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Editorial

Recent Advances in Special Functions and Their Applications

Junesang Choi

Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea; junesang@dongguk.ac.kr

Due to their remarkable properties, a plethora of special functions have been crafted and harnessed across a diverse spectrum of fields spanning centuries. These functions have found their place in a variety of disciplines, including mathematics, physics (quantum mechanics, electrodynamics, thermodynamics, fluid dynamics, and solid-state physics), engineering, statistics, astronomy and astrophysics, computer science, economics and finance, chemistry, biology, geophysics, medicine, materials science, and environmental science. These are just a few examples, and special functions can find applications in various other scientific and engineering disciplines whenever complex mathematical relationships need to be described or solved.

This Special Issue stands as a vibrant platform for the introduction of freshly conceived theories and formulas centered around special functions, alongside their prospective utility across a myriad of research realms.

This Special Issue, in essence, serves as a sequel to the acclaimed "Special Functions and Applications" Special Issue in *Mathematics* during 2019–2020, which saw the successful completion of approximately 40 scholarly contributions.

Contained within this curated compilation are 13 illuminating articles, each thoughtfully outlined as follows:

In the paper [1], novel formulas for Fibonacci polynomials are developed, encompassing higher-order derivatives and recurrent integrals, all expressed in relation to Fibonacci polynomials. These findings are subsequently employed to address connectivity issues bridging the gap between Fibonacci polynomials and orthogonal polynomials. Furthermore, the paper delves into the investigation of inverse cases. In conclusion, the study yields fresh insights into the linear combinations of Fibonacci and orthogonal polynomials, leveraging the previously established moments formula for Fibonacci polynomials.

In various fields of applied mathematics, theoretical and mathematical physics, statistics, and beyond, hypergeometric functions have emerged as essential tools. These functions, whether in single or multiple variables, have found numerous practical applications today. Among the diverse hypergeometric functions, Appell's four functions and Horn's functions have proven exceptionally valuable in solving a wide range of problems spanning both pure and applied mathematics. One can observe the pervasive utility of Appell functions, notably in scientific and chemical domains. Examples include their application in the Hubbell rectangular source and its generalization, non-relativistic theory, and the computation of hydrogen dipole matrix elements. Furthermore, Appell series play a pivotal role in quantum field theory, particularly in the evaluation of Feynman integrals. Additionally, since 1985, the field of computational sciences, encompassing artificial intelligence (AI) and information processing (IP), has harnessed the power of Horn functions as a fundamental concept. Over the years, the scientific community has published a wealth of results, particularly regarding double series involving Appell and Horn functions, in a series of enlightening and beneficial research publications. In a recent study detailed in [2], the authors have uncovered three general transformation formulas linking Appell functions F_2 and F_4, along with two general transformation formulas connecting Appell function F_2 and Horn function H_4. These discoveries were largely inspired by their prior work and naturally exhibit symmetry. The achievement of these results is facilitated by utilizing generalizations of Kummer's second theorem in the integral representation of Appell function

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F_2. As a consequence of their primary findings, a combination of previously established results and novel discoveries has come to light.

Generalized hypergeometric functions, both in single and multiple variables, along with their natural extensions, have found widespread utility in various mathematical contexts and practical applications. The importance of generalized hypergeometric functions in several variables stems from their crucial role in expressing solutions to partial differential equations that frequently emerge in a wide array of problems within mathematical physics. In particular, the Kampé de Fériet function, when considered in two variables, has established its practical significance as a tool for representing solutions across diverse domains, including pure and applied mathematics, statistics, and mathematical physics. A recent contribution by Progri has successfully computed the ${}_2F_2$-generalized hypergeometric function for specific parameter sets, expressing the outcome as the difference between two Kampé de Fériet functions. Building on this groundbreaking work, a subsequent paper [3] by a different set of authors has achieved three significant results relating to a terminating ${}_3F_2$ series with arguments set to 1 and 2. Furthermore, this paper introduces a transformation formula for the ${}_3F_2(z)$-generalized hypergeometric function, revealing its relation to the difference between two Kampé de Fériet functions. The practical application of this newly discovered relationship is also elucidated. The paper's conclusion rounds out the discussion by presenting six reduction formulas applicable to the Kampé de Fériet function, consolidating its value and versatility in mathematical problem solving.

