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Confidence Levels-Based Cubic Fermatean Fuzzy Aggregation Operators and Their Application to MCDM Problems

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Abstract: Assessment specialists (experts) are sometimes expected to provide two types of information: knowledge of rating domains and the performance of rating objects (called confidence levels). Unfortunately, the results of previous information aggregation studies cannot be properly used to combine the two categories of data covered above. Additionally, a significant range of symmetric/asymmetric events and structures are frequently included in the implementation process or practical use of fuzzy systems. The primary goal of the current study was to use cubic Fermatean fuzzy set features to address such situations. To deal with the ambiguous information of the aggregated arguments, we defined information aggregation operators with confidence degrees. Two of the aggregation operators we initially proposed were the confidence cubic Fermatean fuzzy weighted averaging (CCFFWA) operator and the confidence cubic Fermatean fuzzy weighted geometric (CCFFWG) operator. They were used as a framework to create an MCDM process, which was supported by an example to show how effective and applicable it is. The comparison of computed results was carried out with the help of existing approaches.

Keywords: cubic Fermatean fuzzy sets; MCDM; confidence levels; aggregation operators

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1. Introduction

Researchers working in the general area of fuzzy decision-making have drawn inspiration from the Bellman–Zadeh conception of a symmetrical decision model in an uncertain environment, with complete symmetry between constraints and decision variables. A significant range of symmetric/asymmetric events and structures is frequently included in the implementation process or practical use of fuzzy systems. Multi-criteria decision-making (MCDM) is one of the fast-developing active research problems for obtaining conclusive results in a reasonable time. However, due to different restrictions, it is not always possible to express the requirements precisely, hence the corresponding solutions are not always optimal. The intuitionistic fuzzy set (IFS) [1] theory is one of the most effective and promising strategies scholars usually apply to manage the ambiguity and imprecision of information. In this context, different scholars focus more on IFSs for integrating the different alternatives using various aggregation algorithms. The performances of the criteria for alternatives are aggregated throughout the data synthesis process using weighted and ordered weighted aggregation operators (AOs) [2,3]. In an IFS environment, Xu and Yager [4] presented a geometric aggregation operator (GAO) while Xu [5] proposed a weighted averaging aggregation operator (AAO). Wang and Liu [6] proposed Einstein aggregation operators by using Einstein norm operations in the IFS context. Lai et al. [7] presented a matching algorithm based on similarity measures and adaptive weights. Ye [8] proposed an accuracy function (ac) for interval-valued IFS to compare them to interval-valued intuitionistic fuzzy numbers. Garg [9] presented a series of communicating AOs for IFSs. Garg [10,11] introduced interacting geometric operators employing Einstein t-norm and Einstein t-conorm operations to aggregate intuitionistic fuzzy data. Xu et al. [12] introduced the intuitionistic fuzzy Einstein–Choquet integral-based operators for decision-making (DM) problems. According to the results of the research mentioned above, they are legitimate as long as the sum of the membership grades does not exceed one. However, in real life, it is not always possible to communicate one's preferences within this restriction. For instance, if someone were to review an option according to their preferences, they would give it a satisfaction rating of 0.7 and an unsatisfaction rating of 0.6. As a result, the review would be unable to meet the IFS condition, as $0.7 + 0.6 > 1$. Because the effectiveness cannot be tested under these circumstances, the IFS theory has some limits and disadvantages. To address these issues, Yager [13,14] introduced Pythagorean fuzzy sets (PFSs) as an extension of the IFS theory. PFSs relax the limitations of IFS. Furthermore, it has been demonstrated that all intuitionistic fuzzy values are part of Pythagorean fuzzy values, which specifies that PFSs have superior ability to manage ambiguous issues (See Figure 1). Following his pioneering work, scholars are continually attempting to improve PFSs. According to Yager and Abbasov [15], Pythagorean fuzzy grades are subclasses of complex numbers. Moreover, Zhang and Xu [16] provided a method for determining the optimal alternative based on an ideal solution in a Pythagorean fuzzy environment. Yager [14] presented a series of aggregation operators in a PFS environment. Peng and Yang [17] defined some basic operational laws and their related properties for Pythagorean fuzzy numbers. Garg presented correlation and correlation coefficients for PFSs. Geo and Deng [18] proposed a Pythagorean fuzzy generation technique based on probability negativity to handle MCDM problems. Zhang [19] presented the notions of interval-valued PFSs (IVPFSs) by extending PFSs. Some important properties of IVPFSs were presented by Peng and Yang [20]. To relax the limitations of PFSs, Senapati and Yager [21] proposed Fermatean fuzzy sets (FFSs) and some operational laws of FFSs. Senapati and Yager [22] proposed weighted averaging and weighted geometric aggregation operators under an FFS environment. Rani and Mishra proposed interval-valued FFSs.

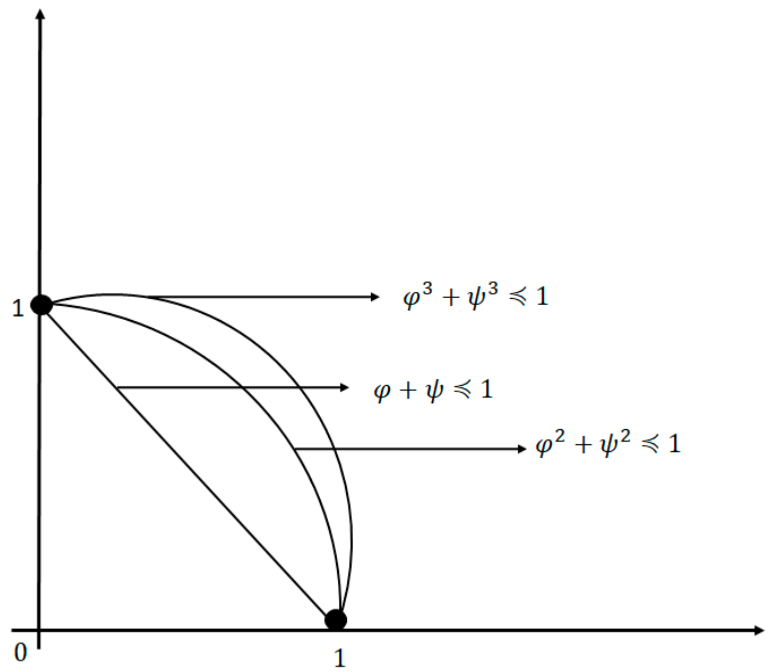

Figure 1. Space analysis between IFS, PFS and FFS.

According to the available research, fuzzy sets, IFS, PFS, and their corresponding implementations are the main topics of all current research. Later on, Jun et al. proposed cubic sets (CSs) by integrating fuzzy sets and interval-valued fuzzy sets. Kaur and Garg [23] presented cubic IFSs and a series of AOs based on t-norm operations. Khan et al. [24,25] suggested CS operations and their characteristics. Abbas et al. [26] proposed cubic PFSs (CPFSs) by combining PFSs and IVPFSs for solving MCDM problems. The flaws and ambiguities of CPFSs were investigated by Amin et al. [27]. Rahim et al. [28] proposed Bonferroni mean aggregation operators under a CPFS environment. Rong and Mishra [29] proposed cubic FFSs and their application in MCDM problems.

Despite the popularity of the aforementioned work, the level of confidence in the criteria was not assessed in any of the studies described above. To put it another way, every researcher has approached the studies with the premise that decision-makers are unquestionably competent in the subjects being investigated. However, these types of prerequisites are only partially accomplished in real-world situations. To compensate for this limitation, decision-makers may examine the alternatives in terms of cubic Fermatean fuzzy numbers (CFFNs) and their associated confidence levels based on their familiarity with the evaluation. As a result, during the evaluation of the alternative in terms of CFFNs, the present study proposes the concept of confidence levels in the optimization processes. First, some basic operations such as P-union (rep. P-intersection), R-union (rep. R-intersection) and so on are defined. Based on these investigations a series of weighted and geometric operators and are proposed in this paper. Additionally, a method to address MCDM issues is suggested. The following is a summary of the study's primary goals:

(1) Define some basic operations of CFFSs and their properties.
(2) Based on these operational laws, propose a series of aggregation operators with confidence levels in a CFFS environment.
(3) Develop a new approach to solve MCDM problems under CFFSs.
(4) Provide an example to evaluate the accuracy and reliability of the proposed approach.
(5) Compare the results of the proposed framework with some existing approaches.

2. Preliminaries

In this section, we briefly present some concepts of PFS, IVPFS, and others to understand the paper.

