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Binomial Series-Confluent Hypergeometric Distribution and Its Applications on Subclasses of Multivalent Functions

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Abstract: Over the past ten years, analytical functions' reputation in the literature and their application have grown. We study some practical issues pertaining to multivalent functions with bounded boundary rotation that associate with the combination of confluent hypergeometric functions and binomial series in this research. A novel subset of multivalent functions is established through the use of convolution products and specific inclusion properties are examined through the application of second order differential inequalities in the complex plane. Furthermore, for multivalent functions, we examined inclusion findings using Bernardi integral operators. Moreover, we will demonstrate how the class proposed in this study, in conjunction with the acquired results, generalizes other well-known (or recently discovered) works that are called out as exceptions in the literature.

Keywords: convolution; p-valent functions; binomial series; confluent hypergeometric function; starlike functions; convex functions; close-to-convex functions

MSC: 30C50; 30C45; 33C20; 33C05; 33C15

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1. Introduction, Definitions and Preliminaries

Assume that $\mathcal{H}$ is the class of univalent holomorphic (regular/analytic) functions in the unit disc

$$\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \tag{1}$$

which have the following form:

$$f(\zeta) = \zeta + \sum_{k\geq 2} \chi_k \zeta^k; (\zeta \in \Delta). \tag{2}$$

Furthermore, the family of univalent functions, as in (2), is indicated by the notation $\mathcal{S}$. In 1907, Köebe studied this family $\mathcal{H}$. It was Bieberbach [1] who made feasible the most well-known finding in function theory, the "Bieberbach conjecture," in 1916. Specifically, if $f \in \mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$. This result was also established for $n = 2$ by him. It is clear that many reputable researchers have used a variety of methods to tackle this problem. Schaeffer and Spencer [2] and Löwner [3] used the variational method and Löwner differential equation, respectively, to settle this hypothesis for $n = 3$. Afterwards, Jenkins [4] used quadratic differentials to prove the same coefficient inequality: $|a_3| \leq 3$. Using the variational technique, Garabedian and Schiffer [5] found that $|a_4| \leq 4$. In an effort to address this issue, numerous intriguing subfamilies of the class $\mathcal{S}$ were developed between 1916 and 1985. Here are some definitions for few basic families:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > 0,\ \ (\zeta \in \Delta) \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \Re\left(\frac{(\zeta f'(\zeta))'}{f'(\zeta)}\right) > 0,\ \ (\zeta \in \Delta) \right\}, \\
\mathcal{C} &= \left\{ f \in \mathcal{S} : \Re\left(\frac{\zeta f'(\zeta)}{h(\zeta)}\right) > 0, (h \in \mathcal{S}^*, \zeta \in \Delta) \right\} \\
\mathit{and} & \\
\mathcal{R} &= \left\{ f \in \mathcal{S} : \Re\big(f'(\zeta)\big) > 0(\zeta \in \Delta) \right\}.
\end{aligned}$$

If a function falls within the range of a given simply connected domain and maps Δ conformally onto an image domain of boundary rotation at most $\kappa\pi$, it is considered to have limited boundary rotation. This is because the function is both locally univalent and analytic. Although Loewner [6] did not use the current terminology, he proposed the idea of functions of bounded boundary rotation for the first time in 1917. Paatero [7] was the one who methodically established their properties and conducted a thorough analysis of the class, demonstrating that $f \in \mathcal{C}$ if and only if

$$f'(\zeta) = exp\left\{ -\int_0^{2\pi} log(1 - \zeta e^{it}) d\mu(t) \right\}$$

where $\mu(t)$ is a real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{3}$$

Let $\mathcal{P}_\kappa(\wp)$ be the class of functions $\mathcal{G} \in \mathcal{H}$ analytic in Δ satisfying the features $\mathcal{G}(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{\mathcal{G}(\zeta)\} - \wp}{1 - \wp} \right| d\theta \leq \kappa\pi, \tag{4}$$

where $\zeta = re^{i\theta}$, $\kappa \geq 2$ and $0 \leq \wp < 1$. If $\wp = 0$, we denote $\mathcal{P}_\kappa(0) \equiv \mathcal{P}_\kappa$. Hence, the class $\mathcal{P}_\kappa$ (defined by Pinchuk [8]) denotes the class of analytic functions $\mathcal{G} \in \mathcal{P}_\kappa$, with $\mathcal{G}(0) = 1$ and will be having a representation

$$\int_0^{2\pi} \left| \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} \right| d\mu(t) \leq \kappa\pi \tag{5}$$

where μ satisfies (3) (for details see [9,10]). From (5), one can easily find that $\mathcal{G} \in \mathcal{P}_\kappa$ can also be written as

$$\mathcal{G}(\zeta) = \left(\frac{\kappa}{4} + \frac{1}{2}\right)\mathcal{G}_1(\zeta) - \left(\frac{\kappa}{4} - \frac{1}{2}\right)\mathcal{G}_2(\zeta); \qquad \mathcal{G}_2, \mathcal{G}_2 \in \mathcal{P}_\kappa; \zeta \in \Delta.$$

The subject encompasses classes of functions that have greatly advanced geometric function theory, such as bounded radius and bounded boundary rotation. The main tools, such as convolution and subordination, have been extensively used to investigate the geometric characteristics of analytic classes, but many unanswered questions remain. Padmanabhan and Parvatham [11] introduced the class $\mathcal{P}_\kappa(\wp)$ as previously mentioned, and they also conducted a detailed analysis of order referring to Roberston [12]. Additionally, these transformations maintain a number of the geometric properties of the $\mathcal{P}$ family, such as the subordination, compactness, convexity, and positivity of the real part. They are iterative and are closely associated with certain families of analytic and univalent functions involving the well-known Salagean and Rucheweyh derivatives (see [13,14]) and have been used effectively, and elegantly too, to characterize them. For higher-order derivatives related to the Gaussian hypergeometric function, they can additionally investigate a suitably generalized derivative operator based on subordinations. For example, using both vortex and source/sink methods, an explicit construction for the complicated potential and stream function of two-dimensional fluid flow issues is given throughout a circular cylinder to illustrate the possible practical consequences of this work. One can define a single sourceâ€™s fluid flow and produce a univalent function in order to turn the sourceâ€™s image into a source for a particular complicated potential (See [15]). Aleman and Constantin [16] recently revealed an astounding relationship between fluid dynamics and univalent function theory. They specifically presented a straightforward technique that uses a univalent harmonic map to directly solve incompressible two-dimensional Euler equations. It finds extensive application in modern mathematical physics, fluid dynamics, nonlinear integrable system theory, and partial differential equation theory, among other applied scientific disciplines.