Special polynomials play a significant role across various fields of study, including mathematics, engineering, and theoretical physics. Differential equations comprise a fundamental framework for addressing numerous problems in these domains. In a noteworthy paper referenced as [4], the authors introduce a novel set of polynomials known as the Lagrange-based hypergeometric Bernoulli polynomials using the generating function approach. They elucidate various algebraic and differential characteristics of this expanded family of Lagrange-based Bernoulli polynomials. Additionally, the paper presents a matrix inversion formula that involves these polynomials and uncovers a generating relationship involving the Stirling numbers of the second kind. Furthermore, it is worth noting that potential applications of these polynomials in the aforementioned fields may spark further investigations in this subject area.

Numerous investigations have delved into the realm of hypergeometric series, specifically ${}_2F_1$ and the generalized hypergeometric series ${}_pF_q$. These inquiries have explored a wide array of aspects, encompassing topics like differential equations, integral representations, analytic continuations, asymptotic expansions, reduction cases, extensions in both single and multiple variables, continued fractions, Riemann's equation, the group associated with the hypergeometric equation, summation techniques, and transformation formulas. Of all the different avenues of inquiry into these functions, the transformation formulas for ${}_2F_1$ and ${}_pF_q$ hold particular significance. Their importance resonates in both practical applications and theoretical contexts. The paper referenced as [5] aims to establish a series of transformation formulas for ${}_pF_q$. These formulas encompass specific cases that include Gauss's and Kummer's quadratic transformation formulas for ${}_2F_1$, as well as their two extensions for ${}_3F_2$. This endeavor leverages a recently introduced sequence and employs techniques commonly applied in the study of ${}_pF_q$ theory, offering valuable contributions to the field.

Recently, Brychkov et al. have made significant contributions by unveiling a set of fresh and captivating reduction formulas for the Humbert functions, which are the confluent hypergeometric functions of two variables. This article, denoted as [6], primarily aims to present an alternative and straightforward method for validating four distinct reduction formulas specifically applicable to the Humbert function ψ_2. In doing so, the authors introduce intriguing series expressions that involve the product of two confluent hypergeometric functions. Furthermore, this endeavor has led to the attainment of numerous novel findings, as well as the reiteration of previously established results, all stemming from the core discoveries presented in this study. It is worth noting that hypergeometric functions in both single and dual variables, along with their confluent counterparts, have a natural occurrence in a broad array of fields such as applied mathematics, statistics, operations research, theoretical and mathematical physics, and engineering mathematics. Consequently, the outcomes detailed in this paper hold the potential to be of significant utility within the aforementioned domains.

The literature pertaining to this field is overloaded with an astonishing multitude of integral formulae, each entailing various special functions. In the work presented in [7], the authors embark on a quest to establish three integral formulas. These formulas exhibit integrands that result from the amalgamation of a generalized hypergeometric series of the form ${}_{p+1}F_p$, with integrands that derive from three distinct Beta function formulae. Among the plethora of particular instances exemplified within our formulas, several are elucidated with utmost clarity. Additionally, the process of proof reveals a captivating inequality that consistently emerges, adding an intriguing layer to the discussion. It is worth emphasizing that the versatility of these three integral formulas allows for their expansion and exploration through the incorporation of a diverse array of more generalized special functions, extending the scope of their applicability and relevance.

Another article [8] delves into the exploration of twisted Catalan numbers and twisted Catalan–Daehee numbers, both of which have their origins in p-adic fermionic integrals and p-adic invariant integrals. Within the article, the author presents explicit identities and characteristics of these twisted numbers and polynomials, utilizing techniques such as p-adic integrals and generating functions.

In [9], the authors investigate the enhanced infinite sum representation of the incomplete gamma function, particularly when the parameters involved are significantly large. Additionally, they analyze the infinite sum and the corresponding Hurwitz–Lerch zeta function at specific values, presenting their findings in a readily accessible tabular format. It is worth noting that a notable characteristic of most Hurwitz–Lerch zeta functions is their distinct asymmetrical distribution of zeros.

In article [10], a range of existence theorems is introduced concerning maximal-type elements within a broad context. The authors investigate multivalued maps that incorporate continuous selections, as well as multivalued maps that meet the criteria set forth by Gorniewicz. Their theory of existence draws from both the author's established principles and new coincidence theory. Notably, the second section unveils a comprehensive coincidence coercive-type result applicable to various map classes. In the third section, the focus shifts to the consideration of majorized maps, unveiling a diverse array of new results pertaining to maximal element types. This exploration of coincidence theory is rooted in the motivation drawn from real-world physical models, where both symmetry and asymmetry play pivotal roles.

