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Fixed Point Theory in Extended Parametric S_b-Metric Spaces and Its Applications

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Abstract: This article introduces the novel concept of an extended parametric S_b-metric space, which is a generalization of both S_b-metric spaces and parametric S_b-metric spaces. Within this extended framework, we first establish an analog version of the Banach fixed-point theorem for self-maps. We then prove an improved version of the Banach contraction principle for symmetric extended parametric S_b-metric spaces, using an auxiliary function to establish the desired result. Finally, we provide illustrative examples and an application for determining solutions to Fredholm integral equations, demonstrating the practical implications of our work.

Keywords: metric space; fixed point; parametric; linear; contraction

MSC: 47H10; 54H25

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1. Introduction

The study of the metric fixed point theory is only a little over a century old, but its applicability is relevant in all the branches of science and engineering. Within the context of an axiomatic framework, it is justifiable to attribute the genesis of the notion of distance to Euclid, and conceivably even to a period preceding him. Frechet [1] proposed the introduction of systematic and standardized measures for distance in the realm of abstract mathematics. The concept of metric space has been employed not only in the field of mathematics but also in the qualitative sciences. For example, one notable generalization of metrics, known as partial metrics, was introduced by Matthews [2] to address specific challenges in the field of domain theory in computer science. In addition to these abstract formulations, the concept of metrics has been expanded and diversified through numerous diverse approaches. Among the various concepts, it is important to draw attention to some of the generalizations that are widely recognized and particularly captivating (refer to [3–9]).

One of the earliest generalizations is the quasi-metric one, which is produced by eliminating the standard metric's symmetry property. Another notion that was presented early on is the concept of semi-metric (proposed by [10]). This type of metric satisfies only the properties of self-distance and symmetry, which are characteristic of the Euclidean metric.

An alternative formulation of the metric concept was derived by substituting the triangle inequality with a modified version. The concept under consideration is referred to as quasi-metric in certain references [11], and as a b-metric in other references [12,13].

Definition 1 ([13])**.** *Let $\mathfrak{W}$ be a nonempty set, define a real-valued function $\rho : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ such that for a given b (real number) ≥ 1, it satisfies the following conditions:*

I. $\rho(g,e) = 0$ *if and only if* $g = e$;
II. $\rho(g,e) = \rho(e,g)$;
III. $\rho(g,e) \leq b[\rho(g,h) + \rho(h,e)]$, *for all* $g,e,h \in \mathfrak{W}$.

Then, ρ *is said to be b-metric and the pair* $(\mathfrak{W}, \rho)$ *is said to be b-metric space.*

It is imperative to note that, in a broad context, the b-metric does not exhibit continuity. Moreover, it is important to note that not every b-metric space may be considered a metric space. In an alternative scenario, assuming the value of b to be equal to 1, one can see that every b-metric space would therefore be considered as metric space.

Branciari [14] proposes a novel approach by altering the triangular inequality in metric spaces to a quadrilateral inequality.

Definition 2 ([14])**.** *Let* $\mathfrak{W}$ *be a non-empty set, define a real-valued function* $\rho : \mathfrak{W} \times \mathfrak{W} \to [0,\infty)$ *such that for all* $g,e \in \mathfrak{W}$ *and all distinct* $h,\sigma \in \mathfrak{W}$*, where* h *and* σ *are different from* g *and* e*, which satisfies*

I. $\rho(g,e) = 0$ *if and only if* $g = e$;
II. $\rho(g,e) = \rho(e,g)$;
III. $\rho(g,e) \leq \rho(g,h) + \rho(h,\sigma) + \rho(\sigma,e)$.

Then ρ *is called a generalized metric (a Branciari distance), and the pair* $(\mathfrak{W}, \rho)$ *is known as GMS (generalized metric space) in the sense of Branciari.*

Remark 1. *In general, a Branciari distance may not be continuous. The topologies of Branciari distance space and metric space are incompatible. Furthermore, every metric is a Branciari distance but the converse does not need to be true.*

Example 1. *Let* $U = \{0,2\}$, $V = \{\frac{1}{n} : n \geq 1\}$ *and* $\mathfrak{W} = U \cup V$. *Define* $\rho : \mathfrak{W}^2 \to [0,\infty)$ *by*

$$\rho(g,e) = \begin{cases} 0 & \text{if } g = e, \\ 1 & \text{if } g \neq e \text{ and either } g,e \in U \text{ or } g,e \neq V, \\ e & \text{if } g \in U \text{ and } e \in V, \\ g & \text{if } g \in V \text{ and } e \in U. \end{cases}$$

Then, ρ *is a Branciari distance on* $\mathfrak{W}$ *but not a metric.*

In the last three decades, another emerging technique in the field of metric extension involves the utilization of the geometric properties of three points, as opposed to the conventional approach which relies on only two points, such as $\mathfrak{D} : \mathfrak{W} \times \mathfrak{W} \times \mathfrak{W} \to [0,+\infty)$. The idea of D-metric [15] and G-metric [16] are the most famous examples of this trend. All the authors have derived the analogue version of the most celebrated result in the history of fixed point theory, precisely known as Banach contraction principle (BCP) [17] in such spaces (see [18–20]). This theory only began to emerge as a distinct field in the late 19th century and early 20th century, when important developments took place, and several new metrics were introduced.Some of them are new, and a few are the generalization of the existence of previous spaces.

