Review

Quantum-to-Classical Coexistence: Wavefunction Decay Kinetics, Photon Entanglement, and Q-Bits

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Abstract: By utilizing a generalized version of the Madelung quantum hydrodynamic framework that incorporates noise, we derive a solution using the path integral method to investigate how a quantum superposition of states evolves over time. This exploration seeks to comprehend the process through which a stable quantum state emerges when fluctuations induced by the noisy gravitational background are present. The model defines the conditions that give rise to a limited range of interactions for the quantum potential, allowing for the existence of coarse-grained classical descriptions at a macroscopic level. The theory uncovers the smallest attainable level of uncertainty in an open quantum system and examines its consistency with the localized behavior observed in large-scale classical systems. The research delves into connections and similarities alongside other theories such as decoherence and the Copenhagen foundation of quantum mechanics. Additionally, it assesses the potential consequences of wave function decay on the measurement of photon entanglement. To validate the proposed theory, an experiment involving entangled photons transmitted between detectors on the moon and Mars is discussed. Finally, the findings of the theory are applied to the creation of larger Q-bit systems at room temperatures.

Keywords: quantum decoherence; quantum entanglement; quantum-to-classical transition; wavefunction collapse kinetics; quantum uncertainty and relativistic locality; measure and decoherence; photon entanglement; pre-measure existing reality; EPR paradox

1. Introduction

The notion that quantum mechanics operates as a probabilistic process can be traced back to the research of Nelson [1] and has endured over time. However, Nelson's hypotheses ultimately fell short due to the imposition of a particular stochastic derivative with time inversion symmetry, which limited its generality. Furthermore, the results of Nelson's theory do not entirely align with those of quantum mechanics, as demonstrated by Von Neumann's proof [2] of the impossibility of reproducing quantum mechanics with theories based on underlying classical probabilistic variables.

The definitive resolution to this question was provided by Kleinert [3] (refer to Appendix A), who utilized the path integral approach to establish that quantum mechanics can be conceptualized as an imaginary time probabilistic process. These imaginary time quantum fluctuations differ from the more commonly understood real-time fluctuations. They result in a "reversible" pseudo-diffusion behavior, which is elucidated by the Madelung quantum hydrodynamic model through the action of the quantum potential.

The distinguishing characteristic of quantum pseudo-diffusion is the impossibility of defining a positive diffusion coefficient. This leads to a significant implication: the quantum evolution within a spatially distributed system may exhibit local entropy reduction over certain spatial domains and reversible deterministic evolution with recurrence time and overall null entropy variation [4].

In this study, the author investigates the quantum imaginary time stochastic process in scenarios where real-time random fluctuations are concurrently in play, leading to irreversible quantum evolution and possibly to decoherence and classical behavior in large-scale systems. The goal of this work is accomplished by extending the Madelung hydrodynamic description of quantum mechanics to its stochastic counterpart.

The primary topics are as follows:

i. Elucidating the formulation of the stochastic quantum hydrodynamic equation of motion as a consequence of fluctuations in spacetime curvature originating from the noisy background of gravitational waves (dark energy).

ii. Exploring the path integral approach to comprehend quantum stochastic dynamics, which describes the progression of quantum superposition states and potentially their transition to stable configurations.

iii. Describing stationary quantum state configurations in the presence of noise and establishing their correspondence with standard quantum mechanical states.

iv. Defining the circumstances under which the zero noise "deterministic" case aligns with conventional quantum mechanics.

v. Identifying the criteria that lead to the emergence of classical behavior within extensive-scale systems.

vi. Generalizing the uncertainty principles within the context of fluctuating quantum systems.

vii. Investigating quantum entanglement, wave function decay, and the measurement process.

viii. Comparing the measurement process as defined by the stochastic quantum hydrodynamic model with the perspectives of decoherence theory and the Copenhagen interpretation of quantum mechanics.

ix. Examine experiments involving entangled photons within the context of finite time decay kinetics of wave functions.

x. Extension of quantum coherence to macroscopic distances to build a system with a large number of Q-bits

2. The Quantum Potential Fluctuations Induced by the Background of Stochastic Gravitational Waves

With the emergence of evidence for dark energy in the form of a gravitational background, whether of relic origin or due to the dynamics of general relativity bodies, the quantum hydrodynamic representation offers the possibility of describing the dynamic fluctuations in a system without the need to introduce an external environment. This approach does not introduce divergent results that contradict established theories such as decoherence and the Copenhagen foundation of quantum mechanics; instead, it enhances and complements our understanding of these theories. The noise arising from spacetime curvature ripples can be incorporated into Madelung's quantum hydrodynamic description by leveraging the foundational assumption of relativity, which allows us to consider the energy associated with a spacetime curvature as virtual mass.

The Madelung quantum hydrodynamic representation transforms the Schrodinger equation [5–7]

$$-i\hbar\partial_t\psi = \left(\frac{\hbar^2}{2m}\partial_i\partial_i - V_{(q)}\right)\psi \tag{1}$$

for the complex wave function $\psi = |\psi|e^{-\frac{iS}{\hbar}}$ into two equations of real variables: the conservation of the mass density $|\psi|^2$

$$\partial_t|\psi|^2 + \partial_i(|\psi|^2\dot{q}_i) = 0 \tag{2}$$

and the motion equation for the momentum $m\dot{q}_i = p_i = \partial_i S$,

$$\ddot{q}_j = -\frac{1}{m}\left(\partial_j V_{(q)} + \partial_j V_{qu_{(n)}}\right) \tag{3}$$

where $S = -\frac{\hbar}{2} ln \frac{\psi}{\psi*}$ and

$$V_{qu} = -\frac{\hbar^2}{2m} \frac{1}{|\psi|} \partial_i \partial_i |\psi|. \tag{4}$$

To introduce the fluctuating virtual mass due to the spacetime background, we assume the following:

1. The virtual mass density fluctuations own the wave function ψ_{vac} and density $|\psi_{vac}|^2$
2. The associated energy density E (of the spacetime background fluctuations) is proportional to $|\psi_{vac}|^2$
3. The virtual mass m_{vac} is defined by the identity $E = m_{vac}c^2|\psi_{vac}|^2$;
4. The virtual mass does not interact with the mass of the physical system (since the gravity is sufficiently weak to be ignored).

Given that, the overall wave function ψ_{tot} reads as

$$\psi_{tot} \cong \psi\psi_{vac} \tag{5}$$

Moreover, since the virtual mass m_{vac} is much smaller than m of the system (i.e., $m_{tot} = m_{vac} + m \cong m$), it follows that (4) reads as

$$
\begin{aligned}
V_{qu_{(n_{tot})}} &= -\frac{\hbar^2}{2m_{tot}} |\psi|^{-1} |\psi_{vac}|^{-1} \partial_i \partial_i |\psi| |\psi_{vac}| \\
&= -\frac{\hbar^2}{2m} \left(|\psi|^{-1} \partial_i \partial_i |\psi| + |\psi|^{-1} |\psi_{vac}|^{-1} \partial_i |\psi_{vac}| \partial_i |\psi| + |\psi_{vac}|^{-1} \partial_i \partial_i |\psi_{vac}| \right)
\end{aligned}
\tag{6}
$$

Furthermore, given the fluctuating mass density of wavelength λ

$$|\psi_{vac}|^2{}_{(\lambda)} \propto cos^2 \frac{2\pi}{\lambda} q \tag{7}$$

related to the wave function of the mass fluctuation

$$\psi_{vac(\lambda)} \propto \pm cos \frac{2\pi}{\lambda} q, \tag{8}$$

we have that the energy fluctuations due to the quantum potential read as

$$\delta \overline{E}_{qu} = \int_V n_{tot(q,t)} \delta V_{qu(q,t)} dV. \tag{9}$$

Following the procedure described in reference [8], the energy fluctuation reads as

$$\delta \overline{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m} \left(\frac{2\pi}{\lambda} \right)^2 \tag{10}$$

and, in 3D space, leads to

$$\delta \overline{E}_{qu(\lambda)} \cong \frac{\hbar^2}{2m} \sum_i (k_i)^2 = \frac{\hbar^2}{2m} |k|^2 \tag{11}$$

The outcome indicated by Equation (11) demonstrates that the energy arising from fluctuations in the mass density of the vacuum rises in inverse proportion to the square of its wave-length λ. Consequently, fluctuations in the quantum potential with extremely short wavelengths (i.e., $\lambda \to 0$) may result in disproportionately large energy, even when the noise amplitudes approach zero (i.e., $T \to 0$). This observation raises concerns about the attainment of the deterministic zero noise limit (2)–(4) that represents quantum mechanics.

Conversely, the assurance of convergence to the deterministic limit (2)–(4) in quantum mechanics for $T \to 0$ stems from the improbability of uncorrelated fluctuations occurring in increasingly shorter distances. Consequently, the necessity for convergence to conventional

quantum mechanics for $T \to 0$ imposes an additional condition on the spatial correlation function of the noise, specifically as $\lambda \to 0$.

The derivation of the shape of the space correlation of noise $G(\lambda)$ is a heavy stochastic calculation [9]. A simpler and more straightforward way to calculate $G(\lambda)$ is obtained by considering the spectrum $S(k)$ of the noise that reads as [8]

$$S_{(k)} \propto p_{\left(\frac{2\pi}{\lambda}\right)} = exp\left[-\left(\frac{k\lambda_c}{2}\right)^2\right]. \tag{12}$$

showing a not-white characteristic, with wavelength λ smaller than λ_c that goes fast to zero. From (12), $G(\lambda)$ reads as

$$G_{(\lambda)} \propto \int\limits_{-\infty}^{+\infty} e^{ik\lambda} S_{(k)} dk \propto \frac{\pi^{1/2}}{\lambda_c} exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right]. \tag{13}$$

where

$$\lambda_c = \sqrt{2}\frac{\hbar}{(mkT)^{1/2}} \tag{14}$$

is the De Broglie length.

Equation (13) reveals that uncorrelated mass density fluctuations on increasingly shorter distances are gradually dampened by the quantum potential. This damping effect facilitates the achievement of conventional "deterministic" quantum mechanics for systems with physical lengths much smaller than the de Broglie wavelength λ_c.

In a sufficiently general case that holds practical significance, where the mass density noise correlation function can be reasonably assumed to be Gaussian, featuring a null correlation time, isotropy throughout space, and independence among different coordinates, it can be postulated in the following form:

$$< \delta n_{(q_\alpha,t)}, \delta n_{(q_\beta+\lambda,t+\tau)} > = < \delta n_{(q_\alpha)}, \delta n_{(q_\beta)} >_{(T)} G(\lambda)\delta(\tau)\delta_{\alpha\beta}, \tag{15}$$

where in systems of physical length $\mathcal{L}$ such as $\frac{\mathcal{L}}{\lambda_c} \ll 1$, $G_{(\lambda)}$ read as

$$lim_{T\to 0}G_{(\lambda)} \propto \frac{1}{\lambda_c} exp\left[-\left(\frac{\lambda}{\lambda_c}\right)^2\right] \cong \frac{1}{\lambda_c}\left(1 - \left(\frac{\lambda}{\lambda_c}\right)^2\right) \cong \frac{1}{\lambda_c} = \frac{1}{\hbar}\sqrt{\frac{mkT}{2}} \tag{16}$$

On this ansatz, Equation (3) assumes the stochastic form [10] (see Appendix B)

$$\ddot{q}_{j_{(t)}} = -\frac{1}{m}\partial_j\left(V_{(q)} + V_{qu_{(\rho)}}\right) - \kappa\dot{q}_{j_{(t)}} + \kappa D^{1/2}\xi_{(t)}. \tag{17}$$

where [8,10]

$$D^{1/2} = \left(\frac{\mathcal{L}}{\lambda_c}\right)\left(\gamma_D\frac{\hbar}{2m}\right)^{1/2} = \sqrt{\gamma_D}\frac{\mathcal{L}}{2}\sqrt{\frac{kT}{2\hbar}} \tag{18}$$

where γ_D is a pure not-zero number [10].

It is worth noting that the probability mass density (PMD) function ρ, defined by the Smolukowski conservation equation stemming from (17), obeys the condition $lim_{T\to 0}\rho = |\psi|^2$, since by (13) and (14), convergence to quantum mechanics is warranted.

The gravitational dark energy introduces the concept of a self-fluctuating system where the noise is an intrinsic property that does not require the presence of an environment.

3. The Schrodinger-Langevin Equation from Stochastic Madelung Quantum Hydrodynamics

Langevin equations, commonly utilized for depicting underlying dynamics, can be obtained through various methods, including Poisson transformation [11] and Fock space

formalism [12]. Exact definitions occasionally can be obtained for nonlinear chemical kinetics and a few other issues. Alternatively, one can approach a Langevin equation from a phenomenological standpoint, where the approximated dynamics are determined beforehand. In this context, achieving a rigorous Langevin description can prove to be quite challenging.

In our current study, we have the advantage of employing an exact microscopic quantum mechanical model.

Generally, assuming for the stochastic case, the complex field is

$$\psi_{(q,t)} = \rho_{(q,t)}^{1/2} e^{i\frac{S_{(q,t)}}{\hbar}}, \tag{19}$$

where ρ obeys the Smolukowski conservation equation [8,13] that, near the deterministic limit of quantum mechanics for $lim\frac{\mathcal{L}}{\lambda_c} \to 0$ with $\rho_{(q,t)}^{1/2} \to |\psi_{(q,t)}|$, reads as

$$\partial_t \rho + \partial_i \rho \dot{q}_i + Q_{diss_{(q,t)}} \cong \partial_t |\psi_{(q,t)}|^2 + \partial_i |\psi_{(q,t)}|^2 \dot{q}_i + Q_{diss_{(q,t)}} = 0 \tag{20}$$

where the diffusional dissipation $Q_{diss_{(q,t)}}$ is detailed in ref. [8]. Moreover, for $lim\frac{\mathcal{L}}{\lambda_c} \to 0$ of quantum mechanics, it follows that the quantum hydrodynamic Equations (2)–(4) read as

$$\begin{aligned}\ddot{q}_i &= \frac{1}{m}\frac{d}{dt}\partial_i S = \partial_t \partial_i S + \frac{1}{m}(\partial_i \partial_j S)\partial_j S = \partial_i\left(\partial_t S + \frac{1}{m}\partial_j S \partial_j S\right) \\ &= -\frac{1}{m}\partial_i\left(V_{(q)} - \frac{\hbar^2}{2m}\frac{1}{|\psi|}\partial_j\partial_j|\psi| + \kappa S - q_j^{1/2}q_j^{1/2}\,m\kappa D^{1/2}\xi_{(t)}\right)\end{aligned} \tag{21}$$

leading to

$$m\partial_t S + \partial_i S \partial_i S = -\begin{pmatrix} -\frac{\hbar^2}{2m}\frac{1}{|\psi|}\partial_j\partial_j|\psi| + V_{(q)} \\ +\kappa S - q_j^{1/2}q_j^{1/2}m\kappa D^{1/2}\xi_{(t)} + C_{(t)} \end{pmatrix} \tag{22}$$

that, with the help of (20) rearranged in the form

$$\partial_t|\psi| = -\frac{1}{m}\partial_i|\psi|\partial_i S - \frac{1}{2m}|\psi|\partial_i\partial_i S + \frac{Q_{diss_{(q,t)}}}{2|\psi|}. \tag{23}$$

leads to the quantum irreversible Langevin-like equation that, for time-independent systems, reads as

$$-i\hbar\partial_t|\psi| = \frac{\hbar^2}{2m}\partial_i\partial_i\psi - \left(V_{(q)} + Const + \kappa S - q_i^{1/2}q_i^{1/2}m\kappa D^{1/2}\xi_{(t)} + i\frac{Q_{diss_{(q,t)}}}{2|\psi|^2}\right)\psi \tag{24}$$

Although, within the frame of the quantum hydrodynamic formalism, quantum mechanics serves as the deterministic limit of the theory with no dissipation, it is interesting to analyze a case when, approaching the zero-noise limit, dissipation is still important.

Close to the zero noise condition, it is possible to characterize the ability of the system to dissipate by the semiempirical parameter α defined by the relation [10] $lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\kappa \cong lim_{T\to 0}\alpha\frac{2kT}{mD}$.

On this ansatz, (24) reads as

$$-i\hbar\partial_t|\psi| = \frac{\hbar^2}{2m}\partial_i\partial_i\psi - \left(V_{(q)} + \alpha\frac{2kT}{mD}S - q_i^{1/2}q_i^{1/2}m\kappa D^{1/2}\xi_{(t)} + i\frac{Q_{diss_{(q,t)}}}{2|\psi|^2}\right)\psi \tag{25}$$

from which we can see that the realization of quantum mechanics is warranted (see Equation (17)) by the condition $lim_{T\to 0}\alpha = 0$. In this case, it can be readily seen that Equation (24) reduces to the Schrodinger equation.