1.1. Confluent Hypergeometric Function (CHF)

De Brangesâ€™ use of the generalized hypergeometric function to prove Bieberbachâ€™s conjecture in 1985 established the link between univalent function theory and hypergeometric functions [17]. CHFs were employed in numerous investigations, after hypergeometric functions were taken into consideration in studies pertaining to univalent functions. Lately, the CHF of the first kind was given by El-Deeb and Catas [18] as below:

$$\begin{aligned} F(\vartheta; v; \zeta) &= 1 + \frac{\vartheta}{v}\zeta + \frac{\vartheta\,(\vartheta+1)}{v\,(v+1)}\frac{\zeta^2}{2!} + \cdots \\ &= \sum_{k\geq 0} \frac{(\vartheta)_k}{(v)_k\,(1)_k}\zeta^k, \quad (\vartheta \in \mathbb{C},\ v \in \mathbb{C}\setminus\{0,-1,-2,\dots\}), \end{aligned}$$

for all finite values of (see [19]) is convergent. It may also be noted as

$$F(\vartheta; v; m) = \sum_{k\geq 0} \frac{(\vartheta)_k}{(v)_k\,(1)_k} m^k, \quad (\vartheta \in \mathbb{C}, v \in \mathbb{C}\setminus\{0,-1,-2,\dots\}),$$

which is convergent for $\vartheta, v, m > 0$.

The confluent hypergeometric distribution (CHD) whose probability mass function

$$P(k) = \frac{(\vartheta)_k}{(v)_k\,k!F(\vartheta; v; m)} m^k,\ (\vartheta, v, m > 0,\ k = 0,1,2,\dots)$$

is given by Porwal and Kumar [20] (see also [21–23]) . Assume $\mathcal{H}_p$ is the class of p-valently holomorphic functions in Δ have the following

$$\mathcal{F}(\zeta) = \zeta^p + \sum_{k\geq p+1} \chi_k \zeta^k, \quad (p \in \mathbb{N} = \{1,2,\dots\},\ \zeta \in \Delta). \tag{6}$$

The convolution of two functions expresses how the geometric representation of one function is improved by the other. The concept clarification covers both the effect function and the calculating method. When one of the functions is moved and inverted, it can be written as the integral, or sum, of the product of the two functions. Let $\Upsilon \in \mathcal{H}_p$ be assumed as

$$\Upsilon(\zeta) := \zeta^p + \sum_{k\geq p+1} \psi_k \zeta^k,\ \ \zeta \in \Delta. \tag{7}$$

then the Hadamard (or convolution) product of $\mathcal{F}$ and Y is assumed as

$$(\mathcal{F} * \Upsilon)(\zeta) := \zeta^p + \sum_{k \geq p+1} \chi_k \psi_k \zeta^k, \ \ \zeta \in \Delta.$$

El-Deeb and Catas [18] introduced a series $\mathcal{I}_p(\vartheta; v; m)$ whose coefficients are probabilities of CHD

$$\mathcal{I}_p(\vartheta; v; m) = \zeta^p + \sum_{k \geq p+1} \frac{(\vartheta)_{k-p} m^{k-p}}{(v)_{k-p} \ (k-p)! F(\vartheta; v; m)} \zeta^k, \ (\vartheta, v, m > 0), \tag{8}$$

and defined a linear operator $Q_p^{\vartheta,v,m}\mathcal{F} : \mathcal{H}_p \to \mathcal{H}_p$ by the convolution product as follows

$$\begin{aligned} Q_p^{\vartheta,v,m}\mathcal{F}(\zeta) &= \mathcal{I}_p(\vartheta; v; m) * \mathcal{F}(\zeta) \\ &= \zeta^p + \sum_{k \geq p+1} \frac{(\vartheta)_{k-p} m^{k-p}}{(v)_{k-p} \ (k-p)! F(\vartheta; v; m)} a_k \, \zeta^k, \ (\vartheta, v, m > 0). \end{aligned}$$

Making use of the binomial series

$$(1-\varrho)^j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \, \varrho^i \quad (j \in \mathbb{N}).$$

El-Deeb and Catas [18] introduced the linear differential operator by fixing $\mathcal{F} \in \mathcal{H}_p$, and $\varrho > 0$; $\vartheta, v, m > 0$; $j \in \mathbb{N}$; $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as below:

$$\mathcal{D}_{p,j}^{\varrho,0,\vartheta,v,m}\mathcal{F}(\zeta) = Q_p^{\vartheta,v,m}\mathcal{F}(\zeta),$$

$$\begin{aligned} \mathcal{D}_{p,j}^{\varrho,1,\vartheta,v,m}\mathcal{F}(\zeta) &= \mathcal{D}_{p,j}^{\varrho,\vartheta,v,m}\mathcal{F}(\zeta) = [1-\varrho]^j Q_p^{\vartheta,v,m}\mathcal{F}(\zeta) + \frac{\zeta}{p}\left[1-(1-\varrho)^j\right]\left(Q_p^{\vartheta,v,m}\mathcal{F}\right)'(\zeta) \\ &= \zeta^p + \sum_{k \geq p+1} \left[1 + \left(\frac{k}{p} - 1\right) c^j(\varrho)\right] \left[\frac{(\vartheta)_{k-p} m^{k-p}}{(v)_{k-p} \ (k-p)! F(\vartheta; v; m)}\right] \chi_k \, \zeta^k \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ \mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) &= \mathcal{D}_{p,j}^{\varrho,\vartheta,v,m}\left(\mathcal{D}_{p,j}^{\varrho,n-1,\vartheta,v,m}\mathcal{F}(\zeta)\right) \\ &= [1-\varrho]^j \mathcal{D}_{p,j}^{\varrho,n-1,\vartheta,v,m}\mathcal{F}(\zeta) + \frac{\zeta}{p}\left[1-(1-\varrho)^j\right]\left(\mathcal{D}_{p,j}^{\varrho,n-1,\vartheta,v,m}\mathcal{F}(\zeta)\right)' \\ &= \zeta^p + \sum_{k \geq p+1} \left[1 + \left(\frac{k}{p} - 1\right) c^j(\varrho)\right]^n \left[\frac{(\vartheta)_{k-p} m^{k-p}}{(v)_{k-p} \ (k-p)! F(\vartheta; v; m)}\right] \chi_k \, \zeta^k \\ &= \zeta^p + \sum_{k \geq p+1} \psi_k \chi_k \, \zeta^k, \end{aligned} \tag{9}$$

where

$$\psi_k = \left[1 + \left(\frac{k}{p} - 1\right) c^j(\varrho)\right]^n \left[\frac{(\vartheta)_{k-p} m^{k-p}}{(v)_{k-p} \ (k-p)! F(\vartheta; v; m)}\right] \tag{10}$$

and

$$c^j(\varrho) = \sum_{i=1}^{j} \binom{j}{i} (-1)^i \, \varrho^i \quad (j \in \mathbb{N}). \tag{11}$$

