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Analytic Invariants of Semidirect Products of Symmetric Groups on Banach Spaces

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Abstract: We consider algebras of polynomials and analytic functions that are invariant with respect to semidirect products of groups of bounded operators on Banach spaces with symmetric bases. In particular, we consider algebras of so-called block-symmetric and double-symmetric analytic functions on Banach spaces $\ell_p(\mathbb{C}^n)$ and the homomorphisms of these algebras. In addition, we describe an algebraic basis in the algebra of double-symmetric polynomials and discuss a structure of the spectrum of the algebra of double-symmetric analytic functions on $\ell_p(\mathbb{C}^n)$.

Keywords: symmetric analytic functions on Banach spaces; algebras of analytic functions; algebraic bases; semidirect product; block-symmetric polynomials

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1. Introduction

Let X be a complex Banach space and S a group of bounded operators on X. A function f on X is said to be S-*symmetric* if it is invariant with respect to the actions of operators in S. We denote by $\mathcal{P}_S(X)$ the algebra of S-symmetric polynomials on X. The algebras of symmetric polynomials on a finite-dimensional linear space are typical objects of the Classic Invariant Theory [1] (see also a survey in [2]), where principal results were obtained for finite groups. In the case of an abstract infinite-dimensional Banach space, we have different problems arising from the topological structures of Banach spaces, and we need different methods (and notations). The authors of [3,4] considered discrete and continual analogues of the group of permutations of variables for abstract Banach spaces with symmetric structures and obtained representations of algebraic bases in the corresponding algebras of symmetric polynomials. Symmetric polynomials with respect to the actions of abstract groups of operators on Banach spaces were investigated in [5–7]. Note that in the case of Banach spaces, it is natural to investigate algebras of symmetric analytic functions (as completions of algebras of symmetric polynomials in some suitable topology) and their spectra.

Algebras of symmetric analytic functions of a bounded type on ℓ_p were considered in [8,9]. These investigations were continued in a number of papers (see, e.g., [10] and the references therein). A continual group of symmetry and the corresponding algebras of symmetric analytic functions on L_∞ were investigated in [11–13]. If the algebra of S-symmetric polynomials $\mathcal{P}_S(X)$ admits an algebraic basis (P_n), $n \in \mathbb{N}$, then any homomorphism $\mathcal{F}$ of $\mathcal{P}_S(X)$ can be defined by its evaluations on polynomials P_n. In other words, any homomorphism can be uniquely determined by the sequence $\mathcal{F}(P_n)$, $n \in \mathbb{N}$. If it is continuous with respect to a uniform topology on $\mathcal{P}_S(X)$, then it can be extended to a corresponding algebra of symmetric analytic functions. Thus, the first important question concerning an algebra of symmetric polynomials is about the existence of a countable algebraic basis (or a generating sequence) of polynomials. The algebras of analytic functions on X, generated by a countable family of polynomials were systematically studied in [10,14–16].

In this paper, we consider the case when S is a semidirect product of two groups of symmetry on a Banach space X. For the case $X = \ell_p(\mathbb{C}^n)$, we obtain an algebraic basis of the algebra of S-symmetric polynomials and apply it for a description of the spectrum of the algebra of S-symmetric analytic functions of a bounded type on $X = \ell_p(\mathbb{C}^n)$.

In Section 2, we give the necessary definitions and preliminary results. Various classes of symmetric polynomials are considered in Section 3. In Section 4, we consider the question how to describe generators of S-symmetric polynomials if S is a semidirect product of two groups acting on X, and we have information about the generators of the symmetric polynomials related to these groups. In Section 5, we apply the obtained results for the corresponding algebras of symmetric analytic functions on ℓ_p, $1 \le p < \infty$ and their spectra.

General information on polynomials and analytic functions on Banach spaces can be found in [17,18].

2. Preliminary Results

Let us denote by $H(X)$ the algebra of all analytic functions on a Banach space X over the field of complex numbers $\mathbb{C}$. Recall that an entire analytic function f on X can be defined as a continuous function such that the restriction of f to a y finite-dimensional subspace of X is analytic. An analytic function f_n is an n-homogeneous (continuous) polynomial if $f_n(\lambda x) = \lambda^n f_n(x)$ for every $x \in X$ and $\lambda \in \mathbb{C}$. A finite sum of homogeneous polynomials is a polynomial. The algebra of all continuous polynomials on X is denoted by $\mathcal{P}(X)$. It is well known that every function $f \in H(X)$ can be represented as a series of n-homogeneous polynomials f_n

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad x \in X$$

which is called the Taylor series of f.

A function $f \in H(X)$ is said to be of a *bounded type* if it is bounded on bounded subsets of X. The algebra of all the entire functions of a bounded type is denoted by $H_b(X)$. This is a Fréchet algebra with respect to the metrizable locally convex topology generated by the following countable family of norms

$$\|f\|_r = \sup_{\|x\| \le r} |f(x)|,$$

where r goes over positive rational numbers. It is known (see, e.g., [19]) that $H_b(X)$ is a proper subalgebra of $H(X)$ providing X is infinite-dimensional. A continuous complex-valued homomorphism $\varphi\colon H_b(X) \to \mathbb{C}$ is called a *character* of $H_b(X)$, and the set of all characters is the *spectrum* of $H_b(X)$. The spectrum of $H_b(X)$ was investigated by many authors [20–27]. In particular, it is known that for every point $x \in X$, the point evaluation functional $\delta_x\colon f \to f(x)$ is a character on $H_b(X)$. Moreover, for every point z of the second dual space X'' of X, we can assign a functional $\widetilde{\delta}_z(f) = \widetilde{f}(z)$, where $\widetilde{f}$ is the Aron–Berner extension [28] of f to X''. In the general case, the functional $\widetilde{\delta}_z$ does not exhaust the spectrum of $H_b(X)$, and it may have a complicated structure. It was a motivation for studying the spectra of the countable generated subalgebras of $H_b(X)$, in particular, the subalgebras of symmetric functions (see, e.g., [10]).

Throughout this paper, we use the notations ℓ_p for the Banach space absolutely summable sequences in power p, $1 \le p < \infty$ and c_{00} for the linear space of all finite sequences.

3. Classes of Symmetric Polynomials

Let $\{P_\alpha\}$ be a family of nonzero polynomials in $\mathcal{P}_S(X)$, where $\mathcal{P}_S(X)$ is the algebra of S-symmetric polynomials for a given group S of bounded linear operators on a Banach space X. We say that $\{P_\alpha\}$ is *algebraically independent* if any finite subset $\{P_{\alpha_1}, \dots, P_{\alpha_n}\}$ is algebraically independent. That is, if $q(t_1, \dots, t_n)$ is a nonzero polynomial of n variables, then

$$q(P_{\alpha_1}(x), \dots, P_{\alpha_n}(x)) \not\equiv 0, \quad x \in X.$$

In other words, any non-trivial algebraic combination of algebraically independent polynomials is nonzero. A family of nonzero polynomials $\{P_\alpha\} \in \mathcal{P}_S(X)$ is a *generating set* if every polynomial $P \in \mathcal{P}_S(X)$ can be represented as an algebraic combination of a finite subset of $\{P_\alpha\}$. An algebraically independent generating set is called an *algebraic basis* of $\mathcal{P}_S(X)$. It is easy to check that any generating set of an algebra of polynomials is an algebraic basis if and only if every polynomial in this algebra can be uniquely represented as a (finite) algebraic combination of elements in the generating set.

