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A New Method for Finding Lie Point Symmetries of First-Order Ordinary Differential Equations

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Abstract: The traditional algorithm for finding Lie point symmetries of ordinary differential equations (ODEs) faces challenges when applied to first-order ODEs. This stems from the fact that for first-order ODEs, unlike higher-order ODEs, the determining equation lacks derivatives, rendering it impossible to decompose into simpler PDEs to be solved for the infinitesimals. Consequently, a common technique for determining Lie point symmetries of first-order ODEs involves making speculative assumptions about the form of the infinitesimal generator. In this study, we propose a novel and more efficient approach for finding Lie point symmetries of first-order ODEs and systems of first-order ODEs. Our method leverages the inherent connection between first-order ODEs and their corresponding second-order counterparts derived through total differentiation. By exploiting this connection, we develop a systematic algorithm for determining Lie point symmetries of a wide range of first-order ODEs. We present the algorithm and provide illustrative examples to demonstrate its effectiveness.

Keywords: first-order ODEs; Lie point symmetry; symmetry methods; total differentiation; integration of ODEs

MSC: 34A05; 34C14; 34A34

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1. Introduction

First-order ODEs and systems of first-order ODEs are prevalent mathematical models utilized to represent diverse physical phenomena across scientific disciplines (see, for example, [1–3]). In physics, they find relevance in describing phenomena like exponential decay, circuit charging and discharging, and radioactive decay. Mechanical systems employ these equations to model simple harmonic motion and damping. In biology, first-order ODEs are employed in population dynamics to simulate population growth and decline, as well as in processes such as enzyme kinetics and drug metabolism. Chemistry employs them to characterize chemical reactions through rate equations. In economics, they serve to model economic growth, consumption patterns, and investment behavior. Engineering fields rely on first-order ODEs to represent control systems, heat transfer, fluid flow, and electrical circuits. Environmental science utilizes these equations to model pollutant dispersion, groundwater flow, and ecological interactions.

Solving first-order ODEs and systems of first-order ODEs can be challenging, which is why methods based on Lie symmetry analysis of differential equations are attractive.

Lie symmetry analysis, which has its origins in the pioneering work of the Norwegian mathematician Sophus Lie, provides very effective techniques for studying and analyzing differential equations (see, for example, [4–8]). Detailed accounts of Lie symmetry analysis of differential equations are found in many books [9–16] and monographs [17,18].

One of the most common applications of Lie symmetry analysis to ODEs is the reduction of order; when an ODE admits a Lie point symmetry, the equation can be reduced to a simpler form. In the case of first-order ODEs, the reduction amounts to complete integration of the equation. Other applications include the determination of invariant solutions, the derivation of conservation laws, the construction of "links" between seemingly different differential equations, and generating new solutions from known solutions. Lie's method for studying differential equations is so versatile that various approaches to solving differential equations can be traced to particular forms of the corresponding symmetries of the differential equations [11,14].

Central to many routines for Lie symmetry analysis of differential equations is the determination of their symmetries. In the case of an ODE or a system of ODEs of order two or more, the infinitesimals of the symmetry generator are found systematically, using Lie's algorithm, a straightforward algorithm that reduces the problem to that of solving an overdetermined system of first-order linear PDEs. However, for first-order ODEs and systems of first-order ODEs, Lie's algorithm does not work, in that the strategy of splitting the determining PDE is inapplicable. A typical approach in this case involves guessing a form of the infinitesimals and then invoking the invariance criterion [19]. The success of this approach depends on the specific form of the ODE. Different forms may require different guessing strategies, which makes the approach less universally applicable.

This article proposes an innovative method for finding Lie point symmetries of first-order ODEs and systems of first-order ODEs. The method exploits the "increase in order" of a first-order ODE or a system of first-order ODEs achieved through total differentiation. Lie point symmetries of a given first-order ODE or a system of first-order ODEs are searched among those of the associated second-order equations for which Lie's algorithm is effective in determining the admitted symmetries. We show that the mapping of first-order ODEs into second-order ODEs through total differentiation provides a systematic algorithm for finding Lie point symmetries of first-order ODEs and systems of first-order ODEs. In related work by Bildik and Açcl [20], the authors only considered basic examples involving simple scalar first-order ODEs, and their exposition did not use the language of total differentiation.