From (9), we obtain that

$$c^j(\varrho)\,\zeta\left(\mathcal{D}^{\varrho,n,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)\right)' = p\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta) - p\Big[1-c^j(\varrho)\Big]\mathcal{D}^{\varrho,n,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta). \tag{12}$$

1.2. Multivalent Functions of Bounded Radius Rotations

Let $\mathcal{P}_{p,\kappa}(\wp)$ be the class of functions $\mathcal{G}:\Delta\to\mathbb{C}$ analytic in Δ satisfying the properties $\mathcal{G}(0)=p$ and

$$\int_0^{2\pi}\left|\frac{\Re\{\mathcal{G}(\zeta)\}-\wp}{p-\wp}\right|d\theta \le \kappa\pi, \tag{13}$$

where $\zeta = re^{i\theta}$, $\kappa \ge 2$ and $0 \le \wp < p$. This class was introduced by Aouf [24] with $\wp = 0$.

Using (13), we obtain $\mathcal{G}(\zeta) \in \mathcal{P}_{p,\kappa}(\wp)$ if and only if there exists $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{P}_{p,\kappa}(\wp)$ such that

$$\mathcal{G}(\zeta) = \left(\frac{\kappa}{4}+\frac{1}{2}\right)\mathcal{G}_1(\zeta) - \left(\frac{\kappa}{4}-\frac{1}{2}\right)\mathcal{G}_2(\zeta)\ \ (\zeta\in\Delta). \tag{14}$$

Remark 1. *We note that:*

(i) $\mathcal{P}_{1,\kappa}(\wp) = \mathcal{P}_\kappa(\wp)$ $(\kappa \ge 2,\ 0\le\wp<1)$ *(see Padmanabhan and Parvatham [11]);*
(ii) $\mathcal{P}_{1,\kappa}(0) = \mathcal{P}_\kappa$ $(\kappa\ge 2)$ *(see Pinchuk [8] and Robertson [12]);*
(iii) $\mathcal{P}_{p,2}(\wp) = \mathcal{P}(p,\wp)$ $(0\le\wp<p,\ p\in\mathbb{N})$, *where* $\mathcal{P}(p,\wp)$ *is the class of functions with a positive real part greater than* $\wp$ *(see [24]);*
(iv) $\mathcal{P}_{p,2}(0) = \mathcal{P}(p)$ $(p\in\mathbb{N})$, *where* $\mathcal{P}(p)$ *is the class of functions with a positive real part (see [24]); and*
(v) $\mathcal{P}_{1,2}(\wp) = \mathcal{P}(\wp)$ $(0\le\wp<1)$, *where* $h(\zeta) = (1-\wp)g(\zeta)+\wp$, $g\in\mathcal{P}$ *and* $\Re\{h(\zeta)\} > \wp$.

Now, for $0 \le \wp, \eta < p$, $p\in\mathbb{N}$ and $\kappa\ge 2$, we define the following classes $\mathcal{S}_{\kappa,p}(\wp)$, $\mathcal{C}_{\kappa,p}(\wp)$ and $K_{\kappa,p}(\eta,\wp)$ of $\mathcal{H}_p$ as below:

$$\mathcal{S}_{\kappa,p}(\wp) = \left\{\mathcal{F}\in\mathcal{H}_p:\ \frac{\zeta\mathcal{F}'(\zeta)}{\mathcal{F}(\zeta)} \in \mathcal{P}_{p,\kappa}(\wp),\ \zeta\in\Delta\right\},$$

$$\mathcal{C}_{\kappa,p}(\wp) = \left\{\mathcal{F}\in\mathcal{H}_p:\ \frac{(\zeta\mathcal{F}'(\zeta))'}{\mathcal{F}'(\zeta)} \in \mathcal{P}_{p,\kappa}(\wp),\ \zeta\in\Delta\right\},$$

and

$$K_{\kappa,p}(\eta,\wp) = \left\{\mathcal{F}:\ \mathcal{F}\in\mathcal{H}_p,\ g\in\mathcal{S}_{2,p}(\wp) \text{ and } \frac{\zeta\mathcal{F}'(\zeta)}{g(\zeta)} \in \mathcal{P}_{p,\kappa}(\wp),\ \zeta\in\Delta\right\}.$$

Obviously, we know that

$$\mathcal{F}(\zeta)\in\mathcal{C}_{\kappa,p}(\wp) \Leftrightarrow \frac{\zeta\mathcal{F}'(\zeta)}{p} \in \mathcal{S}_{\kappa,p}(\wp). \tag{15}$$

1. $\mathcal{S}_{2,p}(\wp) = \mathcal{S}^*_p(\wp)$ $(0\le\wp<p,\ p\in\mathbb{N})$, where $\mathcal{S}^*_p(\wp)$ was introduced by Patil and Thakare [25];
2. $\mathcal{C}_{2,p}(\wp) = \mathcal{C}_p(\wp)$ $(0\le\wp<p,\ p\in\mathbb{N})$, where $\mathcal{C}_p(\wp)$ was introduced by Owa [26];
3. $K_{2,p}(\eta,\wp) = K_p(\eta,\wp)$ $(0\le\wp,\eta<p,\ p\in\mathbb{N})$, where $K_p(\eta,\wp)$ was introduced by Aouf [27].

Paatero [28] noted that when $2\le k\le 4$, $\mathcal{S}_{\kappa,1}$ coincides with $\mathcal{C}_{\kappa,1}$. Pinchuk [8] also demonstrated that functions in $\mathcal{C}_{\kappa,1}$ are close-to-convex in Δ if $2\le k\le 4$, and are therefore univalent.