When physical dissipation is significant at the quantum level, the parameter α remains quite high, close to the quantum limit, such as

$$lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} \alpha = \alpha_0. \tag{26}$$

In this case, it is straightforward to see that Equation (24) converges to quantum Brownian motion. In fact, under condition (26) and by utilizing dimensional considerations, the following identities apply:

$$lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} D = lim_{T \to 0} \gamma_D \left(\frac{\mathcal{L}}{\lambda_c}\right)^2 \frac{\hbar}{2m} = lim_{T \to 0} \gamma_D \mathcal{L}^2 \frac{kT}{4\hbar} = 0 \tag{27}$$

$$lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} \kappa \cong lim_{T \to 0} \alpha \frac{2kT}{mD} = \alpha_0 \frac{8\hbar}{m\gamma_D \mathcal{L}^2} = finite, \tag{28}$$

$$lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} Q_{diss_{(q,t)}} = 0. \tag{29}$$

Thus, by (28) and (29) being $|\frac{Q_{diss_{(q,t)}}}{2|\psi|^2}| << |\kappa S|$, the term $i\frac{Q_{diss_{(q,t)}}}{|\psi|^2}$ can be disregarded in (24), and for $\frac{\mathcal{L}}{\lambda_c} \ll 1$, it follows the conventional Langevin equation for quantum Brownian motion

$$lim_{T \to 0} i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_i \partial_i \psi + \left(V_{(q)} + \kappa S - q_i^{1/2} q_i^{1/2} m\kappa D^{1/2} \xi_{(t)} + C\right)\psi \tag{30}$$

The emergence of the Schrödinger-Langevin equation through the stochastic extension of the quantum hydrodynamic model is noteworthy, showcasing a precise alignment with traditional quantum mechanics.

4. The Quantum Path Integral Motion Equation in the Presence of Stochastic Noise

The Markov process (17) can be described by the Smolukowski equation for the Markov probability transition function (PTF) [13]

$$P_{(q,q_0|t+\tau,t_0)} = \int_{-\infty}^{\infty} P_{(q,z|\tau,t)} \, P_{(z,q_0|t-t_0,t_0)} \, d^r z. \tag{31}$$

where the PTF $P_{(q,z|\tau,t)}$ is the probability that a quantity of probability mass density (PMD) $\rho_{(z,t)}$ in time interval τ is transferred to point q [13].

The conservation of the PMD shows that the PTF displaces the PMD according to the rule [13]

$$\rho(q, t) = \int P(q, z|t, 0)\rho_{(z,0)} d^r z \tag{32}$$

Generally, for the quantum case, Equation (31) cannot be reduced to a Fokker–Planck equation (FPE). The functional dependence of $V_{qu_{(\rho)}}$ by $\rho_{(q,t)}$ and by the PTF $P(q, q_0|t+\tau, t_0)$ produces non-Gaussian terms (see Appendix C).

Nonetheless, if, at initial time, $\rho_{(q,t_0)}$ is stationary (e.g., quantum eigenstate) and close to the long-time final stationary distribution ρ_{eq}, it is possible to assume that the quantum potential is constant in time as a Hamilton potential following the approximation

$$V_{qu} = -(\frac{\hbar^2}{4m})\left(\partial_q \partial_q \left(ln\rho_{eq(q)}\right) + \frac{1}{2}\left(\partial_q \left(ln\rho_{eq(q)}\right)\right)^2\right). \tag{33}$$

The quantum potential being independent by the wave function, the PMD ρ quasi-stationary long-time solutions can be approximately described by the Fokker–Planck equation

$$\partial_t P(q, z|t, 0) + \partial_i P(q, z|t, 0) \cdot v_i = 0 \tag{34}$$

where

$$v = \frac{1}{m\kappa} \partial_q \left(\frac{\hbar^2}{4m} \left(\partial_q \partial_q ln\rho_{eq} - \frac{1}{2} (\partial_q ln\rho_{eq})^2 \right) + V_{(q)} \right) - \frac{D}{2} \partial_q ln\rho_{eq} \tag{35}$$

leading to the final equilibrium of the stationary quantum configuration

$$\frac{1}{m\kappa} \partial_q \left(V_{(q)} - \frac{\hbar^2}{4m} \left(\partial_q \partial_q \left(ln\rho_{eq(q)} \right) + \frac{1}{2} \left(\partial_q \left(ln\rho_{eq(q)} \right) \right)^2 \right) \right) + \frac{D}{2} \partial_q \left(ln\rho_{eq} \right) = 0 \tag{36}$$

In Appendix D, the stationary states of a harmonic oscillator obeying (36) are shown. The results show that the quantum eigenstates are stable and maintain their shape (with a small change in their variance) when subject to fluctuations.

4.1. Evolution of Quantum Superposition of States Submitted to Stochastic Noise

The quantum evolution of not-stationary state superpositions (not considering fast kinetics and jumps) involves the integration of Equation (17) that, removing fast variables, reads as

$$\dot{q} = -\frac{1}{\kappa m} \partial_q \left(V_{(q)} - \frac{\hbar^2}{4m} \left(\partial_q (\partial_q ln\rho) + \frac{1}{2} (\partial_q ln\rho)^2 \right) \right) + D^{1/2} \xi_{(t)} \tag{37}$$

By utilizing both the Smolukowski Equation (37) and the associated conservation Equation (32) for the PMD ρ, it is possible to integrate (37) by using its second-order discrete expansion

$$q_{k+1} \cong q_k - \frac{1}{m\kappa} \partial_k \left(V_{(q_k)} + V_{qu(\rho_{q_k}, t_k)} \right) \Delta t_k - \frac{1}{m\kappa} \frac{d}{dt} \partial_k \left(V_{(q_k)} + V_{qu(\rho_{q_k}, t_k)} \right) \frac{\Delta t_k^2}{2} + D^{1/2} \Delta \mathcal{W}_k \tag{38}$$

where

$$q_k = q_{(t_k)} \tag{39}$$

$$\Delta t_k = t_{k+1} - t_k \tag{40}$$

$$\Delta \mathcal{W}_k = \mathcal{W}_{(t_{k+1})} - \mathcal{W}_{(t_k)} \tag{41}$$

where $\Delta \mathcal{W}_k$ has a Gaussian zero mean and unitary variance which probability function $\mathcal{P}(\Delta \mathcal{W}_k, \Delta t)$, for $\Delta t_k = \Delta t \; \forall \; k$, reads as

$$
\begin{aligned}
lim_{\Delta t \to 0} \mathcal{P}_{(\Delta W_k, \Delta t)} &= lim_{\Delta t \to 0} \frac{D^{-1/2}}{(4\pi\Delta t)^{1/2}} exp - \frac{\Delta \mathcal{W}_k^2}{4\Delta t} \\
&= lim_{\Delta t \to 0} \frac{D^{-1/2}}{(4\pi\Delta t)^{1/2}} exp - \frac{1}{4\Delta t} \frac{(q_{k+1} - <q_{k+1}>)^2}{D} \\
&= (4\pi D \Delta t)^{-1/2} exp - \frac{1}{4\Delta t} \frac{\left(q_{k+1} - q_k - <\dot{\bar{q}}_k> \Delta t - \frac{<\ddot{\bar{q}}_k>}{2} \Delta t^2 \right)^2}{D}
\end{aligned}
\tag{42}
$$

where the midpoint approximation has been introduced

$$\bar{q}_k = \frac{q_{k+1} + q_k}{2}, \tag{43}$$

and where

$$<\dot{\bar{q}}_k> = -\frac{1}{m\kappa} \frac{\partial \left(V_{(\bar{q}_k)} + V_{qu(\rho_{\bar{q}_k}, t_k)} \right)}{\partial \bar{q}_k}. \tag{44}$$

and

$$< \ddot{\bar{q}}_k > = -\frac{1}{2m\kappa}\frac{d}{dt}\frac{\partial\left(V_{(\bar{q}_k)} + V_{qu(\rho_{(\bar{q}_k)},t_k)}\right)}{\partial\bar{q}_k} \tag{45}$$

are the solutions of the deterministic problem

$$< q_{k+1} > \cong < q_k > - \frac{1}{m\kappa}\partial_k\left(V_{(q_k)} + V_{qu(\rho_{q_k},t_k)}\right)\Delta t_k - \frac{1}{m\kappa}\frac{d}{dt}\partial_k\left(V_{(q_k)} + V_{qu(\rho_{q_k},t_k)}\right)\frac{\Delta t_k^2}{2}. \tag{46}$$

As shown in ref. [8], the PTF $\mathcal{P}_{(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)}$ can be achieved after successive "u" steps of approximation and reads as

$$\mathcal{P}_{(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)} = lim_{u\to\infty}\mathcal{P}^{(u)}_{(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)}$$
$$\cong (4\pi D\Delta t)^{-1/2}e^{\frac{\Delta t}{4D}[-(\dot{q}_{k-1} - \frac{<\dot{q}_k>^{(\infty)}+<\dot{q}_{k-1}>}{2})^2 + D(\partial_{q_k}<\dot{q}_k>^{(\infty)}+\partial_{q_{k-1}}<\dot{q}_{k-1}>)]} . \tag{47}$$

and the PMD at the k-th instant reads as

$$\rho^{(\infty)}_{(q_k,k\Delta t)} = \int_{-\infty}^{\infty}\mathcal{P}^{(\infty)}_{(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)}\,\rho_{(q_{k-1},(k-1)\Delta t)}dq_{k-1}. \tag{48}$$

leading to the velocity field

$$< \dot{q}_k >^{(\infty)} = -\frac{1}{m\kappa}\partial_{q_k}\left(V_{(q_k)} - \frac{\hbar^2}{4m}\left(\partial_q\partial_q\left(ln\rho^{(\infty)}\right) + \frac{1}{2}\left(\partial_q\left(ln\rho^{(\infty)}\right)\right)^2\right)\right) \tag{49}$$

Moreover, the continuous limit of the PTF gives

$$P_{(q,q_0|t-t_0,0)} = lim_{\Delta t\to 0}\mathcal{P}^{(\infty)}_{(q_n,q_0|n\Delta t,0)}$$
$$= lim_{\Delta t\to 0}\int_{-\infty}^{\infty}\prod_{k=1}^{n}dq_{k-1}\mathcal{P}^{(\infty)}_{(q_k,q_{k-1}|\Delta t,(k-1)\Delta t)}$$
$$= \int_{q_0}^{q}Dqe^{[-\frac{1}{2D}\sum_{k=1}^{n}<\dot{\bar{q}}_{k-1}>^{(\infty)}\Delta q_k]}e^{-\frac{\Delta t}{4D}[\sum_{k=1}^{n}\left(\frac{q_k-q_{k-1}}{\Delta t}\right)^2 - 2D\frac{\partial<\dot{\bar{q}}_{k-1}>^{(\infty)}}{\partial\bar{q}_{k-1}}+<\dot{\bar{q}}_{k-1}>^{(\infty)2}]} \tag{50}$$
$$= \left(e^{\int_{q_0}^{q}\frac{1}{2D}<\dot{q}>dq}\right)\int_{q_0}^{q}Dqe^{-\frac{1}{4D}\int_{t_0}^{t}dt(\dot{q}^2+<\dot{q}>^2+2D\partial_q<\dot{q}>)}$$

where $< \dot{\bar{q}}_{k-1} >^{(\infty)} = \frac{1}{2}\left(< \dot{q}_k >^{(\infty)} + < \dot{q}_{k-1} >^{(\infty)}\right)$.

The resolution of the recursive Expression (50) offers the advantage of being applicable to nonlinear systems that are challenging to handle using conventional approaches [14–17].

4.2. General Features of Relaxation of the Quantum Superposition of States

The classical Brownian process admits the stationary long-time solution

$$P_{(q,q_{-\infty}|t-t_{-\infty},t_{-\infty})} = lim_{t_0\to-\infty}Ne^{\frac{1}{D}\int_{q_{-\infty}}^{q}<\dot{q}>_{(q'_{(t,t_0)})}dq'} = Ne^{\frac{1}{D}\int_{q_{-\infty}}^{q}K(q')dq'} \tag{51}$$

where $K(q) = -\frac{1}{m\kappa}\frac{\partial V_{(q)}}{\partial q}$, leading to solution [3]

$$P(q,q_0|t-t_0,t_0) = \left(exp\int_{q_0}^{q}\frac{1}{2D}K(q')dq'\right)\int_{q_0}^{q}Dqexp-\frac{1}{4D}\int_{t_0}^{t}dt\left(\dot{q}^2 + K^2(q) + 2D\partial_q K(q)\right) \tag{52}$$

As far as it concerns $< \dot{q} >^{(\infty)}{}_{(q,t)}$ in (50), it cannot be expressed in a closed form, unlike (51), because it is contingent on the particular relaxation path $\rho_{(q,t)}$ the system follows toward the steady state. This path is significantly influenced by the initial conditions $\rho_{(q,t_0)}$, $< \dot{q} >_{(q,t_0)}$ and, consequently, the initial time t_0 at which the quantum superposition of states is subjected to fluctuations.

In addition, from (38), we can see that q_{t_k} depends on the exact sequence of inputs of stochastic noise, since, in classically chaotic systems, very small differences can lead to relevant divergences of the trajectories in a short time. Therefore, in principle, different stationary configurations $\rho_{(q,t=\infty)}$ (i.e., eigenstates) can be reached whenever starting from identical superposition of states. Therefore, in classically chaotic systems, Born's rule can also be applied to the measurement of a single quantum state.

Even if $\mathcal{L} \ll \lambda_c \cup \lambda_{qu}$, it is worth noting that, to have finite quantum lengths λ_c and λ_{qu} (necessary to have the quantum stochastic dynamics) and the quantum decoupled (classical) environment or measuring apparatus), the nonlinearity of the overall system (system–environment) is necessary: Quantum decoherence, leading to the decay of superposition states, is significantly promoted by the widespread classical chaotic behavior observed in real systems.

On the other hand, a perfect linear universal system would maintain quantum correlations on a global scale and would never allow quantum decoupling between the system and the experimental apparatus performing the measure (see Section 5). It should be noted that even the quantum decoupling of the system from the environment would be impossible, as quantum systems function as a unified whole. Merely assuming the existence of separate systems and environments subtly introduces a classical condition into the nature of the overall supersystem.

Furthermore, given that the relationship (A31) and (A38) are valid only in the leading order of approximation of $\dot{q}$ (i.e., during a slow relaxation process with small amplitude fluctuations), in instances of large fluctuations occurring on a timescale much longer than the relaxation period of $\rho_{(q,t)(q,t)}$, transitions may occur that are not captured by (50), potentially leading from a stationary eigenstate to a general superposition of states.

In this case, relaxation will follow again toward another stationary state. The PMD $\rho_{(q,t)}$ (48) describes the relaxation process occurring in the time interval between two large fluctuations rather than the system evolution toward a statistical mixture. Due to the extended timescales associated with these jumping processes, a system comprising a significant number of particles (or independent subsystems) undergoes a gradual relaxation towards a statistical mixture. The statistical distribution of this mixture is dictated by the temperature-dependent behavior of the diffusion coefficient.

5. Emerging Classical Mechanics on Large Size Systems

If the quantum potential is manually eliminated from the quantum hydrodynamic equations of Motion (1)–(3), the classical equation of the motion is revealed [7]. Despite the apparent validity of this approach, it is not mathematically sound, as it alters the fundamental characteristics of the quantum hydrodynamic equations. Specifically, this operation eliminates the stationary configurations (i.e., eigenstates), since it nullifies the balancing force of the quantum potential against the Hamiltonian force [18], crucial for establishing the stationary condition. Consequently, even a small quantum potential is important and cannot be disregarded in the deterministic quantum hydrodynamic model (2)–(4).

Conversely, in the stochastic generalization, it is permissible to accurately disregard the quantum potential in (17) and (37) when its force is significantly inferior to the force noise ϖ, such as $|\frac{1}{m}\partial_i V_{qu(\rho)}| \ll |\varpi_{(T)}|$, that, by using (17), leads to

$$\left|\frac{1}{m}\partial_i V_{qu(\rho)}\right| \ll \kappa \left(\frac{\mathcal{L}}{\lambda_c}\right) \sqrt{\gamma_D \frac{\hbar}{2m}} = \kappa \left(\frac{\mathcal{L}\sqrt{mkT}}{2\hbar}\right) \sqrt{\gamma_D \frac{\hbar}{2m}}, \tag{53}$$

and hence, in a coarse-grained description with an elemental cell side Δq, to

$$\lim_{q \to \Delta q} \left| \partial_i V_{qu(\rho)} \right| \ll m\kappa \left(\frac{\mathcal{L}}{\lambda_c} \right) \sqrt{\gamma_D \frac{\hbar}{2m}} = m\kappa \sqrt{\gamma_D} \frac{\mathcal{L}}{2} \sqrt{\frac{kT}{2\hbar}}, \tag{54}$$

where $\mathcal{L}$ is the physical system length.

In addition, even if the noise $\wp_{(q,t,T)}$ has a zero mean, the mean of the quantum potential fluctuations $\overline{V}_{st(n,S)} \cong \kappa S$ is not null, resulting in the emergence of the dissipative force $-\kappa \dot{q}_{(t)}$ in (17). Consequently, the stochastic sequence of noise inputs disrupts the coherent evolution of the quantum superposition of states. Additionally, upon closer examination of the stochastic noise,

$$\kappa \left(\frac{\mathcal{L}}{\lambda_c} \right) \sqrt{\gamma_D \frac{\hbar}{2m}} \xi_{(t)} \tag{55}$$

we observe that is proportional to the size of the system; for $\frac{\mathcal{L}}{\lambda_c} \to \infty$ in macroscopic systems, (53) is satisfied if

$$\lim_{\frac{q}{\lambda_c} \to \frac{\mathcal{L}}{\lambda_c} = \infty} \left| \frac{1}{m} \partial_i V_{qu(n_{(q)})} \right| < \infty. \tag{56}$$

To attain a comprehensive large-scale description that is entirely free from quantum correlations, we can impose more stringent requirements.

$$\lim_{\frac{q}{\lambda_c} \to \infty} \left| \frac{1}{m} \partial_i V_{qu(\rho_{(q)})} \right| = \lim_{\frac{q}{\lambda_c} \to \infty} \frac{1}{m} \sqrt{\partial_i V_{qu(\rho_{(q)})} \partial_i V_{qu(\rho_{(q)})}} = 0. \tag{57}$$

Therefore, by noting that, in the case of linear systems,

$$\lim_{q \to \infty} V_{qu(q)} \propto q^2, \tag{58}$$

it follows that they cannot lead to the classical mechanics on a macroscopic scale.