Sedghi et al. [21] introduced a new type of generalized metric space, by relaxing the symmetry property, known as S-metric space.

Definition 3 ([21])**.** *Let* $\mathfrak{W}$ *be a non-empty set. Then, a function* $S : \mathfrak{W}^3 \to [0,\infty)$ *is said to be S-metric on* $\mathfrak{W}$ *if for each* $g,e,h,t \in \mathfrak{W}$ *the following conditions hold:*

(i). $S(g,e,h) \geq 0$;
(ii). $S(g,e,h) = 0$ *if and only if* $g = e = h$;
(iii). $S(g,e,h) \leq S(g,g,t) + S(e,e,t) + S(h,h,t)$.

The pair $(\mathfrak{W}, S)$ *is called an S-metric space.*

Example 2. *Let $\mathfrak{W} = \mathbb{R}^n$ and $\|.\|$ be a norm on $\mathfrak{W}$; then, $S(g,e,h) = \|e+h-2g\| + \|e-h\|$ is an S-metric space.*

Sedghi and Dung [22] made the observation that every S-metric space can be considered topologically equal to a metric space. Several researchers have studied the S-metric space as well as developed a number of results pertaining the presence of fixed points [23–25].

On taking motivation from the research conducted by Bakhtin [12] and Sedghi et al. [21], Souayah and Mlaiki [26] initially proposed the notion of an S_b-metric space. Subsequently, Rohen et al. [27] made modifications to the definition of S_b-metric spaces as follows:

Definition 4 ([27])**.** *Let $\mathfrak{W}$ be a non-empty set and b be a real number ≥ 1. A function $S_b : \mathfrak{W}^3 \to [0,\infty)$ be such that for all $g,e,h,t \in \mathfrak{W}$, it satisfies the following conditions:*

(i). $S_b(g,e,h) = 0$ *if and only if* $g = e = h$*;*
(ii). $S_b(g,e,h) \leq b[S_b(g,g,t) + S_b(e,e,t) + S_b(h,h,t)]$.

Then, S_b is said to be S_b-metric on $\mathfrak{W}$ and the pair $(\mathfrak{W}, S_b)$ is said to be S_b-metric spaces.

Hussain et al. [28] gave a definition and analysis of parametric spaces. Subsequently, a year later, the authors extended their study by introducing the concept of parametric b-metric space [29]. In another incremental advancement, Taş and Ozgür [30] proposed the concept of a parametric S-metric space as an extension of the parametric metric space, as follows:

Definition 5 ([30])**.** *Let $\mathfrak{W}$ be a non-empty set. Define a function $P_r : \mathfrak{W}^3 \times (0,\infty) \to [0,\infty)$ such that for all $g,e,h,\sigma \in \mathfrak{W}$ and $\lambda > 0$, it satisfies the following conditions:*

(i). $P_r(g,e,h,\lambda) = 0$ *if and only if* $g = e = h$*;*
(ii). $P_r(g,e,h,\lambda) \leq P_r(g,g,\sigma,\lambda) + P_r(e,e,\sigma,\lambda) + P_r(h,h,\sigma,\lambda)$.

Then, the function P_r is said to be parametric S-metric on $\mathfrak{W}$ and the pair $(\mathfrak{W}, P_r)$ is called parametric S-metric space.

Moreover, Taş and Özgür [31] improved their own idea and introduced the concept of parametric S_b-metric space in 2018.

Definition 6 ([31])**.** *Let $\mathfrak{W}$ be a non-empty set and let $b \geq 1$ be a given real number. Define a function $N : \mathfrak{W}^3 \times (0,\infty) \to [0,\infty)$ such that for all $g,e,h,\sigma \in \mathfrak{W}$ and $\lambda > 0$, it satisfies the following conditions:*

(i). $N(g,e,h,\lambda) = 0$ *if and only if* $g = e = h$*;*
(ii). $N(g,e,h,\lambda) \leq b[N(g,g,\sigma,\lambda) + N(e,e,\sigma,\lambda) + N(h,h,\sigma,\lambda)]$.

Then, the function N is said to be parametric S_b-metric on $\mathfrak{W}$ and the pair $(\mathfrak{W}, N)$ is called parametric S_b-metric space.

Example 3. *Let $\mathfrak{W} = \{\nu \mid \nu : (0,\infty) \to \mathbb{R}$ is a function$\}$. Define a function $N : \mathfrak{W}^3 \times (0,\infty) \to [0,\infty)$ by*

$$N(\nu,e,r,\sigma) = \frac{1}{9}(|\nu(\sigma) - e(\sigma)| + |\nu(\sigma) - r(\sigma)| + |e(\sigma) - r(\sigma)|)^2$$

for each $\sigma > 0$ and for all $\nu,e,r \in \mathfrak{W}$. If $b = 4$, then (X,N) is a parametric S_b-metric space; nonetheless, it is not a parametric S-metric space.

Mlaiki [32] followed the work of Rohen et al. [27] to introduce the concept of extended S_b-metric space as follows:

Definition 7 ([32]). *Let $\mathfrak{W}$ be a non-empty set and $N : \mathfrak{W}^3 \to [1, \infty)$ be a positive real-valued function. Define a function $R_N : \mathfrak{W}^3 \to [1, \infty)$ such that for all $g, e, h, \sigma \in \mathfrak{W}$, it satisfies the following conditions:*

(i). $R_N(g, e, h) = 0$ *if and only if* $g = e = h$;

(ii). $R_N(g, e, h) \leq N(g, e, h)[R_N(g, g, \sigma) + R_N(e, e, \sigma) + R_N(h, h, \sigma)]$.