The remainder of this article unfolds as follows: Section 2 introduces the essential background on Lie symmetry analysis of ODEs, including an explanation of the challenge of finding Lie point symmetries of first-order ODEs. In the same section, we present the algorithm at the core of this article: the algorithm for uncovering Lie point symmetries of first-order ODEs by "mapping" them into associated second-order ODEs via total differentiation. Section 3 presents illustrative examples involving scalar first-order ODEs. In Section 4, we extend the preliminaries of Lie symmetry analysis presented in Section 2 to systems of ODEs and we provide illustrative examples that involve systems of first-order ODEs. Finally, in Section 5, we provide concluding remarks.

2. Preliminaries

We briefly review some background material on Lie symmetry analysis of differential equations relevant to this paper. For the interested reader, fuller accounts of the methods can be found in many books, including [9–16].

Consider an nth-order ODE,

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \tag{1}$$

where

$$y^{(k)} = \frac{d^k y}{dx^k}, \quad k = 1, 2, \ldots, n,$$

and a one-parameter Lie group of point transformations,

$$\begin{aligned} \widetilde{x} &= f(x, y; \varepsilon) = x + \varepsilon\xi(x, y) + O\left(\varepsilon^2\right), \\ \widetilde{y} &= g(x, y; \varepsilon) = y + \varepsilon\eta(x, y) + O\left(\varepsilon^2\right), \end{aligned} \tag{2}$$

with the group parameter ε and the corresponding infinitesimal generator,

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{3}$$

The Lie group (2) is admitted by (1) if and only if

$$X^{(n)}F\left(x,y,y',\ldots,y^{(n)}\right) = 0 \quad \text{whenever} \quad F\left(x,y,y',\ldots,y^{(n)}\right) = 0, \tag{4}$$

where $X^{(n)}$ is the nth extended infinitesimal generator of (3) given by

$$X^{(n)} = X + \eta^{(1)}(x,y,y')\frac{\partial}{\partial y'} + \cdots + \eta^{(k)}\left(x,y,y',\ldots,y^{(k)}\right)\frac{\partial}{\partial y^{(k)}}, \tag{5}$$

where

$$\eta^{(k)}\left(x,y,y',\ldots,y^{(k)}\right) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi, \quad k = 1,2,\ldots,n$$

with

$$\eta^{(0)} = \eta(x,y),$$

and D_x is the total differential operator defined by

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots. \tag{6}$$

We typically refer to either (2) or (3) as a symmetry of the ODE (1) if the invariance condition (4) holds.

The infinitesimal criterion for invariance (4) results in a linear homogeneous differential equation called the *determining equation*. If the order of the ODE (1) is two or greater, then the solution of the determining equation is simplified by the fact that the unknown functions ξ and η depend only on x and y, while the determining equation involves also the derivatives $y',\ldots,y^{(n)}$. Because the determining equation must hold identically in all variables x, y, $y',\ldots,y^{(n)}$, the terms in the equation multiplied by the various monomials in the derivatives $y',\ldots,y^{(n)}$ must be equated to zero. Therefore, the determining equation splits into a large number of elementary linear PDEs for the coefficients ξ and η. The general solution of the system of the elementary PDEs determines the most general forms of the functions ξ and η.

Let us assume that (1) is a first-order ODE and that it can be written in the form

$$y' = \Omega(x,y). \tag{7}$$

Applying the invariance condition, we ascertain that (7) admits a Lie group of point transformations that has an infinitesimal generator (3) if

$$X^{(1)}(y' - \Omega(x,y))|_{(7)} = 0. \tag{8}$$

Equation (8) reduces to the determining PDE

$$\eta_x + (\eta_y - \xi_x)\Omega - \xi_y\Omega^2 - \xi\Omega_x - \eta\Omega_y = 0, \tag{9}$$

which is to be solved for the functions ξ and η. Given the function Ω, this PDE possesses infinitely many (nonzero) solutions ξ and η [11,14,17,21]. Despite this, no universal guidelines exist to facilitate the resolution of Equation (9). Typically, to solve (9) an ansatz is used. One common ansatz is to use $\xi = \alpha(x)$ and $\eta = \beta(x)y + \gamma(x)$, where α, β, γ are taken to be polynomials. Other ad hoc methods have been proposed to deal with particular kinds of first-order ODEs [7,19,22–24].