In general, the greater the strength of the Hamiltonian potential, the more localized the wave function and the more pronounced the quantum potential behavior at infinity. This observation is easily substantiated by considering the MDD.

$$|\psi|^2 \propto e^{-P^k_{(q)}} \tag{59}$$

where $P^k_{(q)}$ is a polynomial of order k.

In order to establish a finite range of interaction for the quantum potential, it is imperative that $k < \frac{3}{2}$. Therefore, linear systems, with $k = 2$, have an infinite range of action of quantum potential.

A concrete illustration arises in the context of solids possessing a quantum lattice. When examining phenomena at intermolecular distances characterized by linear interactions, the quantum behavior becomes apparent (as seen in phenomena like X-ray diffraction). However, when investigating the macroscopic properties, a transition to classical behavior occurs. This shift is attributed to the Lennard-Jones potential, which goes to zero at infinity, resulting in a finite quantum range of interaction (see (60)–(62)). In such cases, classical behavior emerges, exemplified by low-frequency acoustic waves with wavelengths much larger than the linear range of interatomic distances. For instance, when particles interact through the Lennard-Jones potential, the long-distance wave function is given by [19]

$$\lim_{r \to \infty} |\psi| \propto a^{-1/2} \frac{1}{r}, \tag{60}$$

the quantum potential reads as

$$lim_{r\to\infty}V_{qu(\rho)} \cong lim_{q\to\infty}\frac{\hbar^2}{2m}\frac{1}{|\psi|}\partial_r\partial_r|\psi| = \frac{1}{r^2} = \frac{\hbar^2}{m}a|\psi|^2. \tag{61}$$

leading to the quantum force

$$lim_{r\to\infty} - \partial_r V_{qu(\rho)} = lim_{q\to\infty}\frac{\hbar^2}{2m}\partial_r\frac{1}{|\psi|}\partial_r\partial_r|\psi| = \frac{\hbar^2}{2m}\partial_r r\partial_r\partial_r\frac{1}{r} = -2\frac{\hbar^2}{m}\frac{1}{r^3} = 0, \tag{62}$$

So that, by (53) and (57), the macroscopic classical behavior may manifest [20,21] in a phase with adequate sparsity. It is noteworthy that, in Equation (61), the quantum potential underpins the hard sphere potential within the "pseudopotential Hamiltonian model" of the Gross–Pitaevskii equation [22,23], where $\frac{a}{4\pi}$ denotes the boson–boson s-wave scattering length.

By recognizing that, to meet Condition (57), we can appropriately demand that

$$\int_0^\infty r^{-1}|\frac{1}{m}\partial_i V_{qu(\rho_{(q)})}|_{(r,\theta,\varphi)}dr < \infty \quad \forall\theta, \varphi, \tag{63}$$

we can determine the distance λ_{qu} at which the quantum potential is effective [20,21] by the formula

$$\lambda_{qu} = \frac{\lambda_c\int_0^\infty r^{-1}|\partial_i V_{qu(\rho)}|_{(r,\theta,\varphi)}dr}{|\partial_i V_{qu(\rho)}|_{(r=\lambda_c,\theta,\varphi)}} = \lambda_c I_{qu}. \tag{64}$$

where I_{qu} reads

$$I_{qu} = \frac{\int_0^\infty r^{-1}|\partial_i V_{qu(\rho)}|_{(r,\theta,\varphi)}dr}{|\partial_i V_{qu(\rho)}|_{(r=\lambda_c,\theta,\varphi)}}. \tag{65}$$

5.1. Lindemann Constant for Quantum Lattice-to-Classical Fluid Transition

For a L-J gas and its condensed liquid phase, the distance of quantum potential interaction λ_{qu} reads as

$$\lambda_{qu} \cong r_0(1+\varepsilon)\left(1 + \frac{1}{3}\left(\frac{\lambda_c}{d}\right)^4\right) \tag{66}$$

where $r_0(1+\varepsilon)$ is the distance up to which the L-J force is linear ($\varepsilon = \frac{\Delta}{r_0}$) and where r_0 is the molecular equilibrium position.

An empirical validation of practical significance of the quantum potential interaction length is observed during the quantum-to-classical transition in a crystalline solid at melting point. During this transition, the system shifts from a quantum lattice state to a fluid, amorphous classical phase.

If we make the assumption that, within the quantum lattice, the atomic wave function (around the equilibrium distance) has a smaller extent than the quantum potential interaction distance, then it can be deduced that, at the melting point, its variance equals $\lambda_{qu} - r_0$.

Therefore, the Lindemann constant $L_C = \dfrac{\left\{\begin{array}{c}\text{wave function variance}\\ \text{at transition}\end{array}\right\}}{r_0}$ [24] reads as $L_C = \frac{\lambda_{qu}-r_0}{r_0}$, and it can be theoretically calculated, since

$$\lambda_{qu} \approx r_0\left((1+\varepsilon) + \frac{1}{3}\left(\frac{\lambda_c}{r_0(1+\varepsilon)}\right)^3\right) \approx r_0\left((1+\varepsilon) + \frac{1}{3}\left(\frac{\lambda_c}{r_0}\right)^3\right) \tag{67}$$

that, being typically $\varepsilon \approx 0.05 \div 0.1$ and $\frac{\lambda_c}{r_0} \approx 0.8$, leads to

$$L_C = \frac{\lambda_{qu} - r_0}{r_0} \approx 0.217 \div 0.267. \tag{68}$$

A more precise assessment, employing the potential well approximation for molecular interactions [20,21], results in $\lambda_{qu} \cong 1.2357\, r_0$ and yields a value for the Lindemann constant $L_C = 0.2357$ that aligns closely with the measured values, falling within the range of 0.2 to 0.25 [24].

5.2. Fluid–Superfluid ^{4}He Transition

Since the de Broglie distance λ_c is a function of temperature, its influence on the fluid–superfluid transition in monomolecular liquids at very low temperatures, such as for ^{4}He, can be detected. The treatment of this case is detailed in ref. [20,21], where, for the ^{4}He–^{4}He interaction, the potential well is assumed to be

$$V_{(r)} \to \infty \qquad\qquad 0 \le r \le \sigma. \tag{69}$$

$$V_{(r)} = -1.23 \times 10^{-22} \qquad J \;\; \sigma \le r \le \sigma + 2\Delta \tag{70}$$

$$V_{(r)} = 0 \qquad\qquad \sigma + 2\Delta < r. \tag{71}$$

where $\Delta = 1.54 \times 10^{-10}$ m and where $\sigma + \Delta = 3.7 \times 10^{-10}$ m is the mean ^{4}He – ^{4}He atomic distance.

By posing that, at the superfluid transition, the de Broglie length progressively reaches the ^{4}He – ^{4}He atomic distance, being in the length interval

$$\sigma < \lambda_c < \sigma + 2\Delta, \tag{72}$$

we have, for $\lambda_c < \sigma$, that the ratio of superfluid/normal ^{4}He density is approximately null, while, for $\lambda_c \approx \sigma + 2\Delta$, we have almost 100% superfluid ^{4}He. Therefore, at the condition

$$\lambda_c = \sqrt{2}\,\frac{\hbar}{(mkT_{\lambda_c})^{1/2}} = \sigma + \Delta, \tag{73}$$

when the superfluid/normal ^{4}He density ratio is 50%, it follows that the temperature $T_{50\%}$ for the ^{4}He mass of $m_{4\text{He}} = 6.6 \times 10^{-27} kg$ reads as

$$T_{50\%} = \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma + \Delta}\right)^2 = \frac{2 \times 1.113 \times 10^{-68}}{6.6 \times 10^{-27} \times 1.38 \times 10^{-23}}\left(\frac{1}{1.3 \times 10^{-19}}\right) = 1.92\,^\circ K \tag{74}$$

which agrees well with the experimental data in ref. [25] of approximately $1.95\,^\circ K$.

On the other hand, since, by (72) for $\lambda_c = \sigma + 2\Delta$, all the couples of ^{4}He fall into the quantum state, the superfluid ratio of 100% is reached at the temperature

$$T_{100\%} \approx \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma + 2\Delta}\right)^2 = 0.92\,^\circ K \tag{75}$$

well agreeing with the experimental data in ref. [25] of approximately $1.0\,^\circ K$.

Moreover, by utilizing the superfluid ratio of 38% at the λ-point of ^{4}He, the transition temperature T_λ reads as

$$T_\lambda \approx \frac{2\hbar^2}{mk}\left(\frac{1}{\sigma + 0,76\Delta}\right)^2 = 2.20\,^\circ K \tag{76}$$

in good agreement with the measured ^{4}He superfluid transition temperature of 2.17 $^\circ K$.

As a concluding observation, it is noteworthy to mention that there are two methods for establishing macroscopic quantum behavior. The first method entails reducing the temperature, thereby effectively augmenting the de Broglie length. The second method involves intensifying the strength of the Hamiltonian interaction among the particles within the system.

With regard to the latter, it is crucial to highlight that the restricted potency of the Hamiltonian interaction over extended distances serves as the pivotal factor enabling the manifestation of classical behavior. In the analysis of systems dictated by a quadratic or more potent Hamiltonian potential, the interaction range linked to the quantum potential becomes infinite, as depicted in Equation (96). Consequently, attaining a classical phase becomes unachievable, irrespective of the system's size being $\lambda_{qu} = \infty$.

In this specific scenario, we exclusively witness the full realization of classical behavior on a macroscopic scale in systems characterized by interactions that are sufficiently feeble, even weaker than linear interactions, which exhibit classical chaos. In such instances, the quantum potential is incapable of exerting its nonlocal influence over substantial distances.

Hence, classical mechanics arises as a decoherent result of quantum mechanics in the presence of fluctuating spacetime.

5.3. The Coarse-Grained Approach

The PMD current $J_j = \rho \, \dot{q}_j$ reads as

$$
\begin{aligned}
J_j = \rho \, \dot{q}_j = & -\rho\left(\tfrac{m\mathcal{L}^2}{4\alpha}\partial_j\left(V + V_{qu}\right) + \varpi\right) \\
= & -\rho\left(\tfrac{1}{m\kappa}\partial_j\left(V + V_{qu}\right) + D_{jk}^{1/2}\xi_{k(t)}\right)
\end{aligned}
\tag{77}
$$

The macroscopic behavior can be derived through the discrete coarse-grained spatial representation of Equation (77), utilizing local cells with a side length of l, where the mass density dx_j for the j-th cell is expressed as

$$
dx_j = -\frac{m\mathcal{L}^2}{4\alpha}\mathcal{D}'_{jm}x_{(m)}\left(\mathcal{D}_{mk}V_k + \mathcal{D}^{qu}_{mk}V_{qu_k}\right)dt + \mathcal{D}''_{jk}x_{(k)}\Phi_{(k)}dW_{k(t)}
\tag{78}
$$

where

$$
x_j = l^3\rho_{(q_j, t)},
\tag{79}
$$

$$
V_k = V_{(q_k)},
\tag{80}
$$

$$
V_{qu_k} = V_{qu\left(\rho_{(q_k)}\right)},
\tag{81}
$$

$$
\Phi_k = \Phi_{(q_k, t)},
\tag{82}
$$

where

$$
lim_{l\to 0}l^{-6}\overline{\Phi_j, \Phi_k} = \overline{\varpi_{(q_j)}, \varpi_{(q_k)}}_{(T)}G_{(l(k-j))}.
\tag{83}
$$

where $G_{(k-j)}$ is the correlation length of the noise, where the matrices $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, as well as $\mathcal{D}^{qu}$, correspond to the discrete derivatives $\frac{\partial}{\partial q_k}$ at the j-th point.

In general, the quantum potential interaction arising from the k-th cell is contingent upon the magnitude of the Hamiltonian potential $V_{(q)}$.

In a system comprising a large number of particles, configuring the side length $\updownarrow$ to match the mean intermolecular distance $\mathcal{L}$ allows the attainment of a classic phase. This condition is met when $\mathcal{L}$ significantly surpasses the quantum potential length of interaction

λ_{qu}, which, as per Equation (64), is also contingent upon the de Broglie length. Typically, the Lennard-Jones potential (62) yields

$$lim_{\frac{q}{\lambda_{qu}} \to \frac{\mathcal{L}}{\lambda_{qu}} = \infty} - \frac{\partial V_{qu(\rho)}}{\partial q} \approx 2 \frac{\hbar^2}{m \lambda_{qu}{}^3} \frac{1}{\left(\frac{q}{\lambda_{qu}}\right)^3} \approx \frac{kT}{\hbar} \sqrt{\frac{mkT}{2}} I_{qu} \frac{1}{\left(\frac{q}{\lambda_{qu}}\right)^3} = 0, \qquad (84)$$

This results in the nullification of the interaction of the quantum potential originating from the k-th cell with the adjacent cells, rendering $\mathcal{D}_{mk}^{qu}$ diagonal. Consequently, the quantum effects are restricted to each individual molecule cell domain.

In classical systems where $\mathcal{L} \gg \lambda_c \cup \lambda_{qu}$, the correlation length of the noise reads as $G_{(k-j)} \propto \delta_{kj}$, resulting in spatially uncorrelated fluctuations in macroscopic systems. Conversely, in systems with interactions stronger than linear with $\lambda_{qu} \to \infty$, where the quantum potential of each cell extends its influence to others, quantum characteristics manifest in the coarse-grained description. As detailed in Section 5.2, employing the quantum stochastic hydrodynamic model (SQHM) allows the derivation of descriptions for dense phases where quantum effects become apparent on the macroscopic scale.

5.4. Measurement Process and the Finite Distance of Quantum Entanglement

Throughout the process of measurement, consisting of a "deterministic" conventional quantum interaction between the sensing component of the experimental setup and the system being measured, the interaction ceases when the experimental apparatus is moved far away from the system, well beyond the distances λ_c and λ_{qu}.

The interpretation and handling of the "interaction output" are subsequently managed by the measuring apparatus, typically involving a classical, irreversible procedure characterized by a clear arrow of time. This procedure ultimately yields the macroscopic measurement result.

Nonetheless, the phenomenon of decoherence plays a pivotal role in the measurement process. It allows for the emergence of a large-scale classical framework that ensures genuine quantum isolation between the experimental setup and the system, both before and after the measurement event.

This quantum-isolated state, both initially and finally, is essential for determining the conclusion of the measurement and for gathering statistical data from a set of independent repeated measurements.

It is worth emphasizing that, within the framework of the SQHM, the standard condition of moving the measured system to an infinite distance before and after the measurement is not sufficient to guarantee the independence of the system and the measuring apparatus if either $\lambda_c = \infty$ or $\lambda_q = \infty$.

5.5. Minimum Measurement Uncertainty of Quantum Systems in Fluctuating Spacetime Backgrounds

Any quantum theory striving to elucidate the evolution of a physical system across various orders of magnitude in size must inherently provide an explanation for the transition from quantum mechanical properties to the emergent classical behavior observed at larger scales. The fundamental distinctions between the two descriptions lie in the minimum uncertainty principle of quantum mechanics and the finite speed of propagation of interactions and information in local classical relativistic mechanics.

If, at a specific distance $\mathcal{L}_q$, which is smaller than λ_c, a system completely adheres to the conventional principles of quantum mechanics, where its subparts lack distinct identities, then, for an independent observer to gather information about the system, he must be positioned at a distance at least as far as $\mathcal{L}_q$ from the observed system, both before and after the process (refer to Appendix E).

Hence, owing to the finite speed of propagation of interactions and information, the execution of the process cannot be completed within a time shorter than

$$\Delta\tau_{min} > \frac{\mathcal{L}_q}{c} \propto \frac{\lambda_c}{c} \propto \frac{2\hbar}{(2mc^2kT)^{1/2}}. \tag{85}$$

Furthermore, considering the Gaussian noise (18) with the diffusion coefficient proportional to kT, it follows that the mean value of the energy fluctuation is $\delta E_{(T)} = \frac{kT}{2}$ for degree of freedom. Therefore, a nonrelativistic ($mc^2 >> kT$) scalar structureless particle of mass m has an energy variance ΔE

$$\begin{aligned}\Delta E \approx \quad &(< (mc^2 + \delta E_{(T)})^2 - (mc^2)^2 >)^{1/2} \cong (< (mc^2)^2 + 2mc^2\frac{kT}{2} - (mc^2)^2 >)^{1/2}\\ &\cong (mc^2kT)^{1/2}\end{aligned} \tag{86}$$

from which it follows that

$$\Delta E \Delta t > \frac{(mc^2kT)^{1/2}\lambda_c}{c}) \propto \sqrt{2}\hbar, \tag{87}$$

It is worth noting that the product $\Delta E \Delta \tau$ remains constant, as the increase in energy variance precisely compensates for the corresponding decrease in the minimum acquisition time $\Delta \tau$. This holds true due to the growth following the square root of T. The same outcome is attained when establishing the uncertainty relations between the position and momentum of a particle with mass m.