Then, the function R_N is said to be extended S_b-metric on $\mathfrak{W}$ and the pair $(\mathfrak{W}, R_N)$ is called extended S_b-metric space.

Remark 2. *Every S_b-metric space is an extended S_b-metric space ($N(g, e, h) = b \geq 1$), but the converse not always true.*

Furthermore, counter-examples and associated findings regarding the aforementioned spaces are available in [33–39].

As an expansion of the parametric metric space and the S_b-metric space, we present in this article a novel metric space called the extended parametric S_b-metric space. Section 2 contains the definition of an extended parametric S_b-metric space, proof of two Lemma's along with two illustrative examples. In Section 3, analogues of the some well-known fixed point theorems are proved in both extended parametric S_b-metric spaces and in symmetric extended parametric S_b-metric spaces. At last, in Section 4, we make use of our result in order to find the existence of a solution to a Fredholm integral equation.

2. Extended Parametric S_b-Metric Space

This section commences with the definition of the extended parametric S_b-metric space.

Definition 8. *Let $\mathfrak{W}$ be a non-empty set and $N : \mathfrak{W}^3 \to [1, \infty)$ be a positive real-valued function. Define a function $R_N : \mathfrak{W}^3 \times (0, \infty) \to [0, \infty)$ such that for all $g, e, h, \sigma \in \mathfrak{W}$ and $\lambda > 0$, it satisfies the following conditions:*

R_N*-1.* $R_N(g, e, h, \lambda) = 0$ *if and only if* $g = e = h$;

R_N*-2.* $R_N(g, e, h, \lambda) \leq N(g, e, h)[R_N(g, g, \sigma, \lambda) + R_N(e, e, \sigma, \lambda) + R_N(h, h, \sigma, \lambda)]$.

Then, the function R_N is said to be extended parametric S_b-metric (EPS_b) on $\mathfrak{W}$ and the pair $(\mathfrak{W}, R_N)$ is called extended parametric S_b-metric space.

Example 4. *Let $\mathfrak{W} = \mathbb{R}$. Define function $N : \mathfrak{W}^3 \to [1, \infty)$ by*

$$N(g, e, h) = 1 + \mid g \mid + \mid e \mid$$

and a function $R_N : \mathfrak{W}^3 \times (0, \infty) \to [0, \infty)$ by

$$R_N(g, e, h, \lambda) = \lambda^2[\mid g - e \mid + \mid e - h \mid + \mid g - h \mid]$$

for each $g, e, h \in \mathbb{R}$ and $\lambda > 0$. Then, R_N is an extended parametric S_b-metric space.

Example 5. *Let $\mathfrak{W} = C[a, b]$ be the set of all continuous real-valued functions on $[a, b]$. Define function $N : \mathfrak{W}^3 \to [1, \infty)$ by*

$$N(g(\sigma), e(\sigma), h(\sigma)) = \max\{\mid g(\sigma) \mid, \mid e(\sigma) \mid\} + \mid h(\sigma) \mid + 2$$

and function $R_N : \mathfrak{W}^3 \times (0, \infty) \to [0, \infty)$ by

$$R_N(g(\sigma), e(\sigma), h(\sigma), \lambda) = P(\lambda) \sup_{\sigma \in [a,b]} \mid \max\{g(\sigma), e(\sigma)\} - h(\sigma) \mid^2$$

for each $g, e, h \in \mathbb{R}$, where $P : (0, \infty) \to (0, \infty)$ is defined as $P(\lambda) = \lambda$.

Then, the pair $(\mathfrak{W}, R_N)$ is a complete extended parametric S_b-metric space.

Definition 9. *Let $(\mathfrak{W}, R_N)$ be a extended parametric S_b-metric space and let $\{t_n\}$ be a sequence in $\mathfrak{W}$. Then,*

(i). *$\{t_n\}$ converges to g if and only if there exists $n_0 \in N$ such that $R_N(t_n, t_n, g, \lambda) < \epsilon$ for all $n \geq n_0$ and $\lambda > 0$;*
(ii). *$\{t_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} R_N(t_n, t_n, t_m, \lambda) = 0$. for all $\lambda > 0$;*
(iii). *$(\mathfrak{W}, R_N)$ is called complete if every Cauchy sequence is convergent in $\mathfrak{W}$.*

Lemma 1. *Let $(\mathfrak{W}, R_N)$ be a extended parametric S_b-metric space. Then, for each $g, e \in \mathfrak{W}$ and for all $\lambda > 0$,*

$$R_N(g, g, e, \lambda) \leq N R_N(e, e, g, \lambda) \quad \text{and} \quad R_N(e, e, g, \lambda) \leq N R_N(g, g, e, \lambda)$$