The problem of finding symmetries admitted by first-order ODEs is also tackled in the "reverse" order. Given an infinitesimal generator of a Lie group of point transformations, the general first-order ODE that admits the group is determined. Consider, for example, the group with the symmetry generator

$$X = x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{10}$$

Using the once-extended operator

$$X^{(1)} = x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1-y')\frac{\partial}{\partial y'}, \tag{11}$$

one readily finds an invariant $u = y - x$ and a first-order differential invariant $v = x(y' - 1)$. Hence, the general first-order ODE invariant under (10) has the form $v = H(u)$, where H is an arbitrary function. This means that the general form of a first-order ODE that admits a Lie group of point transformations with infinitesimal generator (10) is

$$xy' = x + H(y - x). \tag{12}$$

Using this approach, tables can be (in fact, have been) generated of general forms of first-order ODEs with known Lie point symmetries (see, for example, [25] (p. 59)).

In this article, we propose a straightforward method for obtaining Lie point symmetries of a wide range of first-order ODEs via exploitation of the connection between a given first-order ODE and the second-order ODE derived from it through total differentiation.

Consider a first-order ODE,

$$f(x, y, y') = 0 \tag{13}$$

for some function f. The equation

$$D_x f(x, y, y') = 0, \tag{14}$$

where D_x is the total differential operator defined by (6), is a second-order ODE. We shall refer to (14) as the second-order ODE associated with the first-order ODE (13). Equation (14), being a second-order ODE, possesses k Lie point symmetries, where $k \in \{0, 1, 2, 3, 8\}$ (see [15,26]). Lie point symmetries for Equation (14) can be readily determined using popular software packages [27–30]. A natural question to ask is whether any of the k symmetries (if $k > 0$) admitted by Equation (14) are inherited by Equation (13).

It turns out that, in many instances, this is the case, i.e., among the Lie point symmetries of the associated second-order ODE are Lie point symmetries of the first-order ODE.

To determine which of the symmetries $X_1, \ldots, X_k$ of (14) are inherited by (13), we take a linear combination, $X = \sum_{i=1}^{k} \delta_i X_i$, where δ_is are arbitrary constants, and then apply the invariance condition to determine conditions on the arbitrary constants under which the first-order ODE admits X.

In instances where other symmetries of a given first-order ODE (or system) are known, additional admitted Lie point symmetries can be found systematically by calculating the Lie brackets for all conceivable pairs of the infinitesimal generators. Upon finding new ones through this process, the method is iteratively re-applied to the extended set of infinitesimal generators. This iterative procedure is repeated until no further linearly independent infinitesimal generators are uncovered. This is illustrated in Example 5.

The calculations reported in this article were performed using Mathematica 9.0 [31] and the Mathematica-based package MathLie [27,28]. Several illustrative examples are provided below.

3. Lie Point Symmetries of Scalar First-Order ODEs

Example 1. *Consider the ODE [11] (p. 108),*

$$y' = x - y. \tag{15}$$

The associated second-order ODE obtained is

$$D_x(y' - x + y) = y'' + y' - 1 = 0. \tag{16}$$

Equation (16) admits the following Lie point symmetries:

$$\begin{aligned}
X_1 &= (2 - y + x)\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \\
X_3 &= \left[(x-y)^2 - 2\right]\frac{\partial}{\partial x} + (x - y - 2)\frac{\partial}{\partial y}, \\
X_4 &= 4\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}, \quad X_5 = e^{-x}\frac{\partial}{\partial x}, \\
X_6 &= e^{-x}\left[(x-y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right], \quad X_7 = e^{x}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \\
X_8 &= e^{x}(x-y)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).
\end{aligned} \tag{17}$$