Moreover, if, as the external (quantum disentangled) measuring system, we can acquire information about the spatial position of a particle with precision

$$\Delta L > \mathcal{L}_q. \tag{88}$$

the variance Δp of its momentum $(p^\mu p_\mu)^{1/2} = mc$ due to the fluctuations in the low-velocity limit $mc^2 >> kT/$ reads as

$$\Delta p \approx (< (mc + \frac{\delta E_{(T)}}{c})^2 - (mc)^2 >)^{1/2} \cong (mkT)^{1/2}. \tag{89}$$

leading to uncertainty relation

$$\Delta L \Delta p > \mathcal{L}_q (mkT)^{1/2} \propto \sqrt{2}\hbar \tag{90}$$

Moreover, by (90), the uncertainty condition

$$\Delta L \Delta p > \mathcal{L}_q (2mkT)^{1/2} = \frac{\hbar}{2}. \tag{91}$$

or

$$\Delta E \Delta t > \frac{(2mc^2kT)^{1/2}\mathcal{L}_q}{c} = \frac{\hbar}{2}. \tag{92}$$

requires that $\mathcal{L}_q = \frac{\lambda_c}{2\sqrt{2}}$ represents the physical length below which quantum entanglement is fully effective, and it denotes the minimum distance—both initial and final—between the system and the measuring apparatus. Regarding the theoretical minimum uncertainty of nonrelativistic quantum mechanics (i.e., $c \to \infty$), which is obtainable from the minimum uncertainty (91) and (92) in the limit of zero noise ($\lambda_c \to \infty$), it follows that

$$\Delta\tau_{min} = \frac{\lambda_c}{2c\sqrt{2}} \quad \text{undefined} \tag{93}$$

$$\Delta E \cong (mc^2 kT)^{1/2} = \sqrt{2}\frac{\hbar c}{\lambda_c} \quad \text{undefined,} \tag{94}$$

$$\mathcal{L}_q = \frac{\lambda_c}{2\sqrt{2}} \to \infty \tag{95}$$

$$\Delta p \cong (mkT)^{1/2} \cong \frac{\sqrt{2}\hbar}{\lambda_c} \to 0 \tag{96}$$

but the uncertainty relations

$$\Delta E \Delta t > \frac{\hbar}{2} \tag{97}$$

$$\Delta L \Delta p > \mathcal{L}_q (mkT)^{1/2} = \frac{\hbar}{2} \tag{98}$$

remain finite and constitute the minimum uncertainty of the quantum deterministic limit.

It is interesting to note that, in the relativistic limit, the minimum acquisition time of measure in the conventional quantum limit reads as

$$\Delta \tau_{min} = \frac{\mathcal{L}_q}{c} \to \infty. \tag{99}$$

The output (99) shows that it is not possible to carry out any measurement in the deterministic fully quantum mechanical global system, since its duration is infinite.

Moreover, by increasing the temperature we can localize the system in a domain of smaller physical length ΔL satisfying the choice (88). Nevertheless, the minimum uncertainty (98) in the context of quantum mechanics remains unchanged in the limit.

Given that nonlocality is restricted to domains with a physical length on the order of $\frac{\lambda_c}{2\sqrt{2}}$ and information about a quantum system cannot be transmitted faster than the speed of light (as violating the uncertainty principle), local realism is affirmed in the realm of coarse-grained macroscopic physics, where the domains of order λ_c^3 reduce to a point. Meanwhile, the paradox of "spooky action at a distance" is confined to microscopic distances (smaller than $\frac{\lambda_c}{2\sqrt{2}}$), where quantum mechanics in the low-velocity limit are fully assumed.

It is crucial to note that, in the low-velocity limit of quantum mechanics, the conditions $c \to \infty$ and $\lambda_c \to \infty$ are implicitly incorporated into the theory, resulting in the (apparent) instantaneous transmission of an interaction at an infinite distance.

It is also noteworthy that, in the presence of noise, the measurement indeterminacy experiences a relativistic correction, given that $\Delta E \cong \left(mc^2 kT\left(1 + \frac{kT}{4mc^2}\right)\right)^{1/2}$ leads to the minimum uncertainty in a quantum system, subjected to gravitational background noise ($T > 0$), which reads as

$$\Delta E \Delta t > \frac{\hbar}{2}\left(1 + \frac{kT}{4mc^2}\right)^{1/2} \tag{100}$$

and

$$\Delta L \Delta p > \frac{\hbar}{2}\left(1 + \frac{kT}{4mc^2}\right)^{1/2}. \tag{101}$$

The relativistic correction can become significant for light particles (with $m \to 0$), but in the realm of conventional quantum mechanics for $T = 0$, the uncertainty relations remain unchanged.

5.6. The Noisy Quantum Hydrodynamic Theoryl and the Decoherence Approach

In the framework of the SQHM, ensuring statistically reproducible measurement processes and guaranteeing the complete independence of the measuring apparatus from the measured system (free from quantum potential coupling both before and after the measurement) necessitates a global system with a finite length of quantum potential interaction.

In this scenario, the SQHM suggests that, given the finite speed of light and information transmission, it becomes feasible to conduct the measurement within a finite time interval. As a result, having the finite length of a quantum potential interaction, the subsequent decoherence become essential prerequisites for the execution of the measurement process.

The decoherence theory [26–31] does not aim to address the measurement problem or the collapse of the wave function. Its focus lies in elucidating the transition of a system to the statistical mixture of states, brought about by the leakage of quantum entanglement with the environment. Additionally, while the decoherence process may be prolonged for a microscopic system, the decoherence time τ_d for macroscopic systems composed of n microscopic quantum elements can be very short $\propto \frac{\tau_d}{n}$. However, in the context of decoherence theory, the superposition of states within the global universal wave function persists, maintaining its global coherence.

This puzzle finds a logical resolution through the solution proposed by Poincaré, recently extended to quantum systems [4]. The solution elucidates that irreversible phenomena can manifest in a globally reversible system due to an exceedingly long recurrence time (far surpassing the lifespan of the universe). The global quantum system locally emulates classical behavior, indistinguishable from that of a large-scale classical universe, as the recurrence time significantly exceeds the lifetime of the universe. For instance, Boltzmann's calculated recurrence time for a single cubic centimeter of gas to revert to its initial state spans many trillions of digits, dwarfing the thirteen-digit timespan of the universe.

In the context of Madelung's approach, the Wigner distribution and the quantum hydrodynamic theory are closely connected and do not contradict each other [7]. However, the interpretation of the global system as classical or quantum in nature is ultimately a matter of interpretation. Essentially, we cannot determine whether the noise from the environment is truly random or pseudorandom. In computer simulations, it is widely accepted that any algorithm generating noise will actually produce pseudorandom outputs, but this distinction is not critical when describing irreversible phenomena.

The theory of decoherence can explain macroscopic behavior as a result of dissipative quantum dynamics, but it cannot determine the conditions that allow for a truly classical global system.

In contrast, the SQHM approach offers a criterion for the shift from quantum dynamics to classical dynamics at a suitable macroscopic scale, eliminating the requirement for an external environment. Furthermore, the SQHM demonstrates the potential for a classical global system within a self-fluctuating spacetime, characterized by background curvature fluctuations aligning with the quantum gravitational portrayal of the universe. In this context, gravity is viewed as a source of global decoherence [32,33].

From a conceptual standpoint, the SQHM theory addresses the problematic issue of spontaneous entropy reduction in the global quantum reversible system, which is necessary for the system to return to its initial state, as required by the recurrence theorem. Furthermore, since quantum pseudo-diffusion evolution [34] shows that entropic and antientropic processes occur simultaneously in different regions of a quantum system, the question of why spontaneous antientropic processes are not observed anywhere and at any time in the universe remains unsolved in the context of decoherence theory.

In relation to this aspect, it is important to clarify that, when the environment size is considered infinite, the recurrence time is pushed beyond infinity. This can be easily demonstrated by examining what are known as free Gaussian coherent states. These states are obtained as the limit of eigenstates of harmonic oscillators when the quadratic coefficient of the potential approaches zero, which is equivalent to increasing the size of the system to infinity.

While we can observe the emergence of antientropic behavior in the harmonic Gaussian states after half of the oscillating period time has passed, this behavior is never observed in the coherent states, because their period of oscillation extends to infinity. Consequently, the expansion of the environment to infinity is a tricky mathematical proce-

dure that eliminates the unavoidable antientropic behavior of conventional (deterministic) quantum mechanics, which contradicts physical evidence.

5.7. The SQHM and the Copenhagen Foundations of Quantum Mechanics

The path integral solution of the SQHM (50) is specific and applies to the small noise limit before the occurrence of a significant fluctuation. It characterizes the "microscopic stage" of the decoherence process at the de Broglie physical length scale.

Furthermore, the SQHM characterizes quantum-to-classical transition through the utilization of two physical lengths, λ_c and λ_{qu}, treating quantum mechanics as the asymptotic behavior for $lim\lambda_c \to \infty$. As a result, it provides additional insights into the measurement process.

While the measurement process can be asymptotically described as a quantum interaction between the system and the measurement apparatus, marginal decoherence effects persist during its realization, attributed to

i.   need for real decoupling at the initial and final state of the measure between the system and the measuring apparatus and

ii.   utilization of classical experimental equipment for the collection and treatment of data.

The marginal decoherence is ignored or disregarded, because the classical equipment is mistakenly assumed to be decoupled at infinity, while the assumption of a perfect global quantum interaction (which extends itself at infinity, such as $lim\lambda_c \to \infty$ and $lim\lambda_{qu} \to \infty$) does not allow the realization of such conditions.

To describe the decoherence due to external apparatus V_{ext}, the SQHM reads as

$$\dot{p}_j = -\partial_j \left(V + V_{ext} + V_{qu_{(n)}} \right) - m\kappa\dot{q}_j + m\omega_j. \tag{102}$$

where

$$V_{ext} = 0 \text{ for } t < t_0 \cup t > t_0 + \Delta\tau \tag{103}$$

In principle, marginal decoherence, characterized by a time constant τ_d, could influence the measurement if τ_d is comparable to the measurement duration time $\Delta\tau$. The absence of marginal effects is considered in the analysis as a specific instance, namely, when the measurement is sufficiently fast, such as $\frac{\Delta\tau}{\tau_d} \to 0$.

From a broader perspective, the SQHM reveals that the steady state following relaxation is contingent upon the system's initial configuration:

$$|\psi_{(q,t)}> = \sum_n a_n e^{\frac{E_n(t-t_0)}{\hbar}} |\psi_{n(q)}> \tag{104}$$

at moment t_0, enabling the system to potentially attain any eigenstate within the superposition.

Considering that the quantum superposition of energy eigenstates exhibits a cyclic evolution, characterized by a recurrence time T, the probability of relaxation to the i-th energy eigenstate in the SQHM model can be expressed as follows:

$$P(\psi \to \psi_i) = lim_{N\to\infty} \frac{N_i}{N} \tag{105}$$

where N is the total number of time intervals $\Delta t = \frac{T}{N}$ centered around the instants t_{0_j} (with $0 \leq t_{0_j} < T$ and $j = 1, \ldots, N$) in which the system is initially submitted to fluctuations, and N_i is the number of times the *i*-th energy eigenstate is reached in the final steady state.

Moreover, since the eigenstates are stable and stationary, it also follows that the transition probability between the *k*-th and the *i*-th eigenstates reads as

$$P(\psi_k \to \psi_i) = lim_{N\to\infty} \frac{N_i}{N} = \delta_{ki} \tag{106}$$

Since the finite quantum lengths λ_c and λ_{qu}, allowing quantum decoupling between the system and the measuring apparatus, necessarily implies "marginal decoherence", it

follows that the output of the measure is produced in a finite time lapse (larger than $\Delta\tau_{min}$ of (85)) due to the wave function decay time.

Regarding the Copenhagen interpretation of quantum mechanics, the measurement is a process that triggers the collapse of the wave function, and the result (e.g., the energy value E_n for the state (104)) is delineated by the transition probability given by

$$\widetilde{P}(\psi \to \psi_i) = \frac{|a_i|^2}{\sum\limits_{j=1}^{j_{max}} |a_j|^2}, \tag{107}$$

That, for the i-th eigenstate, reads as

$$\widetilde{P}(\psi_k \to \psi_i) = \delta_{ki}, \tag{108}$$

To analyze the interconnection between the wavefunction decay and the wavefunction collapse, we assume, as a starting point, that they are different phenomena and have independent realizations.

In the first instance, we can assume that the wavefunction decay (with characteristic time τ_d) occurs first and then the wavefunction collapses (with characteristic time τ_c) during the measurement process. Without loss of generality, we can assume $\Delta\tau_{min} \le \tau_d \ll \tau_c \ll \Delta\tau$, and therefore, in this case, we obtain

$$\begin{aligned} lim_{\frac{\tau_d}{\Delta\tau} \to 0} P^{(tot)}{}_{(\psi \to \psi_n)} &= \sum_i P_{(\psi \to \psi_i)} \widetilde{P}_{(\psi_i \to \psi_n)} \\ &= \left(\sum_i lim_{N \to \infty} \frac{N_i}{N} \right) \delta_{in} = lim_{N \to \infty} \frac{N_n}{N} \end{aligned} \tag{109}$$

On the other hand, for the Copenhagen interpretation, the measure on a quantum state with $\Delta\tau \gg \tau_c$ by (107)–(109), it follows that

$$\widetilde{P}(\psi \to \psi_n) = \frac{|a_n|^2}{\sum\limits_{k=1}^{t} |a_k|^2} = lim_{\frac{\tau_c}{\Delta\tau} \to 0} P^{(tot)}(\psi \to \psi_n) = lim_{N \to \infty} \frac{N_n}{N} \tag{110}$$

The outputs (109) and (110) show that the wavefunction collapse beyond the duration of the wavefunction decay is ineffective for the output of the measure. Furthermore, we can infer that, since after the wavefunction decays, the system has already reached its final steady eigenstate, and the wavefunction collapse does not occur at all since it does not affect the eigenstates.

Therefore, we can shorten the measure duration $\Delta\tau$ up to the wavefunction decay time τ_d without changing the result (109) for the measure, leading to the relation

$$\widetilde{P}(\psi \to \psi_n) = \frac{|a_n|^2}{\sum\limits_{k=1}^{t} |a_k|^2} = lim_{N \to \infty} \frac{N_n}{N} \tag{111}$$

On the other hand, by considering in the second instance that the wavefunction collapse τ_c is much shorter than the wavefunction decoherence, such as $\Delta\tau_{min} \le \tau_c \ll \tau_d \ll \Delta\tau$ (i.e., $\frac{\tau_d}{\tau_c} \to \infty$), the output reads as

$$\begin{aligned} lim_{\frac{\tau_d}{\Delta\tau} \to 0} P^{(tot)}{}_{(\psi \to \psi_n)} &= \widetilde{P}_{(\psi \to \psi_i)} P_{(\psi_i \to \psi_n)} \\ &= \left(\sum_i \frac{|a_i|^2}{\sum\limits_{k=1}^{t} |a_k|^2} \right) \left(lim_{N \to \infty} \frac{N_n}{N} \right) = \sum_i \frac{|a_i|^2}{\sum\limits_{k=1}^{t} |a_k|^2} \delta_{in} = \frac{|a_n|^2}{\sum\limits_{k=1}^{t} |a_k|^2} \end{aligned} \tag{112}$$

showing that the wave function decay, due to the marginal decoherence, does not impact the measurement, even if it might proceed beyond the wave function collapse. More precisely, we can affirm that, since the wavefunction decay does not affect the eigenstates, it does not occur at all after the wavefunction collapse.

Therefore, both the wavefunction collapse and the wavefunction decoherence occur together only during the time of the measure $\Delta\tau > \Delta\tau_{min}$. In the case of (112), we might shorten the measure duration time $\Delta\tau$ to τ_c, as the wavefunction decay finishes at the end of the measure, Identities (105) and (112) also lead to

$$P^{(tot)}(\psi \to \psi_n) = \frac{|a_n|^2}{\sum\limits_{k=1}^{t} |a_k|^2} = lim_{N \to \infty} \frac{N_n}{N} \tag{113}$$

The proof of (113) can be validated by the numerical output of the motion Equation (37).

The SQHM through Identity (113) furnishes the linkage between the collapse and the decay of the wave function generated by the marginal decoherence, possibly showing that they are the same phenomenon.

5.8. Stochastic Quantum Hydrodynamics, EPR Paradox and Completeness of Quantum Mechanics

Several unresolved facets persist within quantum theories. The EPR paradox [35] and the completeness of quantum mechanics represent fundamental aspects that do not find a clear placement within a comprehensive theoretical framework.

The Copenhagen interpretation assumes quantum theory as a closed system, requiring acceptance of its probabilistic foundations regardless of the validity of the principle of causality. A notable challenge to this perspective was presented by the Einstein–Podolsky–Rosen objection, articulated in a renowned paper [35], which argued the EPR paradox analytically.

It is therefore noteworthy to consider the perspective of the SQHM concerning these irreconcilable aspects. Initially, the SQHM highlights that quantum theory, despite its mathematical precision, remains incomplete with respect to its foundational postulates. Specifically, the SQHM underscores that the measurement process is not explicated within the deterministic "Hamiltonian" framework of standard quantum mechanics. Instead, it manifests as a phenomenon comprehensively described within the framework of a quantum stochastic generalized approach.

The SQHM reveals that quantum mechanics represents the deterministic (zero noise) limit of a broader stochastic theory. From this standpoint, quantum mechanics defines the deterministic evolution of the "probabilistic wave". Moreover, the SQHM suggests that the term "probabilistic" is inaccurately introduced, arising from the inherent probabilistic nature of the measurement process, as the standard quantum mechanics itself cannot fully describe its output. Given the capacity of the SQHM to describe both wavefunction decay and the measurement process, thereby achieving a comprehensive quantum theory, the term "state wave" is a more appropriate substitute for the expression "probabilistic wave". The SQHM theory restores the principle of determinism into the quantum theory, leading to a deterministic evolution of a "state wave", explaining the probabilistic output coming from the fluctuations of the gravitational background. This acknowledges a curved spacetime characterized by stochastic fluctuations in both curvature and geodesic distance.