Let now Y be a Banach space with a (linear) symmetric countable Schauder basis (e_n), $\dim Y \leq \infty$, and let Z be an arbitrary Banach space. Suppose that S_Y and S_Z are groups of bounded operators on Z and Y, respectively. We denote by $X = Y(Z)$ the space of elements

$$x = (x_1, x_2, \ldots, x_n, \ldots), \quad x_n \in Z,$$

such that

$$\|x_1\|_Z e_1 + \|x_2\|_Z e_2 + \cdots + \|x_n\|_Z e_n + \cdots \in Y$$

with

$$\|x\|_X := \big\|(\|x_1\|_Z e_1 + \|x_2\|_Z e_2 + \cdots + \|x_n\|_Z e_n + \cdots)\big\|_Y.$$

We will write, also, the vector $x = (x_1, x_2, \ldots, x_n, \ldots)$ as a formal sum

$$x = \sum_{n=1}^{\dim Y} x_n \mathbf{e}_n$$

understudying that x_n are vectors in Z. Here, $\dim Y$ means the topological dimension, that is, the sum is finite or countable.

The following technical result is probably known.

Remark 1. *Let τ be a bounded linear operator from Y to Y,*

$$\tau(e_m) = \sum_{n=1}^{\dim Y} b_n^m e_n$$

for some numbers b_n^m. If $b_n^m \geq 0$, then there exists a bounded linear operator $\widetilde{\tau}\colon Y(Z) \to Y(Z)$ by

$$\widetilde{\tau}(x) = \widetilde{\tau}\Big(\sum_{n=1}^{\dim Y} x_n \mathbf{e}_n\Big) = \sum_{m=1}^{\dim Y} \sum_{n=1}^{\dim Y} x_m b_n^m e_n,$$

and $\|\tau\| = \|\widetilde{\tau}\|$.

Proof. Let

$$y = \sum_{n=1}^{\dim Y} y_n e_n \in Y.$$

Then,

$$\tau(y) = \sum_{m=1}^{\dim Y} y_m \sum_{n=1}^{\dim Y} b_n^m e_n = \sum_{n=1}^{\dim Y} \Big(\sum_{m=1}^{\dim Y} y_m b_n^m\Big) e_n.$$

Here, we change the order of summation because the series converges unconditionally, because (e_n) is a symmetric basis. Hence,

$$\widetilde{\tau}(x) = \sum_{n=1}^{\dim Y} \Big(\sum_{m=1}^{\dim Y} x_m b_n^m\Big) e_n,$$

and from the assumption that all $b_n^m \geq 0$, it follows that

$$\|\widetilde{\tau}(x)\|_X = \left\| \sum_{n=1}^{\dim Y} \left\| \sum_{m=1}^{\dim Y} x_m b_n^m \right\|_Z e_n \right\|_Y \leq \left\| \sum_{n=1}^{\dim Y} \left(\sum_{m=1}^{\dim Y} \|x_m\|_Z b_n^m \right) e_n \right\|_Y = \|\tau(y)\|,$$

for $y = (\|x_1\|_Z, \|x_2\|_Z, \ldots,)$. Because for this y, $\|x\|_X = \|y\|_Y$, it follows that $\|\widetilde{\tau}\| \leq \|\tau\|$. On the other hand, because $\widetilde{\tau}$ is an extension of τ, we have $\|\widetilde{\tau}\| \geq \|\tau\|$. So $\|\widetilde{\tau}\| = \|\tau\|$. □

Using groups S_Y and S_Z, it is possible to construct a group of symmetries on X in different ways.

Definition 1. *A function f on X is called separately S_Z-symmetric if for every $\sigma \in S_Z$ and $n \leq \dim Y$,*

$$f(x_1, x_2, \ldots, \sigma(x_n), \ldots) = f(x_1, x_2, \ldots, x_n, \ldots).$$

A function f on X is called block S_Y-symmetric if for every $\tau \in S_Y$, the operator $\widetilde{\tau}$ is well defined and continuous on X and $f(\widetilde{\tau}(x)) = f(x)$ for every $x \in X$.

We say that a function f on X is (S_Y, S_Z)-symmetric or double symmetric if it is both separately S_Z-symmetric and block S_Y-symmetric.

Let us recall the definition of the semidirect product of two groups. For a given group G, we denote by $\operatorname{Aut} G$ the group of automorphisms of G. Let Ψ be a group homomorphism from a group H to $\operatorname{Aut} G$, that is, $\Psi\colon h \to \Psi_h \in \operatorname{Aut} G$, $h \in H$. The *(outer) semidirect product* $G \rtimes_\Psi H = G \rtimes H$ of G and H with respect to Ψ is the direct product $G \times H$ endowed with the group operation

$$(g_1, h_1)(g_2, h_2) = (g_1 \Psi_{h_1}(g_2), h_1 h_2).$$

Let $\widetilde{S}_Z$ be the minimal group of operators on X generated by operators

$$\widetilde{\sigma}_n = (x_1, \ldots, x_n, \ldots) \mapsto (x_1, \ldots, \sigma(x_n), \ldots), \quad \sigma \in S_Z, \quad n \in \mathbb{N}.$$

Proposition 1.

(i) *A function f on $X = Y(Z)$ is separately S_Z-symmetric if and only if it is $\widetilde{S}_Z$-symmetric.*

(ii) *A function f on X is (S_Y, S_Z)-symmetric if and only if it is $S_Y \rtimes_\Psi \widetilde{S}_Z$-symmetric, where $\Psi\colon \tau \mapsto \widetilde{\tau}$.*

(iii) *A function f on X is block S_Y-symmetric if and only if it is $S_Y \rtimes_\Psi \widetilde{I}_Z$-symmetric, where $\widetilde{I}_Z$ is a trivial subgroup of $\widetilde{S}_Z$ consisting of the identity map.*

Proof. Item (i) follows from the definition of separately S_Z-symmetric functions. The operator $\Psi(\tau) = \Psi_\tau = \widetilde{\tau}$ belongs to the group of automorphisms $\operatorname{Aut} \widetilde{S}_Z$ so that $\Psi_\tau(\widetilde{\sigma}) = \widetilde{\sigma} \circ \widetilde{\tau}$, $\widetilde{\sigma} \in \widetilde{S}_Z$. By the definition of (S_Y, S_Z)-symmetric functions, f is (S_Y, S_Z)-symmetric if and only if it is invariant with respect to the action of $\widetilde{\sigma} \circ \widetilde{\tau}$ for all $\widetilde{\sigma} \in \widetilde{S}_Z$ and $\tau \in S_Y$. Thus, item (ii) is proved. Finally, item (iii) is a partial case of (ii) if σ is the identity operator. □

Example 1. *Let $X = \ell_p$, $1 \leq p < \infty$ and S_{ℓ_p} is the group of all permutations of the basis vectors $e_n = (\underbrace{0, \ldots, 0, 1}_{n}, 0, \ldots)$. In other words, every permutation σ on the set of positive integers $\mathbb{N}$ acts as a linear operator on ℓ_p (which we denote by the same symbol $\sigma \in S_{\ell_p}$) by*

$$\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots) = \sum_{n=1}^{\infty} x_{\sigma(n)} e_n = \sum_{n=1}^{\infty} x_n e_{\sigma^{-1}(n)}.$$

It is well known [3,4] that polynomials

$$F_k = \sum_{n=1}^{\infty} x_n^k, \quad k \geq \lceil p \rceil$$

form an algebraic basis in the algebra of all S_{ℓ_p}-symmetric polynomials on ℓ_p (which are called symmetric polynomials), where $\lceil p \rceil$ is the minimal integer, which is greater than or equal to p. We denote by $\mathcal{P}_s(\ell_p)$ the algebra of the symmetric polynomials on ℓ_p.