Proof. Using the condition (R_N-2) of Definition 8, we obtain

$$\begin{aligned} R_N(g, g, e, \lambda) &\leq N[R_N(g, g, g, \lambda) + R_N(g, g, g, \lambda) + R_N(e, e, g, \lambda)] \\ &\leq N[2R_N(g, g, g, \lambda) + R_N(e, e, g, \lambda)] \\ &\leq N R_N(e, e, g, \lambda) \end{aligned}$$

and

$$\begin{aligned} R_N(e, e, g, \lambda) &\leq N[R_N(e, e, e, \lambda) + R_N(e, e, e, \lambda) + R_N(g, g, e, \lambda)], \\ &\leq N[2R_N(e, e, e, \lambda) + R_N(g, g, e, \lambda)] \\ &\leq N R_N(g, g, e, \lambda) \end{aligned}$$

Hence, the proof. □

Lemma 2. *Let $(\mathfrak{W}, R_N)$ be a extended parametric S_b-metric space. If $\{t_n\}$ converges to g, then g is unique.*

Proof. Since $\{t_n\}$ converges to g, $\lim_{n\to\infty} t_n = g$. On the contrary, assume that the limit g is not unique. Therefore, there exists some $e \in \mathfrak{W}$ such that $\lim_{n\to\infty} t_n = e$, with $g \neq e$. Thus, for each $\epsilon > 0$ and for all $\lambda > 0$, we can choose $n_1, n_2 \in \mathbb{N}$ and $n \geq \{n_1, n_2\}$ such that

$$R_N(t_n, t_n, g, \lambda) < \frac{\epsilon}{4N} \quad \text{and} \quad R_N(t_n, t_n, e, \lambda) < \frac{\epsilon}{2N} \tag{1}$$

Let us set $n_0 = \max\{n_1, n_2\}$, and the condition (R_N-2) of Definition 8 and Lemma 1 implies that

$$\begin{aligned} R_N(g, g, e, \lambda) &\leq N[2R_N(g, g, t_n, \lambda) + R_N(e, e, t_n, \lambda)] \\ &\leq N[2R_N(t_n, t_n, g, \lambda) + R_N(t_n, t_n, e, \lambda)] \\ &< N[2\frac{\epsilon}{4N} + \frac{\epsilon}{2N}] \qquad \text{[on using (1)]} \\ &< N[\frac{\epsilon}{2N} + \frac{\epsilon}{2N}] = \epsilon. \end{aligned}$$

which implies that $R_N(g, g, e, \lambda) = 0$. Thus, we have $g = e$. □

This leads to the following important result.

Lemma 3. *Let $(\mathfrak{W}, R_N)$ be a extended parametric S_b-metric space. If $\{t_n\}$ converges to g, then $\{t_n\}$ is Cauchy.*

Definition 10. *Let $(\mathfrak{W}, R_N)$ be a extended parametric S_b-metric space. Then,*

(i). *The diameter of a subset $\mathcal{Y}$ of $\mathfrak{W}$ is defined as*

$$\text{diam}(\mathcal{Y}) := \sup\{R_N(g, e, h, \lambda) \mid g, e, h \in \mathfrak{W},\ \lambda > 0\}.$$

(ii). *For $g \in \mathfrak{W}$ and $\epsilon > 0$, we can define a ball $B(g, \epsilon)$ as follows:*

$$B(g, \epsilon) = \{e \in X \mid R_N(g, g, e, \lambda) \leq \epsilon,\ \lambda > 0\}.$$

3. Main Results

In this discussion, we will begin by presenting and demonstrating the analogous form of the Banach fixed point theorem in the context of extended parametric S_b-metric space.

Theorem 1. *Consider a complete extended parametric S_b-metric space $(\mathfrak{W}, R_N)$, where R_N is a continuous function. Let f be a self-mapping on $\mathfrak{W}$ satisfying the following condition: for all $g, e, h \in \mathfrak{W}$ and $\lambda > 0$*

$$R_N(fg, fe, fh, \lambda) \leq \theta R_N(g, e, h, \lambda), \tag{2}$$

where $0 \leq \theta < \frac{1}{2}$ and for any $g_0 \in \mathfrak{W}$, we have

$$\lim_{n,m\to\infty} N(f^n g_0, f^n g_0, f^m g_0) < \frac{1}{2\theta}. \tag{3}$$

Then, f has a unique fixed point $\nu \in \mathfrak{W}$. Moreover, for every $h \in \mathfrak{W}$, we have $\lim_{n\to\infty} f^n h = \nu$.

Proof. Since $\mathfrak{W}$ is a non-empty set and f is a self-map on $\mathfrak{W}$, we can choose a $g_0 \in \mathfrak{W}$ such that $fg_0 = g_1$. Continuing like this, we can define a sequence $\{g_n\}$ of iterates as follows:

$$\begin{aligned} g_1 &= fg_0, \\ g_2 &= fg_1 = f^2 g_0, \\ &\vdots \\ g_n &= fg_{n-1} = f^n g_0. \end{aligned} \tag{4}$$

Let us substitute $g = g_{n-1}$, $e = g_{n-1}$ and $h = g_n$ in Equation (2), and we have

$$\begin{aligned} R_N(g_n, g_n, g_{n+1}, \lambda) &= R_N(fg_{n-1}, fg_{n-1}, fg_n, \lambda) \\ &\leq \theta R_N(g_{n-1}, g_{n-1}, g_n, \lambda). \end{aligned}$$

Again, from Equation (2), we have

$$\begin{aligned} R_N(g_{n-1}, g_{n-1}, g_n, \lambda) &= R_N(fg_{n-2}, fg_{n-2}, fg_{n-1}, \lambda) \\ &\leq \theta R_N(g_{n-2}, g_{n-2}, g_{n-1}, \lambda). \end{aligned}$$