Furthermore, it is noteworthy to observe that the SQHM addresses the lingering question of preexisting reality before measurement. In contrast, the Copenhagen interpretation posits that only the measurement process allows the system to decay into a stable state, establishing a persistent reality over time. Consequently, it remains indeterminate within this framework whether a persistent reality exists prior to measurement.

At this juncture, the SQHM introduces a subtle innovation: the world is capable of self-decay through macroscopic-scale decoherence, resulting in stable macroscopic eigenstates. These states, being stable with respect to fluctuations, establish a persistent reality that exists prior to measurement.

Regarding the EPR paradox, the SQHM demonstrates that, in a perfect quantum deterministic universe, achieving two completely decoupled parts of the system is not feasible. Instead, this condition can only be realized within a classical system—a self-fluctuating quantum system with finite distances, where the interaction of the quantum potential extends.

In summary, the SQHM reveals the following key points:

i.   Classical reality emerges at the macroscopic level, persisting as a preexisting reality before measurement.
ii.   The measurement process is feasible in a classical macroscopic world, because we can have independent systems, namely the system and the measuring apparatus.
iii. Determinism is acknowledged within standard quantum mechanics under the condition of zero noise.
iv.   Locality and causality are achieved at the macroscopic scale, where quantum nonlocal domains condense to punctual domains.
v.   The maximum light speed of the propagation of information aligns with quantum uncertainty.

Moreover, if we consider the SQHM as a complete theory, it should be capable of elucidating the "hidden variable" theory of Bohm that endeavors to restore the determinism of quantum mechanics by introducing the concept of a pilot wave. The fundamental concept posits that, in addition to the particles themselves, there exists a "guidance" or influence from the pilot wave function that dictates the behavior of the particles. Although this pilot wave function is not directly observable, it does impact the measurement probabilities of the particles.

On the other hand, the SQHM posits that quantum mechanics represents the deterministic limit of a broader quantum stochastic theory. In a perfect quantum universe, the measurement process cannot be executed. Furthermore, measurement becomes feasible in a macroscopically classical universe generated through quantum decoherence induced by the fluctuating gravitational background. While the Bohm theory attributes the indeterminacy of the measurement process to the undeterminable pilot wave, the SQHM attributes it to the fluctuating gravitational background (difficult to determine due to its predominantly early-generation nature during the Big Bang and the weak force of gravity). This indeterminacy is compounded by the inherently chaotic nature of the classical law of motion, further contributing to the uncertainty of measurement outcomes.

6. Measurement of Photon Entanglement on Large Distances

Once we establish a unified theory where quantum mechanics governs at the microscopic level, classical physics prevails at the macroscopic scale and fluctuations seamlessly coexist, we understand how the range of effectiveness of quantum entanglement is important in obtaining the classical independence of two distinct systems, as discussed in the EPR paradox. Therefore, it is of paramount importance to test the theoretical predictions in measures on entangled photons, since the length of effectiveness of their quantum potential extends itself at macroscopic distances (see Section 6.1) typical of the macroscopic world.

As the time of wave function decay plays a crucial role in the measuring process, the output of quantum entanglement experiments is necessarily influenced by this. While it is challenging to precisely measure the time it takes for information to transfer itself via entangled photons in a laboratory, this parameter can be more accurately controlled in experiments conducted over planetary distances (see Figure 1).
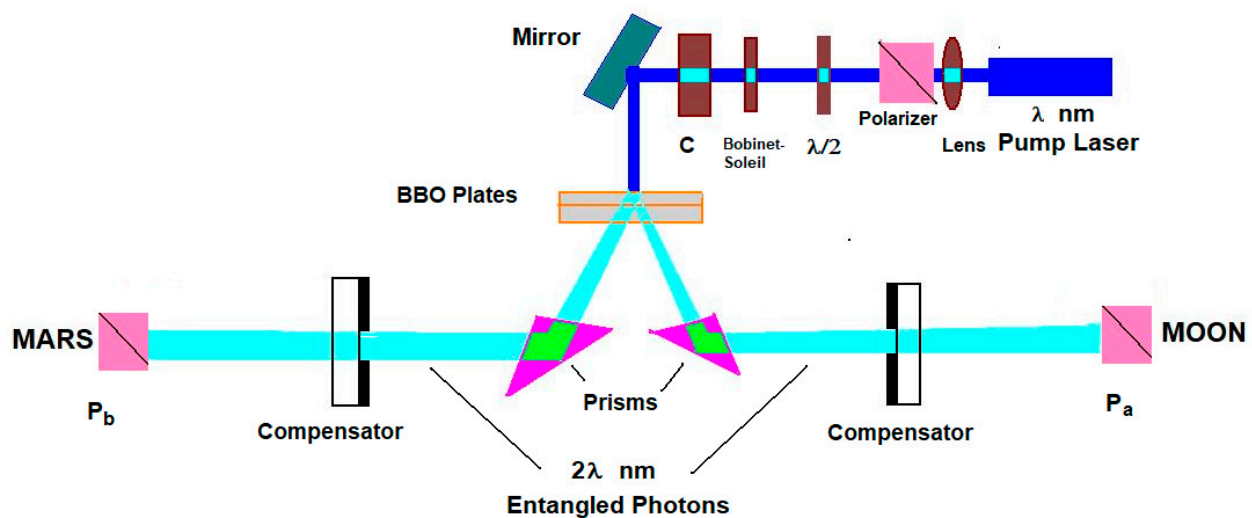
Figure 1. Schematic illustration of the experimental setup.

SQHM theory aligns with the idea that relativistic causality and quantum nonlocality are compatible, a concept that is already supported by relativistic quantum field theory. To check this, we analyze the output of two entangled photon experiments traveling in opposite directions in the state

$$|\psi> = \frac{1}{\sqrt{2}}|H_1, H_2> + e^{i\varphi}|V_1, V_2>,\tag{114}$$

where V and H are vertical and horizontal polarizations, respectively, and ϕ is a constant phase coefficient. Photons "one" and "two" impact polarizers P_a (Alice) and P_b(Bob) with polarization axes positioned at angles α and β relative to the horizontal axis, respectively. For our purpose, we can assume $\phi = 0$.

The probability that photon "two" also passes through Bob's polarizer is $P(\alpha, \beta) = \frac{1}{2}\cos^2(\alpha - \beta)$.

As widely held by the majority of the scientific community in quantum mechanics physics, when photon "one" passes through polarizer P_a with its axes at an angle of α, the state of photon "two" instantaneously collapses to a linear polarized state at the same angle α, resulting in the combined state $|\alpha_1, \alpha_2> = |\alpha_1>|\alpha_2>$. In the context of the SQHM, we posit that the collapse is not instantaneous, and following the Copenhagen quantum mechanics standpoint, we assert rigorously that the state of photon "two" is not defined before its measurement at the polarizer P_b.

Therefore, after photon "one" passes through polarizer P_a, from the standpoint of SQHM, we assume that the combined state is $|\alpha_1, S> = |\alpha_1>|QP_1, S_2>$, where the state $|QP_1, S_2>$ represents the state of photon "two" in the interaction with the residual quantum potential field QP_1 generated by photon "one" at polarizer P_a. When the residual part of the two entangled photons $|QP_1, S_2>$ also passes through Bob's polarizer, it makes the transition $|QP_1, S_2> \rightarrow |\beta_2>$ with probability $P(\alpha, \beta) = \frac{1}{2}\cos^2(\alpha - \beta)$.

Since, in the SQHM, the wave function collapse is not instantaneous but takes a finite time interval, we designate these time intervals as Δt_1 and Δt_2 for the two photons, respectively.

Furthermore, we posit that the measurement interval τ_m commences upon the detection of the first entangled photon by the polarizer photon counter system (at time t_1) and concludes when the second entangled photon is detected at the second polarizer photon counter system (at time $t_2 = t_1 + \tau_m$). Additionally, we presume that the detection of each photon promptly occurs upon the completion of its wavefunction collapse.

The optimal approach for conducting the experiment involves maximizing the separation distance, denoted as $\mathcal{L}$, between the two polarizer photon counter systems. To meet this condition, we can contemplate situating the photon source on Earth, positioning

the first polarizer photon counter system on the moon and situating the second system on Mars.

For instance, let us consider a scenario where the moon, Earth and Mars are aligned. In this case, it follows that the distance $\mathcal{L}$ between the two polarizer photon counter systems is determined by $\mathcal{L} = \mathcal{D}_{e-ma} + \mathcal{D}_{e-mo}$, from which we obtain that $t_1 = \frac{\mathcal{D}_{e-mo}}{c} + \Delta t_1$ and $t_2 = \frac{\mathcal{D}_{e-ma}}{c} + \Delta t_2$, where the Earth–Mars and Earth–moon distances are denoted as $\mathcal{D}_{e-ma}$ and $\mathcal{D}_{e-mo}$, respectively.

To determine the modality by which the quantum potential transfers the information of the first photon measure to the second photon, we assume that

i. The quantum potential interaction propagates at the speed of light;
ii. The quantum potential has a spatial extension equal to the physical length λqu.

At the time $t = 0$, the first photon arrives and interacts at the moon polarizer; at time $t_1 = \Delta t_1$, its wavefunction decays at the final measured state, the second photon arrives on Mars at $t = \frac{\mathcal{D}_{e-ma} - \mathcal{D}_{e-mo}}{c}$, and the quantum potential signal generated by the first photon interaction (at $t = 0$) reaches the second polarizer on Mars at the time

$$t = \frac{\mathcal{D}_{e-mo} + \mathcal{D}_{e-ma}}{c}. \tag{115}$$

In addition, at the time

$$t_2 = \frac{\mathcal{D}_{e-ma} - \mathcal{D}_{e-mo}}{c} + \Delta t_2 \tag{116}$$

The second photon on Mars decays to its final state and is measured.

Thus, assuming no superluminal transmission of information, it follows that

$$t_2 = \frac{1}{c}(\mathcal{D}_{e-ma} - \mathcal{D}_{e-mo}) + \Delta t_2 > \frac{1}{c}(\mathcal{D}_{e-mo} + \mathcal{D}_{e-ma}) \tag{117}$$

and therefore,

$$\Delta t_2 > \frac{2\mathcal{D}_{e-mo}}{c} \cong 2.55s. \tag{118}$$

where $\frac{\mathcal{D}_{e-ma} - \mathcal{D}_{e-mo}}{c}$ must be measured by utilizing parallel nonentangled photons.

Moreover, in order for the two photons to be quantum entangled, it must also hold that

$$\lambda_{qu} > 2\mathcal{D}_{e-mo} + c\Delta t_1. \tag{119}$$

The value λ_{qu} in (119) can be obtained by knowing the spectral composition of the photon (see Appendix F). In this case, when photon "one" interacts with the polarizer on the moon, it is still entangled with photon "two" traveling toward Mars.

In this scenario, photons "one" and "two" form a single quantum system that interacts with the polarizer on the moon when photon "one" reaches it. After interacting with the polarizer on the moon, photon "two" travels to Mars, carrying in its quantum potential the information about the polarization of photon "one". At this point, we essentially have an "entangled interaction" of the two photons with the first polarizer.

Equation (118) has been derived by assuming that the decay of photon "two" to the final measured state begins upon its arrival at the polarizer on Mars and that the quantum potential effect, resulting from photon "one", starts to affect photon "two" after its arrival at the polarizer on Mars (but not during its travel), since it is assumed that the quantum potential interaction does not travel faster than light.

It is worth mentioning that the quantum entanglement in the low-velocity limit of quantum mechanics may be superluminal, since, in that limit, the velocity of light goes to infinity, but that is not so for the photon dynamics.

However, if

$$\Delta t_2 < \frac{2\mathcal{D}_{e-mo}}{c} \cong 2.55s. \tag{120}$$

this implies that the photon wave function's decay on Mars, leading to the polarizer photon counter interaction, takes place prior to the arrival of the quantum potential interaction (information) from the first photon detection on the moon. If the measurement output adheres to the quantum correlation law, this contravenes relativistic local causality.

Furthermore, to detect the entangled interaction of photons with the moon polarizer, a counter experiment can be conducted by enclosing the moon polarizer detector in a Faraday cage (approximately 300 m long), with a diaphragm that closes as soon as the photon enters the cage and before it reaches the moon polarizer. The closure of the diaphragm forces the two-photon system to interact with the macroscopic environment before interacting with the moon polarizer. As a result, the photons should decay into decoherent states before their detection and show stochastic polarization states.

6.1. The Quantum Potential Range of Photon Interactions

Given the photon wavefunction,

$$\psi = \int b_k |\psi_k| cos[k_\mu q^\mu] d^4k = \int A_k cos[(kq - \omega t)] dk \tag{121}$$

With Gaussian spectrum $A_k \propto e^{-\frac{(k-k_0)^2}{2\Delta k}}$, the quantum correlation length

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{q_{max}} q^{-1} \left| \partial_q \left(\frac{\partial_q \partial_q |\psi|}{|\psi|} \right) \right|_{(r,\theta,\varphi)} dq}{\left| \partial_q \left(\frac{\partial_q \partial_q |\psi|}{|\psi|} \right) \right|_{(q=\lambda_c,\theta,\varphi)}} \tag{122}$$

reads as (see Appendix F)

$$\begin{aligned} \lambda_{qu} &= 2\lambda_c \frac{\int\limits_0^\infty x^{-1} \left| \left(-\Delta k^4 x \right) \right|_{(x,\theta,\varphi)} dx}{\left| \left(-\Delta k^4 x \right) \right|_{(x=\lambda_c,\theta,\varphi)}} \\ &= \frac{1}{2} \int\limits_0^\infty dx = \infty \end{aligned} \tag{123}$$

The same result is obtained if we use the superposition of the two photon fields in calculating the quantum coherence length λ_{qu}.

This output allows the possibility that the two entangled photons undergo an "entangled interaction" when interacting with the first polarizer on the moon, generating the "synchronized decay" at the second polarizer on Mars.

6.2. Discussion

The SQHM theory is in line with the concept that relativistic causality and quantum nonlocality can coexist, a notion already substantiated by the relativistic quantum field theory. It elucidates the synchronization of photon polarization separated by a large distance without necessitating faster-than-light arguments. This is accomplished by leveraging the infinite length of the interaction provided by the quantum potential, which effectively binds the two photons into a unique quantum system.

The quantum potential, with its extensive range of interaction $\lambda_{qu} = \infty$, establishes a connection between both photons, one situated on the moon and the other traveling towards Mars. At the first photon interaction with the polarizer, the quantum potential changes and will influence the evolution of the other photon and its interaction at the second polarizer. This connection allows for the synchronization of both final states. There

is no concealed mechanism at play that involves the two entangled photons, even when they are separated by a distance greater than the de Broglie length λ_c.

For the SQHM, there exists a state wave before the measurement is conducted by the polarizers. This state wave comprises two entangled photons, giving rise to an aggregate quantum potential. Subsequently, when the measurement occurs, the polarizers alter this state wave, resulting in the production of the polarized state as an output.

If, for any reason, the polarizers at both ends become desynchronized for a duration longer than the wave function decay time of the photon upon which the initial polarizer is applied, the SQHM theory anticipates that the measurement of the first photon will influence the second photon, regardless of the temporary desynchronization of the polarizers. According to SQHM, this correlation will endure, as it is inherently embedded in the quantum potential, even if the polarizers remain unsynchronized for a period surpassing the collapse of the wave function of the initial photon.

The polarization detection of two quantum-entangled photons is a specific measurement process conducted at two separate moments in time. During the first measurement, only a portion of the quantum system is affected (and destroyed) and not the entire system. However, during this initial measurement, the "two-photon system" transitions into the measured state as it interacts with the polarizer. Subsequently, when the second photon is detected on Mars, its quantum state is found to be coherent with the one measured at the first photon.

The underlying challenge lies in understanding how the information, which may also be encoded within the quantum potential, can effectively transmit from the first photon to the second photon as the second photon travels towards Mars. If information does not propagate faster than the light speed, it becomes difficult to explain how the second photon's wave function collapses before the quantum potential energy manages to reach the second polarizer. Therefore, in this case, the second photon's wavefunction decay time will result in a detection delay time relative to the arrival of a nonentangled photon that was simultaneously launched from Earth. In this context, the time of application of the first polarizer is possibly defined by another nonentangled photon simultaneously launched from Earth to the moon.

7. Extending Quantum Coherence to Obtain a Large Number of Entangled Q-Bits

If we examine the two quantum coherence lengths, denoted as λ_c and λ_{qu}, we can develop methodologies to extend quantum coherence over macroscopic distances and achieve a system of a large number of Q-bits. The first proposed approach, advocated by λ_c, involves lowering the temperature, a method already in use but one that imposes significant constraints on the maintenance of quantum computers. A more intriguing possibility arises from analyzing λ_{qu}, where an extension of the effective range of quantum potential can be utilized to enhance the physical dimensions of the Q-bit system.

One initial insight gleaned from this analysis is that extending the linear range of interparticle interactions as much as possible can enhance the system size at which quantum behavior remains observable. Furthermore, given that the quantum potential distance of interaction becomes larger as the Hamiltonian interparticle interaction strengthens, we can extend the range of quantum potential interactions by reducing the spatial dimensions of intermolecular interactions. This can be achieved through materials composed of two-dimensional structures, such as graphene, or linear one-dimensional ones, as found in polymers. Notably, the superconducting phase of polymers may endure even at relatively high temperatures. In light of the stochastic quantum hydrodynamic theory, superconducting polymer research appears to be more promising in achieving room temperature quantum computers. The use of linear systems for Q-bits can also gain advantage by the utilization of photon fields, since λ_{qu} is infinite among entangled photons.