Example 2. *Let $X = L_p[0,1]$, $1 \le p \le \infty$ and $S_{L_p[0,1]}$ is the group of operators on $L_p[0,1]$ generated by the measurable automorphisms of $[0,1]$ that preserve the Lebesgue measure on $[0,1]$. That is, if $\sigma\colon [0,1] \to [0,1]$ is a measure-preserving measurable automorphism, then it acts as an operator on $L_p[0,1]$ (which we denote by the same symbol $\sigma \in S_{L_p[0,1]}$) by*

$$\sigma(x(t)) = x \circ \sigma(t), \quad x(t) \in L_p[0,1].$$

According to [3,11], polynomials

$$R_k(x) = \int_{[0,1]} (x(t))^k dt, \quad k \le \lfloor p \rfloor$$

form an algebraic basis in the algebra $\mathcal{P}_s(L_p[0,1])$ of all $S_{L_p[0,1]}$-symmetric polynomials on $S_{L_p[0,1]}$. Here, $\lfloor p \rfloor$ is the maximal integer that is less than or equal to p. Hence, $\mathcal{P}_s(L_p[0,1])$ is finitely generated if $p < \infty$ and countably generated if $p = \infty$. Further results about symmetric and block-symmetric polynomials on L_∞ can be found in [29–31] and the cited literature therein.

Example 3. *Let $Y = \ell_p$ and $Z = \mathbb{C}^n$. Any element $x \in X = \ell_p(\mathbb{C}^n) = \mathbb{C}^n \otimes \ell_p$ can be represented as*

$$x = \left(x^{(1)}, \dots, x^{(n)}\right) = \sum_{k=1}^{n} \sum_{j=1}^{\infty} x_j^{(k)} e_j,$$

where $\sum_{j=1}^{\infty} x_j^{(k)} e_j = (x_1^{(k)}, \dots, x_j^{(k)}, \dots) \in \ell_p$ for $k = 1, \dots, n$. Let $S_Y = S_{\ell_p}$ be the group of permutations of the basis vectors in ℓ_p and let $S_Z = S_n$ be the group of permutations of the basis vectors in $\mathbb{C}^n$. In [32,33], it is shown that the algebra of the block S_{ℓ_p}-symmetric polynomials on X admits an algebraic basis of the so-called power block-symmetric polynomials

$$H^{\mathbf{k}}(x) = H^{k_1,k_2,\dots,k_n}(x) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |\mathbf{k}| \ge \lceil p \rceil}}^{n} \left(x_j^{(r)}\right)^{k_r}, \tag{1}$$

where $\mathbf{k} = (k_1, k_2, \dots, k_n)$ is a multi-index, $|\mathbf{k}| = k_1 + k_2 + \dots + k_n$.

The algebras of block-symmetric polynomials and analytic functions on $\ell_p(\mathbb{C}^n)$ were studied, also, in [34,35].

4. Generators in Algebras of Double-Symmetric Polynomials

The following theorem generalizes a result in [36] about separately symmetric polynomials on ℓ_1.

Theorem 1. *Let $\dim Y = m < \infty$ and Z be an arbitrary Banach space. Suppose that the algebra of S_Z-symmetric polynomials admits a finite or countable family of generators $\{P_k\}$. Then, the algebra of all separately S_Z-symmetric polynomials on $X = Y(Z)$ has a family of generators $\{P_k^{(j)}\}$, $j = 1, \dots, m$, where*

$$P_k^{(j)}(x) = P_k^{(j)}(x_1, \dots, x_m) = P_k(x_j), \quad x = (x_1, \dots, x_m) \in X.$$

If $\{P_k\}$ is an algebraic basis, then $P_k^{(j)}$ is an algebraic basis as well.

Proof. If $\dim Y = 1$, then the statement is trivial. Suppose it is true for $\dim Y = m - 1$. Let P be a separately S_Z-symmetric polynomial on X. Then, $P(x) = P(x_1, \dots, x_m)$ can be considered as an S_Z-symmetric polynomial of x_m with coefficients in the field $\mathbb{K}_{m-1}$ of separately S_Z-symmetric rational functions of $(x_1, \ldots, x_{m-1})$. From the Classical Invariant Theory (see, e.g., p. 12 in [1]), it is known that polynomials $\{P_k\} = \{P_k^{(m)}\}$ form a family of generators in the algebra of S_Z-symmetric polynomials over the field $\mathbb{K}_{m-1}$. That is, P can be represented in the form

$$P(x_1, \ldots, x_m) = \sum a_{k_1 \ldots k_r}(x_1, \ldots, x_{m-1}) \left[P_1^{(m)}(x_m)\right]^{k_1} \cdots \left[P_r^{(m)}(x_m)\right]^{k_r}, \tag{2}$$

where $k_1 \deg P_1 + \cdots + k_r \deg P_r \le \deg P$, and $a_{k_1 \ldots k_r}(x_1, \ldots, x_{m-1})$ are separately S_Z-symmetric rational functions of $(x_1, \ldots, x_{m-1})$. But on the left side of Equation (2) there is a polynomial, so on the right side must be a polynomial too. Because polynomials $P_k^{(m)}$ do not depend on $(x_1, \ldots, x_{m-1})$ and rational functions $a_{k_1 \ldots k_r}$ do not depend on (x_m), the functions $a_{k_1 \ldots k_r}$ are actually polynomials, and we know that they are separately S_Z-symmetric. By the induction assumption, all polynomials $a_{k_1 \ldots k_r}$ can be represented by algebraic combinations of $\{P_k^{(j)}\}$, $j = 1, \ldots, m-1$.