2.1. PFSs, IVPFSs, and CPFSs

Definition 1 Ref. [13]**.** *Let F be a non-empty finite set. A PFS over element $t \in F$ is defined as*

$$A = \{\langle t,\ \varphi_A(t),\ \psi_A(t)\rangle | t \in F\}, \tag{1}$$

where $\varphi_A(t) \in [0,\ 1]$ and $\psi_A(t) \in [0, 1]$ are the membership and non-membership function of an element $t \in F$ such that $(\varphi_A(t))^2 + (\psi_A(t))^2 \leq 1$.

For convenience, Zhang and Xu [16] called $\langle \varphi_A(t),\ \psi_A(t)\rangle$ a PFN denoted by $\langle \varphi_A, \psi_A\rangle$. The score function of A can be calculated as $sc(A) = \varphi_A^2 - \psi_A^2$.

Definition 2 Ref. [19]**.** *For a non-empty set F, an IVPFS over an element $t \in F$ is defined as follows:*

$$B = \{\langle t, \widetilde{\varphi}_B(t),\ \widetilde{\psi}_B(t)\rangle | t \in F\}, \tag{2}$$

where $\widetilde{\varphi}_B(t)$ and $\widetilde{\psi}_B(t)$ are interval-valued fuzzy numbers representing the interval membership and non-membership grades of set B, respectively. Let $\widetilde{\varphi}_B(t) = [\widetilde{\varphi}_B^L(t), \widetilde{\varphi}_B^U(t)]$ and $\widetilde{\psi}_B(t) = [\widetilde{\psi}_B^L(t), \widetilde{\psi}_B^U(t)]$ then IVPFS can be written as $B = \{\langle t, [\widetilde{\varphi}_B^L(t), \widetilde{\varphi}_B^U(t)], [\widetilde{\psi}_B^L(t), \widetilde{\psi}_B^U(t)]\rangle | t \in F\}$.

For convenience, we denote these pairs as $\langle[\widetilde{\varphi}_B^L, \widetilde{\varphi}_B^U], [\widetilde{\psi}_B^L, \widetilde{\psi}_B^U]\rangle$ and call this an interval-valued PFN (IVPFN). We also set, $0 \le \widetilde{\varphi}_B^L, \widetilde{\varphi}_B^U, \widetilde{\psi}_B^L, \widetilde{\psi}_B^U \le 1$ such that $(\widetilde{\varphi}_B^U)^2 + (\widetilde{\psi}_B^U)^2 \le 1$. The score function of B can be calculated as $sc(B) = \frac{1}{2}\left((\widetilde{\varphi}_B^L)^2 + (\widetilde{\varphi}_B^U)^2 - (\widetilde{\psi}_B^L)^2 - (\widetilde{\varphi}_B^U)^2\right)$.

Definition 3 Refs. [26,27]**.** *Let F be a non-empty finite set. A CPFS over an element $t \in F$ is defined as*

$$C = \left\{\langle t, \widetilde{\mathcal{B}}_C(t), \mathcal{A}_C(t)\rangle | t \in F\right\}, \tag{3}$$

where $\widetilde{\mathcal{B}}_C(t) = \left(\left[\widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^L(t), \widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^U(t)\right], \left[\widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^L(t), \widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^U(t)\right]\right)$ *represents an IVPFS while* $\mathcal{A}_C(t) = \left(\varphi_{\widetilde{\mathcal{B}}_C}(t), \psi_{\widetilde{\mathcal{B}}_C}(t)\right)$ *represents a PFS. We also set,* $0 \le \widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^L(t)$, $\widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^U(t)$, $\widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^L(t)$, $\widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^U(t) \le 1$ *such that* $\left(\widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^U(t)\right)^2 + \left(\widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^U(t)\right)^2 \le 1$.

For convenience, we denote the pairs as $\langle\left[\widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^L, \widetilde{\varphi}_{\widetilde{\mathcal{B}}_c}^U\right], \left[\widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^L, \widetilde{\psi}_{\widetilde{\mathcal{B}}_c}^U\right], \varphi_{\widetilde{\mathcal{B}}_C}, \psi_{\widetilde{\mathcal{B}}_C}\rangle$ and call this a CPFN.

Definition 4 Ref. [28]**.** *Let* $C_1 = \left(\langle\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right], \left[\psi_{C_1}^L, \psi_{C_1}^U\right]\rangle, \langle\varphi_{C_1}, \psi_{C_1}\rangle\right)$ *be a CPFN, then the score function is defined under R-order as*

$$sc(C_1) = \frac{\left(\varphi_{C_1}^L\right)^2 + \left(\varphi_{C_1}^L\right)^2 - \left(\psi_{C_1}^L\right)^2 - \left(\psi_{C_1}^L\right)^2}{2} + (\psi_{C_1})^2 - (\varphi_{C_1})^2, \tag{4}$$

and for P-order as

$$sc(C_1) = \frac{\left(\varphi_{C_1}^L\right)^2 + \left(\varphi_{C_1}^L\right)^2 - \left(\psi_{C_1}^L\right)^2 - \left(\psi_{C_1}^L\right)^2}{2} + (\varphi_{C_1})^2 - (\psi_{C_1})^2, \tag{5}$$

where $-2 \le sc(\beta_1) \le 2$.

Definition 5 Ref. [28]**.** *Let* $C_1 = \left(\langle\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right], \left[\psi_{C_1}^L, \psi_{C_1}^U\right]\rangle, \langle\varphi_{C_1}, \psi_{C_1}\rangle\right)$ *be a CPFN, then the accuracy function is defined as*

$$ac(C_1) = \frac{\left(\varphi_{C_1}^L\right)^2 + \left(\varphi_{C_1}^L\right)^2 + \left(\psi_{C_1}^L\right)^2 + \left(\psi_{C_1}^L\right)^2}{2} + (\varphi_{C_1})^2 + (\psi_{C_1})^2, \tag{6}$$

where $0 \le ac(C_1) \le 2$.

Definition 6 Ref. [28]**.** *Let* $C_1 = \left(\langle\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right], \left[\psi_{C_1}^L, \psi_{C_1}^U\right]\rangle, \langle\varphi_{C_1}, \psi_{C_1}\rangle\right)$ *and* $C_2 = \left(\langle\left[\varphi_{C_2}^L, \varphi_{C_2}^U\right], \left[\psi_{C_2}^L, \psi_{C_2}^U\right]\rangle, \langle\varphi_{C_2}, \psi_{C_2}\rangle\right)$ *be two CPFSs in F. Then:*

- *(Equality):* $C_1 = C_2$, *if and only if* $\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right] = \left[\varphi_{C_2}^L, \varphi_{C_2}^U\right]$, $\psi_{C_1}^L, \psi_{C_1}^U = \psi_{C_2}^L, \psi_{C_2}^U$, $\varphi_{C_1} = \varphi_{C_2}$ *and* $\psi_{C_1} = \psi_{C_2}$;
- *(P-order):* $C_1 \subseteq_P C_2$ if $\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right] \subseteq \left[\varphi_{C_2}^L, \varphi_{C_2}^U\right]$, $\left[\psi_{C_1}^L, \psi_{C_1}^U\right] \supseteq \left[\psi_{C_2}^L, \psi_{C_2}^U\right]$, $\varphi_{C_1} \le \varphi_{C_2}$ *and* $\psi_{C_1} \ge \psi_{C_2}$;
- *(R-order):* $C_1 \subseteq_R C_2$ if $\left[\varphi_{C_1}^L, \varphi_{C_1}^U\right] \subseteq \left[\varphi_{C_2}^L, \varphi_{C_2}^U\right]$, $\left[\psi_{C_1}^L, \psi_{C_1}^U\right] \supseteq \left[\psi_{C_2}^L, \psi_{C_2}^U\right]$, $\varphi_{C_1} \ge \varphi_{C_2}$ *and* $\psi_{C_1} \le \psi_{C_2}$.