It is worth highlighting that the SQHM itself can serve as a potential source for rapid Q-bits simulations capable of obviating the necessity for the physical instantiation of Q-bit systems.

8. Conclusions

The SQHM presents a framework for characterizing the behavior of quantum systems in a physical vacuum, incorporating a fluctuating metric. This model posits that the spatial noise spectrum is non-white, featuring a correlation function defined by the de Broglie characteristic length. Consequently, effective quantum entanglement emerges, persisting in systems where the physical length is significantly smaller than this characteristic length.

Nonlocal quantum interactions may extend beyond the de Broglie length, reaching a finite distance in nonlinear weakly bonded systems. The dynamics of such systems can be described by the Schrodinger-Langevin equation, deviating from the deterministic bounds of quantum mechanics.

As the physical length of the system increases, classical physics may predominate when the scale surpasses the range of interaction of the quantum potential. The long-distance characteristics of the quantum potential govern the existence of a coarse-grained classical large-scale description.

The SQHM further elucidates that the minimum uncertainty during the measurement process tends to approach the quantum uncertainty relations in the absence of noise. The principle of minimum uncertainty holds when interactions and information propagate no faster than the speed of light, aligning with the relativistic macroscopic locality and nonlocal quantum interactions at the microscale.

The SQHM posits that the interaction between a quantum system and the gravitational background induces a gradual loss of coherence, resulting in classical-like behavior and outcomes nearly identical to those predicted by the decoherence approach. A key distinction lies in the self-generation of fluctuations in the SQHM, intrinsic to the spacetime characteristics originating at the Big Bang, without the reliance on an external environment. Within the SQHM framework, weak quantum potentials fail to sustain coherence in the presence of fluctuations as a drag force emerges, leading to decoherence.

This phenomenon is observable in macroscopic systems, such as those comprised of molecules and atoms interacting through long-range weak potentials, as seen in the Lennard-Jones gas phase. In such scenarios, the impacts of decoherence become more pronounced, and the quantum characteristics of the system become increasingly challenging to discern as the system's size and complexity grow. The SQHM offers a valuable framework for comprehending the intricate interplay between quantum mechanics and classical behavior in such systems.

The stochastic quantum hydrodynamic model proposes that classical mechanics can manifest in a quantum system when the physical length of the system far exceeds the range of interaction of the quantum potential λ_{qu}.

When the eigenstates of a quantum system experience fluctuations, their stationary configurations undergo slight perturbations but persist as stationary and are closely aligned with those predicted by quantum mechanics. However, when the system evolves in a superposition of states, fluctuations induce the relaxation of the superposition to the stationary configuration of one of the eigenstates composing the superposition. This results in the emergence of classical mechanics on a large scale, influenced by the temperature dependence of the de Broglie length, as observed in phenomena like the ^{4}He fluid–superfluid transition. Additionally, this classical emergence is influenced by the extension of the quantum potential range of interactions, as seen in events such as the solid–fluid transition occurring at the melting point of the crystal lattice.

The model offers a comprehensive path integral solution that can be derived in a recursive manner. Additionally, it encompasses conventional quantum mechanics as the deterministic limit of the theory.

In accordance with the stochastic quantum hydrodynamic model, decoherence is deemed essential for a quantum measurement to transpire within finite time intervals, playing a role in the execution, data collection and management of the measuring apparatus. The model posits that the wavefunction collapse phenomenon in the Copenhagen

interpretation of quantum mechanics may align with the concept of wavefunction decay. This decay process is characterized by unique kinetics and occurs within a finite timespan.

With the premise that wavefunction decay occurs within a finite timeframe during the measurement process, a thought experiment is conducted to scrutinize the measurement of the photon entanglement. The SQHM posits the existence of an underlying wave state prior to the polarizer measurements. This preexisting wave state consists of two entangled photons, which give rise to an aggregate quantum potential. When a measurement takes place, the polarizers modify this preexisting wave state, resulting in the emergence of polarized states as the outcome. According to the SQHM framework, the process of polarization detection for two quantum-entangled photons occurs over two distinct steps. During the first one, only a portion of the quantum system is affected. However, during the first part of the measurement, the "two-photon system" transitions into the measured state as it interacts with the polarizer. Consequently, when the second photon is detected on Mars, its quantum state is found to be coherent with the state measured for the first photon. The quantum hydrodynamic approach posits that the information encoded within the quantum potential can effectively transfer from the first photon to the second one, resulting in a delay of the second photon's detection due to its prolonged wavefunction decay time.

The model reveals that canonical quantum mechanics is applicable only in a perfectly static universal spacetime. In real-world scenarios, due to fluctuations in the spacetime background caused by the Big Bang and other cosmological sources, the quantum evolution of mass densities encounters a resistance force while traveling through spacetime with oscillating metrics. This phenomenon shows some analogy to the effect observed with the Higgs Boson, where its field imparts inertia to elementary particles as they move through it.

The description offered by the stochastic quantum hydrodynamic theory envisions a situation where classical mechanics emerges on a macroscopic scale within a spacetime characterized by fluctuations in the curvature. This portrayal seamlessly aligns with the quantum gravitational representation of the cosmos, where gravity serves as the catalyst for universal decoherence.

The model shows that, if reversible quantum mechanics are realized in a static vacuum, the measurement process cannot take a finite time to occur.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Quantum Mechanics as an Imaginary Time Stochastic Process

The Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_i\partial_i\psi + U_{(q)}\psi \tag{A1}$$

in the quantum integral path representation is equivalent to the FPE [3] that, in the unidimensional case, reads as

$$\partial_t P_{(q,t|\ q_a,t_a)} = -\partial_q K(q) P_{(q,t|\ q_a,t_a)} + D\partial_q\partial_q P_{(q,t|\ q_a,t_a)} \ . \tag{A2}$$

where the solution

$$P_{(q,t|\ q_a,t_a)} = \sqrt{\frac{P_{s(q)}}{P_{s(q_a)}}} \int_{q_a}^{v} Dq \ exp\left[-\frac{1}{2D}\int_{t_a}^{t} dt\left(\frac{1}{2}\dot{q}_{(t)}^2 + \frac{1}{2}K_{(q)} + DK'_{(q)}\right)\right], \tag{A3}$$

with

$$P_{s(q)} = P(q, q_{-\infty}|t - t_{-\infty}, t_{-\infty})$$

$$= lim_{t_0 \to -\infty} N \exp\frac{1}{D} \int\limits_{q_{-\infty}}^{q} < \dot{q} >_{(q'_{(t,t_0)})} dq' = N \exp\frac{1}{D} \int\limits_{q_{-\infty}}^{q} K(q')dq' \text{ ,} \tag{A4}$$

and

$$K'_{(q)} = \partial_q K_{(q)}, \tag{A5}$$

lead to the imaginary time $\tau = it$ evolution amplitude

$$\psi_{(q,t)} = \psi_{(q,0)} \int Dq \ exp\left[-\frac{1}{\hbar}\int_0^{-i\tau} d\tau\left(\frac{M}{2}\dot{q}^2_{(\tau)} + V_{(q(\tau))}\right)\right] = \sqrt{\frac{P_{s(q_a)}}{P_{s(q)}}} P_{(q,t|\ q_a,t_a)}. \tag{A6}$$

By comparing (A6) with (A2), we can see that the quantum particle subject to the potential

$$V_{(q)} = \frac{m}{2}K^2_{(q)} + \frac{\hbar}{2}\partial_q K_{(q)}. \tag{A7}$$

is equivalent to the Brownian particle of mass m subject to friction with coefficient $\beta = \kappa m$, obeying the stochastic motion equation

$$\dot{q} = -\frac{1}{\beta}\partial_q U_{(q)} + D^{1/2}\xi_{(t)} \tag{A8}$$

where

$$\partial_q U_{(q)} = -m\kappa K_{(q)} \text{ and where } D = \frac{\hbar}{2m}.$$

The pseudo-diffusional evolution of the quantum mass density $|\psi|^2$ described by the imaginary time stochastic process (A8) [3] (i.e., deterministic) quantum hydrodynamic formalism is described by the motion of particle density $n_{(q,t)} = |\psi|^2_{(q,t)}$ with the velocity $\dot{q}_i = \frac{1}{m}\partial_i S_{(q,t)}$ governed by the following equations [5–7]:

$$\partial_t n_{(q,t)} + \partial_i(n_{(q,t)}\dot{q}) = 0, \tag{A9}$$

$$\dot{q}_i = \frac{1}{m}\partial_i S_{(q,t)} = \frac{p_i}{m} \tag{A10}$$

$$p_i = -\partial_i(H + V_{qu}),) \tag{A11}$$

where, for a nonrelativistic particle in an external potential,

$$H = \frac{p_i p_i}{2m} + V_{(q)} \tag{A12}$$

and

$$S = \int\limits_{t_0}^{t} dt(\frac{p_i p_i}{2m} - V_{(q)} - V_{qu}), \tag{A13}$$

where V_{qu} is the quantum pseudopotential that reads as

$$V_{qu} = -(\frac{\hbar^2}{2m})n^{-1/2}\partial_i\partial_i n^{1/2}. \tag{A14}$$

Expression (A14) leads to the motion equation

$$
\begin{aligned}
\dot{q} &= \frac{1}{m}\frac{\partial S}{\partial q_i} = \frac{1}{m}\int_{t_0}^{t}(\partial_i\frac{p_i p_i}{2m} - \nabla\left(V_{(q)} - V_{qu}\right))dt \\
&= -\frac{1}{m}\int_{t_0}^{t}\partial_i\left(V_{(q)} - V_{qu}\right)dt = -\frac{1}{m}\int_{t_0}^{t}\frac{\partial}{\partial q_i}\left(V_{(q)} + \frac{\hbar^2}{2m}n^{-1/2}\partial_i\partial_i n^{1/2}\right)dt
\end{aligned}
\tag{A15}
$$

and to the stationary equilibrium condition ($\dot{q} = 0$)

$$
-\frac{1}{m}\int_{t_0}^{t}dt(\partial_i\left(V_{(q)} - V_{qu}\right)) = 0
\tag{A16}
$$

that, by using (A14), reads as

$$
\partial_i\left(V_{(q)} + \frac{\hbar^2}{2m}n^{-1/2}\partial_i\partial_i n^{1/2}\right) = 0.
\tag{A17}
$$

By comparing the expansive term $\frac{\hbar^2}{2m}\partial_i\left(n^{-1/2}\partial_i\partial_i n^{1/2}\right)$ in (A17), for example, with the chemical diffusion described by the Fick equation, we can observe that the quantum imaginary stochastic process (A8) gives rise to a "pseudo-diffusional" process propelled by the quantum potential

$$
V_{qu_{(n)}} \propto n^{-1/2}\partial_i\partial_i n^{1/2} = \partial_i\partial_i lnn - \frac{1}{2}(\partial_i lnn)(\partial_i lnn)
\tag{A18}
$$

whereas irreversible chemical diffusion is driven by the potential $V_{(n)} \propto lnn$. The fundamental distinction between the two lies in the fact that, in the latter case, the diffusion coefficient can be defined positive, whereas, for the former case, it cannot.

Both the (real time) thermal stochastic process and the quantum (imaginary time) stochastic process show that stationary states (i.e., $\dot{q} = 0$) occur when the force of the driving potential, which accounts for the effect of the fluctuations, exactly counterbalances, point by point, the one given by the Hamiltonian potential $V_{(q)}$.

The further basic difference between the two is that, in the quantum case, more than one stationary state can possibly exist (i.e., the quantum eigenstates).

Therefore, beyond the analogy, the two processes have fundamental differences.

For instance, the quantum "imaginary time" diffusion, concerning the mass distribution of Gaussian state, leads to the "ballistic" expansion with the variance Δq^2 following the time law

$$
\Delta q^2 \propto t^2,
\tag{A19}
$$

while the spreading of the same initial mass distribution, submitted to thermal fluctuations, follows the thermal expansion with the law

$$
\Delta q^2 \propto t.
\tag{A20}
$$

Notably, the "real-time" stochastic dynamics are dissipative, while the "imaginary time" stochastic dynamics of quantum mechanics are reversible and "deterministic". The ballistic behavior of the Gaussian wavefunction (A19) can be obtained from the limit of a particle in a harmonic potential in the limit of zero amplitude [11].

This dissipative-to-conservative switching of characteristics from the real to the imaginary time stochastic process is analogous to the rheology of elastic solids, where the real elastic constants describe conservative elastic dynamics while the imaginary elastic constants account for the viscous dissipative behavior.

In the quantum deterministic evolution, there is no loss of information with reversible dynamics; the superposition of states (with their complex configuration) is maintained over time and never relaxes to a simpler (higher entropy) stationary configuration.

Appendix B. Stochastic Generalization of the Madelung Quantum Hydrodynamic Model

In the presence of curvature fluctuations, the mass distribution density (MDD) $|\psi|^2 = \widetilde{n}$ becomes a stochastic function that we can ideally pose $\widetilde{n} = \overline{n} + \delta n$, where δn is the fluctuating part, and $\overline{n}$ is the regular part that obeys the limit condition

$$lim_{T \to 0}\widetilde{n} = lim_{T \to 0}\overline{n} = |\psi|^2. \tag{A21}$$

The characteristics of the Madelung quantum potential that, in the presence of stochastic noise, fluctuates can be derived by generally posing that it is composed of the regular part $\overline{V_{qu(n)}}$ (to be defined) plus the fluctuating part V_{st}, such as

$$V_{qu_{(\widetilde{n})}} = -\frac{\hbar^2}{2m}\widetilde{n}^{-1/2}\partial_i\partial_i\widetilde{n}^{1/2} = \overline{V_{qu(\widetilde{n})}} + V_{st}. \tag{A22}$$

where the stochastic part of the quantum potential V_{st} leads to the force noise

$$-\partial_i V_{st} = m\varpi_{(q,t,T)}. \tag{A23}$$

where the noise correlation function reads as

$$\begin{aligned} \lim_{T \to 0} <\varpi_{(q_\alpha,t)}, \varpi_{(q_\beta+\lambda,t+\tau)}> \quad &= \lim_{T \to 0} <\varpi_{(q_\alpha)}, \varpi_{(q_\beta)}>_{(T)} \ G(\lambda)\delta(\tau)\delta_{\alpha\beta} \\ &\approx \frac{<\varpi_{(q_\alpha)}, \varpi_{(q_\beta)}>_{(T)}}{\lambda_c}\delta(\tau)\delta_{\alpha\beta} \end{aligned} \tag{A24}$$

with

$$\lim_{T \to 0} <\varpi_{(q_\alpha)}, \varpi_{(q_\beta)}>_{(T)} = 0 \tag{A25}$$

In addition, the regular part $\overline{V_{qu(\widetilde{n})}}$ for microscopic systems ($\frac{\mathcal{L}}{\lambda_c} \ll 1$), without loss of generality, can be rearranged as

$$\overline{V_{qu(\widetilde{n})}} = -\frac{\hbar^2}{2m}\overline{(\widetilde{n}^{-1/2}\partial_i\partial_i\widetilde{n}^{1/2})} = -\frac{\hbar^2}{2m}\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} + \overline{\Delta V} = V_{qu_{(\rho)}} + \overline{\Delta V} \tag{A26}$$

where $\rho_{(q,t)}$ is the probability mass density function (PMD) associated with the stochastic process that, in the deterministic limit, obeys the condition $lim_{T \to 0}\rho_{(q,t)} = lim_{T \to 0}\widetilde{n} = lim_{T \to 0}\overline{n} = |\psi|^2$.