This paperâ€™s novel findings are motivated by the excellent outcomes lately achieved through the integral and derivative operators in the field of geometric function theory. Our inspiration to further investigate the binomial series-confluent hypergeometric distribution was sparked by reading about the applications of an operator on new subclasses of univalent functions and how they relate to classical theories of differential subordination and superordination [15,16,29–33] and the references cited therein. This led us to consider the idea of introducing and studying new subclasses of univalent functions in Δ with bounded boundary rotation. Using the operator given in (9), we familiarize the new subclasses of the class $\mathcal{H}_p$ as below:

$$\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp) = \left\{ \mathcal{F} \in \mathcal{H}_p : \mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) \in \mathcal{S}_{\kappa,p}(\wp),\ \zeta \in \Delta \right\}, \tag{16}$$

$$\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp) = \left\{ \mathcal{F} \in \mathcal{H}_p : \mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) \in \mathcal{C}_{\kappa,p}(\wp),\ \zeta \in \Delta \right\} \tag{17}$$

and

$$K_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\eta,\wp) = \left\{ \mathcal{F} \in \mathcal{H}_p : \mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) \in K_{\kappa,p}(\eta,\wp),\ \zeta \in \Delta \right\}. \tag{18}$$

It is easy to understand that

$$\mathcal{F}(\zeta) \in \mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp) \Leftrightarrow \frac{\zeta\mathcal{F}'(\zeta)}{p} \in \mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp). \tag{19}$$

In the following section, for functions belonging to this newly-defined analytic function classes, $\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$, $\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$ and $K_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\eta,\wp)$, motivated by the earlier studies on bounded boundary rotations and differential subordination [3,11,24,30,31,33,34] and using following Lemmas 1–3, we discuss inclusion properties involving the differential operator $\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}$ and integral operator $\mathcal{J}_{\varepsilon,p}$.

2. Inclusion Properties Involving the Operator $\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}$

To establish our major results, we will need the following lemmas.

Lemma 1 ([34,35]). *Assume that $\Phi(u,v)$ is the complex valued function, $\Phi : \Delta \to \mathbb{C}$, $\Delta \subset \mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that $\Phi(u,v)$ satisfies the following conditions:*

(i) $\Phi(u,v)$ is continuous in a domain Δ;
(ii) $(1,0) \in \Delta$ and $\Re\{\Phi(1,0)\} > 0$;
(iii) $\Re\{\Phi(iu_2,v_1)\} \le 0$ for all $(iu_2,v_1) \in \Delta$ and such that $v_1 \le -\frac{1}{2}(1+u_2^2)$.

Let $h(\zeta) = 1 + \sum_{m=1}^{\infty} c_m\zeta^m$, be regular in Δ such that $(h(\zeta),\zeta h'(\zeta)) \in \Delta$ for all $\zeta \in \Delta$. If

$$\Re\left\{\Phi(h(\zeta),\zeta h'(\zeta))\right\} > 0 \quad (\zeta \in \Delta),$$

then

$$\Re\{h(\zeta)\} > 0 \quad (\zeta \in \Delta).$$

Lemma 2 ([36]). *Let $p(\zeta)$ be analytic in Δ with $p(0) = 1$ and $\Re(p(\zeta)) > 0$, $\zeta \in \Delta$. Then, for $s > 0$ and $\hbar \neq -1$ (complex),*

$$\Re\left\{ p(\zeta) + \frac{s\zeta p'(\zeta)}{p(\zeta)+\hbar} \right\} > 0, \text{ for } |\zeta| < \mathcal{R}_0,$$

where R_0 is given by

$$\mathcal{R}_0 = \frac{|\hbar+1|}{\sqrt{Q + (Q^2 - \left|\hbar^2 - 1\right|)^{\frac{1}{2}}}}, \; Q = 2(s+1)^2 + |\hbar|^2 - 1.$$

Lemma 3 ([37]). *Let $\mathcal{F}$ to be starlike in Δ and Φ to be convex. If Υ is analytic in Δ with $\Upsilon(0) = 1$ is attained, $\frac{\Phi * \Upsilon\mathcal{F}}{\Phi * \mathcal{F}}$ is contained in the convex hull of $\Upsilon(\Delta)$.*

Unless otherwise mentioned, we assume throughout this paper that: $\kappa \geq 2$, $p, j \in \mathbb{N}; n \in \mathbb{N}_0;$ $\varrho > 0;$ $\vartheta, v, m > 0$ and $\zeta \in \Delta$ and the power is the principal values.

Theorem 1. *For $0 \leq \aleph < \wp < p$, we have $\mathbf{B} < 0$ and for $\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) \neq 0$, then*

$$\mathcal{S}_{\kappa,p,j}^{\varrho,n+1,\vartheta,v,m}(\wp) \subset \mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\aleph),$$

where $\aleph$ is given by

$$\aleph = \frac{2\left[pc^j(\varrho) - 2\wp\left(pc^j(\varrho) - p\right)\right]}{\sqrt{\left[2p - c^j(\varrho)(2p+2\wp-1)\right]^2 + 8c^j(\varrho)\left(pc^j(\varrho) - 2\wp\left(pc^j(\varrho) - p\right)\right) + \left(2p - c^j(\varrho)(2p+2\wp-1)\right)}}. \tag{20}$$

Proof. Let $\mathcal{F} \in \mathcal{S}_{\kappa,p,j}^{\varrho,n+1,\vartheta,v,m}(\wp)$ and let

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)} = \mathcal{G}(\zeta) = (p-\aleph)h(\zeta) + \aleph, \tag{21}$$

where

$$h(\zeta) = \left(\frac{\kappa}{4} + \frac{1}{2}\right)h_1(\zeta) - \left(\frac{\kappa}{4} - \frac{1}{2}\right)h_2(\zeta) \tag{22}$$

and $h_i \in \mathcal{P}(\wp); (i = 1, 2)$ are analytic in Δ with $h_i(0) = 1, \; (i = 1, 2)$.
Using (12) and (21), we have

$$p\frac{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)} - p\left(1 - c^j(\varrho)\right) = (p-\aleph)c^j(\varrho)h(\zeta) + \aleph c^j(\varrho). \tag{23}$$

Differentiating (23) logarithmically with respect to ζ, we have

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)} - \wp = \aleph - \wp + (p-\aleph)h(\zeta) + \frac{(p-\aleph)c^j(\varrho)\zeta h'(\zeta)}{(p-\aleph)c^j(\varrho)h(\zeta) + c^j(\varrho)(\aleph - p) + p}. \tag{24}$$