Definition 7 Ref. [27]**.** *For the CPFNs* $C_i = \left(\langle\left[\varphi_{C_i}^L, \varphi_{C_i}^U\right], \left[\psi_{C_i}^L, \psi_{C_i}^U\right]\rangle, \langle\varphi_{C_i}, \psi_{C_i}\rangle\right)$ $(1, 2, 3, 4)$ *we have:*

(a) *If* $C_1 \subseteq_P C_2$ *and* $C_2 \subseteq_P C_3$ *then* $C_1 \subseteq_P C_3$;

(b) *If* $C_1 \subseteq_P C_2$ *then* $C_2^c \subseteq_P C_1^c$*;*
(c) *If* $C_1 \subseteq_P C_2$ *and* $C_1 \subseteq_P C_3$ *then* $C_1 \subseteq_P C_2 \cap C_3$*;*
(d) *If* $C_1 \subseteq_P C_2$ *and* $C_3 \subseteq_P C_4$ *then* $C_1 \cup C_3 \subseteq_P C_2 \cup C_4$ *and* $C_1 \cap C_3 \subseteq_P C_2 \cap C_4$*;*
(e) *If* $C_1 \subseteq_P C_2$ *and* $C_3 \subseteq_P C_2$ *then* $C_1 \cup C_3 \subseteq_P C_2$*;*
(f) *If* $C_1 \subseteq_R C_2$ *and* $C_2 \subseteq_R C_3$ *then* $C_1 \subseteq_R C_3$*;*
(g) *If* $C_1 \subseteq_R C_2$ *then* $C_2^c \subseteq_R C_1^c$*;*
(h) *If* $C_1 \subseteq_R C_2$ *and* $C_1 \subseteq_R C_3$ *then* $C_1 \subseteq_R C \cap C_3$*;*
(i) *If* $C_1 \subseteq_R C_2$ *and* $C_3 \subseteq_R C_4$ *then* $C_1 \cup C_3 \subseteq_R C_2 \cup C_4$ *and* $C_1 \cap C_3 \subseteq_R C_2 \cap C_4$*;*
(j) *If* $C_1 \subseteq_R C_2$ *and* $C_3 \subseteq_R C_2$ *then* $C_1 \cup C_3 \subseteq_R C_2$.

2.2. FFSs, IVFFSs, and CFFSs

Definition 8 Ref. [21]**.** *Let F be a non-empty set and $t \in F$. The FFS over element t is defined as*

$$\mathcal{F} = \{\langle t, \varphi_{\mathcal{F}}(t), \psi_{\mathcal{F}}(t)\rangle | t \in F\}, \tag{7}$$

where $\varphi_{\mathcal{F}}(t) \in [0,\ 1]$ and $\psi_{\mathcal{F}}(t) \in [0, 1]$ are the membership and non-membership function of an element $t \in F$ such that $(\varphi_{\mathcal{F}}(t))^3 + (\psi_{\mathcal{F}}(t))^3 \in 1$.

For convenience, Senapati and Yager [21] called $\langle \varphi_{\mathcal{F}}(t),\ \psi_{\mathcal{F}}(t)\rangle$ an FFN denoted by $\langle \varphi_{\mathcal{F}}, \psi_{\mathcal{F}}\rangle$. The score function of A can be calculated as $sc(A) = \varphi_{\mathcal{F}}^3 - \psi_{\mathcal{F}}^3$.

Definition 9 Ref. [30]**.** *For a non-empty set F, an interval-valued FFS (IVFFS) over an element $t \in F$ is defined as follows:*

$$\mathcal{G} = \{\langle t, \widetilde{\varphi}_{\mathcal{G}}(t),\ \widetilde{\psi}_{\mathcal{G}}(t)\rangle | t \in F\}, \tag{8}$$

where $\widetilde{\varphi}_{\mathcal{G}}(t)$ and $\widetilde{\psi}_{\mathcal{G}}(t)$ are interval-valued fuzzy numbers representing the interval membership and non-membership grades of the set $\mathcal{G}$ repectively. Let $\widetilde{\varphi}_{\mathcal{G}}(t) = \left[\widetilde{\varphi}_{\mathcal{G}}^L(t), \widetilde{\varphi}_{\mathcal{G}}^U(t)\right]$ and $\widetilde{\psi}_{\mathcal{G}}(t) = \left[\widetilde{\psi}_{\mathcal{G}}^L(t), \widetilde{\psi}_{\mathcal{G}}^U(t)\right]$ then IVPFS can be written as $\mathcal{G} = \left\{\langle t, \left[\widetilde{\varphi}_{\mathcal{G}}^L(t), \widetilde{\varphi}_{\mathcal{G}}^U(t)\right], \left[\widetilde{\psi}_{\mathcal{G}}^L(t), \widetilde{\psi}_{\mathcal{G}}^U(t)\right]\rangle | t \in F\right\}$.

For convenience, we denote these pairs as $\langle \left[\widetilde{\varphi}_{\mathcal{G}}^L, \widetilde{\varphi}_{\mathcal{G}}^U\right], \left[\widetilde{\psi}_{\mathcal{G}}^L, \widetilde{\psi}_{\mathcal{G}}^U\right]\rangle$ and call this an interval-valued FFN (IVFFN). We also set, $0 \leq \widetilde{\varphi}_{\mathcal{G}}^L, \widetilde{\varphi}_{\mathcal{G}}^U, \widetilde{\psi}_{\mathcal{G}}^L, \widetilde{\psi}_{\mathcal{G}}^U \leq 1$ such that $\left(\widetilde{\varphi}_{\mathcal{G}}^U\right)^3 + \left(\widetilde{\psi}_{\mathcal{G}}^U\right)^3 \leq 1$. The score function of $\mathcal{G}$ can be calculated as $sc(B) = \frac{1}{2}\left((\widetilde{\varphi}_B^L)^3 + (\widetilde{\varphi}_B^U)^3 - (\widetilde{\psi}_B^L)^3 - (\widetilde{\varphi}_B^U)^3\right)$.

Definition 10 Ref. [29]**.** *Let F be a non-empty finite set. A CPFS $\mathcal{L}$ over an element $t \in F$ is defined as*

$$\mathcal{L} = \{\langle t, \mathcal{G}_{\mathcal{L}}(t), \mathcal{F}_{\mathcal{L}}(t)\rangle | t \in F\}, \tag{9}$$

where $\mathcal{G}_{\mathcal{L}}(t) = \left([\widetilde{\varphi}_{\mathcal{L}}^L(t), \widetilde{\varphi}_{\mathcal{L}}^U(t)], [\widetilde{\psi}_{\mathcal{L}}^L(t), \widetilde{\psi}_{\mathcal{L}}^U(t)]\right)$ represents an IVFFS while $\mathcal{F}_{\mathcal{L}}(t) = (\varphi_{\mathcal{L}}(t), \psi_{\mathcal{L}}(t))$ represents PFS. We also set, $0 \leq \widetilde{\varphi}_{\mathcal{L}}^L(t),\ \widetilde{\varphi}_{\mathcal{L}}^U(t),\ \widetilde{\psi}_{\mathcal{L}}^L(t),\ \widetilde{\psi}_{\mathcal{L}}^U(t) \leq 1$ such that $(\widetilde{\varphi}_{\mathcal{L}}^U(t))^2 + (\widetilde{\psi}_{\mathcal{L}}^U(t))^2 \leq 1$.

For convenience, we denote the pairs as $\langle [\widetilde{\varphi}_{\mathcal{L}}^L, \widetilde{\varphi}_{\mathcal{L}}^U], [\widetilde{\psi}_{\mathcal{L}}^L, \widetilde{\psi}_{\mathcal{L}}^U], \varphi_{\mathcal{L}}, \psi_{\mathcal{L}}\rangle$ and call this a CPFN.

3. New Operational Laws and Aggregation Operators under CFFSs with Confidence Levels

In this section, the existing operations defined by Rong et al. [29] are modified. Furthermore, the order relations such as P-order and R-order of CFFNs are presented. Finally, based on these modified operations some series aggregation operators with confidence levels are proposed.

3.1. Modified Operations of CFFSs

Definition 11. *For a family of CFFS* $\{\mathcal{L}_i,\ i \in \Delta\}$, *it follows that*

(a) (*P-union*): $\bigcup^{P}_{i\in\Delta} \mathcal{L}_i = \left(\begin{array}{c} \langle \begin{bmatrix} max_{i\in\Delta}\left(\varphi^{L}_{\mathcal{L}_i}\right), \\ max_{i\in\Delta}\left(\varphi^{U}_{\mathcal{L}_i}\right) \end{bmatrix}, \begin{bmatrix} min_{i\in\Delta}\left(\psi^{L}_{i}\right), \\ min_{i\in\Delta}\left(\psi^{U}_{\mathcal{L}_i}\right) \end{bmatrix} \rangle, \\ \langle max_{i\in\Delta}\mu_{\mathcal{L}_i}, min_{i\in\Delta}\nu_{\mathcal{L}_i} \rangle \end{array} \right)$;