If $\{P_k\}$ is an algebraic basis, then by the induction assumption, polynomials $a_{k_1 \ldots k_r}$ have unique representations by $\{P_k^{(j)}\}$, $j < m$. Suppose that

$$P(x_1, \ldots, x_m) = \sum b_{k_1 \ldots k_r}(x_1, \ldots, x_{m-1}) \left[P_1^{(m)}(x_m)\right]^{k_1} \cdots \left[P_r^{(m)}(x_m)\right]^{k_r}$$

is another representation of P. Then,

$$\sum (a_{k_1 \ldots k_r} - b_{k_1 \ldots k_r})(x_1, \ldots, x_{m-1}) \left[P_1^{(m)}(x_m)\right]^{k_1} \cdots \left[P_r^{(m)}(x_m)\right]^{k_r} \equiv 0.$$

If there is $(y_1, \ldots, y_{m-1})$ such that

$$\sum (a_{k_1 \ldots k_r} - b_{k_1 \ldots k_r})(y_1, \ldots, y_{m-1}) \left[P_1^{(m)}(x_m)\right]^{k_1} \cdots \left[P_r^{(m)}(x_m)\right]^{k_r}$$

is a non-trivial algebraic combination, then it contradicts the algebraic independence of $\{P_k\}$. Thus, $a_{k_1 \ldots k_r} - b_{k_1 \ldots k_r} \equiv 0$ for all polynomials $a_{k_1 \ldots k_r}$, and so (2) is a unique representation of P. Hence, $\{P_k^{(j)}\}$, $j = 1, \ldots, m$ is an algebraic basis. $\square$

Corollary 1. *Let $X = Y(Z)$ be as in Example 3. Then, there is an algebraic basis of separately S_Z-symmetric polynomials of the form*

$$F_k^{(j)}(x) = F_k^{(j)}(x^{(1)}, \ldots, x^{(n)}) = F_k(x^{(j)}) = \sum_{i=1}^{\infty} \left(x_i^{(j)}\right)^k, \quad x = (x^{(1)}, \ldots, x^{(n)}) \in X,$$

for $j = 1, \ldots, n$.

Proof. It is enough to apply Theorem 1 to Example 3, taking into account the algebraic basis in Example 1. $\square$

Let $X = \ell_p(\mathbb{C}^n)$ for some $1 \le p < \infty$. That is, any vector $x \in X$ can be represented as $x = (x^{(1)}, \ldots, x^{(n)})$, where $x^{(k)} = (x_1^{(k)}, \ldots, x_j^{(k)}, \ldots) \in \ell_p$, $k = 1, \ldots, n$. The space $\ell_p(\mathbb{C}^n)$ is a Banach space with respect to the norm

$$\|x\|_{\ell_p} = \left(\sum_{j=1}^{\infty} \sum_{k=1}^{n} |x_j^{(k)}|^p\right)^{1/p}.$$

Let us define the following mapping $\mathcal{F}_{\ell_p,n}$ on $\ell_p(\mathbb{C}^n)$, $\mathcal{F}_{\ell_p,n} = (\mathcal{F}_{\ell_p,n}^{(1)}, \ldots, \mathcal{F}_{\ell_p,n}^{(j)}, \ldots)$, where

$$\mathcal{F}_{\ell_p,n}^{(j)}(x) = (F_1(x_j), \ldots, F_n(x_j)) = \left(\sum_{i=1}^{n} x_j^{(i)}, \ldots, \sum_{i=1}^{n} \left[x_j^{(i)}\right]^n\right). \tag{3}$$

In other words, we can write

$$\mathcal{F}_{\ell_p,n}\colon \left(\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \\ \cdots \\ x_1^{(n)} \end{pmatrix} \cdots \begin{pmatrix} x_j^{(1)} \\ x_j^{(2)} \\ \cdots \\ x_j^{(n)} \end{pmatrix} \cdots \right) \rightsquigarrow \left(\begin{pmatrix} F_1(x_1) \\ F_2(x_1) \\ \cdots \\ F_n(x_1) \end{pmatrix} \cdots \begin{pmatrix} F_1(x_j) \\ F_2(x_j) \\ \cdots \\ F_n(x_j) \end{pmatrix} \cdots \right).$$

Proposition 2. *The mapping $\mathcal{F}_{\ell_p,n}$ is a continuous polynomial map from $\ell_p(\mathbb{C}^n)$ to itself.*

Proof. Clearly, $\mathcal{F}_{\ell_p,n}$ is a polynomial. Note that $|F_k(x_j)| \le n\|x_j\|^k_{\ell_p}$. Thus, we have

$$\big\|\mathcal{F}_{\ell_p,n}(x)\big\|_{\ell_p} = \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{n}|F_k(x_j)|^p\Big)^{1/p} \le n\sum_{k=1}^{n}\|x\|^k_{\ell_p} < \infty.$$

From here, it follows that $\mathcal{F}_{\ell_p,n}(x) \in \ell_p(\mathbb{C}^n)$ for every $x \in \ell_p(\mathbb{C}^n)$. Also, the inequality shows that $\mathcal{F}_{\ell_p,n}$ is bounded on bounded subsets and so it is continuous. □

Let us denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ the algebra of all block-symmetric (that is, block S_{ℓ_p}-symmetric) polynomials on $\ell_p(\mathbb{C}^n)$ and by $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ the algebra of all double-symmetric (that is, (S_{ℓ_p}, S_n)-symmetric) polynomials on $\ell_p(\mathbb{C}^n)$, where S_{ℓ_p} and S_n are as in Example 3. The composition operator $C_{\mathcal{F}_{\ell_p,n}}(P) = P \circ \mathcal{F}_{\ell_p,n}$ is a homomorphism from $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ to $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$. Indeed, if P is a block-symmetric polynomial, then

$$P \circ \mathcal{F}_{\ell_p,n} = P(\mathcal{F}^{(1)}_{\ell_p,n}(x_1), \ldots, \mathcal{F}^{(j)}_{\ell_p,n}(x_j), \ldots)$$

is (S_{ℓ_p}, S_n)-symmetric because $\mathcal{F}_{\ell_p,n}$ is S_n-symmetric.

Let us consider the partial case if $X = \mathbb{C}^m(\mathbb{C}^n)$. Then, we have the mapping $\mathcal{F}_{m,n}$ instead of $\mathcal{F}_{\ell_p,n}$,

$$\mathcal{F}_{m,n} = (\mathcal{F}^{(1)}_{m,n}, \ldots, \mathcal{F}^{(j)}_{m,n}).$$

Thus, the composition operator $C_{\mathcal{F}_{m,n}}$ is a homomorphism from $\mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n))$ to $\mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$.

Theorem 2. *The composition operator $C_{\mathcal{F}_{m,n}}$ is an isomorphism of algebras $\mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n))$ and $\mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$.*

Proof. It is enough to show that $C_{\mathcal{F}_{m,n}}\colon \mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n)) \to \mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$ is bijective. Clearly, if $C_{\mathcal{F}_{m,n}}(P) = 0$, then $P = 0$, that is, the composition operator is injective. Let $Q \in \mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$, and then Q is separately S_n-symmetric. By Corollary 1, the double-symmetric polynomial Q can be represented us an algebraic combination of polynomials

$$F_k^{(j)}(x) = F_k^{(j)}(x_1, \ldots, x_m) = F_k(x_j) = \sum_{i=1}^{n}\big[x_j^{(i)}\big]^k.$$

In other words, Q is of the form $Q(x) = P\big(\mathcal{F}_{m,n}(x)\big)$ for some polynomial P. Indeed, let

$$q(t_1^{(1)}, \ldots, t_1^{(n)}, t_2^{(1)}, \ldots, t_2^{(n)}, \ldots, t_m^{(1)}, \ldots, t_m^{(n)})$$

be a polynomial of mn variables such that

$$Q(x) = q\big(F_1(x_1), \ldots, F_n(x_1), F_1(x_2), \ldots, F_n(x_2), \ldots, F_1(x_m), \ldots, F_n(x_m)\big).$$