Combining the above two inequalities and repeating the process n times, we obtain

$$R_N(g_n, g_n, g_{n+1}, \lambda) \leq \theta^2 R_N(g_{n-2}, g_{n-2}, g_{n-1}, \lambda) \leq \cdots \leq \theta^n R_N(g_0, g_0, g_1, \lambda)$$

This implies that

$$R_N(g_n, g_n, g_{n+1}, \lambda) \leq \theta^n R_N(g_0, g_0, g_1, \lambda) \tag{5}$$

This proves that the sequence $\{g_n\}$ is a Cauchy sequence in $\mathfrak{W}$. Indeed, for all $m > n$, $m, n \in \mathbb{N}$, and when using inequality (2), condition (R_N-2) of Definition 8; we obtain

$$\begin{aligned} R_N(g_n, g_n, g_m, \lambda) \leq\ & N(g_n, g_n, g_m)(2\theta)^n R_N(g_0, g_0, g_1, \lambda) \\ & + N(g_n, g_n, g_m) N(g_{n+1}, g_{n+1}, g_m)(2\theta)^{n+1} R_N(g_0, g_0, g_1, \lambda) \\ & \vdots \\ & + N(g_n, g_n, g_m) N(g_{n+1}, g_{n+1}, g_m) \cdots N(g_{m-1}, g_{m-1}, g_m) \\ & \quad (2\theta)^{m-1} R_N(g_0, g_0, g_1, \lambda). \end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
&R_N(g_n, g_n, g_m, \lambda) \\
&\leq R_N(g_0, g_0, g_1, \lambda) \begin{bmatrix} N(g_1, g_1, g_m) N(g_2, g_2, g_m) \cdots \\ \quad N(g_{n-1}, g_{n-1}, g_m) N(g_n, g_n, g_m)(2\theta)^n \\ + N(g_1, g_1, g_m) N(g_2, g_2, g_m) \cdots \\ \quad N(g_n, g_n, g_m) N(g_{n+1}, g_{n+1}, g_m)(2\theta)^{n+1} \\ \vdots \\ + N(g_1, g_1, g_m) N(g_2, g_2, g_m) \cdots \\ \quad N(g_{m-2}, g_{m-2}, g_m) N(g_{m-1}, g_{m-1}, g_m)(2\theta)^{m-1} \end{bmatrix} \quad (6) \\
&\leq R_N(g_0, g_0, g_1, \lambda) \sum_{j=n}^{m-1} (2\theta)^j \prod_{i=1}^{j} N(g_i, g_i, g_m)
\end{aligned}
$$

Suppose we have a series

$$\mathbb{B} = \sum_{n=1}^{\infty} (2\theta)^n \prod_{i=1}^{n} N(g_i, g_i, g_m)$$

and its partial sum

$$\mathbb{B}_n = \sum_{j=1}^{n} (2\theta)^j \prod_{i=1}^{j} N(g_i, g_i, g_m).$$

When using Equation (3) and when applying ratio test, we obtain that the series

$$\sum_{n=1}^{n} (2\theta)^n \prod_{i=1}^{n} N(g_i, g_i, g_m)$$

converges. Hence, from (6), for $m > n$ we have

$$R_N(g_n, g_n, g_m, \lambda) \leq R_N(g_0, g_0, g_1, \lambda)[\mathbb{B}_{m-1} - \mathbb{B}_n].$$

Thus, $R_N(g_n, g_n, g_m, \lambda) \to 0$ as $n, m \to \infty$. The completeness of $\mathfrak{W}$ implies that there exist some $\nu \in \mathfrak{W}$ such that

$$\lim_{n \to \infty} g_n = \nu = f g_{n-1}. \quad (7)$$

Next, we prove that ν is a fixed point of f. Again, from Equation (2) and when using condition (R_N-2) of Definition 8, we obtain

$$
\begin{aligned}
R_N(\nu, \nu, f\nu, \lambda) &\leq N(\nu, \nu, f\nu)[2R_N(\nu, \nu, g_{n+1}, \lambda) + R_N(f\nu, f\nu, g_{n+1}, \lambda)] \\
&\leq N(\nu, \nu, f\nu)[2R_N(\nu, \nu, g_{n+1}, \lambda) + R_N(f\nu, f\nu, f g_n, \lambda)] \quad (8) \\
&\leq N(\nu, \nu, f\nu)[2R_N(\nu, \nu, g_{n+1}, \lambda) + K R_N(\nu, \nu, g_n, \lambda)].
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain

$$R_N(\nu, \nu, f\nu, \lambda) = 0.$$

This is possible only if $f\nu = \nu$. Hence, ν is a fixed point of f.