(b) (*P-intersection*): $\bigcap^{R}_{i\in\Delta} \mathcal{L}_i = \left(\begin{array}{c} \langle \begin{bmatrix} min_{i\in\Delta}\left(\varphi^{L}_{\mathcal{L}_i}\right), \\ min_{i\in\Delta}\left(\varphi^{U}_{\mathcal{L}_i}\right) \end{bmatrix}, \begin{bmatrix} max_{i\in\Delta}\left(\psi^{L}_{\mathcal{L}_i}\right), \\ max_{i\in\Delta}\left(\psi^{U}_{\mathcal{L}_i}\right) \end{bmatrix} \rangle, \\ \langle min_{i\in\Delta}\mu_{\mathcal{L}_i}, max_{i\in\Delta}\nu_{\mathcal{L}_i} \rangle \end{array} \right)$;

(c) (*R-union*): $\bigcup^{R}_{i\in\Delta} \mathcal{L}_i = \left(\begin{array}{c} \langle \begin{bmatrix} max_{i\in\Delta}\left(\varphi^{L}_{\mathcal{L}_i}\right), \\ max_{i\in\Delta}\left(\varphi^{U}_{\mathcal{L}_i}\right) \end{bmatrix}, \begin{bmatrix} min_{i\in\Delta}\left(\psi^{L}_{\mathcal{L}_i}\right), \\ min_{i\in\Delta}\left(\psi^{U}_{\mathcal{L}_i}\right) \end{bmatrix} \rangle, \\ \langle min_{i\in\Delta}\mu_{\mathcal{L}_i}, max_{i\in\Delta}\nu_{\mathcal{L}_i} \rangle \end{array} \right)$;

(d) (*R-intersection*): $\bigcup^{P}_{i\in\Delta} \mathcal{L}_i = \left(\begin{array}{c} \langle \begin{bmatrix} max_{\mathcal{L}_i\in\Delta}\left(\varphi^{L}_{\mathcal{L}_i}\right), \\ max_{i\in\Delta}\left(\varphi^{U}_{\mathcal{L}_i}\right) \end{bmatrix}, \begin{bmatrix} min_{i\in\Delta}\left(\psi^{L}_{\mathcal{L}_i}\right), \\ min_{i\in\Delta}\left(\psi^{U}_{\mathcal{L}_i}\right) \end{bmatrix} \rangle, \\ \langle min_{i\in\Delta}\varphi_{\mathcal{L}_i}, max_{i\in\Delta}\psi_{\mathcal{L}_i} \rangle \end{array} \right)$.

Definition 12. *Let* $\mathcal{L}_1 = \left(\langle\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right],\ \left[\psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_1}\right]\rangle,\ \langle\varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1}\rangle\right)$ *and* $\mathcal{L}_2 = \left(\langle\left[\varphi^{L}_{\mathcal{L}_2},\ \varphi^{U}_{\mathcal{L}_2}\right],\ \left[\psi^{L}_{\mathcal{L}_2}, \psi^{U}_{\mathcal{L}_2}\right]\rangle,\ \langle\varphi_{\mathcal{L}_2},\ \psi_{\mathcal{L}_2}\rangle\right)$ *be two CFFSs in F. Then*

(a) (*Equality*): $\mathcal{L}_1 = \mathcal{L}_2$, *if and only if* $\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right] = \left[\varphi^{L}_{2},\ \varphi^{U}_{2}\right]$, $\left[\psi^{L}_{\mathcal{L}_1},\ \psi^{U}_{\mathcal{L}_1}\right] = \left[\psi^{L}_{\mathcal{L}_2},\ \psi^{U}_{\mathcal{L}_2}\right]$, $\varphi_{\mathcal{L}_1} = \varphi_{\mathcal{L}_2}$ *and* $\psi_{\mathcal{L}_1} = \psi_{\mathcal{L}_2}$:

(b) (*P-order*): $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *if* $\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right] \subseteq \left[\varphi^{L}_{\mathcal{L}_2},\ \varphi^{U}_{\mathcal{L}_2}\right]$, $\left[\psi^{L}_{\mathcal{L}_1},\ \psi^{U}_{\mathcal{L}_1}\right] \supseteq \left[\psi^{L}_{\mathcal{L}_2},\ \psi^{U}_{\mathcal{L}_2}\right]$, $\varphi_{\mathcal{L}_1} \le \varphi_{\mathcal{L}_2}$ *and* $\psi_{\mathcal{L}_1} \ge \psi_{\mathcal{L}_2}$;

(c) (*R-order*): $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *if* $\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right] \subseteq \left[\varphi^{L}_{\mathcal{L}_2},\ \varphi^{U}_{\mathcal{L}_2}\right]$, $\left[\psi^{L}_{\mathcal{L}_1},\ \psi^{U}_{\mathcal{L}_1}\right] \supseteq \left[\psi^{L}_{\mathcal{L}_2},\ \psi^{U}_{\mathcal{L}_2}\right]$, $\varphi_{\mathcal{L}_1} \ge \varphi_{\mathcal{L}_2}$ *and* $\psi_{\mathcal{L}_1} \le \psi_{\mathcal{L}_2}$.

Definition 13. *Let* $\mathcal{L}_1 = \left(\langle\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right],\ \left[\psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_1}\right]\rangle,\ \langle\varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1}\rangle\right)$ *be a CFFN, then the score function is defined under R-order as*

$$sc(\mathcal{L}_1) = \frac{\left(\varphi^{L}_{\mathcal{L}_1}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3 - \left(\psi^{L}_{\mathcal{L}_1}\right)^3 - \left(\psi^{U}_{\mathcal{L}_1}\right)^3}{2} + \left(\psi^{3}_{\mathcal{L}_1} - \varphi^{3}_{\mathcal{L}_1}\right), \tag{10}$$

and for P-order as

$$sc(\mathcal{L}_1) = \frac{\left(\varphi^{L}_{\mathcal{L}_1}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3 - \left(\psi^{L}_{\mathcal{L}_1}\right)^3 - \left(\psi^{U}_{\mathcal{L}_1}\right)^3}{2} + \left(\varphi^{3}_{\mathcal{L}_1} - \psi^{3}_{\mathcal{L}_1}\right), \tag{11}$$

where $-2 \le sc(\mathcal{L}_1) \le 2$.

Definition 14. *Let* $\mathcal{L}_1 = \left(\langle\left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right],\ \left[\psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_1}\right]\rangle,\ \langle\varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1}\rangle\right)$ *be a CFFN, then the accuracy function is defined under R-order as*

$$ac(\mathcal{L}_1) = \frac{\left(\varphi^{L}_{\mathcal{L}_1}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3 + \left(\psi^{L}_{\mathcal{L}_1}\right)^3 + \left(\psi^{U}_{\mathcal{L}_1}\right)^3}{2} + \left(\varphi^{3}_{\mathcal{L}_1} + \psi^{3}_{\mathcal{L}_1}\right), \tag{12}$$

where $0 \leq ac(\mathcal{L}_1) \leq 2$.

Theorem 1. For the CPFNs $\mathcal{L}_i = \left(\langle\left[\varphi^L_{\mathcal{L}_i},\ \varphi^U_{\mathcal{L}_i}\right],\ \left[\psi^L_{\mathcal{L}_i}, \psi^U_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)$ (1, 2, 3, 4) we have:

(a) *If* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *and* $\mathcal{L}_2 \subseteq_P \mathcal{L}_3$ *then* $\mathcal{L}_1 \subseteq_P \mathcal{L}_3$;
(b) *If* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *then* $\mathcal{L}_2^c \subseteq_P \mathcal{L}_1^c$;
(c) *If* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *and* $\mathcal{L}_1 \subseteq_P \mathcal{L}_3$ *then* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2 \cap \mathcal{L}_3$;
(d) *If* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *and* $\mathcal{L}_3 \subseteq_P \mathcal{L}_4$ *then* $\mathcal{L}_1 \cup \mathcal{L}_3 \subseteq_P \mathcal{L}_2 \cup \mathcal{L}_4$ *and* $\mathcal{L}_1 \cap \mathcal{L}_3 \subseteq_P \mathcal{L}_2 \cap \mathcal{L}_4$;
(e) *If* $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ *and* $\mathcal{L}_3 \subseteq_P \mathcal{L}_2$ *then* $\mathcal{L}_1 \cup \mathcal{L}_3 \subseteq_P \mathcal{L}_2$;
(f) *If* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *and* $\mathcal{L}_2 \subseteq_R \mathcal{L}_3$ *then* $\mathcal{L}_1 \subseteq_R \mathcal{L}_3$;
(g) *If* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *then* $\mathcal{L}_2^c \subseteq_R \mathcal{L}_1^c$;
(h) *If* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *and* $\mathcal{L}_1 \subseteq_R \mathcal{L}_3$ *then* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2 \cap \mathcal{L}_3$;
(i) *If* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *and* $\mathcal{L}_3 \subseteq_R \mathcal{L}_4$ *then* $\mathcal{L}_1 \cup \mathcal{L}_3 \subseteq_R \mathcal{L}_2 \cup \mathcal{L}_4$ *and* $\mathcal{L}_1 \cap \mathcal{L}_3 \subseteq_R \mathcal{L}_2 \cap \mathcal{L}_4$;
(j) *If* $\mathcal{L}_1 \subseteq_R \mathcal{L}_2$ *and* $\mathcal{L}_3 \subseteq_R \mathcal{L}_2$ *then* $\mathcal{L}_1 \cup \mathcal{L}_3 \subseteq_R \mathcal{L}_2$.