Then, for

$$P(x) = q(x_1^{(1)}, \ldots, x_1^{(n)}, x_2^{(1)}, \ldots, x_2^{(n)}, \ldots, x_m^{(1)}, \ldots, x_m^{(n)})$$

we have that $Q(x) = P\big(\mathcal{F}_{m,n}(x)\big)$. Moreover, because Q is also block S_m-symmetric, and for every j, $\mathcal{F}^{(j)}_{m,n}$ maps $\mathbb{C}^n$ onto $\mathbb{C}^n$, it follows that q is invariant with respect to permutations $(x_i^{(1)}, \ldots, x_i^{(n)}) \rightsquigarrow (x_j^{(1)}, \ldots, x_j^{(n)})$, $1 \le i, j \le m$. Hence, P is block S_m-symmetric. Therefore, every $Q \in \mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$ is of the form $Q = C_{\mathcal{F}_{m,n}}(P)$ for some $P \in \mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n))$. Thus, $\mathcal{F}_{m,n}$ is bijective. $\square$

Note that every double-symmetric polynomial is block symmetric but not every block-symmetric polynomial is double symmetric. That is, $\mathcal{P}_{ds}(\mathbb{C}^m(\mathbb{C}^n))$ is a proper subalgebra of $\mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n))$, which is isomorphic to $\mathcal{P}_{vs}(\mathbb{C}^m(\mathbb{C}^n))$.

Example 4. *Let $X = \mathbb{C}^2(\mathbb{C}^2)$. Then, every element $x \in X$ can be represented as*

$$x = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Every block-symmetric polynomial is invariant with respect to operator

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix}.$$

It is known (see, e.g., [37]) that polynomials

$$\begin{aligned} h_1 = H^{1,0} &= x_1 + x_2; \\ h_2 = H^{0,1} &= y_1 + y_2; \\ h_3 = H^{2,0} &= x_1^2 + x_2^2; \\ h_4 = H^{0,2} &= y_1^2 + y_2^2; \\ h_5 = H^{1,1} &= x_1 y_1 + x_2 y_2. \end{aligned} \tag{4}$$

form a minimal generating set that, however, is algebraically dependent. The isomorphism $\mathcal{F} = \mathcal{F}_{2,2}$ from $\mathcal{P}_{vs}(\mathbb{C}^2(\mathbb{C}^2))$ to $\mathcal{P}_{ds}(\mathbb{C}^2(\mathbb{C}^2))$ is defined as

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 \end{pmatrix}.$$

Thus, the generating polynomials in $\mathcal{P}_{ds}(\mathbb{C}^2(\mathbb{C}^2))$, $N^{i,j} = H^{i,j} \circ \mathcal{F}$ can be written as

$$\begin{aligned} N^{1,0} &= x_1 + y_1 + x_2 + y_2; \\ N^{0,1} &= x_1^2 + y_1^2 + x_2^2 + y_2^2; \\ N^{2,0} &= (x_1 + y_1)^2 + (x_2 + y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1 y_1 + x_2 y_2); \\ N^{0,2} &= (x_1^2 + y_1^2)^2 + (x_2^2 + y_2^2)^2 = x_1^4 + x_2^4 + y_1^4 + y_2^4 + 2(x_1^2 y_1^2 + x_2^2 y_2^2); \\ N^{1,1} &= (x_1 + y_1)(x_1^2 + y_1^2) + (x_2 + y_2)(x_2^2 + y_2^2) = x_1^3 + x_2^3 + y_1^3 + y_2^3 + x_1^2 y_1 + x_2^2 y_2 + x_1 y_1^2 + x_2 y_2^2. \end{aligned}$$

Because every polynomial in $\mathcal{P}_{ds}(\mathbb{C}^2(\mathbb{C}^2))$ is block symmetric, polynomials N^{k_1,k_2}, $1 \le k_1 + k_2 \le 2$, can be represented as an algebraic combination of polynomials (4):

$$N^{1,0} = h_1 + h_2;$$

$$N^{0,1} = h_3 + h_4;$$

$$N^{2,0} = h_3 + h_4 + 2h_5;$$

$$N^{0,2} = -\frac{1}{2}(h_1)^4 + (h_1)^2 h_3 + \frac{1}{2}(h_3)^2 - \frac{1}{2}(h_2)^4 + (h_2)^2 h_4 + \frac{1}{2}(h_4)^2 - (h_1)^2(h_2)^2 \\ + \frac{1}{2}(h_1)^2 h_4 + h_1 h_2 h_5 + \frac{1}{2}(h_2)^2 h_3 + (h_5)^2,$$

because, according to [37],

$$-\frac{1}{2}(h_1)^2 h_4 + h_1 h_2 h_5 + h_3 h_4 - \frac{1}{2}(h_2)^2 h_3 - (h_5)^2 \equiv 0;$$

$$N^{1,1} = -\frac{1}{2}(h_1)^3 + \frac{3}{2}h_1 h_3 - \frac{1}{2}(h_2)^3 + \frac{3}{2}h_2 h_4 - \frac{1}{2}(h_1)^2 h_2 + \frac{1}{2}h_3 h_2 + h_5 h_1 - \frac{1}{2}(h_2)^2 h_1 + \frac{1}{2}h_4 h_1 + h_5 h_2.$$

Example 5. *Let $X = \mathbb{C}^3(\mathbb{C}^2)$. Then, every element $x \in X$ can be represented as*

$$x = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Polynomials

$$\begin{aligned} H^{1,0} &= x_1 + x_2 + x_3; \\ H^{0,1} &= y_1 + y_2 + y_3; \\ H^{2,0} &= x_1^2 + x_2^2 + x_3^2; \\ H^{0,2} &= y_1^2 + y_2^2 + y_3^2; \\ H^{1,1} &= x_1 y_1 + x_2 y_2 + x_3 y_3; \\ H^{3,0} &= x_1^3 + x_2^3 + x_3^3; \\ H^{0,3} &= y_1^3 + y_2^3 + y_3^3; \\ H^{1,2} &= x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2; \\ H^{2,1} &= x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3. \end{aligned} \tag{5}$$

form a minimal set of generating polynomials in $\mathcal{P}_{vs}(\mathbb{C}^2(\mathbb{C}^2))$. Let

$$\mathcal{F} = \mathcal{F}_{3,2} \colon \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{pmatrix}.$$