Given the quantum hydrodynamic equation of motion (3) for the fluctuating MDD

$$\begin{aligned} \widetilde{n} &= \overline{n} + \delta n, \\ \ddot{q}_i &= -\frac{1}{m}\partial_i\left(V_{(q)} + V_{qu_{(\widetilde{n})}}\right) \end{aligned} \tag{A27}$$

we can rearrange it as

$$\ddot{q}_j = -\frac{1}{m}\partial_j\left(V_{(q)} + V_{qu_{(\rho)}} + \frac{\hbar^2}{2m}\left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2}\right)\right) \tag{A28}$$

The term

$$\partial_j\left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2}\right) \tag{A29}$$

that, in the deterministic case, is null, since

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0 \text{ or } T\to 0} \widetilde{n} = \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0 \text{ or } T\to 0} (\overline{n} + \delta n) = \overline{n} \equiv \rho, \tag{A30}$$

generates an additional acceleration in the motion Equation (A28), which close to the stationary condition ($\dot{q} \to 0$), can be developed in the series approximation and reads as

$$\partial_j \left(\frac{1}{\rho^{1/2}} \partial_i \partial_i \rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}} \partial_i \partial_i \widetilde{n}^{1/2} \right) \cong A_{0_j} + A_{1_j}\dot{q} + \ldots + A_{n_j}\dot{q}^n + O(\frac{\mathcal{L}}{\lambda_c}). \tag{A31}$$

Moreover, since near the limiting condition (A30), we can pose

$$\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \widetilde{n}_{(q)} = \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \overline{n}_{(q)} + \delta n_{(q,t)} \cong \rho_{(q)} + \varepsilon_{(q)}\delta n_{(t)}, \tag{A32}$$

with $\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\varepsilon_{(q)} = 1$ and where $\varepsilon_{(q)}$ is smooth with $\frac{\partial^2 \varepsilon^{1/2}}{\partial q_\beta \partial q_\beta}$ finite (since $\lambda_c \gg \mathcal{L} \neq 0$), it follows that

$$\begin{aligned}
&\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \partial_j \left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2} \right) \\
&\cong \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0} \partial_j \left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2}\left(1 + \frac{\varepsilon_{(q)}\delta n_{(t)}}{2\rho}\right) \right), \\
&\cong \delta n_{(t)}\partial_j\left(-\frac{1}{\rho^{1/2}}\partial_i\partial_i\frac{\varepsilon_{(q)}}{2\rho^{1/2}} \right)
\end{aligned} \tag{A33}$$

Moreover, given that at the stationary condition (i.e., $<\dot{q}> = \lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\dot{q}_{(q,\tau)}d\tau = 0$), it holds

$$\begin{aligned}
\overline{n}_{(q)(T=0)} &= \lim_{T\to 0}<\widetilde{n}_{(q)}> = \lim_{T\to 0}\lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\widetilde{n}_{(q,\tau)}d\tau \\
&= \widetilde{n}_{(q)(T=0)} + \lim_{T\to 0}\lim_{\Delta t\to\infty}\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t\frac{\Delta t}{2}}\delta n_{(t)}d\tau \\
&\cong \overline{n}_{(q)(T=0)} + \lim_{T\to 0}<\delta n_{(t)}>_{(T)}
\end{aligned} \tag{A34}$$

and thus

$$\lim_{T\to 0 \text{ or } \frac{\mathcal{L}}{\lambda_c}\to 0}<\delta n_{(t)}>_{(T)} = 0, \tag{A35}$$

the mean $<A_{0_j}>$, by (A33), reads as

$$\begin{aligned}
\lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}<A_{0_j}> &= \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}\lim_{\dot{q}\to 0}<\partial_j\left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2}\right)> \\
&\cong \lim_{\frac{\mathcal{L}}{\lambda_c}\to 0}<\delta n_{(t)}>\partial_j\left(-\frac{1}{\rho^{1/2}}\partial_i\partial_i\frac{\varepsilon_{(q)}}{2\rho^{1/2}}\right) = 0
\end{aligned} \tag{A36}$$

Therefore, the general form of the stochastic term A_0 with zero mean noise, with null correlation time (see (A24)), reads as

$$A_{0_j} = m\kappa D_j^{1/2}\xi_{(t)}. \tag{A37}$$

Therefore, at the leading order in $\dot{q}$, sufficiently close to the deterministic limit of quantum mechanics ($\frac{\mathcal{L}}{\lambda_c} \ll 1$), for the isotropic case $D_i = D \forall i$, we obtain that

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\partial_j\left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2}\right) &\cong m\kappa D_j^{1/2}\xi_{(t)} + A_{1_{jk}}\dot{q}_k + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right) \\
&\cong \left(\partial_j m\kappa D_i^{1/2}\xi_{(t)}q_i + \partial_k\kappa_{jk}S\right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right) \\
&\cong \partial_j\left(m\kappa D_i^{1/2}\xi_{(t)}q_i + \kappa S\right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right) \\
&\cong \partial_j\left(m\kappa D^{1/2}\xi_{(t)}q_i^{1/2}q_i^{1/2} + \kappa S\right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c}\right)^2\right)
\end{aligned}
\tag{A38}
$$

The first-order approximation (A38) allows us to write (A27) as the Markovian process

$$
\ddot{q}_{j_{(t)}} = -\kappa\dot{q}_{j_{(t)}} - \frac{1}{m}\partial_j\left(V_{(q)} + V_{qu_{(\rho)}}\right) + \kappa D^{1/2}\xi_{(t)}.
\tag{A39}
$$

where

$$
\rho_{(q,t)} = \int\limits_{-\infty}^{+\infty} \mathcal{N}_{(q,p,t)}d^3p,
\tag{A40}
$$

where

$$
\mathcal{N}(q,\dot{q},t) = \int P(q,\dot{q},z,\dot{z},|t,0)\mathcal{N}_{(z,\dot{z},0)}d^3z\,d^3\dot{z}
\tag{A41}
$$

where $P(q,\dot{q},z,\dot{z},|t,0)$ is the probability transition function of the Smolukowski conservation equation

$$
P(q,p,q_0,p_0|(t'+\tau-t_0),t_0) = \int\limits_{-\infty}^{\infty} P(q,p,q',p'|\tau,t')P(q',p',q_0,p_0|t'-t_0,t_0)d^3q'\,d^3p'
\tag{A42}
$$

of the Markovian process (A39).

Moreover, by comparing (A27) in the form

$$
\ddot{q}_i = -\frac{1}{m}\partial_i\left(V_{(q)} + \overline{V_{qu(\widetilde{n})}} + V_{st}\right) = -\frac{1}{m}\partial_i\left(V_{(q)} + V_{qu_{(\rho)}} + \overline{\Delta V} + V_{st}\right)
\tag{A43}
$$

with (A39), it follows that

$$
\overline{\Delta V} = \frac{\hbar^2}{2m}\left(\frac{1}{\rho^{1/2}}\partial_i\partial_i\rho^{1/2} - \overline{\left(\frac{1}{\widetilde{n}^{1/2}}\partial_i\partial_i\widetilde{n}^{1/2}\right)}\right) \cong \kappa S
\tag{A44}
$$

and that

$$
V_{st} = q_i^{1/2}q_i^{1/2}\kappa D^{1/2}\xi_{\alpha(t)}.
\tag{A45}
$$

Generally, it must be observed that the validity of (A39) is not general, since, as shown in refs. [14–17], the friction coefficient $\beta = m\kappa$ is never constant, but only in the case of a linear harmonic oscillator.

In addition, since to have quantum decoupling with the environment, nonlinear interaction is necessary (see relations (57) and (58)), actually, the linear case with the β constant cannot be rigorously assumed, except for the case $\beta = 0$ that corresponds to the deterministic limit of the theory, namely, conventional quantum mechanics.

Appendix C. The Markovian Noise Approximation in the Presence of the Quantum Potential

Once the infinitesimal (dark) gravitational fluctuations have broken the quantum coherence on a cosmological scale (e.g., for baryonic particles with mass $m \sim 10^{-(30 \div 27)}Kg$, it is enough a temperature of $T \gg 10^{-40}\frac{4\hbar^2}{mk}{}^\circ K \sim 10^{-100\circ}K$ to have $\lambda_c \ll 10^{20}m$), it follows that the resulting universe can acquire the classical behavior and can be divided into quantum

decoupled subparts (in weak gravity regions with low curvature, since Newtonian gravity is sufficiently feeble for satisfying condition (54)). In this context, we can postulate the existence of the classical environment.

Thus, for a mesoscale quantum system in contact with a classical environment, it is possible to consider the stochastic process

$$\dot{q}_i = \frac{p}{m}, \tag{A46}$$

$$\dot{p}_j = -\partial_j \left(V_{(q)} + V_{qu_{(\rho)}} + \frac{\hbar^2}{2m} \left(\frac{1}{\rho^{1/2}} \partial_i \partial_i \rho^{1/2} - \frac{1}{\tilde{n}^{1/2}} \partial_i \partial_i \tilde{n}^{1/2} \right) \right), \tag{A47}$$

In the presence of the quantum potential, the evolution of the stochastic MDD $\tilde{n}_{(q,t)}$ from the initial configuration, determined by (38), depends on the exact random sequence of the force inputs of the Markovian noise.

The probabilistic phase space mass density $\mathcal{N}_{(q,p,t)}$ of the Smolukowski equation

$$P(q, p, q_0, p_0 | (t' + \tau - t_0), t_0) = \int_{-\infty}^{\infty} P(q, p, q', p' | \tau, t\prime) P(q', p', q_0, p_0 | t' - t_0, t_0) d^3 q' d^3 p', \tag{A48}$$

for the Markovian process (A47), is somehow indefinite, since the quantum potential, being nonlocal, also depends on $\mathcal{N}_{(q,p,t)}$ at the subsequent instants (see (50)–(53)).

Even if the connection (A31)

$$
\begin{aligned}
lim_{\frac{\mathcal{L}}{\lambda_c} \to 0} - \frac{\hbar^2}{2m} \partial_j \left(\frac{1}{\rho^{1/2}} \partial_i \partial_i \rho^{1/2} - \frac{1}{\tilde{n}^{1/2}} \partial_i \partial_i \tilde{n}^{1/2} \right) &\cong m\kappa D_j^{1/2} \xi_{(t)} + A_{1jk} \dot{q}_k + O\left(\left(\frac{\mathcal{L}}{\lambda_c} \right)^2 \right) \\
&\cong \left(\partial_j m\kappa D_i^{1/2} \xi_{(t)} q_i + \partial_k \kappa_{jk} S \right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c} \right)^2 \right) \\
&\cong \partial_j \left(m\kappa D^{1/2} \xi_{(t)} q_i^{1/2} q_i^{1/2} + \kappa S \right) + O\left(\left(\frac{\mathcal{L}}{\lambda_c} \right)^2 \right) \\
&\cong m\kappa D^{1/2} \xi_{(t)} + \partial_j \kappa S
\end{aligned}
, \tag{A49}
$$

between the stochastic function $\tilde{n}_{(q,t)}$ and the distribution $\rho_{(q,t)} = \int \mathcal{N}(q, p, t) d^3 p$ cannot be generally defined, in the small noise approximation, it introduces the linkage between $\tilde{n}_{(q,t)}$ and ρ through $\dot{q}_{\alpha_{(\rho)}}$ and leads to the motion equation

$$\dot{p}_i = -m\kappa \dot{q}_{j_{(t)}} - \partial_i \left(V_{(q)} + V_{qu_{(\rho)}} \right) + m\kappa D^{1/2} \xi_{(t)} \tag{A50}$$

where $V_{qu_{(\rho)}}$ in (A50) is a function of the PMD $\rho_{(q,t)}$ instead of the MDD $\tilde{n}_{(q,t)}$.

It is worth mentioning that the applicability of (A50) is not general, but it is strictly subjected to the condition of being applied to small-scale systems with $\frac{\mathcal{L}}{\lambda_c} \ll 1$ that, being close to the quantum deterministic condition, admit stationary states (the fluctuating homology of the quantum eigenstates with $< \dot{q}_\alpha >= 0$). In this case, for sufficiently slow kinetics (i.e., close to the stationary condition $< \dot{q}_\alpha > \to 0$), it is possible to assume that the collection of all possible MDDs $\tilde{n}_{(q,t)}$ approximately reproduces the PMD $\rho_{(q,t)}$.

Furthermore, it is useful to note that by using the method of Pontryagin [13], the Smolukowski equation can be transformed into the differential conservation equation for the PTF $P(q, z | \tau, t)$:

$$\partial_t P_{(q,z|t,0)} + \partial_i P_{(q,z|t,0)} \mathcal{V} = 0, \tag{A51}$$

In the stochastic case, (A51) is the analogous of (2) as a conservation equation.

It is noteworthy that in the classical case (i.e., $V_{qu_{(n)}} = const = 0$), the Gaussian character of the PTF is warranted by the property that the cumulants are higher than two [13]

$$C_{im_1 \dots m_n}^{(n)} = lim_{\tau \to 0} \frac{1}{\tau} \int (y_i - q_i)(y_{m_1} - q_{m_1}) \dots (y_{m_n} - q_{m_n}) P_{(y,q|\tau,t)} d^{3h} y \tag{A52}$$

and are null in the current

$$
\begin{aligned}
P_{(q,z|t,0)} \mathcal{V}_i =\ & -P_{(q,z|t,0)} \frac{1}{\beta} \partial_i \left(V_{(q)} + V_{qu}{}_{(P(q,q_0|t+\tau,t_0))} \right) - \frac{1}{2} \partial_m D_{im} P_{(q,z|t,0)} \\
& + \dots + \frac{1}{n!} \sum_{n=2}^{\infty} \partial_{m_1} \dots \partial_{m_n} C_{im_1 \dots m_n}^{(n)} P_{(q,z|t,0)}
\end{aligned}
\tag{A53}
$$

This condition is satisfied in classical problems, since the continuity of the Hamiltonian potential leads to velocities that remain finite as $\tau \to 0$, leading to a nonzero contribution just for the first term $\frac{(y_i - q_i)}{\tau}$.

In the quantum case, since the quantum potential depends on the derivatives of $\tilde{n}_{(q,t)} = \bar{n}_{(q,t)} + \delta n \approx |\psi|^2 + \delta$ and since, in macroscopic systems $\frac{\mathcal{L}}{\lambda_c} \gg 1$, the spatial correlation function of noise $G(\lambda)$ (16) tends to the delta-function $\delta(\lambda)$ (i.e., white noise spectrum), very high spikes of quantum force are possible on very close points $(y_i - q_i)$ as $\tau \to 0$. This behavior can give finite nonzero contributions even in the limit of infinitesimal time intervals. In this case, in the limit of $\tau \to 0$, cumulants higher than two contribute to the probability transition function $P(y, q | \tau, t)$.

Such spiking quantum potential contributions grow as a function of $\frac{\mathcal{L}}{\lambda_c}$ for $\frac{\mathcal{L}}{\lambda_c} \gg 1$ we may have jumping processes $\mathcal{N}(q, p, t)$.

Given the conservation Equation (A51) for the phase space density

$$
\begin{aligned}
& \frac{\partial \mathcal{N}_{(q,p,t)}}{\partial t} + \frac{\partial \mathcal{N}_{(q,p,t)} \mathcal{V}_i}{\partial x_i} \\
&= \frac{\partial \mathcal{N}_{(q,p,t)}}{\partial t} + \frac{\partial \mathcal{N}_{(q,p,t)} \bar{\bar{x}}_i}{\partial x_i} + \frac{\partial \left(\frac{1}{2} \frac{\partial C_{im}^{(1)} \mathcal{N}_{(q,p,t)}}{\partial x_m} + \dots + \frac{1}{n!} \sum_{n=2}^{\infty} \frac{\partial^n C_{im \dots l}^{(n)} \mathcal{N}_{(q,p,t)}}{\partial x_m \dots \partial x_l} \right)}{\partial x_i} \\
&= \frac{\partial \mathcal{N}}{\partial t} - \frac{\partial \mathcal{N} \bar{\dot{q}}_\alpha}{\partial q_\alpha} - \frac{\partial \mathcal{N} \bar{\dot{p}}_\beta}{\partial p_\beta} + \frac{\partial \left(\frac{1}{2} \frac{\partial C_{im}^{(1)} \mathcal{N}_{(q,p,t)}}{\partial x_m} + \dots + \frac{1}{n!} \sum_{n=2}^{\infty} \frac{\partial^n C_{im \dots l}^{(n)} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial x_m \dots \partial x_l}_{n-terms}} \right)}{\partial x_i} = 0
\end{aligned}
\tag{A54}
$$

and

$$
\mathcal{V}_i = \begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \bar{x}_i - \frac{1}{2 P_{(x,z|t,0)}} \frac{\partial D_{im} P_{(x,z|t,0)}}{\partial x_m} + \dots + \frac{1}{n! P_{(x,z|t,0)}} \sum_{h=2}^{\infty} \frac{\partial^k C_{im \dots l}^{(k)} P_{(x,z|t,0)}}{\underbrace{\partial x_m \dots \partial x_l}_{(2k-terms)}}
\tag{A55}
$$

$$
\bar{x}_i = \begin{pmatrix} \bar{\dot{q}}_\beta \\ \bar{p} = m \bar{\ddot{q}}_{\alpha(t)} \end{pmatrix} = \begin{pmatrix} \dot{q}_\beta \\ -\frac{\partial (V_{(q)} + V_{qu})}{\partial q_\alpha} \end{pmatrix}
\tag{A56}
$$

$$
\dot{x}_i = \begin{pmatrix} \dot{q}_\beta \\ \dot{p} = m \ddot{q}_{\alpha(t)} \end{pmatrix} = \begin{pmatrix} p_\beta / m \\ -\frac{\partial (V_{(q)} + V_{qu})}{\partial q_\alpha} + D_p^{1/2} \xi_{\alpha(t)} \end{pmatrix}
\tag{A57}
$$

where

$$
V_{qu} = -\frac{\hbar^2}{2m} \frac{1}{\rho^{1/2}} \partial_\gamma \partial_\gamma \rho^{1/2},
\tag{A58}
$$

$$
\rho_{(q,t)} = \int \mathcal{N}(q, p, t) d^{3h} p,
\tag{A59}
$$

by integrating it over the momenta, we obtain

$$\partial_t \int \mathcal{N} d^{3h}p - \partial_\alpha \int \mathcal{N} \dot{q}_\alpha d^{3h}p - <\dot{p}>_{(q,t)} \beta \int_{-\infty}^{\infty} \frac{\partial \mathcal{N}}{\partial p_\beta} d^{3h}p$$

$$+ \int \frac{\partial \left(\frac{1}{2} \frac{\partial C_{im}^{(1)} \mathcal{N}_{(q,p,t)}}{\partial x_m} + \ldots + \frac{1}{n!} \sum_{n=2}^{\infty} \frac{\partial^n C_{im\ldots l}^{(n)} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial x_m \ldots \partial x_l}_{n-terms}} \right)}{\partial x_i} d^{3h}p = 0 \tag{A60}$$

that, with the condition $lim_{p\to\infty}\mathcal{N}_{(q,p,t)} = 0$ and by posing

$$<\dot{q}_\alpha> = \frac{\int \mathcal{N} \dot{q}_\alpha d^{3h}p}{\int \mathcal{N} d^{3h}p}, \tag{A61}$$

leads to

$$\partial_t n - \partial_\alpha n <\dot{q}_\alpha> = -\int \frac{\partial \left(\frac{1}{2} \frac{\partial C_{im}^{(1)} \mathcal{N}_{(q,p,t)}}{\partial x_m} + \ldots + \frac{1}{n!} \sum_{n=2}^{\infty} \frac{\partial^n C_{im\ldots l}^{(n)} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial x_m \ldots \partial x_l}_{n-terms}} \right)}{\partial x_i} d^{3h}p \tag{A62}$$

where $C_{im}^{(1)} = p \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{D}\delta_{\alpha\beta} \end{pmatrix}$, so that (A62) can be rearranged as

$$\partial_t n = <\dot{q}_\alpha> \partial_\alpha n + n \partial_\alpha <\dot{q}_\alpha> + Q_{diss_{(q,t)}} - \tag{A63}$$

where

$$Q_{diss_{(q,t)}} = \int \left(\frac{\partial \left(\begin{matrix} 0 \\ \frac{1}{2} \frac{\partial \widetilde{D}\delta_{\alpha\chi} \mathcal{N}_{(q,p,t)}}{\partial p_\chi} + \ldots + \frac{1}{n!} \sum_{h=2}^{\infty} \frac{\partial^k C_{\alpha\chi\ldots\varepsilon}^{(k)} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial p_\chi \ldots \partial p_\varepsilon}_{(k-terms)}} \end{matrix} \right)}{\partial p_\alpha} \right) d^{3h}p$$

$$= \int \frac{\partial}{\partial p_\alpha} \left(\begin{matrix} 0 \\ \frac{1}{2} \frac{\partial \widetilde{D} \mathcal{N}_{(q,p,t)}}{\partial p_\alpha} + \ldots + \frac{1}{n!} \sum_{h=2}^{\infty} \frac{\partial^k C_{\alpha\chi\ldots\varepsilon}^{(k)} \mathcal{N}_{(q,p,t)}}{\underbrace{\partial p_\chi \ldots \partial p_\varepsilon}_{(k-terms)}} \end{matrix} \right) d^{3h}p \tag{A64}$$

describes the compressibility of the mass density distribution as a consequence of dissipation.