Proof. (a) Since $\mathcal{L}_1 = \left(\langle\left[\varphi^L_{\mathcal{L}_1},\ \varphi^U_{\mathcal{L}_1}\right],\ \left[\psi^L_{\mathcal{L}_1}, \psi^U_{\mathcal{L}_1}\right]\rangle,\ \langle\varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1}\rangle\right)$, $\mathcal{L}_2 = \left(\langle\left[\varphi^L_{\mathcal{L}_2},\ \varphi^U_{\mathcal{L}_2}\right],\ \left[\psi^L_{\mathcal{L}_2}, \psi^U_{\mathcal{L}_2}\right]\rangle,\ \langle\varphi_{\mathcal{L}_2},\ \psi_{\mathcal{L}_2}\rangle\right)$, and $\mathcal{L}_3 = \left(\langle\left[\varphi^L_{\mathcal{L}_3},\ \varphi^U_{\mathcal{L}_3}\right],\ \left[\psi^L_{\mathcal{L}_3}, \psi^U_{\mathcal{L}_3}\right]\rangle,\ \langle\varphi_{\mathcal{L}_3},\ \psi_{\mathcal{L}_3}\rangle\right)$ be CPFNs. Using Definition 12, if $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ then $\left[\varphi^L_{\mathcal{L}_1},\ \varphi^U_{\mathcal{L}_1}\right] \subseteq \left[\varphi^L_{\mathcal{L}_2},\ \varphi^U_{\mathcal{L}_2}\right]$, $\left[\psi^L_{\mathcal{L}_1},\ \psi^U_{\mathcal{L}_1}\right] \supseteq \left[\psi^L_{\mathcal{L}_2},\ \psi^U_{\mathcal{L}_2}\right]$, $\varphi_{\mathcal{L}_1} \leq \varphi_{\mathcal{L}_2}$, and $\psi_{\mathcal{L}_1} \geq \psi_{\mathcal{L}_2}$. Similarly, if $\mathcal{L}_2 \subseteq_P \mathcal{L}_3$, then $\left[\varphi^L_{\mathcal{L}_2},\ \varphi^U_{\mathcal{L}_2}\right] \subseteq \left[\varphi^L_{\mathcal{L}_3},\ \varphi^U_{\mathcal{L}_3}\right]$, $\left[\psi^L_{\mathcal{L}_2},\ \psi^U_{\mathcal{L}_2}\right] \supseteq \left[\psi^L_{\mathcal{L}_3},\ \psi^U_{\mathcal{L}_3}\right]$, $\varphi_{\mathcal{L}_2} \leq \varphi_{\mathcal{L}_3}$, and $\psi_{\mathcal{L}_2} \geq \psi_{\mathcal{L}_3}$ which implies that $\left[\varphi^L_{\mathcal{L}_1},\ \varphi^U_{\mathcal{L}_1}\right] \subseteq \left[\varphi^L_{\mathcal{L}_2},\ \varphi^U_{\mathcal{L}_2}\right] \subseteq \left[\varphi^L_{\mathcal{L}_3},\ \varphi^U_{\mathcal{L}_3}\right]$; $\left[\psi^L_{\mathcal{L}_1},\ \psi^U_{\mathcal{L}_1}\right] \supseteq \left[\psi^L_{\mathcal{L}_2},\ \psi^U_{\mathcal{L}_2}\right] \supseteq \left[\psi^L_{\mathcal{L}_3},\ \psi^U_{\mathcal{L}_3}\right]$; $\varphi_{\mathcal{L}_1} \leq \varphi_{\mathcal{L}_2} \leq \varphi_{\mathcal{L}_3}$; and $\psi_{\mathcal{L}_1} \geq \psi_{\mathcal{L}_2} \geq \psi_{\mathcal{L}_3}$ and hence $\left[\varphi^L_{\mathcal{L}_1},\ \varphi^U_{\mathcal{L}_1}\right] \subseteq \left[\varphi^L_{\mathcal{L}_3},\ \varphi^U_{\mathcal{L}_3}\right]$; $\left[\psi^L_{\mathcal{L}_1},\ \psi^U_{\mathcal{L}_1}\right] \supseteq \left[\psi^L_{\mathcal{L}_3},\ \psi^U_{\mathcal{L}_3}\right]$; $\varphi_{\mathcal{L}_1} \leq \varphi_{\mathcal{L}_3}$; and $\psi_{\mathcal{L}_1} \geq \psi_{\mathcal{L}_3}$. Therefore, if $\mathcal{L}_1 \subseteq_P \mathcal{L}_2$ and $\mathcal{L}_2 \subseteq_P \mathcal{L}_3$, then $\mathcal{L}_1 \subseteq_P \mathcal{L}_3$. Similarly, for the others. □

Definition 15. *Let* $\mathcal{L} = (\langle[\varphi^L_{\mathcal{L}}, \varphi^U_{\mathcal{L}}],\ [\psi^L_{\mathcal{L}},\ \psi^U_{\mathcal{L}}]\rangle,\ \langle\varphi_{\mathcal{L}}, \psi_{\mathcal{L}}\rangle)$*and* $\mathcal{L}_i = \left(\langle\left[\varphi^L_{\mathcal{L}_i},\ \varphi^U_{\mathcal{L}_i}\right],\ \left[\psi^L_{\mathcal{L}_i}, \psi^U_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)$ $(i = 1, 2)$ *be the collections of CFFNs, and* $\zeta \succ 0$ *be a real number then*

(a) $$\mathcal{L}_1 \oplus \mathcal{L}_2 = \left(\begin{array}{c} \langle \left[\begin{array}{c} \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\varphi^L_{\mathcal{L}_i}\right)^3\right)}, \\ \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\varphi^U_{\mathcal{L}_i}\right)^3\right)} \end{array}\right],\ \left[\prod_{i=1}^2 \psi^L_{\mathcal{L}_i},\ \prod_i^2 \psi^U_{\mathcal{L}_i}\right]\rangle, \\ \langle \prod_{i=1}^2 \varphi_{\mathcal{L}_i},\ \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\psi_{\mathcal{L}_i}\right)^3\right)}\ \rangle \end{array}\right);$$

(b) $$\mathcal{L}_1 \otimes \mathcal{L}_2 = \left(\begin{array}{c} \langle \left[\prod_{i=1}^2 \varphi^L_{\mathcal{L}_i},\ \prod_i^2 \varphi^U_{\mathcal{L}_i}\right],\ \left[\begin{array}{c} \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\psi^L_{\mathcal{L}_i}\right)^3\right)}, \\ \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\psi^U_{\mathcal{L}_i}\right)^3\right)} \end{array}\right]\rangle, \\ \langle \sqrt[3]{1 - \prod_{i=1}^2 \left(1 - \left(\varphi_{\mathcal{L}_i}\right)^3\right)}, \prod_{i=1}^2 \psi_{\mathcal{L}_i}\rangle \end{array}\right);$$

(c) $$\zeta\mathcal{L} = \left(\begin{array}{c} \langle \left[\begin{array}{c} \sqrt[3]{1 - \left(1 - (\varphi^L_{\mathcal{L}})^3\right)^{\zeta}}, \\ \sqrt[3]{1 - \left(1 - (\varphi^U_{\mathcal{L}})^3\right)^{\zeta}} \end{array}\right],\ \left[(\psi^L_{\mathcal{L}})^{\zeta},\ (\psi^U_{\mathcal{L}})^{\zeta}\right]\rangle, \\ \langle \varphi^{\zeta}_{\mathcal{L}},\ \sqrt[3]{1 - (1 - \psi^3_{\mathcal{L}})^{\zeta}}\rangle \end{array}\right);$$

$$\text{(d)} \quad \mathcal{L}^{\zeta} = \left(\begin{array}{c} \langle \left[\left(\varphi^{L}_{\mathcal{L}}\right)^{\zeta},\ \left(\varphi^{U}_{\mathcal{L}}\right)^{\zeta} \right],\ \left[\begin{array}{c} \sqrt[3]{1-\left(1-\left(\psi^{L}_{\mathcal{L}}\right)^{3}\right)^{\zeta}}, \\ \sqrt[3]{1-\left(1-\left(\psi^{U}_{\mathcal{L}}\right)^{3}\right)^{\zeta}} \end{array} \right] \rangle, \\ \langle \sqrt[3]{1-\left(1-\varphi^{3}_{\mathcal{L}}\right)^{\zeta}}, \varphi^{\zeta}_{\mathcal{L}} \rangle \end{array} \right).$$