The article [11] establishes a series of finite sum identities. These identities encompass reciprocals of binomial and central binomial coefficients, as well as harmonic, Fibonacci, and Lucas numbers. Some of these findings reaffirm existing knowledge, while others introduce novel results.

The paper [12] introduces (p,q)-cosine Euler polynomials, exploring various properties and identities derived from these polynomials. Additionally, the authors employ computer-based methods to determine approximate roots for (p,q)-cosine Euler polynomials and present corresponding circle equations.

In [13], the authors introduce a novel class of polynomials known as two-variable q-generalized tangent-based Apostol-type Frobenius–Euler polynomials by incorporating two well-established polynomials. Their work encompasses a comprehensive exploration of these polynomials, offering a plethora of properties and formulas. These include explicit expressions, series representations, summation formulas, addition formulas, as well as q-derivative and q-integral formulas. Additionally, they present various specific instances of these new polynomials and their associated formulas, thoughtfully organized in two tables. Moreover, the authors employ computer-aided programs such as Mathematica or Matlab to visualize graphs of select instances of these new polynomials. This graphical representation allows for a multifaceted examination of the distribution and positioning of polynomials' zeros from different perspectives. In conclusion, their investigation naturally gives rise to numerous observations and questions that further enrich the field of study.

Conflicts of Interest: The author has no conflict of interest.

References

1. Abd-Elhameed, W.M.; Ahmed, H.M.; Napoli, A.; Kowalenko, V. New Formulas Involving Fibonacci and Certain Orthogonal Polynomials. *Symmetry* **2023**, *15*, 736. [CrossRef]
2. Kim, I.; Rathie, A.K. A Note on Certain General Transformation Formulas for the Appell and the Horn Functions. *Symmetry* **2023**, *15*, 696. [CrossRef]
3. Kim, I.; Paris, R.B.; Rathie, A.K. Some New Results for the Kampé de Fériet Function with an Application. *Symmetry* **2022**, *14*, 2588. [CrossRef]
4. Albosaily, S.; Quintana, Y.; Iqbal, A.; Khan, W.A. Lagrange-Based Hypergeometric Bernoulli Polynomials. *Symmetry* **2022**, *14*, 1125. [CrossRef]
5. Qureshi, M.I.; Choi, J.; Shah, T.R. Certain Generalizations of Quadratic Transformations of Hypergeometric and Generalized Hypergeometric Functions. *Symmetry* **2022**, *14*, 1073. [CrossRef]
6. Mohammed, A.O.; Kilicman, A.; Awad, M.M.; Rathie, A.K.; Rakha, M.A. Another Method for Proving Certain Reduction Formulas for the Humbert Function ψ_2 Due to Brychkov et al. with an Application. *Symmetry* **2022**, *14*, 868. [CrossRef]
7. Choi, J.; Kurumujji, S.K.; Kilicman, A.; Rathie, A.K. Certain Integral Formulae Associated with the Product of Generalized Hypergeometric Series and Several Elementary Functions Derived from Formulas for the Beta Function. *Symmetry* **2022**, *14*, 389. [CrossRef]
8. Lim, D. Some Explicit Expressions for Twisted Catalan-Daehee Numbers. *Symmetry* **2022**, *14*, 189. [CrossRef]
9. Reynolds, R.; Stauffer, A. A Note on the Summation of the Incomplete Gamma Function. *Symmetry* **2021**, *13*, 2369. [CrossRef]
10. O'Regan, D. Maximal Type Elements for Families. *Symmetry* **2021**, *13*, 2269. [CrossRef]
11. Batir, N.; Sofo, A. Finite Sums Involving Reciprocals of the Binomial and Central Binomial Coefficients and Harmonic Numbers. *Symmetry* **2021**, *13*, 2002. [CrossRef]
12. Ryoo, C.S.; Kang, J.Y. Some Symmetric Properties and Location Conjecture of Approximate Roots for (p,q)-Cosine Euler Polynomials. *Symmetry* **2021**, *13*, 1520. [CrossRef]
13. Yasmin, G.; Islahi, H.; Choi, J. q-Generalized Tangent Based Hybrid Polynomials. *Symmetry* **2021**, *13*, 791. [CrossRef]