Thus, combining $\mathcal{F}$ and (5), we can represent generating polynomials in $\mathcal{P}_{ds}(\mathbb{C}^3(\mathbb{C}^2))$, $N^{i,j} = H^{i,j} \circ \mathcal{F}$ as

$$\begin{aligned} N^{1,0} &= x_1 + y_1 + x_2 + y_2 + x_3 + y_3; \\ N^{0,1} &= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2; \\ N^{2,0} &= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 + 2(x_1 y_1 + x_2 y_2 + x_3 y_3); \\ N^{0,2} &= x_1^4 + x_2^4 + x_3^4 + y_1^4 + y_2^4 + y_3^4 + 2(x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2); \\ N^{1,1} &= x_1^3 + x_2^3 + x_3^3 + y_1^3 + y_2^3 + y_3^3 + x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 + x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2; \\ N^{3,0} &= x_1^3 + x_2^3 + x_3^3 + y_1^3 + y_2^3 + y_3^3 + 3(x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3) + 3(x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2); \\ N^{0,3} &= x_1^6 + x_2^6 + x_3^6 + y_1^6 + y_2^6 + y_3^6 + 3(x_1^4 y_1^2 + x_2^4 y_2^2 + x_3^4 y_3^2) + 3(x_1^2 y_1^4 + x_2^2 y_2^4 + x_3^2 y_3^4); \\ N^{1,2} &= x_1^5 + x_2^5 + x_3^5 + y_1^5 + y_2^5 + y_3^5 + x_1^4 y_1 + x_2^4 y_2 + x_3^4 y_3 + x_1 y_1^4 \\ &\quad + x_2 y_2^4 + x_3 y_3^4 + 2(x_1^3 y_1^2 + x_2^3 y_2^2 + x_3^3 y_3^2) + 2(x_1^2 y_1^3 + x_2^2 y_2^3 + x_3^2 y_3^3); \\ N^{2,1} &= x_1^4 + x_2^4 + x_3^4 + y_1^4 + y_2^4 + y_3^4 + 2(x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2) \\ &\quad + 2(x_1 y_1^3 + x_2 y_2^3 + x_3 y_3^3) + 2(x_1^3 y_1 + x_2^3 y_2 + x_3^3 y_3). \end{aligned}$$

Let us recall that a linear operator A on $\mathbb{C}^k$ is a *pseudoreflection* if it is an invertible operator such that it is not the identity map, has a finite multiplicative order and the fixed subspace $V_A := \{x \in \mathbb{C}^k \colon A(x) = x\}$ has the dimension $k-1$. It is well known [38,39] that the algebra of G-symmetric polynomials on $\mathbb{C}^k$ for a finite group G of linear operators on $\mathbb{C}^k$ has an algebraic basis and is isomorphic to the algebra of all polynomials on $\mathbb{C}^k$ if and only if it is generated by pseudoreflections. Thus, for example, the generators of the algebra of block-symmetric polynomials on $(\mathbb{C}^m(\mathbb{C}^n))$ are algebraically dependent if both n and m are greater than 1. The situation is different in infinite-dimensional cases. As we mentioned above, polynomials (1) form an algebraic basis in the algebra of block-symmetric polynomials on $\ell_p(\mathbb{C}^n.)$ Moreover, in Corollary 7 in [32], the following result was obtained.

Theorem 3 ([32])**.** *Let $\mathbf{k}_1, \ldots, \mathbf{k}_s$ be multi-indexes such that $|\mathbf{k}_s| \geq 1$ for every $j \in \{1, \ldots, s\}$. Then, there exists $m \in \mathbb{N}$ such that for every $r > m$ polynomials $H^{\mathbf{k}_1}, \ldots, H^{\mathbf{k}_s}$ as in (1) are algebraically independent on $\mathbb{C}^r(\mathbb{C}^n)$.*

Theorem 4. *The composition operator $C_{\mathcal{F}_{\ell_p,n}}(P) = P \circ \mathcal{F}_{\ell_p,n}$ is an isomorphism from the algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ of block-symmetric polynomials on $\ell_p(\mathbb{C}^n)$ to the algebra of double-symmetric polynomials $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ on $\ell_p(\mathbb{C}^n)$. Polynomials*

$$N^{\mathbf{k}} = H^{\mathbf{k}} \circ \mathcal{F}_{\ell_p,n}, \quad |\mathbf{k}| \geq \lceil p \rceil$$

form an algebraic basis in $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$.

Proof. Let us show first that polynomials $N^{\mathbf{k}}$, $|\mathbf{k}| \geq \lceil p \rceil$ are algebraically independent on $\ell_p(\mathbb{C}^n)$. It is well known (see, e.g., [8]) that the mapping

$$(t_1, \ldots, t_n) \rightsquigarrow \Big(\sum_{i=1}^n t_i, \sum_{i=1}^n t_i^2, \ldots, \sum_{i=1}^n t_i^n \Big)$$

is a surjection onto $\mathbb{C}^n$. Let us suppose that there is a non-trivial polynomial of a finite number of variables, $q(t_1, \ldots, t_s)$ such that

$$q(N^{\mathbf{k}_1}(x), \ldots, N^{\mathbf{k}_s}(x)) \equiv 0.$$

Because q is non-trivial and $H^{\mathbf{k}}$ are algebraically independent on $\ell_p(\mathbb{C}^n)$,

$$q(H^{\mathbf{k}_1}(x), \ldots, H^{\mathbf{k}_s}(x)) \not\equiv 0.$$

From the continuity of polynomials $H^{\mathbf{k}}$, it follows that there is an open set $U \subset X$ such that

$$q(H^{\mathbf{k}_1}(y), \ldots, H^{\mathbf{k}_s}(y)) \neq 0$$

for every $y \in U$. Because the subspace of finite sequences is dense in $\ell_p(\mathbb{C}^n)$, we can choose $y = (y_1, \ldots, y_j, \ldots) \in U$, $y_j = (y_j^{(1)}, \ldots, y_j^{(n)})$ such that $y_j = 0$ for every j that is greater than a number j_0. Let us take $(u_j^{(1)}, \ldots, u_j^{(n)})$ so that

$$y_j^{(k)} = \sum_{i=1}^n \big[u_j^{(i)}\big]^k, \quad j \in \mathbb{N}, \quad k \in \{1, \ldots, n\}.$$

Thus, the vector $u = (u_1, \ldots, u_j, \ldots)$, $u_j = (u_j^{(1)}, \ldots, u_j^{(n)}) \in \mathbb{C}^n$ has only a finite number of nonzero coordinates and so belongs to $\ell_p(\mathbb{C}^n)$. On the other hand, $y = \mathcal{F}_{\ell_p,n}(u)$. Thus,

$$q(N^{\mathbf{k}_1}(u), \ldots, N^{\mathbf{k}_s}(u)) = q(H^{\mathbf{k}_1}(y), \ldots, H^{\mathbf{k}_s}(y)) \neq 0.$$

A contradiction. Hence, polynomials $N^{\mathbf{k}}$, $|\mathbf{k}| \geq \lceil p \rceil$ are algebraically independent.