Further, assume that there exist $e, \nu \in \mathfrak{W}$, with $e \neq \nu$ such that $f\nu = \nu$ and $fe = e$ and we claim that $e = \nu$. Then, suppose not.
Therefore, from Equation (2) for all $\lambda > 0$, we have

$$
\begin{aligned}
0 < R_N(\nu, \nu, e, \lambda) &= R_N(f\nu, f\nu, fe, \lambda) \\
&\leq \theta R_N(\nu, \nu, e, \lambda) \\
&< R_N(\nu, \nu, e, \lambda)
\end{aligned}
$$

which leads to a contradiction. Hence, $e = \nu$. This establishes the uniqueness of fixed point and hence the result. □

Example 6. *In continuation with Example 5, let us define a self-map f on $\mathfrak{W}$ by*

$$fg = \frac{g}{\sqrt{5}}$$

for all $g \in \mathfrak{W}$. Then, f satisfies the inequality (2) with $\theta = 1/5$. Moreover, we define for every $g \in \mathfrak{W}$

$$f^n g = \frac{g}{5^{\frac{n}{2}}}.$$

Thus,

$$\lim_{n,m\to\infty} N(f^n g, f^n g, f^m g) = \lim_{n,m\to\infty} (\frac{g}{2^n} + \frac{g}{2^m} + 2) < \frac{5}{2}.$$

Thus, all the conditions of Theorem 1 are satisfied. Also, 0 is the unique fixed point of f.

Example 7. *Let $\mathfrak{W} = [0, 1)$. Define function $N : \mathfrak{W}^3 \to [1, \infty)$ by*

$$N(g, e, h) = \max\{g, e\} + h + 1$$

and a function $R_N : \mathfrak{W}^3 \times (0, \infty) \to [0, \infty)$ by

$$R_N(g, e, h, \lambda) = \lambda(\max\{g, e\} - h)^2$$

for each $g, e, h \in \mathbb{R}$ and $\lambda > 0$. Then, R_N is an extended parametric S_b-metric space. Define a self-map f on X, by

$$fg = g^3.$$

Note that

$$R_N(fg, fe, fh, \lambda) = \lambda(\max\{g^3, e^3\} - h^3)^2 \leq \frac{1}{3} R_N(g, e, h).$$

On the other hand, for every $g \in X$, define

$$f^n = g^{3n}$$

Thus,

$$\lim_{n,m\to\infty} \theta(f^n g, f^n g, f^m g) < \frac{3}{2}.$$

Therefore, all the conditions of Theorem 1, are satisfied. Here, 0 is the fixed point of f, which is unique.

3.1. Symmetric Extended Parametric S_b-Metric Space

Let us first start with the definition of symmetric extended parametric S_b-metric space as follows:

Definition 11. *An extended parametric S_b-metric space (X, R_N) is said to be symmetric if it satisfies the following condition:*

$$R_N(g, g, e, \lambda) = R_N(e, e, g, \lambda) \text{ for all } g, e \in X,\ \lambda > 0. \tag{9}$$

We next present a nice refinement of the Banach contraction principle in symmetric extended parametric S_b-metric space with the help of an auxiliary function ϕ.

Theorem 2. *Consider a symmetric complete extended parametric S_b-metric space $(\mathfrak{W}, R_N)$, where R_N is a continuous function. Let f be a self-mapping on $\mathfrak{W}$ that satisfies the following condition:*

$$R_N(fg, fe, fh, \lambda) \leq \phi[R_N(g, e, h, \lambda)] \tag{10}$$

for all $g, e, h \in \mathfrak{W}$ and $\lambda > 0$, where $\phi : [0, +\infty) \to [0, +\infty)$ is an increasing function such that for each fixed $\sigma > 0$, $\lim_{n\to\infty} \phi^n(\sigma) = 0$.

Furthermore, assume that there exist $r > 1$ such that for every $g, g_0 \in \mathfrak{W}$, we have

$$\lim_{n\to\infty} N(g_n, g_n, g) < \frac{r}{2}.$$

Then, f has a unique fixed point in $\mathfrak{W}$.

Proof. Assume $g \in \mathfrak{W}$. For $\epsilon > 0$ and $n \in \mathbb{N}$, let $\phi^n(\epsilon) < \frac{\epsilon}{2r}$.

Furthermore, for $l \in \mathbb{N}$, let $G = f^n$ and $g_l = G^l(g)$. Clearly, G is continuous. Then, for any $g, e, \in \mathfrak{W}$ and $\alpha = \phi^n$, we have

$$\begin{aligned} R_N(Gg, Gg, Ge, \lambda) &= R_N(f^n g, f^n g, f^n e, \lambda) \\ &\le \phi^n(R_N(g, g, e, \lambda)) \\ &= \alpha(R_N(g, g, e, \lambda). \end{aligned}$$

Thus, as l tends to infinity, this implies that $R_N(g_{l+1}, g_{l+1}, g_l, \lambda)$ tends to zero. Therefore, assume that l such that

$$R_N(g_{l+1}, g_{l+1}, g_l, \lambda) < \frac{\epsilon}{2r}. \tag{11}$$

Also, $g_l \in B(g_l, \epsilon)$ implies that, $B(g_l, \epsilon) \neq \phi$. Therefore, for all $h \in B(g_l, \epsilon)$, we have

$$\begin{aligned} R_N(Gh, Gh, Gg_l, \lambda) &\le \alpha(R_N(h, h, g_l, \lambda)) \\ &\le \alpha(R_N(g_l, g_l, h, \lambda)) \\ &\le \alpha(\epsilon) = \phi^n(\epsilon) < \frac{\epsilon}{2r} < \frac{\epsilon}{r}. \end{aligned} \tag{12}$$

Thus,

$$\begin{aligned} R_N(g_l, g_l, Gh, \lambda) &\le N(g_l, g_l, Gh) \begin{bmatrix} R_N(g_l, g_l, g_{l+1}, \lambda) \\ +R_N(g_l, g_l, g_{l+1}, \lambda) \\ +R_N(Gh, Gh, g_{l+1}, \lambda) \end{bmatrix} \\ &= N(g_l, g_l, Gh)[2R_N(g_l, g_l, g_{l+1}, \lambda) + R_N(Gh, Gh, g_{l+1}, \lambda)] \\ &\le N(g_l, g_l, Gh)[2\frac{\epsilon}{2r} + \frac{\epsilon}{r}]. \end{aligned}$$

When taking the limit in the above inequality as $l \to \infty$, we obtain

$$R_N(g_l, g_l, Gh, \lambda) \le \epsilon$$

Hence, G maps $B(g_l, \epsilon)$ to itself.