To determine which of these symmetries are admitted by first-order ODE (15), we take the linear combination $X = \sum_{i=1}^{8} \delta_i X_i$, where δ_is are arbitrary constants, and then apply the invariance condition, i.e., we solve the equation

$$X^{(1)}(y' - x + y)|_{(15)} = 0 \tag{18}$$

for constants δ_is. We determine that Equation (18) holds if and only if

$$\delta_1 = \delta_3 = \delta_5 = \delta_6 = 0, \quad \delta_2 = -\delta_4, \quad \delta_8 = -\delta_7. \tag{19}$$

This means that the following symmetries of (16) are admitted by the first-order ODE (15):

$$X_4 - X_2, \quad X_7 - X_8. \tag{20}$$

Example 2. *Consider the nonlinear ODE [32] (p. 28),*

$$x^4 y y' + 2x^3 y^2 + x = 0. \tag{21}$$

The associated second-order ODE is

$$x^4 (y')^2 + x^3 y(xy'' + 8y') + 6x^2 y^2 + 1 = 0, \tag{22}$$

and admits the following Lie point symmetries:

$$\begin{aligned}
X_1 &= x^2\frac{\partial}{\partial x} - \frac{1}{2}\left(\frac{1}{xy} + 3xy\right)\frac{\partial}{\partial y}, \\
X_2 &= x\frac{\partial}{\partial x} - \frac{1}{4}\left(\frac{3}{x^2 y} + 7y\right)\frac{\partial}{\partial y}, \\
X_3 &= \frac{\partial}{\partial x} - \left(\frac{3}{2x^3 y} + \frac{2y}{x}\right)\frac{\partial}{\partial y}, \\
X_4 &= x^3\left(x^2y^2 + 1\right)\frac{\partial}{\partial x} - \frac{1}{2}\left(3x^2y^2 + 1\right)\left(x^2 y + \frac{1}{y}\right)\frac{\partial}{\partial y}, \\
X_5 &= x^2\left(x^2y^2 + 3\right)\frac{\partial}{\partial x} - 2\left(x^3y^3 + 3xy + \frac{1}{xy}\right)\frac{\partial}{\partial y}, \\
X_6 &= x\frac{\partial}{\partial x} - \left(\frac{1}{x^2 y} + 2y\right)\frac{\partial}{\partial y}, \\
X_7 &= \frac{\partial}{\partial x} - \left(\frac{2}{x^3 y} + \frac{2y}{x}\right)\frac{\partial}{\partial y}, \quad X_8 = \frac{1}{x^4 y}\frac{\partial}{\partial y}.
\end{aligned} \tag{23}$$

If set $X = \sum_{i=1}^{8} \delta_i X_i$*, then the invariance condition*

$$X^{(1)}\left(x^4 y y' + 2x^3 y^2 + x\right)|_{(21)} = 0 \tag{24}$$

holds if and only if

$$\delta_3 = -2\delta_7, \quad \delta_5 = -\tfrac{1}{2}\delta_2. \tag{25}$$

Therefore, the following symmetries are admitted by the first-order ODE (21):

$$X_1 - \tfrac{1}{2}X_5, \quad X_2, \quad X_4, \quad X_6, \quad -2X_3 + X_7, \quad X_8. \tag{26}$$

Example 3. *Consider the nonlinear ODE [24],*

$$y' = \frac{(y - x\ln x)^2}{x^2} + \ln x. \tag{27}$$

The associated second-order ODE is

$$x^3 y'' + 2xy(1 - y') + 2\ln x[x^2(y' - 1) - xy] - 2y^2 + x^2 = 0 \tag{28}$$

and is found to admit the following Lie point symmetries:

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + (y + x)\frac{\partial}{\partial y}, \\ X_2 &= x(1 - 2\ln x)\frac{\partial}{\partial x} - [(2x + y)2\ln x - 3y]\frac{\partial}{\partial y}. \end{aligned} \tag{29}$$