We already observed that $C_{\mathcal{F}_{\ell_p,n}}$ is a homomorphism. Clearly that $C_{\mathcal{F}_{\ell_p,n}}$ is injective. So, we need to show that it is surjective. Let $Q \in \mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ and $\deg Q = d$. There is a finite number of polynomials $N^{\mathbf{k}_1}, \ldots, N^{\mathbf{k}_r}$ such that $\deg N^{\mathbf{k}_i} = |\mathbf{k}_i| \leq d$. Thus, if Q is an algebraic combination of polynomials $N^{\mathbf{k}}$, $|\mathbf{k}| \geq \lceil p \rceil$, then Q is an algebraic combination of polynomials $N^{\mathbf{k}_1}, \ldots, N^{\mathbf{k}_r}$. We denote by Q_m the restriction of Q to $\mathbb{C}^m(\mathbb{C}^n)$. Suppose that m is large enough so that the restriction of $H^{\mathbf{k}_1}, \ldots, H^{\mathbf{k}_r}$ to $\mathbb{C}^m(\mathbb{C}^n) \subset \ell_p(\mathbb{C}^n)$ is algebraically independent. Such a number must exist by Theorem 3. Then, the restriction of $N^{\mathbf{k}_1}, \ldots, N^{\mathbf{k}_r}$ to $\mathbb{C}^m(\mathbb{C}^n)$ is algebraically independent as well. By Theorem 2, there exists a polynomial q_m of r variables such that

$$Q_m(x) = q_m(N^{\mathbf{k}_1}(x), \ldots, N^{\mathbf{k}_r}(x)), \quad x \in \mathbb{C}^m(\mathbb{C}^n).$$

Note that if $s > m$, then $q_s = q_m$; otherwise,

$$Q_m(x) = q_s(N^{\mathbf{k}_1}(x), \ldots, N^{\mathbf{k}_r}(x)), \quad x \in \mathbb{C}^m(\mathbb{C}^n)$$

will be a different representation of Q_m that contradicts the algebraic independence of $N^{\mathbf{k}_1}, \ldots, N^{\mathbf{k}_r}$ on $\mathbb{C}^m(\mathbb{C}^n)$. Hence, the restriction of Q to the dense subspace

$$c_{00}(\mathbb{C}^n) = \bigcup_{s \geq m} \mathbb{C}^m(\mathbb{C}^n) \subset \ell_p(\mathbb{C}^n)$$

has the representation

$$Q(x) = q_m(N^{\mathbf{k}_1}(x), \ldots, N^{\mathbf{k}_r}(x)), \quad x \in c_{00}(\mathbb{C}^n).$$

By the continuity of Q, this representation is true for every $x \in \ell_p(\mathbb{C}^n)$. Thus, $C_{\mathcal{F}_{\ell_p,n}}$ is surjective. Therefore, it is an isomorphism from $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ to $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$, and polynomials $N^{\mathbf{k}} = H^{\mathbf{k}} \circ \mathcal{F}_{\ell_p,n}$ form an algebraic basis in $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$. $\square$

5. Algebras of Symmetric Analytic Functions and Their Spectra

Proposition 3. *The polynomial mapping* $\mathcal{F}_{\ell_p,n} \colon \ell_p(\mathbb{C}^n) \to \ell_p(\mathbb{C}^n)$ *is not surjective whenever* $n > 1$.

Proof. If $n = 1$, then $\mathcal{F}_{\ell_p,n}$ is just the identity map and so it is surjective. Let $n \in \mathbb{N}$, $n > 1$. We construct a vector y in $\ell_p(\mathbb{C}^n)$, which does not belong to the range of $\mathcal{F}_{\ell_p,n}$. Set

$$y = \left(\begin{pmatrix} 0 \\ \cdots \\ 0 \\ (-1)^{n+1} \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \cdots \\ 0 \\ \frac{(-1)^{n+1}}{j^{2/p}} \end{pmatrix} \cdots \right).$$

Clearly, $y \in \ell_p(\mathbb{C}^n)$. Let us suppose that there is a vector $u = \left(u_j^{(k)}\right)$, $k = 1, \ldots, n$, $j \in \mathbb{N}$ such that $y = \mathcal{F}_{\ell_p,n}(u)$. Then, for every j, the coordinates $u_j^{(k)}$ must satisfy equations

$$\left[u_j^{(1)}\right]^m + \left[u_j^{(2)}\right]^m + \cdots + \left[u_j^{(n)}\right]^m = \begin{cases} 0 & \text{if } m \leq n, \\ \frac{(-1)^{n+1}}{j^{2/p}} & \text{if } m = n. \end{cases}$$

It is easy to check that the set of all roots of the system can be written as

$$\left\{ \frac{\alpha_0}{j^{2/np}}, \ldots, \frac{\alpha_{n-1}}{j^{2/np}} \right\},$$

where $\alpha_0, \ldots, \alpha_{n-1}$ are roots of 1. Hence, up to permutations of coordinates $u_j^{(k)}$ for every fixed j, the vector u can be represented as

$$u = \left(\begin{pmatrix} \alpha_0 \\ \cdots \\ \alpha_{n-2} \\ \alpha_{n-1} \end{pmatrix} \cdots \begin{pmatrix} \frac{\alpha_0}{j^{2/np}} \\ \cdots \\ \frac{\alpha_{n-2}}{j^{2/np}} \\ \frac{\alpha_{n-1}}{j^{2/np}} \end{pmatrix} \cdots \right).$$

But $u \notin \ell_p(\mathbb{C}^n)$, because $\|u\|_{\ell_p} = \infty$ for every $n \geq 2$ and $1 \leq p < \infty$. Therefore, $\mathcal{F}_{\ell_p,n}$ is not surjective because y is not in the range of $\mathcal{F}_{\ell_p,n}$. $\square$

Let us denote by $H_{bvs}(\ell_p(\mathbb{C}^n))$ the closure of $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ and by $H_{bds}(\ell_p(\mathbb{C}^n))$ the closure of $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ in $H_b(\ell_p(\mathbb{C}^n))$. Thus, both $H_{bvs}(\ell_p(\mathbb{C}^n))$ and $H_{bds}(\ell_p(\mathbb{C}^n))$ are Fréchet algebras with respect to the topology of uniform converges on bounded subsets of $\ell_p(\mathbb{C}^n)$.

Theorem 5. *The mapping $C_{\mathcal{F}_{\ell_p,n}}(f) = f \circ \mathcal{F}_{\ell_p,n}$, $f \in H_{bvs}(\ell_p(\mathbb{C}^n))$ is a continuous homomorphism from $H_{bvs}(\ell_p(\mathbb{C}^n))$ to $H_{bds}(\ell_p(\mathbb{C}^n))$ with a dense range.*

Proof. By using Theorem 4, the injective homomorphism $C_{\mathcal{F}_{\ell_p,n}}$ is well defined on the dense subset $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$. It is well known that a composition operator with an analytic map of a bounded type is a continuous operator from $H_b(X)$ to itself. Moreover, for every $f \in H_{bvs}(\ell_p(\mathbb{C}^n))$, the range

$$C_{\mathcal{F}_{\ell_p,n}}(f) = C_{\mathcal{F}_{\ell_p,n}}\Big(\sum_{i=0}^{\infty} f_i\Big) = \sum_{i=0}^{\infty} C_{\mathcal{F}_{\ell_p,n}}(f_i)$$

belongs to $H_{bds}(\ell_p(\mathbb{C}^n))$. Thus, $C_{\mathcal{F}_{\ell_p,n}}$ is a continuous homomorphism from $H_{bvs}(\ell_p(\mathbb{C}^n))$ to $H_{bds}(\ell_p(\mathbb{C}^n))$. On the other hand, the range of $C_{\mathcal{F}_{\ell_p,n}}$ contains the dense subset $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ of $H_{bds}(\ell_p(\mathbb{C}^n))$. $\square$

Note that any double-symmetric analytic function is block symmetric as well. Thus, $H_{bds}(\ell_p(\mathbb{C}^n))$ is a closed subspace of $H_{bvs}(\ell_p(\mathbb{C}^n))$.