Theorem 2. *For two* CFFNs $\mathcal{L}_1 = \left(\langle \left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right],\ \left[\psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_1}\right] \rangle,\ \langle \varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1} \rangle\right)$ *and* $\mathcal{L}_2 = \left(\langle \left[\varphi^{L}_{\mathcal{L}_2},\ \varphi^{U}_{\mathcal{L}_2}\right],\ \left[\psi^{L}_{\mathcal{L}_2}, \psi^{U}_{\mathcal{L}_2}\right] \rangle,\ \langle \varphi_{\mathcal{L}_2},\ \psi_{\mathcal{L}_2} \rangle\right)$*, provided* $\zeta \succ 0$ *is a real number, then* $\mathcal{L}_1 \oplus \mathcal{L}_2$*,* $\mathcal{L}_1 \otimes \mathcal{L}_2$*,* $\mathcal{L}^{\zeta}$*, and* $\zeta\mathcal{L}_1$ *are also CFFNs.*

Proof. Since $\mathcal{L}_1 = \left(\langle \left[\varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}\right],\ \left[\psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_1}\right] \rangle,\ \langle \varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_1} \rangle\right)$ and $\mathcal{L}_2 = \left(\langle \left[\varphi^{L}_{\mathcal{L}_2},\ \varphi^{U}_{\mathcal{L}_2}\right],\ \left[\psi^{L}_{\mathcal{L}_2}, \psi^{U}_{\mathcal{L}_2}\right] \rangle,\ \langle \varphi_{\mathcal{L}_2},\ \psi_{\mathcal{L}_2} \rangle\right)$ are two CFFNs such that $0 \leq \varphi^{L}_{\mathcal{L}_1},\ \varphi^{U}_{\mathcal{L}_1}, \psi^{L}_{\mathcal{L}_1}, \psi^{U}_{\mathcal{L}_i},\ \varphi^{L}_{\mathcal{L}_2}, \varphi^{U}_{\mathcal{L}_2},\ \psi^{L}_{\mathcal{L}_2}, \psi^{U}_{\mathcal{L}_2} \leq 1$ and $\left(\varphi^{U}_{\mathcal{L}_1}\right)^3 + \left(\psi^{U}_{\mathcal{L}_1}\right)^3 \leq 1$ this implies that $0 \leq \left(1-\left(\varphi^{L}_{\mathcal{L}_1}\right)^3\right)\left(1-\left(\varphi^{L}_{\mathcal{L}_1}\right)^3\right) \leq 1$ and hence $0 \leq \sqrt[3]{\left(\varphi^{L}_{\mathcal{L}_1}\right)^3 + \left(\varphi^{L}_{\mathcal{L}_2}\right)^3 - \left(\varphi^{L}_{\mathcal{L}_1}\right)^3\left(\varphi^{L}_{\mathcal{L}_i}\right)^3} \leq 1$. Similarly, we can prove that $0 \leq \sqrt[3]{\left(\varphi^{L}_{\mathcal{L}_2}\right)^3 + \left(\varphi^{L}_{\mathcal{L}_2}\right)^3 - \left(\varphi^{L}_{\mathcal{L}_2}\right)^3\left(\varphi^{L}_{\mathcal{L}_2}\right)^3} \leq 1$, $0 \leq \psi^{L}_{\mathcal{L}_1}\psi^{L}_{\mathcal{L}_2} \leq 1$ and $0 \leq \psi^{U}_{\mathcal{L}_1}\psi^{U}_{\mathcal{L}_2} \leq 1$. We also set, $0 \leq \varphi_{\mathcal{L}_1},\ \psi_{\mathcal{L}_2},\ \varphi_{\mathcal{L}_1}, \psi_{\mathcal{L}_2} \leq 1$ and $\varphi^3_{\mathcal{L}_1} + \psi^3_{\mathcal{L}_1} \leq 1$, $\varphi^3_{\mathcal{L}_2} + \psi^3_{\mathcal{L}_2} \leq 1$, which implies that $\varphi_{\mathcal{L}_1}\varphi_{\mathcal{L}_2} \leq 1$ and $\sqrt[3]{\varphi^3_{\mathcal{L}_1} + \varphi^3_{\mathcal{L}_2} - \varphi^3_{\mathcal{L}_1}\mu^3_{\mathcal{L}_2}} \leq 1$.

Finally, we have

$$\begin{array}{c} \sqrt[3]{\left(\varphi^{U}_{\mathcal{L}_1}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_2}\right)^3 - \left(\varphi^{U}_{\mathcal{L}_1}\right)^3\left(\varphi^{U}_{\mathcal{L}_2}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3\left(\varphi^{U}_{\mathcal{L}_2}\right)^3} \\ = \sqrt[3]{1-\left(1-\left(\varphi^{U}_{\mathcal{L}_1}\right)^3\right)\left(1-\left(\varphi^{U}_{\mathcal{L}_2}\right)^3\right) + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3\left(\varphi^{U}_{\mathcal{L}_2}\right)^3} \\ \leq \sqrt[3]{1-\left(\varphi^{U}_{\mathcal{L}_1}\right)^3\left(\varphi^{U}_{\mathcal{L}_2}\right)^3 + \left(\varphi^{U}_{\mathcal{L}_1}\right)^3\left(\varphi^{U}_{\mathcal{L}_2}\right)^3} \\ \leq 1, \end{array}$$

and

$$\begin{array}{c} \sqrt[3]{\varphi_{\mathcal{L}_1}\varphi_{\mathcal{L}_2} + \psi^3_{\mathcal{L}_1} + \psi^3_{\mathcal{L}_2} - \left(\psi^{U}_{\mathcal{L}_1}\right)^3\left(\psi^{U}_{\mathcal{L}_1}\right)^3} \\ = \sqrt[3]{\varphi_{\mathcal{L}_1}\varphi_{\mathcal{L}_2} + 1 - \left(1-\left(\varphi^{U}_{\mathcal{L}_1}\right)^3\right)\left(1-\left(\varphi^{U}_{\mathcal{L}_2}\right)^3\right)} \\ \leq \sqrt[3]{\varphi_{\mathcal{L}_1}\varphi_{\mathcal{L}_2} + 1 - \varphi_{\mathcal{L}_1}\varphi_{\mathcal{L}_2}} \\ \leq 1. \end{array}$$

Therefore, $\mathcal{L}_1 \oplus \mathcal{L}_2$ is a CFFN. Furthermore, for any positive real number ψ and CFFN $\beta = \left(\langle \left[\varphi^{L}_{\mathcal{L}},\ \varphi^{U}_{\mathcal{L}}\right],\ \left[\psi^{L}_{\mathcal{L}},\ \psi^{U}_{\mathcal{L}}\right] \rangle,\ \langle \varphi_{\mathcal{L}}, \psi_{\mathcal{L}} \rangle\right)$, we have $0 \leq \varphi^{\zeta} \leq 1$, $0 \leq \sqrt[3]{1-\left(1-\left(\varphi_{\mathcal{L}}\right)^3\right)^{\zeta}} \leq 1$, $0 \leq \left(\psi^{L}_{\mathcal{L}}\right)^{\zeta}\left(\psi^{U}_{\mathcal{L}}\right)^{\zeta} \leq 1$ and $0 \leq \sqrt[3]{1-\left(1-\left(\varphi^{L}_{\mathcal{L}}\right)^3\right)^{\zeta}\left(1-\left(\varphi^{U}_{\mathcal{L}}\right)^3\right)^{\zeta}} \leq 1$. Hence $\zeta\mathcal{L}$ is also a CFFN. Similarly, we can prove that $\mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{L}^{\zeta}$ are also CFFNs. □

3.2. Cubic Fermatean AOs with Confidence Levels

In the available studies, all scholars have approached the studies by taking the postulation that decision-makers are confident in using the estimated objects. However, these kinds of prerequisites are only partially met in real-world situations. To address this problem, in this section we propose a set of averaging and geometric operators with different confidence levels in a cubic Fermatean fuzzy environment. Those are named confidence cubic Fermatean fuzzy weighted averaging (CCFFWA) operator and confidence cubic Fermatean fuzzy weighted geometric (CCFFWG) operator.