Upon invoking the invariance condition, we determine that $X = \delta_1 X_1 + \delta_2 X_2$ *is admitted by (27), provided that* $\delta_2 = 0$*. This means that only* X_1 *is admitted by (27) from the symmetries admitted by the associated second-order ODE (28).*

Example 4. *Consider the nonlinear ODE [19]*

$$y' = be^{axy}x^a + \frac{(x^2 - 1)y}{x} - \frac{1}{x^2} + \ln x + c, \tag{30}$$

where a, b, *and* c *are constants. The associated second-order ODE*

$$x^3\left(y'' - y\right) - abx^{a+2}e^{axy}\left(x^2 y' + xy + 1\right) - \left(x^2 - x^4\right)y' + xy - x^2 - 2 = 0 \tag{31}$$

admits only one Lie point symmetry, namely

$$X = \frac{1}{x}\frac{\partial}{\partial x} - \left(\frac{y}{x^2} + \frac{1}{x^3}\right)\frac{\partial}{\partial y}. \tag{32}$$

It happens that Equation (30), the original first-order ODE, also admits this symmetry.

Example 5. *The ODE [14]*

$$y' = -\frac{y + 2x}{x}, \tag{33}$$

has the following associated second-order ODE:

$$y'' = \frac{y - xy'}{x^2}. \tag{34}$$

Equation (34) is found to admit the following eight Lie point symmetries:

$$
\begin{aligned}
&X_1 = y\frac{\partial}{\partial x} - \frac{y^2}{x}\frac{\partial}{\partial y}, \quad X_2 = x^2 y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}, \\
&X_3 = \left(x + \frac{1}{x}\right)\frac{\partial}{\partial y}, \quad X_4 = \left(x - \frac{1}{x}\right)\frac{\partial}{\partial y}, \quad X_5 = y\frac{\partial}{\partial y}, \\
&X_6 = \left(x + \frac{1}{x}\right)\frac{\partial}{\partial x} - y\left(\frac{1}{x^2} + 3\right)\frac{\partial}{\partial y}, \\
&X_7 = \left(x - \frac{1}{x}\right)\frac{\partial}{\partial x} + y\left(\frac{1}{x^2} - 3\right)\frac{\partial}{\partial y}, \\
&X_8 = x\left(x^2 + 1\right)\frac{\partial}{\partial x} + y\left(x^2 + 3\right)\frac{\partial}{\partial y}.
\end{aligned} \tag{35}
$$

Applying the invariance condition, we determine that $X = \sum_{i=1}^{8} \delta_i X_i$ is admitted by (33), provided that

$$
\delta_2 = \delta_8, \quad \delta_3 = 2\delta_1 - \delta_4 + \delta_5 - 4\delta_6 + 4\delta_7 + 2\delta_8. \tag{36}
$$

This leads to the following symmetries of the first-order ODE (33):

$$
\begin{aligned}
&X_1 + 2X_3, \quad X_4 - X_3, \quad X_3 + X_5, \\
&X_6 - 4X_3, \quad 4X_3 + X_7, \quad X_2 + 2X_3 + X_8.
\end{aligned} \tag{37}
$$

Remark 1. *It is noteworthy that the symmetry*

$$
Z = \frac{1}{y + 2x}\frac{\partial}{\partial x}, \tag{38}
$$

which is also admitted by (33) (reported in [14]), is not admitted by the associated second-order ODE (34), and cannot be represented by the symmetries in (37). Therefore, additional Lie point symmetries of the first-order ODE (33) can be found by calculating Lie brackets as explained in Section 2. For example, taking $Y = X_4 - X_3$ and Z, we obtain this new symmetry of (33),

$$
[Y, Z] = \frac{2}{x(y + 2x)}\left(\frac{1}{y + 2x}\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}\right).
$$