Now, we show that $h \in \mathcal{P}_k(\wp)$ or $h_i(\zeta) \in \mathcal{P}(\wp), \; i = 1, 2$. From (22) and (24), we have

$$\begin{aligned}\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)} - \wp = {} & \left(\frac{\kappa}{4} + \frac{1}{2}\right)\left\{\aleph - \wp + (p-\aleph)h_1(\zeta) + \frac{(p-\aleph)c^j(\varrho)\zeta h_1'(\zeta)}{(p-\aleph)c^j(\varrho)h(\zeta) + c^j(\varrho)(\aleph-p) + p}\right\} \\ & - \left(\frac{\kappa}{4} - \frac{1}{2}\right)\left\{\aleph - \wp + (p-\aleph)h_2(\zeta) + \frac{(p-\aleph)c^j(\varrho)\zeta h_2'(\zeta)}{(p-\aleph)c^j(\varrho)h(\zeta) + c^j(\varrho)(\aleph-p) + p}\right\}\end{aligned}$$

and this implies that

$$\Re\left\{\aleph - \wp + (p-\aleph)h_i(\zeta) + \frac{(p-\aleph)c^j(\varrho)\zeta h_i'(\zeta)}{(p-\aleph)c^j(\varrho)h_i(\zeta) + c^j(\varrho)(\aleph-p) + p}\right\} > 0 \; (\zeta \in \Delta; \; i = 1, 2).$$

By fixing $u = h_i(\zeta)$ and $v = \zeta h_i^{'}(\zeta)$ we let $\Phi(u,v)$. Thus

$$\Phi(u,v) = \aleph - \wp + (p-\aleph)u + \frac{(p-\aleph)c^j(\varrho)v}{(p-\aleph)c^j(\varrho)u + c^j(\varrho)(\aleph - p) + p}.$$

Then

(i) $\Phi(u,v)$ is continuous in $\Delta = \left(\mathbb{C} \backslash \frac{c^j(\varrho)(\aleph-p)+p}{(p-\aleph)c^j(\varrho)}\right) \times \mathbb{C}$;

(ii) $(1,0) \in \Delta$ and $\Re\{\Phi(1,0)\} = p - \wp > 0$;

(iii) For all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$,

$$\begin{aligned} \Re\{\Phi(iu_2,v_1)\} &= \Re\left\{(\aleph-\wp) + (p-\wp)iu_2 + \frac{(p-\aleph)c^j(\varrho)v_1}{(p-\aleph)c^j(\varrho)iu_2 + c^j(\varrho)(\aleph-p)+p}\right\} \\ &= (\aleph-\wp) + \frac{(p-\aleph)c^j(\varrho)\left(c^j(\varrho)(\aleph-p)+p\right)v_1}{(p-\aleph)^2(c^j(\varrho))^2u_2^2 + \left(c^j(\varrho)(\aleph-p)+p\right)^2} \\ &\leq (\aleph-\wp) - \frac{(p-\aleph)c^j(\varrho)\left(c^j(\varrho)(\aleph-p)+p\right)(1+u_2^2)}{2\left[(p-\aleph)^2(c^j(\varrho))^2u_2^2 + \left(c^j(\varrho)(\aleph-p)+p\right)^2\right]} \\ &= \frac{\mathbf{A} + \mathbf{B}u_2^2}{2\mathbf{C}}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= 2(\aleph-\wp)\left(c^j(\varrho)(\aleph-p)+p\right)^2 - (p-\aleph)c^j(\varrho)\left(c^j(\varrho)(\aleph-p)+p\right), \\ \mathbf{B} &= 2(\aleph-\wp)(p-\aleph)^2(c^j(\varrho))^2 - (p-\aleph)c^j(\varrho)\left(c^j(\varrho)(\aleph-p)+p\right), \\ \mathbf{C} &= (p-\aleph)^2(c^j(\varrho))^2u_2^2 + \left(c^j(\varrho)(\aleph-p)+p\right)^2. \end{aligned}$$

We note that $\Re\{\Phi(iu_2,v_1)\} < 0$ if and only if $\mathbf{A} \leq 0$, $\mathbf{B} < 0$ and $\mathbf{C} > 0$. From $\mathbf{A} \leq 0$, we obtain $\aleph$ as given by (20) and from $0 \leq \aleph < \wp < p$, we have $\mathbf{B} < 0$. Therefore, applying Lemma 1, $h_i(\zeta) \in \mathcal{P}$ $(i = 1,2)$ and consequently $h(\zeta) \in \mathcal{P}_\kappa(\wp)$ for $\zeta \in \Delta$. This completes the proof of Theorem 1. □

Theorem 2. *For $0 \leq \aleph \leq \wp < p$, then*

$$\mathcal{C}_{\kappa,p,j}^{\varrho,n+1,\vartheta,v,m}(\wp) \subset \mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\aleph),$$

where $\aleph$ is given by (20).

Proof. Let

$$\begin{aligned}\mathcal{F} &\in \mathcal{C}_{\kappa,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\wp) \Rightarrow \mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{F}(\zeta) \in \mathcal{C}_{\kappa,p}(\wp) \\ &\Rightarrow \frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{F}(\zeta)\right)'}{p} \in \mathcal{C}_{\kappa,p}(\wp) \\ &\Rightarrow \mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\left(\frac{\zeta\mathcal{F}'(\zeta)}{p}\right) \in \mathcal{C}_{\kappa,p}(\wp) \\ &\Rightarrow \frac{\zeta\mathcal{F}'(\zeta)}{p} \in \mathcal{S}_{\kappa,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\wp) \subset \mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,\upsilon,m}(\aleph) \\ &\Rightarrow \mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(\frac{\zeta\mathcal{F}'(\zeta)}{p}\right) \in \mathcal{S}_{\kappa,p}(\aleph) \\ &\Rightarrow \mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{F}(\zeta) \in \mathcal{C}_{\kappa,p}(\aleph) \\ &\Rightarrow \mathcal{F} \in \mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,\upsilon,m}(\aleph)\end{aligned}$$

which completes the proof. □

Theorem 3. *Let* $0 \le \aleph \le \wp < p$. *Then,*

$$K_{\kappa,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\eta,\wp) \subset K_{\kappa,p,j}^{\varrho,n,\vartheta,\upsilon,m}(\eta,\aleph).$$