Since $g_l \in B(g_l, \epsilon)$, this implies that $Gg_l \in B(g_l, \epsilon)$.
Consequently, for all $m \in \mathbb{N}$, we obtain

$$G^m g_n \in B(g_l, \epsilon)$$

Therefore for all $p \ge l$, $g_p \in B(g_l, \epsilon)$. Hence, $R_N(g_m, g_m, g_l, \lambda) < \epsilon$ for all $m, p > l$. This results in g_n being a Cauchy sequence. When using the completeness of $\mathfrak{W}$, we can find $\nu \in \mathfrak{W}$ such that $g_l \to \nu$ as $l \to \infty$.
Since G is continuous,

$$\nu = \lim_{l\to\infty} g_{l+1} = \lim_{l\to\infty} g_l = G(\nu).$$

Furthermore, assume that ν and ν_1 are two distinct points of $\mathfrak{W}$ such that $G(\nu) = \nu$ and $G(\nu_1) = \nu_1$. Since $\alpha(\sigma) = \phi^n(\sigma)$ for all $\sigma > 0$, from (10)

$$
\begin{aligned}
R_N(\nu,\nu,\nu_1,\lambda) &= R_N(G\nu,G\nu,G\nu_1,\lambda) \\
&\leq \phi^n R_N(\nu,\nu,\nu_1,\lambda) \\
&= \alpha(R_N(\nu,\nu,\nu_1,\lambda)) \\
&< R_N(\nu,\nu,\nu_1,\lambda).
\end{aligned}
$$

Thus, $R_N(\nu,\nu,\nu,\lambda) = 0$ that is $\nu = \nu_1$. Alternatively, $f^{nl+r}(g) = G^l(f^r(g)) \to \nu$ as $l \to \infty$, and so $f^m g \to \nu$ as $m \to \infty$ for every g. That is, $\nu = \lim_{n\to\infty} f g_m = f(\nu)$. Hence, the proof. □

3.2. Fixed Point Result for Orbitally Lower Semi-Continuous Function

Definition 12. *Let f be a self-map defined on non-empty set $\mathfrak{W}$ and $g_0 \in \mathfrak{W}$. Define the orbit of g_0 as*

$$O(g_0) = g_0, fg_0, f^2 g_0, \cdots.$$

A function $P : \mathfrak{W} \to \mathbb{R}$ is said to be f-orbitally lower semi-continuous at $\sigma \in \mathfrak{W}$ if $< g_n > \subset O(g_0)$ and $g_n \to \sigma$ as $n \to \infty$ implies $P(\sigma) \leq lim_{n\to\infty}$ if $P(g_n)$.

Theorem 3. *Consider a complete extended parametric S_b-metric space $(\mathfrak{W}, R_N)$, where R_N is a continuous function. Let f be a self-mapping on $\mathfrak{W}$ satisfying the following assumptions:*

$$R_N(fg, fe, f^2h, \lambda) \leq \theta[R_N(g, e, fh, \lambda)] \tag{13}$$

for all $g, e, h \in \mathfrak{W}; \lambda > 0$, where $0 \leq \theta < \frac{1}{2}$ and for every $g_0 \in \mathfrak{W}$ we have

$$\lim_{n,m\to\infty} N(g_n, g_n, g_m) < \frac{1}{2\theta}.$$

Then, the sequence $\{f^n(g_0)\}$ converges to some $\nu \in \mathfrak{W}$.

Moreover, ν is a fixed point of f if and only if $P(g) = R_N(g,g,fg)$ is f-orbitally lower semi-continuous at ν.

Proof. Since $\mathfrak{W}$ is a non empty set and f is a self-map on $\mathfrak{W}$, we can therefore choose a $g_0 \in \mathfrak{W}$ such that $fg_0 = g_1$. Continuing like this, we can define a sequence $\{g_n\}$ of iterates as follows:

$$
\begin{aligned}
g_1 &= fg_0, \\
g_2 &= fg_1 = f^2 g_0, \\
&\vdots \\
g_n &= fg_{n-1} = f^n g_0.
\end{aligned}
$$

Building upon the previous argument presented in the proof of Theorem 1, it can be derived that the sequence $\{g_n\}$ is a Cauchy sequence. The completeness property of $\mathfrak{W}$ means that $< g_n >$ converges to some $\nu \in \mathfrak{W}$.
P is f-orbitally lower semi-continuous at ν. Therefore,

$$
\begin{aligned}
R_N(\nu,\nu,f\nu,\lambda) = P(\nu) &\leq \lim_{n\to\infty} \inf P(g_n) \\
&= \lim_{n\to\infty} R_N(g_n, g_n, g_{n+1}, \lambda) \\
&\leq \lim_{n\to\infty} \inf \theta^n R_N(g_0, g_0, g_1, \lambda) = 0.
\end{aligned}
$$

Thus, $f\nu = \nu$.