Example 6. *The nonlinear ODE,*

$$
y^2 y' = x \sin^2 x, \tag{39}
$$

has the following associated second-order ODE:

$$
2y\left(y'\right)^2 + y^2 y'' - x\sin(2x) - \sin^2 x = 0. \tag{40}
$$

Equation (40) is found to admit the following eight Lie point symmetries:

$$
\begin{aligned}
&X_1 = \frac{8y^3 - 6x^2 + 3\cos(2x) + 6x\sin(2x)}{y^2}\frac{\partial}{\partial y}, \\
&X_2 = 2\frac{\partial}{\partial x} - \frac{x\cos(2x)}{y^2}\frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + \frac{x^2\sin^2 x}{y^2}\frac{\partial}{\partial y}, \\
&X_4 = 24x^2\frac{\partial}{\partial x} + \frac{x(8y^3 + 6x^2 + (3 - 12x^2)\cos(2x) + 6x\sin(2x))}{y^2}\frac{\partial}{\partial y}, \\
&X_5 = \frac{1}{y^2}\frac{\partial}{\partial y}, \quad X_6 = \frac{x}{y^2}\frac{\partial}{\partial y}, \\
&X_7 = 48x\frac{\partial}{\partial x} + \frac{8y^3 + 18x^2 + (3 - 12x^2)\cos(2x) + 6x\sin(2x)}{y^2}\frac{\partial}{\partial y}, \\
&X_8 = 4\frac{\partial}{\partial x} + \frac{x(3 + \cos(2x) - 2x\sin(2x))}{y^2}\frac{\partial}{\partial y}.
\end{aligned} \tag{41}
$$

Applying the invariance condition, we determine that $X = \sum_{i=1}^{8} \delta_i X_i$ *is admitted by (39), provided that*

$$\delta_4 = \delta_7 = \delta_8 = 0, \quad \delta_6 = \delta_2. \tag{42}$$

This leads to the following symmetries of the first-order ODE (39):

$$X_1, \quad X_2 + X_6, \quad X_3, \quad X_5. \tag{43}$$

Example 7. *The Bernoulli equation [33,34],*

$$y' = y + y^{-1}e^x, \tag{44}$$

has associated with it the second-order ODE:

$$y'' = \left(\frac{e^x}{y^2} + 1\right)y' - \frac{e^x}{y}, \tag{45}$$

which admits the Lie point symmetries

$$X_1 = e^x\frac{\partial}{\partial x} + ye^x\frac{\partial}{\partial y}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{46}$$

The symmetries given in Equation (46) are also determined, using the invariance condition, to be symmetries of the Bernoulli Equation (44).

4. Lie Point Symmetries of Systems of First-Order ODEs

Systems of ODEs featuring a single independent variable and multiple dependent variables are prevalent in various applications, including mathematical models for infectious disease propagation, species competition dynamics, and predator–prey interactions. Just as with scalar ODEs, Lie symmetry analysis can be effectively employed to analyze such systems, provided admitted Lie point symmetries can be found.

The challenge of finding Lie point symmetries for scalar first-order ODEs, as discussed in Section 2, applies to systems of first-order ODEs. This section is dedicated to extending the procedure introduced in Section 2 to systems of first-order ODEs.

The preliminaries of Lie symmetry analysis presented in Section 2 extend naturally to the study of systems of ODEs. Let us consider a system of ODEs consisting of one dependent variable t and m (with $m \geq 2$) dependent variables $u(t) = (u^1(t), u^2(t), \ldots, u^m(t))$,

$$F_\alpha(t, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{47}$$

where $u_{(k)}$ denotes all kth order derivatives of u with respect to t. Henceforth, we will adhere to the convention of summation over repeated indices. We say that (47) admits a Lie group of point transformations with an infinitesimal generator,

$$X = \xi(t, u)\frac{\partial}{\partial t} + \eta^\alpha(t, u)\frac{\partial}{\partial u^\alpha}, \tag{48}$$

if and only if for all $\alpha = 1, \ldots, m$,

$$X^{(k)}F_\alpha(t, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0 \text{ whenever (47) holds}, \tag{49}$$