Proof. Let $\mathcal{F} \in K_{\kappa,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\eta,\wp)$. Then, there exists $\mathcal{G}(\zeta) \in \mathcal{S}_{2,p}(\wp) \equiv S_p^*(\wp)$ such that

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{F}(\zeta)\right)'}{H(\zeta)} \in \mathcal{P}_{p,k}(\eta). \tag{25}$$

Then $H(\zeta) = \mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{G}(\zeta) \in \mathcal{S}_{2,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\wp)$.
We set

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)} = \mathcal{H}(\zeta) = (p-\eta)h(\zeta) + \eta, \tag{26}$$

where $h(\zeta)$ is given by (22). By using (12) in (25), we obtain

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,\upsilon,m}\mathcal{G}(\zeta)} = \frac{\zeta c^j(\varrho)\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(\zeta\mathcal{F}'(\zeta)\right)\right)' + p[1-c^j(\varrho)]\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(\zeta\mathcal{F}'(\zeta)\right)}{\zeta c^j(\varrho)\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)\right)' + p[1-c^j(\varrho)]\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)}. \tag{27}$$

Also, $\mathcal{H}(\zeta) \in \mathcal{S}_{2,p,j}^{\varrho,n+1,\vartheta,\upsilon,m}(\wp)$ and by using Theorem 1, with $\kappa = 2$, we have $\mathcal{H}(\zeta) \in \mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}(\aleph)$.
Therefore, we can write

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)} = \mathcal{H}_0(\zeta) = (p-\aleph)q(\zeta) + \aleph \quad (q \in \mathcal{P}_\kappa),$$

where $q(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \ldots$ is analytic in Δ and $q(0) = 1$. By differentiating (26) with respect to ζ, we obtain

$$\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(\zeta\mathcal{F}'(\zeta)\right)\right)' = \zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)\right)'\mathcal{H}(\zeta) + \zeta\mathcal{H}'(\zeta)\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}\mathcal{G}(\zeta)$$

then

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\left(\zeta\mathcal{F}'(\zeta)\right)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{G}(\zeta)}=\zeta\mathcal{H}'(\zeta)+\mathcal{H}_0(\zeta)\mathcal{H}(\zeta). \tag{28}$$

From (27) and (28), we obtain

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{G}(\zeta)}=\frac{c^j(\varrho)\left[\zeta\mathcal{H}'(\zeta)+\mathcal{H}_0(\zeta)\mathcal{H}(\zeta)\right]+p\left[1-c^j(\varrho)\right]\mathcal{H}(\zeta)}{c^j(\varrho)\mathcal{H}_0(\zeta)+p\left[1-c^j(\varrho)\right]}$$

so that

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{G}(\zeta)}=\mathcal{H}(\zeta)+\frac{\zeta c^j(\varrho)\mathcal{H}'(\zeta)}{c^j(\varrho)\mathcal{H}_0(\zeta)+p\left[1-c^j(\varrho)\right]}. \tag{29}$$

Let

$$\mathcal{H}(\zeta)=\left(\tfrac{\kappa}{4}+\tfrac{1}{2}\right)\{(p-\eta)h_1(\zeta)+\eta\}-\left(\tfrac{\kappa}{4}-\tfrac{1}{2}\right)\{(p-\eta)h_2(\zeta)+\eta\}$$

and

$$c^j(\varrho)\mathcal{H}_0(\zeta)+p\left[1-c^j(\varrho)\right]=(p-\aleph)c^j(\varrho)q(\zeta)+\left(\aleph c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right).$$

We want to show that $\mathcal{H}\in\mathcal{P}_{p,\kappa}(\eta)$ or $h_i\in\mathcal{P}$ for $i=1,2$. Then, $\Re\{c^j(\varrho)\mathcal{H}_0(\zeta)+p[1-c^j(\varrho)]\}>0$. From (26) and (29), we have

$$\begin{aligned}\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{G}(\zeta)}-\eta&=\left(\tfrac{\kappa}{4}+\tfrac{1}{2}\right)\left[(p-\eta)h_1(\zeta)+\tfrac{(p-\eta)\zeta h_1'(\zeta)}{(p-\eta)c^j(\varrho)q(\zeta)+\left(\eta c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)}\right]\\&\quad-\left(\tfrac{\kappa}{4}-\tfrac{1}{2}\right)\left\{(p-\eta)h_2(\zeta)+\tfrac{(p-\eta)\zeta h_2'(\zeta)}{(p-\eta)c^j(\varrho)q(\zeta)+\left(\eta c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)}\right\}\end{aligned}$$

and this implies that

$$\Re\left\{(p-\eta)h_i(\zeta)+\tfrac{(p-\eta)\zeta h_i'(\zeta)}{(p-\eta)c^j(\varrho)q(\zeta)+\left(\eta c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)}\right\}>0\ (\zeta\in\Delta,\ i=1,2).$$

By fixing $u=h_i(\zeta)$ and $v=\zeta h_i'(\zeta)$, we let $\Phi(u,v)$. Thus,

$$\Phi(u,v)=(p-\eta)u+\frac{(p-\eta)v}{(p-\eta)c^j(\varrho)q(\zeta)+\left(\eta c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)}. \tag{30}$$

Then,

(i) $\Phi(u,v)$ is continuous in $\Delta=\mathbb{C}\times\mathbb{C}$;
(ii) $(1,0)\in\Delta$ and $\Re\{\Phi(1,0)\}=p-\eta>0$;
(iii) For all $(iu_2,v_1)\in\Delta$ such that $v_1\leq-\dfrac{1}{2}(1+u_2^2)$,

$$
\begin{aligned}
\Re\{\Phi(iu_2,v_1)\} &= \Re\left\{(p-\eta)iu_2+\frac{(p-\eta)v_1}{(p-\eta)c^j(\varrho)q(\zeta)+\left(\eta c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)}\right\}\\
&= \frac{(p-\eta)\left[(p-\aleph)c^j(\varrho)q_1+\aleph c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right]v_1}{\left[(p-\aleph)c^j(\varrho)q_1(\zeta)+\left(\aleph c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)\right]^2+\left((p-\aleph)c^j(\varrho)\right)^2q_2^2}\\
&\leq -\frac{(p-\eta)\left[(p-\aleph)c^j(\varrho)q_1+\aleph c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right](1+u_2^2)}{2\left\{\left[(p-\aleph)c^j(\varrho)q_1(\zeta)+\left(\aleph c^j(\varrho)+p\left[1-c^j(\varrho)\right]\right)\right]^2+\left((p-\aleph)c^j(\varrho)\right)^2q_2^2\right\}}<0.
\end{aligned}
$$