3.2.1. Weighted Averaging Operators

Definition 16. *A CCFFWA operator is a mapping* $\text{CCFFWA} : \Gamma^p \to \Gamma$ *defined as*

$$\text{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p) = \sigma_1(\xi_1\mathcal{L}_1) \oplus \sigma_2(\xi_2\mathcal{L}_2) \oplus \ldots \oplus \sigma_p(\xi_p\mathcal{L}_p) \tag{13}$$

where Γ *is the collection of CPFNs with confidence level* $\mathcal{L}_i = \left(\langle\left[\varphi^L_{\mathcal{L}_i},\ \varphi^U_{\mathcal{L}_i}\right],\ \left[\psi^L_{\mathcal{L}_i}, \psi^U_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle, \sigma_i\right)$ *for* $i = 1,2,\ldots,p$; $\xi = (\xi_1,\ \xi_2,\ldots,\xi_p)^T$ *is the weight vector of* ξ_i *such that* $\xi_i > 0$ *and* $\sum_{i=1}^n \xi_i = 1$; *and* σ_i *are the confidence levels of the CFFNs* $\mathcal{L}_i$.

Theorem 3. *For the group of CCFFNs* $\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n$, *the value obtained via CCFFWA is a CFFN, which can be calculated as*

$$\text{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p) = \left(\begin{array}{c} \langle\left[\begin{array}{c} \sqrt[3]{1-\prod_{i=1}^p\left(1-\left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^p\left(1-\left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \end{array}\right], \left[\begin{array}{c} \prod_{i=1}^p\left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^p\left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \end{array}\right]\rangle, \\ \langle\prod_{i=1}^p(\varphi_{\mathcal{L}_i})^{\xi_i\sigma_i}, \sqrt[3]{1-\prod_{i=1}^p\left(1-(\psi_{\mathcal{L}_i})^3\right)^{\xi_i\sigma_i}}\rangle \end{array}\right). \tag{14}$$

Proof. We apply induction principle on $\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_P$

Step 1 For $p = 2$, using Definition 15, we get

$$\text{CCFFWA}(\mathcal{L}_1,\mathcal{L}_2) = \sigma_1\xi_1\mathcal{L}_1 \oplus \sigma_2\xi_2\mathcal{L}_2$$

$$= \left(\begin{array}{c} \langle\left[\begin{array}{c} \sqrt[3]{1-\left(1-\left(\varphi^L_{\mathcal{L}_1}\right)^3\right)^{\sigma_1\xi_1}\left(1-\left(\varphi^L_{\mathcal{L}_2}\right)^3\right)^{\sigma_2\xi_2}}, \\ \sqrt[3]{1-\left(1-\left(\varphi^U_{\mathcal{L}_1}\right)^3\right)^{\sigma_1\xi_1}\left(1-\left(\varphi^L_{\mathcal{L}_2}\right)^3\right)^{\sigma_2\xi_2}} \end{array}\right], \left[\begin{array}{c} \left(\psi^L_{\mathcal{L}_1}\right)^{\sigma_1\xi_1}\left(\psi^L_{\mathcal{L}_2}\right)^{\sigma_2\xi_2}, \\ \left(\psi^U_{\mathcal{L}_1}\right)^{\sigma_1\xi_1}\left(\psi^U_{\mathcal{L}_2}\right)^{\sigma_2\xi_2} \end{array}\right]\rangle, \\ \langle(\varphi_{\mathcal{L}_1})^{\sigma_1\xi_1}(\varphi_{\mathcal{L}_2})^{\sigma_2\xi_2}, \sqrt[3]{1-\left(1-\psi^3_{\mathcal{L}_1}\right)^{\sigma_1\xi_1}\left(1-\psi^3_{\mathcal{L}_1}\right)^{\sigma_1\xi_1}}\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \langle\left[\begin{array}{c} \sqrt[3]{1-\prod_{i=1}^2\left(1-\left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^2\left(1-\left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \end{array}\right], \left[\begin{array}{c} \prod_{i=1}^2\left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^2\left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \end{array}\right]\rangle, \\ \langle\prod_{i=1}^2(\varphi_{\mathcal{L}_i})^{\xi_i\sigma_i}, \sqrt[3]{1-\prod_{i=1}^2\left(1-(\psi_{\mathcal{L}_i})^3\right)^{\xi_i\sigma_i}}\rangle \end{array}\right).$$

Hence, it holds for $P = 2$.

Step 2 Assume Equation (14) holds for $p = \kappa$, then

$$\text{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_k) = \left(\begin{array}{c} \langle\left[\begin{array}{c} \sqrt[3]{1-\prod_{i=1}^k\left(1-\left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^k\left(1-\left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \end{array}\right], \left[\begin{array}{c} \prod_{i=1}^k\left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^k\left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \end{array}\right]\rangle, \\ \langle\prod_{i=1}^k(\varphi_{\mathcal{L}_i})^{\xi_i\sigma_i}, \sqrt[3]{1-\prod_{i=1}^k\left(1-(\psi_{\mathcal{L}_i})^3\right)^{\xi_i\sigma_i}}\rangle \end{array}\right). \tag{15}$$

Step 3 For $p = \kappa + 1$, we have

$$\mathrm{CCFFWA}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_\kappa, \mathcal{L}_{\kappa+1}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots . \oplus \mathcal{L}_\kappa \oplus \mathcal{L}_{\kappa+1}$$

$$= \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \sqrt[3]{1 - \prod_{i=1}^{\kappa} \left(1 - \left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}}, \\ \sqrt[3]{1 - \prod_{i=1}^{\kappa} \left(1 - \left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \end{array} \right], \left[\begin{array}{c} \prod_{i=1}^{\kappa} \left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \\ \prod_{i=1}^{\kappa} \left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i \sigma_i} \end{array} \right] \right\rangle, \\ \left\langle \prod_{i=1}^{\kappa} \left(\varphi_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \sqrt[3]{1 - \prod_{i=1}^{\kappa} \left(1 - \left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \right\rangle \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \sqrt[3]{1 - \left(1 - \left(\varphi^L_{\mathcal{L}_{\kappa+1}}\right)^3\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}}}, \\ \sqrt[3]{1 - \left(1 - \left(\varphi^U_{\mathcal{L}_{\kappa+1}}\right)^3\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}}} \end{array} \right], \left[\begin{array}{c} \left(\psi^L_{\mathcal{L}_{\kappa+1}}\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}}, \\ \left(\psi^U_{\mathcal{L}_{\kappa+1}}\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}} \end{array} \right] \right\rangle, \\ \left\langle \left(\varphi_{\mathcal{L}_{\kappa+1}}\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}}, \sqrt[3]{1 - \left(1 - \left(\psi_{\mathcal{L}_{\kappa+1}}\right)^3\right)^{\sigma_{\kappa+1} \xi_{\kappa+1}}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \sqrt[3]{1 - \prod_{i=1}^{\kappa+1} \left(1 - \left(\varphi^3_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}}, \\ \sqrt[3]{1 - \prod_{i=1}^{\kappa+1} \left(1 - \left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \end{array} \right], \left[\begin{array}{c} \prod_{i=1}^{\kappa+1} \left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \\ \prod_{i=1}^{\kappa+1} \left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i \sigma_i} \end{array} \right] \right\rangle, \\ \left\langle \prod_{i=1}^{\kappa+1} \left(\varphi_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \sqrt[3]{1 - \prod_{i=1}^{\kappa+1} \left(1 - \left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \right\rangle \end{array} \right).$$

As a result, the result is valid for $p = \kappa + 1$, and hence

$$\mathrm{CCFFWA}(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_p) = \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \sqrt[3]{1 - \prod_{i=1}^{p} \left(1 - \left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}}, \\ \sqrt[3]{1 - \prod_{i=1}^{p} \left(1 - \left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \end{array} \right], \left[\begin{array}{c} \prod_{i=1}^{p} \left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \\ \prod_{i=1}^{p} \left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i \sigma_i} \end{array} \right] \right\rangle, \\ \left\langle \prod_{i=1}^{p} \left(\varphi_{\mathcal{L}_i}\right)^{\xi_i \sigma_i}, \sqrt[3]{1 - \prod_{i=1}^{p} \left(1 - \left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} \right\rangle \end{array} \right).$$