Appendix D. Harmonic Oscillator Eigenstates in Fluctuating Spacetime

In the case of linear systems

$$V_{(q)} = \frac{m\omega^2}{2}q^2, \tag{A65}$$

the equilibrium condition, referring to the stationary configuration of the eigenstates, leads to

$$\frac{\hbar^2}{4m}\partial_q\left(\partial_q\partial_q ln\rho_{(q)} + \frac{1}{2}\left(\partial_q ln\rho_{(q)}\right)^2\right) = m\kappa\frac{D}{2}\partial_q ln\rho + m\omega^2 q \tag{A66}$$

that is satisfied by the solution $\rho = \rho_0 exp\left[-\frac{q^2}{\Delta q^2}\right]$, where $\rho_{0(q)}$ is defined by the relation

$$\frac{\hbar^2}{4m^2}\partial_q\left(\partial_q\partial_q\left(ln\rho_0 - \frac{q^2}{\Delta q^2}\right) + \frac{1}{2}\left(\partial_q\left(ln\rho_0 - \frac{q^2}{\Delta q^2}\right)\right)^2\right) = \kappa\frac{D}{2}\partial_q\left(ln\rho_0 - \frac{q^2}{\Delta q^2}\right) + \omega^2 q. \quad (A67)$$

For the fundamental eigenstate

$$lim_{T\to 0}\rho_0^{1/2}exp\left[-\frac{q^2}{2\Delta q^2}\right] = \left(\frac{1}{4\pi}\frac{m\omega}{\hbar}\right)^{1/4}exp\left[-\frac{q^2}{2\Delta q_0^2}\right] \quad (A68)$$

from (A67), it follows that

$$\frac{1}{\Delta q^2} = -\frac{m^2}{2\hbar^2}\kappa D + \frac{m\omega}{\hbar}\sqrt{1 + \left(\frac{m}{2\hbar\omega}\kappa D\right)^2} \quad (A69)$$

which is close to the quantum mechanical state (i.e., $D \to 0$, $\kappa \to 0$ with $\omega \gg \frac{m}{\hbar}\kappa D$), leads to

$$\frac{1}{\Delta q^2} \cong \frac{m\omega}{\hbar} - \frac{m^2}{2\hbar^2}\kappa D + \frac{1}{8}\frac{m^3}{\hbar^3\omega}(\kappa D)^2 \cong \frac{m\omega}{\hbar} - \frac{m^2}{2\hbar^2}\kappa D \cong \frac{m\omega}{\hbar}\left(1 - \alpha\frac{kT}{\hbar\omega}\right). \quad (A70)$$

and to the distribution

$$\rho_{eq} = \rho_0 exp\left[-\frac{m\omega}{\hbar}\left(1 - \alpha\frac{kT}{\hbar\omega}\right)q^2\right] \quad (A71)$$

where

$$\rho_0 = \left[\frac{1}{4\pi}\frac{m\omega}{\hbar}\left(1 - \alpha\frac{kT}{\hbar\omega}\right)\right]^{1/2}. \quad (A72)$$

From (A71) and (A72) it can be observed that, within the limit of small fluctuations, the mass density distribution of the fundamental eigenstate does not lose its Gaussian form but gains a small increase in its variance following the law

$$\Delta q^2 = \Delta q_0^2\left(1 + \alpha\frac{kT}{\hbar\omega}\right) \quad (A73)$$

where

$$\Delta q_0^2 = \left(\frac{\hbar}{m\omega}\right). \quad (A74)$$

The result (A71) satisfies the condition initially stated.

Moreover, in the presence of fluctuation, the energy E_0 of the fundamental stationary state reads as

$$\begin{aligned}
E_0 = <\psi_0|H|\psi_0> \quad &= \int_{-\infty}^{\infty}\rho_{(q,t)}\left[\frac{m}{2}\dot{q}^2 + \frac{m\omega^2}{2}q^2 + V_{qu}\right]dq \\
&= \int_{-\infty}^{\infty}\rho_{(q,t)}\left[\frac{m\omega^2}{2}q^2 + V_{qu}\right]dq \\
&= \int_{-\infty}^{\infty}\rho_{(q,t)}\left[\frac{m\omega^2}{2}q^2 - \frac{m\omega^2}{2}\left(1 - \alpha\frac{kT}{\hbar\omega}\right)q^2 + (n + \frac{1}{2})\hbar\omega\right]dq \\
&= \int_{-\infty}^{\infty}\rho_{(q,t)}\left[\alpha\frac{\omega mkT}{2\hbar}q^2 + \frac{\hbar\omega}{2}\right]dq \\
&= \frac{\hbar\omega}{2} + \alpha\frac{\omega mkT}{2\hbar}\int_{-\infty}^{\infty}q^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}exp\left[-\left(\frac{m\omega}{\hbar} - \frac{2\alpha mkT}{\hbar^2}\right)q^2\right]dq \\
&= \frac{\hbar\omega}{2} + \alpha\frac{\omega mkT}{2\hbar}\left(\frac{1}{1 + \alpha\frac{kT}{\hbar\omega}}\right)\Delta q^2 \\
&\cong \frac{\hbar\omega + \alpha kT}{2}
\end{aligned} \quad , \quad (A75)$$

showing an energy increase of αkT directly connected to the dissipation parameter α.

As far as it concerns the energy variance of the fundamental stationary state

$$\Delta E_0 = <\psi_0|(E-E_0)^2|\psi_0>^{1/2},$$ (A76)

in the presence of fluctuation, it reads as

$$
\begin{aligned}
(\Delta E_0)^2 &= \int_{-\infty}^{\infty} \rho_{(q,t)} \left[\left(\tfrac{m}{2}\dot{q}^2 + \tfrac{m\omega^2}{2}q^2 + V_{qu} \right) - \left(\tfrac{\hbar\omega+\alpha kT}{2} \right) \right]^2 dq \\
&= \int_{-\infty}^{\infty} \rho_{(q,t)} \left[\alpha \tfrac{\omega mkT}{2\hbar}q^2 - \tfrac{\alpha kT}{2} \right]^2 dq \\
&= \left(\tfrac{\alpha kT}{2} \right)^2 \int_{-\infty}^{\infty} \rho_{(q,t)} \left[\tfrac{\omega m}{\hbar}q^2 - \tfrac{1}{2} \right]^2 dq = \left(\tfrac{\alpha kT}{2} \right)^2 \left(-1 + \tfrac{\omega m}{\hbar} \int_{-\infty}^{\infty} \rho_{(q,t)} \left(2q^2 + \tfrac{\omega m}{\hbar}q^4 \right) dq \right) \\
&\cong \left(\tfrac{\alpha kT}{2} \right)^2 \left(-1 + 1 + \left(\tfrac{\omega m}{\hbar} \right)^2 \int_{-\infty}^{\infty} \rho_{(q,t)}q^4 dq \right) \cong \tfrac{3}{8} \left(\tfrac{\alpha kT}{2} \right)^2
\end{aligned}
$$ (A77)

which allows us to measure the dissipation parameter α by the formula

$$\alpha \cong 2\sqrt{\frac{8}{3}}\frac{\Delta E_0}{kT}.$$ (A78)

For higher eigenstates the eigenvalues of the Hamiltonian read as

$$E_j \cong (j+\frac{1}{2})\hbar\omega - (1+\Delta_j)\frac{\alpha kT}{2}$$ (A79)

where

$$\Delta_j = \frac{\Delta q^2}{\left(\Delta q_j^2 - \Delta q_{j-1}^2\right)} - 1$$ (A80)

where Δq_j^2 is the wave function variance of the j-th eigenstate, and Δq^2 is the variance of the fundamental eigenstate.

It is noteworthy that the parameter α can also be experimentally evaluated by measuring the energy gap $\Delta E_j = E_j - E_{j-1}$ between eigenstates through the relation

$$\alpha \cong 2\frac{\Delta q^2}{\left(\Delta q_j^2 - \Delta q_{j-1}^2\right)}\frac{\hbar\omega - \left(E_{j+1} - E_j\right)}{kT}$$ (A81)

Appendix E. Quantum Decoupling of the Measuring Apparatus at the Initial and Final Times

The preparation of the experimental measurement needs, at the initial state, that the system and the apparatus are decoupled. This is commonly assumed by posing the two systems at infinity with the tacit assumption that no interaction at all exists between them. Nonetheless, since the fully quantum supersystem acts as a whole (i.e., $\lambda_c \to \infty$), the nonlocal quantum potential interaction acts everywhere, even at infinite distance. Thus, the decoupling between the measuring apparatus and the system cannot be strictly assumed in a perfect quantum supersystem. Conversely, given that the quantum potential can acquire a finite range of interaction in sufficiently weakly bounded macroscopic systems in the presence of random noise, the initial decoupling can actually be possible. Thus, it follows that large-scale decoherence is necessary for performing the process of measurement.

Appendix F. The Nonlocal Quantum Potential Length of Interaction of the Photon

To calculate the length λ_{qu} of the nonlocal quantum interaction of the photon, which wavefunction reads

$$\psi = \int b_k |\psi_k| cos[k_\mu q^\mu] d^4 k = \int A_k cos[(kq - \omega t)] dk, \tag{A82}$$

we observe that the Klein-Gordon equation for the photon can be rearranged as

$$p_\mu \, p^\mu = \partial_\mu S \, \partial^\mu S = \hbar^2 \frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} = m V_{qu} \tag{A83}$$

From (A83), we see that the quantum properties for the photon are contained in the term

$$\hbar^2 \frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} \tag{A84}$$

Moreover, since the quantum potential length of interaction is independent of multiplicative constants, along the direction of propagation of the photon, it reads as

$$\lambda_{qu} = \lambda_c \frac{\int_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)|\psi|}{|\psi|} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)|\psi|}{|\psi|} \right) \right|_{(x=\lambda_c)}} \tag{A85}$$

$$\lambda_{qu} = \lambda_c \frac{\int_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)\sqrt{\left(\int A_k cos(kq - \omega t) dk\right)\left(\int A^*{}_k cos(kq - \omega t) dk\right)}}{\sqrt{\left(\int A_k cos(kq - \omega t)\right)\left(\int A^*{}_k cos(kq - \omega t)\right)}} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)\sqrt{\left(\int A_k cos(kq - \omega t)\right)\left(\int A^*{}_k cos(kq - \omega t)\right)}}{\sqrt{\left(\int A_k cos(kq - \omega t)\right)\left(\int A^*{}_k cos(kq - \omega t)\right)}} \right) \right|_{(x=\lambda_c)}} \tag{A86}$$

where $x = q - ct$. Furthermore, for the Gaussian spectrum $A_k \propto e^{-\frac{(k-k_0)^2}{2\Delta k^2}}$, it follows that

$$\lambda_{qu} = \lambda_c \frac{\int_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} cos(kq - \omega t) dk\right)}{\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} cos(kq - \omega t) dk\right)} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q \partial_q - \frac{1}{c^2} \partial_t \partial_t\right)\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} cos(kq - \omega t) dk\right)}{\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} cos(kq - \omega t) dk\right)} \right) \right|_{(x=\lambda_c)}}. \tag{A87}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} \cos(kx)dk\right)}{\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} \cos(kx)dk\right)} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} \cos(kx)dk\right)}{\left(\int e^{-\frac{(k-k_0)^2}{2\Delta k^2}} \cos(kx)dk\right)} \right) \right|_{(x=\lambda_c)}} \tag{A88}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x=\lambda_c)}} \tag{A89}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\partial_q\left(e^{-\frac{\Delta k^2(q-ct)^2}{2}}\partial_q e^{ik_0(q-ct)}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} + \frac{\partial_q\left(e^{ik_0(q-ct)}\partial_q e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} - \frac{1}{c^2}\frac{\partial_t\left(e^{-\frac{\Delta k^2(q-ct)^2}{2}}\partial_t e^{ik_0(q-ct)}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} - \frac{1}{c^2}\frac{\partial_t\left(e^{ik_0(kq-\omega t)}\partial_t e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x=\lambda_c)}} \tag{A90}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{\left(-ik_0 e^{ik_0(q-ct)}\Delta k^2(q-ct)e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} - \frac{\left(e^{-\frac{\Delta k^2(q-ct)^2}{2}} k_0^2 e^{ik_0(q-ct)}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} + \frac{\left(-ik_0 e^{ik_0(q-ct)}\Delta k^2(q-ct)e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} + \frac{\left(e^{ik_0(q-ct)}\left(\Delta k^4(q-ct)^2 e^{-\frac{\Delta k^2(q-ct)^2}{2}} + \Delta k^2 e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} - \frac{1}{c^2}\frac{\partial_t\left(e^{-\frac{\Delta k^2(q-ct)^2}{2}}\partial_t e^{ik_0(q-ct)}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} - \frac{1}{c^2}\frac{\partial_t\left(e^{ik_0(q-ct)}\partial_t e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}}{e^{-\frac{\Delta k^2(kq-\omega t)^2}{2}} e^{ik_0(q-ct)}} \right) \right|_{(x=\lambda_c)}} \tag{A91}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\frac{-2ik_0\Delta k^2(q-ct) - k_0^2 + \Delta k^4(q-ct)^2 + \Delta k^2}{+\frac{2}{c^2}\frac{\left(ick_0 e^{ik_0(q-ct)}\partial_t e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}} + k_0^2 - \frac{1}{c^2}\frac{\left(e^{ik_0(q-ct)}\partial_t\partial_t e^{-\frac{\Delta k^2(q-ct)^2}{2}}\right)}{e^{-\frac{\Delta k^2(q-ct)^2}{2}} e^{ik_0(q-ct)}}} \right) \right|_{(x)} dx}{\left| \partial_x \left(\frac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(kq-\omega t)^2}{2}} e^{ik_0(kq-\omega t)}}{e^{-\frac{\Delta k^2(kq-\omega t)^2}{2}} e^{ik_0(kq-\omega t)}} \right) \right|_{(x=\lambda_c)}} \tag{A92}$$

$$\lambda_{qu} = \lambda_c \frac{\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\begin{array}{c} -2ik_0\Delta k^2(q-ct) - k_0{}^2 + \Delta k^4(q-ct)^2 + \Delta k^2 \\ +2ik_0\Delta k^2(q-ct) + k_0{}^2 + \Delta k^2 + \Delta k^4(q-ct)^2 \end{array} \right) \right|_{(x)} dx}{\left| \partial_x \left(\dfrac{\left(\partial_q\partial_q - \frac{1}{c^2}\partial_t\partial_t\right)e^{-\frac{\Delta k^2(q-ct)^2}{2}}e^{ik_0(q-ct)}}{e^{-\frac{\Delta k^2(q-ct)^2}{2}}e^{ik_0(q-ct)}} \right) \right|_{(x=\lambda_c)}} \tag{A93}$$

$$\lambda_{qu} = \lambda_c \frac{2\int\limits_0^{x_{max}} x^{-1} \left| \partial_x \left(\Delta k^4(q-ct)^2 + \Delta k^2 \right) \right|_{(x)} dx}{\left| \partial_x \left(\Delta k^4(q-ct)^2 + \Delta k^2 \right) \right|_{(x=\lambda_c)}} \tag{A94}$$

$$\begin{aligned} \lambda_{qu} &= \lambda_c \frac{2\int\limits_0^{x_{max}} x^{-1} \left| 2\Delta k^4(q-ct) \right|_{(x)} dx}{\left| 2\Delta k^4(q-ct) \right|_{(x=\lambda_c)}} \\ &= 2\lambda_c \frac{\int\limits_0^{\infty} x^{-1} \left| \left(-2\Delta k^4 x\right) \right|_{(x)} dx}{\left| \left(-2\Delta k^4 x\right) \right|_{(x=\lambda_c)}} \\ &= \frac{1}{\Delta k^4} \int\limits_0^{\infty} 2\Delta k^4 dx = 2\int\limits_0^{\infty} dx = \infty \end{aligned} \tag{A95}$$

Considering the superposition of the two photon fields, the results (A95) do not change.

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