.M.; Zhao, M.J.; Chen, K.; Wang, Z.X. Experimental determination of entanglement with a single measurement. *Phys. Rev. A* **2009**, *80*, 032320 [CrossRef]
52. Marshall, A.W.; Olkin, I.; Arnold, B.C. *Inequalities: Theory of Majorization and Its Applications*; Springer: Berlin/Heidelberg, Germany, 1979.
53. Nielsen, M.A.; Kempe, J. Separable states are more disordered globally than locally. *Phys. Rev. Lett.* **2001**, *86*, 5184. [CrossRef]
54. Rudolph, O. Computable cross-norm criterion for separability. *Lett. Math. Phys*. **2004**, *70*, 57–64. [CrossRef]
55. Rudolph, O. Further results on the cross norm criterion for separability. *Quantum Inf. Process.* **2005**, *4*, 219–239. [CrossRef]
56. Chen, K.; Wu, L.A. A matrix realignment method for recognizing entanglement. *Quantum Inf. Comput.* **2003**, *3*, 193–202. [CrossRef]
57. Rudolph, O. Some properties of the Computable Cross Norm Criterion for Separability. *Phys. Rev. A*. **2003**, *67*, 032312. [CrossRef]
58. Bloch, F. Nuclear induction. *Phys. Rev.* **1946**, *70*, 460. [CrossRef]
59. Hioe, F.T.; Eterly, J.H. N-level coherence vector and higher conservation laws in quantum optics and quantum mechanics. *Phys. Rev. Lett.* **1981**, *47*, 838. [CrossRef]
60. de Vicente, J. Separability criteria based on the Block representation of density matrices. *Quant. Inf. Comput.* **2007**, *7*, 624.
61. Fano, U. Pairs of two-level systems. *Rev. Mod. Phys.* **1983**, *55*, 855. [CrossRef]
62. Shang, J.; Asadian, A.; Zhu, H.; Gühne, O. Enhanced entanglement criterion via symmetric informationnally complete measurement. *Phys. Rev. A* **2018**, *98*, 022309. [CrossRef]

63. Bengtsson, I.; Zyczkowski, K. *Geometry of Quantum States: An Introduction to Quantum Entanglement*; Cambridge University Press: Cambridge, UK, 2006.

64. Choi, M.D. Positive linear maps. *Proc. Sympos. Pure Math.* **1982**, *38*, 583.

65. St∅rmer, E. Decomposable positive maps on C^*-algebras. *Proc. Amer. Math. Soc.* **1982**, *86*, 402.

66. Terhal, B. A family of indecomposable positive linear maps based on entangled states. *Lin. Algebr. Appl.* **2000**, *323*, 61. [CrossRef]

67. Lewenstein, M.; Kraus, B.; Cirac, J.I.; Horodecki, P. Optimization of entanglement witnesses. *Phys. Rev. A* **2000**, *62*, 052310. [CrossRef]

68. Gühne, O.; Toth, G. Entanglement detection. *Phys. Rep.* **2009**, *474*, 1–6. [CrossRef]

69. Ganguly, N.; Adhikari, S.; Majumdar, A. Common entanglement witnesses and their characteristics. *Quant. Inf. Pro.* **2013**, *12*, 425–436. [CrossRef]

70. Hyllus, P.; Gühne, O.; Bruss, D.; Lewenstein, M. Relations between entanglement witnesses and Bell inequalities. *Phys. Rev. A* **2005**, *72*, 012321. [CrossRef]

71. Krammer, P. Characterizing entanglement with geometric entanglement witnesses. *J. Phys. A Math. Theor.* **2009**, *42*, 065305. [CrossRef]

72. Hofmann, H.F.; Takeuchi, S. Violation of local uncertainty relations as a signature of entanglement. *Phys. Rev. A* **2003**, *68*, 032103. [CrossRef]

73. Gühne, O.; Lewenstein, M. Entropic uncertainty relations and entanglement. *Phys. Rev. A* **2004**, *70*, 022316. [CrossRef]

74. Gühne, O. Characterizing entanglement via uncertainty relations. *Phys. Rev. Lett.* **2004**, *92*, 117903. [CrossRef] [PubMed]

75. Gühne, O. Entanglement criteria based pn local uncertainty relations are strictly stronger than the computable cross norm criteria. *Phys. Rev. A* **2006**, *74*, 010301. [CrossRef]

76. Li, J.-L.; Qiao, C.-F. A necessary and sufficient criterion for the separability of quantum state. *Sci. Rep.* **2018**, *8*, 1442. [CrossRef]

77. Horn, R.A.; Johnson, C.R. *Topics in Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1991.

78. Horn, R.A.; Johnson, C.R. *Matrix Analysis*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2013.

79. Bercovici, H.; Collins, B.; Dykema, K.; Li, W.S. Characterization of singular numbers of products of operators in matrix algebras and finite von Neumann algebras. *Bull. Sci. Math.* **2015**, *139*, 400–419. [CrossRef]

80. Bhatia, R. *Matrix Analysis*; Springer: New York, NY, USA, 1997.

81. Dai Pra, P.; Pavon, M.; Sahasrabudhe, N. A note on the geometric interpretation of Bell's inequalities. *Lett. Math. Phys.* **2013**, *103*, 1165–1170. [CrossRef]

82. Reid, M.D. Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplifications. *Phys. Rev. A* **1989**, *40*, 913. [CrossRef]

83. Duan, L.; Giedke, G.; Cirac, J.I.; Zoller, P. Inseparability criterion for continuous variable systems. *Phys. Rev. Lett.* **2000**, *84*, 2722. [CrossRef]

84. Tripathi, V.; Radhakrishnan, C.; Byrnes, T. Covariance matrix entanglement criterion for an arbitrary set of operators. *New J. Phys.* **2020**, *22*, 073055. [CrossRef]

85. Balachandran, A.; Govindarajan, T.; De Queiroz, A.R.; Reyes-Lega, A. Algebraic theory of entanglement. *II Nuovo C.* **2013**, *36*, 27–33.

86. Osterloh, A.; Amico, L.; Falci, G.; Fazio, R. Scaling of entanglement close to a quantum phase transition. *Nature* **2002**, *416*, 608–610. [CrossRef]

87. Zhang, J.; Vala, J.; Sastry, S.; Whaley, K.B. Geometric theory of nonlocal two-qubit operations. *Phys. Rev.* **2003**, *A 67*, 042313. [CrossRef]