The spectra of algebras $H_{bvs}(\ell_p(\mathbb{C}^n))$ were considered in [35]. The situation in the case $H_{bds}(\ell_p(\mathbb{C}^n))$ is similar. In particular, for every $x \in \ell_p(\mathbb{C}^n)$, we can assign a character δ_x (so-called a point evaluation functional) on $H_{bds}(\ell_p(\mathbb{C}^n))$ by

$$\delta_x(f) = f(x), \quad f \in H_{bds}(\ell_p(\mathbb{C}^n)).$$

Clearly, $\delta_x = \delta_y$ if and only $P(x) = P(y)$ for every polynomial $P \in H_{bds}(\ell_p(\mathbb{C}^n))$. Also, like in the symmetric and block-symmetric cases (c.f. [8,9,35]), there are characters that are not of the form δ_x.

Example 6. *Let p be a positive integer and $(v_{(m)})$ be a sequence in $H_{bds}(\ell_p(\mathbb{C}^n))$,*

$$v_{(m)} = \left(\underbrace{\begin{pmatrix} \frac{1}{m^{1/p}} \\ 0 \\ \cdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{m^{1/p}} \\ 0 \\ \cdots \\ 0 \end{pmatrix}}_{m} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \cdots \right).$$

The sequence $(v_{(m)})$ is bounded and $\|v_{(m)}\|_{\ell_p} = 1$. Note that $H_{bds}(\ell_p(\mathbb{C}^n))$ is a projective limit of Banach algebras of uniformly continuous double-symmetric analytic functions $H_{uds}(B^r_{\ell_p})$ on *balls in ℓ_p, centered at the origin and of a radius $r > 0$. In other words, $H_{uds}(B^r_{\ell_p})$ is the closure of $H_{bds}(\ell_p(\mathbb{C}^n))$ with respect to the norm*

$$\|f\|_r = \sup_{\|x\|_{\ell_p} \le r} |f(x)|.$$

Thus, the spectrum of $H_{bds}(\ell_p(\mathbb{C}^n))$ is the inductive limit of the spectra of $H_{uds}(B^r_{\ell_p})$, $r > 0$, which are a compact topological space with respect to the Gelfand topology (c.f. [20]). Thus, for any bounded sequence (x_m), the sequence of characters (δ_{x_m}) has a cluster point in the spectrum. Let ψ be a cluster point of $(\delta_{v_{(m)}})$ in the spectrum of $H_{bds}(\ell_p(\mathbb{C}^n))$. Taking a subsequence, if necessary, we may assume that $f(v_{(m)}) \to \psi(f)$ as $m \to \infty$ for every $f \in H_{bds}(\ell_p(\mathbb{C}^n))$. Let $u_{(m)} = \mathcal{F}_{\ell_p,n}(v_{(m)})$. Then,

$$u_{(m)} = \left(\underbrace{\begin{pmatrix} \frac{1}{m^{1/p}} \\ \frac{1}{m^{2/p}} \\ \cdots \\ \frac{1}{m^{n/p}} \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{m^{1/p}} \\ \frac{1}{m^{2/p}} \\ \cdots \\ \frac{1}{m^{n/p}} \end{pmatrix}}_{m} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \cdots \right).$$

This sequence is bounded by the continuity of the polynomial map $\mathcal{F}_{\ell_p,n}$. Actually, it is easy to check that $\|u_{(m)}\|_{\ell_p} \le \pi^2/6$. For every multi-index $\mathbf{k} = (k_1, \ldots, k_n)$, $|\mathbf{k}| \ge p$, we have

$$H^{\mathbf{k}}(u_{(m)}) = \frac{m}{m^{\frac{k_1 + 2k^2 + \cdots + nk_n}{p}}} = \begin{cases} 1 & \text{if } \mathbf{k} = (p, 0, \ldots, 0), \\ \text{tends to } 0 \text{ as } m \to 0 & \text{otherwise.} \end{cases}$$

Because $u_{(m)}$ is bounded, the sequence $\delta_{u_{(m)}}$ has a cluster point ϕ, and

$$\phi(H^{\mathbf{k}}) = \begin{cases} 1 & \text{if } \mathbf{k} = (p, 0, \ldots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

If there is a point $x \in \ell_p(\mathbb{C}^n)$ such that $\psi = \delta_x$, then $\phi = \delta_y$ for $y = \mathcal{F}_{\ell_p,n}(x)$. But, according to [35], such a point y does not exist. Thus, ψ is not a point evaluation functional.

6. Discussion and Conclusions

We considered the analytic functions on a Banach space X that are symmetric with respect to a semidirect product of groups of operators on X. The main examples are algebras of polynomials and analytic functions on $\ell_p(\mathbb{C}^n)$ such that every function $f(x) = f\big(x_j^{(k)}\big)$ is invariant with respect to the permutation of indexes $j \in \mathbb{N}$, and for every fixed j, it is invariant with respect to the permutations of indexes $k \in \{1, \ldots, n\}$. We proved that the algebra of polynomials $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ is isomorphic to the algebra of block-symmetric polynomials $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ for which we do not assume the invariance with respect to the permutations of indexes k. This result may be considered as an infinite-dimensional generalization of the fact that the map

$$P(t_1, \ldots, t_n) \rightsquigarrow P\Big(\sum_{k=1}^n t_k, \ldots, \sum_{k=1}^n t_k^n\Big)$$

is an isomorphism between the algebra of all polynomials on $\mathbb{C}^n$ and symmetric polynomials on $\mathbb{C}^n$. However, we can not extend the isomorphism $C_{\mathcal{F}_{\ell_p,n}}$ of algebras $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ and $\mathcal{P}_{ds}(\ell_p(\mathbb{C}^n))$ to their completions $H_{bvs}(\ell_p(\mathbb{C}^n))$ and $H_{bds}(\ell_p(\mathbb{C}^n))$. Moreover, the fact that $\mathcal{F}_{\ell_p,n}$ is not surjective suggests to us that we start to look for a counterexample.

Further investigations can be continued in different directions. First, we can try to replace $\ell_p(\mathbb{C}^n))$ with $\ell_p(\ell_q)$. Note that even for the case $p = q = 1$, we know almost nothing about block-symmetric and double-symmetric polynomials. Another direction is the spectrum of $H_{bds}(\ell_p(\mathbb{C}^n))$. In this paper, we observed that the spectrum contains characters that are not point evaluation functionals. But the set of point evaluation functionals is interesting itself, because it may admit non-trivial algebraic structures (see, e.g., [40]). Note that in [40], using symmetric polynomials on ℓ_1, some applications in Cryptography were proposed. In [41], possible applications of symmetric and block-symmetric polynomials on $\mathbb{C}^m(\mathbb{C}^n)$ in neural networks and blockchain technologies were considered. Our further investigation will be devoted to a generalization of this approach for the cases of block-symmetric and double-symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

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