By applying Lemma 1, we have $\Re\{h_i(\zeta)\} > 0$ for $(i=1,2)$ and consequently $\mathcal{H}(\zeta)\in\mathcal{P}_{p,\kappa}(\eta)$ for $\zeta\in\Delta$. This completes the proof of Theorem 3. □

Theorem 4. *Let $\mathcal{F}(\zeta)\in\mathcal{S}^{\varrho,n,\vartheta,v,m}_{\kappa,p,j}(\wp)$, then $\mathcal{F}(\zeta)\in\mathcal{S}^{\varrho,n+1,\vartheta,v,m}_{\kappa,p,j}(\wp)$, for*

$$|\zeta|<r_0=\frac{|\hbar+1|}{\sqrt{A+\left(A^2-\left|\hbar^2-1\right|\right)^{\frac{1}{2}}}},\ A=2(s+1)^2+|\hbar|^2-1 \tag{31}$$

with $\hbar=\frac{\wp+p\left[\frac{1}{c^j(\varrho)}-1\right]}{p-\wp}\neq-1$ and $s=\frac{1}{p-\wp}$. This radius is the best possible.

Proof. Let

$$\frac{\zeta\left(\mathcal{D}^{\varrho,n,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{\varrho,n,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)}=\mathcal{H}(\zeta)=(p-\wp)h(\zeta)+\wp. \tag{32}$$

From (12) and (32), we have

$$\frac{p}{c^j(\varrho)}\frac{\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)}{\mathcal{D}^{\varrho,n,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)}=(p-\wp)h(\zeta)+\wp+\frac{p}{c^j(\varrho)}\left[1-c^j(\varrho)\right]. \tag{33}$$

Differentiating (33) logarithmically with respect to ζ, we have

$$\frac{1}{p-\wp}\left\{\frac{\zeta\left(\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)}-\wp\right\}=h(\zeta)+\frac{\left(\frac{1}{p-\wp}\right)\zeta h'(\zeta)}{h(\zeta)+\frac{\wp+p\left[\frac{1}{c^j(\varrho)}-1\right]}{p-\wp}}, \tag{34}$$

where $h(\zeta)$ given by (22), then

$$
\begin{aligned}
\frac{1}{p-\wp}\left\{\frac{\zeta\left(\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)\right)'}{\mathcal{D}^{\varrho,n+1,\vartheta,v,m}_{p,j}\mathcal{F}(\zeta)}-\wp\right\} &= \left(\frac{\kappa}{4}+\frac{1}{2}\right)\left\{h_1(\zeta)+\frac{\left(\frac{1}{p-\wp}\right)\zeta h_1'(\zeta)}{h_1(\zeta)+\frac{\wp+p\left[\frac{1}{c^j(\varrho)}-1\right]}{p-\wp}}\right\}\\
&\quad -\left(\frac{\kappa}{4}-\frac{1}{2}\right)\left\{h_2(\zeta)+\frac{\left(\frac{1}{p-\wp}\right)\zeta h_2'(\zeta)}{h_2(\zeta)+\frac{\wp+p\left[\frac{1}{c^j(\varrho)}-1\right]}{p-\wp}}\right\}
\end{aligned}
$$

where $\Re\{h_i(z)\} > 0$ for $(i = 1, 2)$. By using Lemma 2, with $\eta = \frac{\wp + p\left[\frac{1}{c^j(\varrho)} - 1\right]}{p - \wp} \neq -1$ and $s = \frac{1}{p-\wp}$, we obtain

$$\Re\left\{h_1(\zeta) + \frac{s\zeta h_i'(\zeta)}{h_i(\zeta) + \hbar}\right\} > 0, \text{ for } |\zeta| < r_0,$$

and r_0 is given by (31). Thus, $\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n+1,\vartheta,v,m}\mathcal{F}(\zeta)} \in \mathcal{P}_{p,\kappa}(\wp)$ and consequently $\mathcal{F}(\zeta) \in \mathcal{S}_{\kappa,p,j}^{\varrho,n+1,\vartheta,v,m}(\wp)$ for $|\zeta| < r_0$ and this radius is best possible.
□

3. Inclusion Properties Based on Bernardi Integral Operator $\mathcal{J}_{\varepsilon,p}$

Choi et al. [38] gave the definition of the generalized Bernardi operator as below:

$$\mathcal{J}_{\varepsilon,p}\mathcal{F}(\zeta) = \frac{\varepsilon + p}{\zeta^{\varepsilon}} \int_0^z t^{\varepsilon-1}\mathcal{F}(t)dt \ \ (\varepsilon > -p), \tag{35}$$

which satisfies the following relationship:

$$\zeta\left(\mathcal{J}_{\varepsilon,p}\mathcal{F}(\zeta)\right)' = (\varepsilon + p)\mathcal{F}(\zeta) - \varepsilon\mathcal{J}_{\varepsilon,p}\mathcal{F}(\zeta). \tag{36}$$

Theorem 5. *Let* $0 \le \wp < p$, $\kappa \ge 2$ *and* $\mathcal{F} \in \mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$. *Then,* $\mathcal{J}_{\varepsilon,p}(\mathcal{F}) \in \mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$ $(\varepsilon \ge 0)$.