where $X^{(k)}$ is the kth extended infinitesimal generator given by

$$\mathrm{X}^{(k)} = \xi(t,u)\frac{\partial}{\partial t} + \eta^{\alpha}(t,u)\frac{\partial}{\partial u^{\alpha}} + \eta^{\alpha}_{t}(t,u,u_{(1)})\frac{\partial}{\partial u^{\alpha}_{t}} + \cdots + \eta^{\alpha}_{\underbrace{tt\cdots t}_{k\text{ times}}}(t,u_{(1)},u_{(2)},\ldots,u_{(k)})\frac{\partial}{\partial u^{\alpha}_{\underbrace{tt\cdots t}_{k\text{ times}}}}, \quad k=1,2,\ldots, \tag{50}$$

with the extended infinitesimals given by

$$\eta^{\alpha}_{t} = D_t\eta^{\alpha} - (D_t\xi)u^{\alpha}_{t},$$

and

$$\eta^{\alpha}_{\underbrace{tt\cdots t}_{k\text{ times}}} = D_t\eta^{\alpha}_{\underbrace{tt\cdots t}_{(k-1)\text{ times}}} - (D_t\xi)u^{\alpha}_{\underbrace{tt\cdots t}_{k\text{ times}}}, \quad k=2,3,\ldots,$$

where D_t is the total differential operator with respect to t defined by

$$D_t = \frac{\partial}{\partial t} + u^{\alpha}_{t}\frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{tt}\frac{\partial}{\partial u^{\alpha}_{t}} + \cdots + u^{\alpha}_{\underbrace{tt\cdots t}_{(n+1)\text{ times}}}\frac{\partial}{\partial u^{\alpha}_{\underbrace{tt\cdots t}_{n\text{ times}}}} + \cdots. \tag{51}$$

If the system (47) is first-order, i.e., $k=1$, we increase the system's order to second-order by applying the total differential operator (51). We obtain the corresponding system of second-order ODEs,

$$D_tF_{\alpha}(t,u_{(1)}) = 0, \quad \alpha = 1,\ldots,m. \tag{52}$$

Lie point symmetries of the original first-order ODE are then searched among the Lie point symmetries of (52). We shall now provide three illustrative examples.

Example 8. *Among the mathematical models considered in Nucci and Leach [8] is the system of ODEs*

$$\begin{aligned} u' + u^2 - uv &= 0 \\ v' + auv + v^2 &= 0, \end{aligned} \tag{53}$$

where a is a constant, with one independent variable t, and two dependent variables u and v. The associated second-order system of ODEs

$$\begin{aligned} u'' + (2u-v)u' - uv' &= 0 \\ v'' + avu' + v'(au+2v) &= 0 \end{aligned} \tag{54}$$

is found to admit two Lie point symmetries

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \qquad \mathrm{X}_2 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{55}$$

Taking the linear combination $X = \delta_1 X_1 + \delta_2 X_2$, and then applying the invariance condition, i.e.,

$$\mathrm{X}^{(1)}\left(u' + u^2 - uv\right)|_{(53)} = 0 \tag{56}$$

$$\mathrm{X}^{(1)}\left(v' + auv + v^2\right)|_{(53)} = 0, \tag{57}$$

we obtain that both Equations (56) and (57) hold for all δ_1 and δ_2. Therefore, the Lie point symmetries in (55) are both inherited by the first-order system (53).

Example 9. *The simultaneous first-order ODEs*

$$u' = 3u + v, \quad v' = v - u, \tag{58}$$

are typical of models of two species of animals that compete for survival in a given habitat, where $u(t)$ and $v(t)$ are their respective populations at time t. Associated with (58) is the system of second-order ODEs,

$$u'' = 3u' + v', \quad v'' = -u' + v'. \tag{59}$$

The system (59) is found to admit the following Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_3 = v\frac{\partial}{\partial u} - (u+2v)\frac{\partial}{\partial v}, \\ X_4 &= \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad X_5 = \frac{\partial}{\partial u} - 3\frac{\partial}{\partial v}, \quad X_6 = e^{2t}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right). \end{aligned} \tag{60}$$

It is easily shown that if $X = \sum_{i=1}^{6} \delta_i X_i$ then

$$X^{(1)}(u' - 3u - v)|_{(59)} = 0 \tag{61}$$

$$X^{(1)}(v' - v + u)|_{(59)} = 0, \tag{62}$$

provided that $\delta_4 = \delta_5 = 0$. This means that all the symmetries in (60) except X_4 and X_5 are admitted by the system (58).