Conversely, assume that $fv = v$ and $< g_n > \subset O(g_0)$ with $g_n \to v$ as $n \to \infty$. Therefore,

$$P(v) = R_N(v, v, fv, \lambda) = 0 \leq R_N(g_n, g_n, g_{n+1}, \lambda) = \lim_{n\to\infty} P(g_n).$$

This completes the proof of the Theorem 3. □

Remark 3. *Our following proved results should be noted:*

1. *Theorem 1 is a generalization of the result of Banach [17] in extended parametric S_b-metric space.*
2. *Theorem 2 and Theorem 3 are the extension of the result obtained by Boyd and Wong [40] and Mlaiki [32] in extended parametric S_b-metric space.*

4. Application: Existence of the Solution of Fredholm Integral Equations

In this section, we examine the presence of a solution for a Fredholm integral equation utilizing the outcomes established in Section 3.

Let $\mathfrak{W}$ denote the set $C[a,b]$ consisting of all real-valued continuous functions defined on the closed and bounded interval $[a,b]$ in the real number system $\mathcal{R}$.

For a real no $\lambda > 0$ and for all $g, e, h \in [a,b]$, define $R_N : \mathfrak{W}^3 \times (0,\infty) \to [0,\infty)$ by

$$R_N(g(\sigma), e(\sigma), h(\sigma), \lambda) = \lambda \sup_{\sigma\in[a,b]} \mid \max\{g(\sigma), e(\sigma)\} - h(\sigma) \mid^2$$

and $N : \mathfrak{W}^3 \to [1,\infty)$ by

$$N(g(\sigma), e(\sigma), h(\sigma)) = \max\{\mid g(\sigma) \mid, \mid e(\sigma) \mid\} + h(\sigma) + 1.$$

It is evident that $(\mathfrak{W}, R_N)$ is a complete extended parametric S_b-metric space. We apply Theorem 1 to establish the existence of the solution of Fredholm type defined by

$$g(\sigma) = P(\sigma) + \int_a^b L(\sigma, r, g(r)) \tag{14}$$

for all $\sigma, r \in [a,b]$. Function $g(\sigma) \in [a,b]$ is a solution of Equation (14).

Theorem 4. *The integral equation defined in (14) has a unique solution $g(\sigma) \in [a,b]$, if it satisfies the following assumptions:*

(i). $P : [a,b] \to \mathbb{R}$ *is continuous;*
(ii). $L : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous;*
(iii). for every $\sigma, r \in [a,b]$,

$$\mid L(\sigma, r, g(r)) - L(\sigma, r, fg(r)) \mid \leq \frac{1}{2} \mid g(r) - fg(r) \mid .$$

Proof. $\mathfrak{W} = C[a,b]$ consists of all real-valued continuous functions defined on the closed and bounded interval $[a,b]$ in the real number system $\mathcal{R}$.
Define a map $f : \mathfrak{W} \to \mathfrak{W}$, for all $\sigma, r \in [a,b]$ by

$$fg(\sigma) = \int_a^b L(\sigma, r, g(r))dr + P(\sigma)$$

Also,

$$f(fg(\sigma)) = \int_a^b L(\sigma, r, fg(r))dr + P(\sigma)$$

Therefore,

$$
\begin{aligned}
fg(\sigma) - f(fg(\sigma)) &= \int_a^b L(\sigma, r, g(r))dr + P(\sigma) - \int_a^b L(\sigma, r, fg(r))dr - P(\sigma) \\
&= \int_a^b [L(\sigma, r, g(r))dr - L(\sigma, r, fg(r))]dr. \qquad (15)
\end{aligned}
$$

Consider

$$
\begin{aligned}
R_N(fg(\sigma), fg(\sigma), f^2x(\sigma), \lambda) &= \lambda \mid fg(\sigma) - f(fg(\sigma)) \mid^2 \\
&\leq \lambda(\int_a^b \mid L(\sigma, r, g(r)) - L(\sigma, r, fg(r)) \mid)^2 \\
&= \lambda(\frac{1}{2} \mid g(r) - fg(r) \mid)^2 \\
&\leq \frac{\lambda}{4} R_N(g(\sigma), g(\sigma), fg(\sigma), \lambda).
\end{aligned}
$$

For every λ, $0 < \lambda < 4$, $\lambda/4 < 1$ and hence all the conditions of Theorem 1 are satisfied. Therefore, map f has a unique fixed point. Thus, there exists a unique solution for (14). □

5. Conclusions

In the present study, we started with the novel concept of extended parametric S_b-metric space, supported by suitable examples. Furthermore, three lemmas were proven in order to establish the convergence, uniqueness, and Cauchy behavior of sequences in these spaces. Additionally, we proved three theorems. Theorem 1 is the analogous counterpart of the Banach fixed point result, Theorem 2 is a refined form of the Banach fixed point result in symmetric extended parametric S_b-metric space, and Theorem 3 is derived for orbitally lower semi-continuous maps. Lastly, the obtained results are utilized to establish the existence and uniqueness of a solution for an integral equation.

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