Proof. Let

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta)} = \mathcal{H}(\zeta) = (p - \wp)h(\zeta) + \wp, \tag{37}$$

where $h(\zeta)$ given by (22). Using (36), we have

$$\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta)\right)' = (\varepsilon + p)\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta) - \varepsilon\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta). \tag{38}$$

From (37) and (38), we have

$$(\varepsilon + p)\frac{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta)} = (p - \wp)h(\zeta) + \wp + \varepsilon. \tag{39}$$

Differentiating (39) logarithmically with respect to ζ and multiplying by ζ, we have

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)} - \wp = (p - \wp)h(\zeta) + \frac{(p - \wp)\zeta h'(\zeta)}{(p - \wp)h(\zeta) + \wp + \varepsilon}. \tag{40}$$

Now, we show that $\mathcal{H}(\zeta) \in \mathcal{P}_{p,\kappa}(\wp)$ or $h_i \in \mathcal{P}$ for $i = 1, 2$. From (22) and (40), we have

$$\begin{aligned} \frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)\right)'}{\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}(\zeta)} - \wp = {} & \left(\frac{\kappa}{4} + \frac{1}{2}\right)\left\{(p - \wp)h_1(\zeta) + \frac{(p - \wp)\zeta h_1'(\zeta)}{(p - \wp)h_1(\zeta) + \wp + \varepsilon}\right\} \\ & - \left(\frac{\kappa}{4} + \frac{1}{2}\right)\left\{(p - \wp)h_2(\zeta) + \frac{(p - \wp)\zeta h_2'(\zeta)}{(p - \wp)h_2(\zeta) + \wp + \varepsilon}\right\} \end{aligned}$$

and this implies that

$$\Re\left\{(p-\wp)h_2(\zeta)+\frac{(p-\wp)\zeta h_i^{'}(\zeta)}{(p-\wp)h_i(\zeta)+\wp+\varepsilon}\right\}>0\ (\zeta\in\Delta;\ i=1,2).$$

We form the function $\Phi(u,v)$ by choosing $u=h_i(\zeta)$ and $v=\zeta h_i^{'}(\zeta)$. Thus,

$$\Phi(u,v)=(p-\wp)u+\frac{(p-\wp)v}{(p-\wp)u+\wp+\varepsilon}. \tag{41}$$

Clearly, conditions (i), (ii) and (iii) of Lemma 1 are satisfied. By applying Lemma 1, we have $\Re\{h_i(\zeta)\}>0$ for $(i=1,2)$ and consequently $\mathcal{J}_{\varepsilon,p}(\mathcal{F})\in\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$ for $\zeta\in\Delta$. This completes the proof of Theorem 5. □

Theorem 6. *Let* $0\le\wp<p$, $\kappa\ge2$ *and* $\mathcal{F}\in\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$. *Then* $\mathcal{J}_{\varepsilon,p}(\mathcal{F})\in\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$ $(\varepsilon\ge0)$.

Proof. Let

$$\mathcal{F}\in\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)\Leftrightarrow\frac{\zeta\mathcal{F}^{'}(\zeta)}{p}\in\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp).$$

By applying Theorem 5, it follows that

$$\mathcal{J}_{\varepsilon,p}\left(\frac{\zeta\mathcal{F}^{'}(\zeta)}{p}\right)\in\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)\Leftrightarrow\frac{\zeta\left(\mathcal{J}_{\varepsilon,p}(\mathcal{F})\right)^{'}}{p}\in\mathcal{S}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)\Leftrightarrow\mathcal{J}_{\varepsilon,p}(\mathcal{F})(\zeta)\in\mathcal{C}_{\kappa,p,j}^{\varrho,n,\vartheta,v,m}(\wp)$$

which perceptibly proves Theorem 6. □

4. Inclusion Properties by Convolution

Theorem 7. *Let* Ψ *be a convex function and* $\mathcal{F}\in\mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,v,m}\left(p\wp^{'}\right)$. *Then,* $\mathbb{G}\in\mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,v,m}\left(p\wp^{'}\right)$, *where* $\mathbb{G}=\mathcal{F}*\Psi$ *and* $0\le\wp^{'}<1$.

Proof. To show that $\mathbb{G}=\mathcal{F}*\Psi\in\mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,v,m}\left(p\wp^{'}\right)$ $\left(0\le\wp^{'}<1\right)$, it sufficient to show that $\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}}$ is contained in the convex hull of $R(\Delta)$. Now,

$$\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}}=\frac{\Psi*R\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}}{\Psi*\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}} \tag{42}$$

where $R=\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathcal{F}}$ is analytic in Δ and $R(0)=1$. From Lemma 3, we can see that $\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}}$ is contained in the convex hull of $R(\Delta)$, since $\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}}$ is analytic in Δ and

$$R(\Delta)\subseteq\Omega=\left\{w:\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}w(\zeta)\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}w(\zeta)}\in\mathcal{P}(\wp^{'})\right\} \tag{43}$$

then $\frac{\zeta\left(\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}\right)^{'}}{p\mathcal{D}_{p,j}^{\varrho,n,\vartheta,v,m}\mathbb{G}}$ lies in Ω,; this implies that $\mathbb{G}=\mathcal{F}*\Psi\in\mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,v,m}\left(p\wp^{'}\right)$. □

Theorem 8. *Let Ψ be a convex function and $\mathcal{F} \in \mathcal{C}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right)$. Then, $\mathbb{G} \in \mathcal{C}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right)$, where $\mathbb{G} = \mathcal{F} * \Psi$ and $0 \leq \wp' < 1$.*

Proof. Let $\mathcal{F} \in \mathcal{C}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right)$, then by using (15), we have

$$\frac{\zeta\mathcal{F}'(\zeta)}{p} \in \mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right)$$

and hence by using Theorem 7, we obtain

$$\begin{aligned} \frac{\zeta\mathcal{F}'(\zeta)}{p} * \Psi(\zeta) \quad &\in \quad \mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right) \\ &\Rightarrow \quad \frac{\zeta(\mathcal{F} * \Psi)'(\zeta)}{p} \in \mathcal{S}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right). \end{aligned}$$

Now, applying (15) again, we obtain $\mathbb{G} = \mathcal{F} * \Psi \in \mathcal{C}_{2,p,j}^{\varrho,n,\vartheta,\upsilon,m}\left(p\wp'\right)$, which evidently proves Theorem 8. □

5. Conclusions

In our present investigation, we have made use of a certain combination binomial series and confluent hypergeometric function with a view to introducing a new subclass of multivalent functions in the open unit disk. For functions belonging to this newly defined analytic function class, we have discussed inclusion properties involving the differential operator $\mathcal{D}_{p,j}^{\varrho,n,\vartheta,\upsilon,m}$ and integral operator $\mathcal{J}_{\varepsilon,p}$. Further, we discussed radius problems and derived certain inclusion results under convolution. Theorems of differential subordination and superordination for multivalent analytic functions arising in two-dimensional potential flow problems can be discussed later. Additionally, based on the combination of binomial series and confluent hypergeometric functions, these operators were utilized to investigate univalent function theory and fluid dynamics by a differential subordination technique [15,16], dynamic inequalities with general kernels [39–41] and by describing various classes of bi-univalent functions [42–45] with bounded boundary .

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