Example 10. *The following coupled system of three linear first-order ODEs is also typical of systems commonly encountered in applications involving mathematical modeling with differential equations:*

$$u' = -2u + v + w, \quad v' = u - 3v, \quad w' = 2v - w. \tag{63}$$

The three dependent variables $u(t)$, $v(t)$, and $w(t)$ are related in such a way that their rates of change are determined by linear combinations of the variables themselves. The associated system of second-order ODEs is

$$u'' = -2u' + v' + w', \quad v'' = y' - 3v', \quad w'' = 2u' - w', \tag{64}$$

and admits the following Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = 3\frac{\partial}{\partial u} + \frac{\partial}{\partial v} + 2\frac{\partial}{\partial w}, \quad X_3 = 3\frac{\partial}{\partial u} + 2\frac{\partial}{\partial v} + 4\frac{\partial}{\partial w}, \\ X_4 &= 3\frac{\partial}{\partial u} + \frac{\partial}{\partial v} + 5\frac{\partial}{\partial w}, \quad X_5 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \\ X_6 &= (v+w)\frac{\partial}{\partial u} + (u-v)\frac{\partial}{\partial v} + (2v+w)\frac{\partial}{\partial w}, \\ X_7 &= 2(v+w)\frac{\partial}{\partial u} + w\frac{\partial}{\partial v} + (2u+2v+w)\frac{\partial}{\partial w}, \\ X_8 &= e^{-\left(\frac{\sqrt{5}+3}{2}\right)t}\left(\frac{\partial}{\partial u} + \frac{\sqrt{5}+3}{2}\frac{\partial}{\partial v} - \left(\sqrt{5}+1\right)\frac{\partial}{\partial w}\right), \\ X_9 &= e^{\frac{\sqrt{5}-3}{2}t}\left(\frac{\partial}{\partial u} - \frac{\sqrt{5}-3}{2}\frac{\partial}{\partial v} + \left(\sqrt{5}-1\right)\frac{\partial}{\partial w}\right), \\ X_{10} &= e^{-3t}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right). \end{aligned} \tag{65}$$

We determine that $X = \sum_{i=1}^{10} \delta_i X_i$ is admitted by the system (63), provided that $\delta_2 = \delta_3 = \delta_4 = 0$. Therefore, the symmetries X_1, X_5, X_6, X_7, X_8, X_9, and X_{10} are admitted by the system of first-order ODEs (63).

5. Concluding Remarks

In the study of first-order ODEs and systems of first-order ODEs, Lie symmetry analysis emerges as a powerful toolkit, providing an array of useful routines and algorithms.

Central to Lie symmetry analysis of such equations is the determination of Lie point symmetries of the equations. In this context, our proposed methodology for finding admitted Lie point symmetries stands as a noteworthy contribution to the existing techniques.

As elaborated in the introduction and preliminary sections, Lie's algorithm—the primary mechanism for finding Lie point symmetries of ODEs—falls short when applied to first-order ODEs. However, a given first-order ODE can be viewed as a first integral of the second-order ODE obtained by applying the total differential operator to the given first-order equation. By considering symmetries of the associated second-order ODE, it becomes possible to search for Lie point symmetries of the original first-order equation among the symmetries of the second-order ODE.

Our methodology effectively finds symmetries of many first-order ODEs and systems of first-order ODEs. We have illustrated this through examples involving seven scalar first-order ODEs and three systems of first-order ODEs. This approach to finding Lie point symmetries of first-order ODEs and systems of first-order ODEs potentially enables the determination of Lie point symmetries in previously intractable cases. We hope this will open up new avenues for applying Lie symmetry analysis to complex systems of first-order ODEs that arise in diverse fields, such as the modeling of the spread of infectious diseases, competition between species, and predator–prey relationships.

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