The proof is completed. □

Example 1. *Let* $\mathcal{L}_1 = (\langle [0.4, 0.6], [0.3.0.7], (0.3, 0.5) \rangle; 0.8)$, $\mathcal{L}_2 = (\langle [0.5, 0.6], [0.4.0.5], (0.2, 0.4) \rangle; 0.7)$, *and* $\mathcal{L}_3 = (\langle [0.2, 0.3], [0.4.0.5], (0.7, 0.2) \rangle; 0.6)$ *be three CFFNs with confidence levels and* $\xi = (0.25\sigma, 0.35, 0.4)$ *be their corresponding weight vector then*

$$\sqrt[3]{1 - \prod_{i=1}^{3} \left(1 - \left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} = \left(1 - \left(\begin{array}{c} \left(1 - (0.4)^3\right)^{(0.25)(0.8)} \times \left(1 - (0.5)^3\right)^{(0.35)(0.7)} \\ \times \left(1 - (0.2)^3\right)^{(0.4)(0.6)} \end{array} \right)\right)^{\frac{1}{3}} = 0.3602;$$

$$\sqrt[3]{1 - \prod_{i=1}^{3} \left(1 - \left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i \sigma_i}} = \left(1 - \left(\begin{array}{c} \left(1 - (0.6)^3\right)^{(0.25)(0.8)} \times \left(1 - (0.6)^3\right)^{(0.35)(0.7)} \\ \times \left(1 - (0.3)^3\right)^{(0.4)(0.6)} \end{array} \right)\right)^{\frac{1}{3}} = 0.477;$$

$$\prod_{i=1}^{3} \left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i \sigma_i} = (0.3)^{(0.25)(0.8)} \times (0.4)^{(0.35)(0.7)} \times (0.4)^{(0.4)(0.6)} = 0.5040;$$

$$\prod_{i=1}^{3} \left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i \sigma_i} = (0.7)^{(0.25)(0.8)} \times (0.5)^{(0.35)(0.7)} \times (0.6)^{(0.4)(0.6)} = 0.6653;$$

$$\prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} = (0.3)^{(0.25)(0.8)} \times (0.2)^{(0.35)(0.7)} \times (0.7)^{(0.4)(0.6)} = 0.4864;$$

$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} = \left(1-\left(\begin{array}{c}\left(1-(0.5)^3\right)^{(0.25)(0.8)} \times \left(1-(0.4)^3\right)^{(0.35)(0.7)} \\ \times\left(1-(0.2)^3\right)^{(0.4)(0.6)}\end{array}\right)\right)^{1/3} = 0.3526.$$

Thus, using Equation (14) we get

$$\begin{aligned}\mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2, \mathcal{L}_3) &= \left(\begin{array}{c}\left\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{3}\left(1-\left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^{3}\left(1-\left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right], \left[\begin{array}{c}\prod_{i=1}^{3}\left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^{3}\left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\end{array}\right]\right\rangle, \\ \left\langle\prod_{i=1}^{3}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}, \sqrt[3]{1-\prod_{i=1}^{3}\left(1-\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}\right\rangle\end{array}\right) \\ &= \left(\begin{array}{c}\langle[0.3602, 0.4770], [0.5040.0.6653]\rangle, \\ \langle 0.4864,\ 0.3526\rangle\end{array}\right).\end{aligned}$$

According to Theorem 3, the CCFFWA operator fulfils the certain properties listed below.

Property 1. *For $\mathcal{L}_i = \mathcal{L}$ $i = 1, 2, \ldots, p$, where $\mathcal{L} = \left(\left\langle\left[\varphi^L_{\mathcal{L}},\ \varphi^u_{\mathcal{L}}\right],\ \left[\psi^L_{\mathcal{L}},\ \psi^U_{\mathcal{L}}\right]\right\rangle,\ \langle\varphi_{\mathcal{L}}, \psi_{\mathcal{L}}\rangle\right)$, it follows that* $\mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_n) = \mathcal{L}$. *This property is called idempotency.*

Proof. As $\xi_i \succ 0$, $\sum_{I=1}^{n}\xi_i = 1$ and $\xi_i = \xi$ for all i, then

$$\begin{aligned}\mathrm{CCFFWA}(\mathcal{L},\ \mathcal{L}, \ldots, \mathcal{L}) &= \left(\begin{array}{c}\left\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^L_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^U_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right], \left[\begin{array}{c}\prod_{i=1}^{p}\left(\psi^L_{\mathcal{L}}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^{p}\left(\psi^U_{\mathcal{L}}\right)^{\xi_i\sigma_i}\end{array}\right]\right\rangle, \\ \left\langle\prod_{i=1}^{p}\left(\varphi_{\mathcal{L}}\right)^{\xi_i\sigma_i}, \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}}\right\rangle\end{array}\right) \\ &= \left(\begin{array}{c}\left\langle\left[1-\left(1-\left(\varphi^L_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}, 1-\left(1-\left(\varphi^U_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}\right], \left[\left(\psi^L_{\mathcal{L}}\right)^{\xi_i\sigma_i}, \left(\psi^U_{\mathcal{L}}\right)^{\xi_i\sigma_i}\right]\right\rangle, \\ \left\langle\left(\varphi_{\mathcal{L}}\right)^{\xi_i\sigma_i}, 1-\left(1-\left(\psi_{\mathcal{L}}\right)^3\right)^{\xi_i\sigma_i}\right\rangle\end{array}\right) \\ &= \left(\left\langle\left[\varphi^L_{\mathcal{L}},\ \varphi^u_{\mathcal{L}}\right],\ \left[\psi^L_{\mathcal{L}},\ \psi^U_{\mathcal{L}}\right]\right\rangle,\ \langle\varphi_{\mathcal{L}}, \psi_{\mathcal{L}}\rangle\right) \\ &= \mathcal{L}.\end{aligned}$$

□

Property 2. *Let* $\mathcal{L}_i = \left(\left\langle\left[\varphi^L_{\mathcal{L}_i},\ \varphi^U_{\mathcal{L}_i}\right],\ \left[\psi^L_{\mathcal{L}_i}, \psi^U_{\mathcal{L}_i}\right]\right\rangle,\ \left\langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\right\rangle\right)$ *and* $\widetilde{\mathcal{L}}_i = \left(\left\langle\left[\widetilde{\varphi}^L_{\mathcal{L}_i},\ \widetilde{\varphi}^U_{\mathcal{L}_i}\right],\ \left[\widetilde{\psi}^L_{\mathcal{L}_i}, \widetilde{\psi}^U_{\mathcal{L}_i}\right]\right\rangle,\ \left\langle\widetilde{\varphi}_{\mathcal{L}_i},\ \widetilde{\psi}_{\mathcal{L}_i}\right\rangle\right)$ *be CCFFNs where* $(i = 1, 2, \ldots, p)$, *such that* $\mathcal{L}_i \leq \widetilde{\mathcal{L}}_i$, *then*

$$\mathrm{CPFWA}(\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_n) \leq \mathrm{CCFFWA}\left(\widetilde{\mathcal{L}}_1,\ \widetilde{\mathcal{L}}_2, \ldots, \widetilde{\mathcal{L}}_p\right). \tag{16}$$

This property is called monotonicity.

Proof. First let us express the term of CCFFN as follows:

$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}}=\alpha,\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}=\beta,$$
$$\prod_{i=1}^{p}\left(\psi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}=\gamma,\ \prod_{i=1}^{p}\left(\psi_{\mathcal{L}_i}^{U}\right)^{\xi_i\sigma_i}=\delta,$$
$$\prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}=\varepsilon,\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}=\zeta,$$
$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\widetilde{\varphi}_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}}=\widetilde{\alpha},\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\widetilde{\varphi}_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}=\widetilde{\beta},$$
$$\prod_{i=1}^{p}\left(\widetilde{\psi}_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}=\widetilde{\gamma},\ \prod_{i=1}^{p}\left(\widetilde{\psi}_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}=\widetilde{\delta},$$
$$\prod_{i=1}^{p}\left(\widetilde{\varphi}_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}=\widetilde{\varepsilon}\text{ and }\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\widetilde{\psi}_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}=\widetilde{\zeta}.$$

Also, $\mathcal{L}_i \leq \widetilde{\mathcal{L}}_i$ for all i, then we have $\varphi_{\mathcal{L}_i}^{L} \leq \widetilde{\varphi}_{\mathcal{L}_i}^{L}$, $\varphi_{\mathcal{L}_i}^{U} \leq \widetilde{\varphi}_{\mathcal{L}_i}^{U}$, $\psi_{\mathcal{L}_i}^{L} \geq \widetilde{\psi}_{\mathcal{L}_i}^{L}$, $\psi_{\mathcal{L}_i}^{U} \geq \widetilde{\psi}_{\mathcal{L}_i}^{U}$; $\varphi_{\mathcal{L}_i} \geq \widetilde{\varphi}_{\mathcal{L}_i}$ and $\psi_{\mathcal{L}_i} \leq \widetilde{\psi}_{\mathcal{L}_i}$, then we have $\alpha \leq \widetilde{\alpha}$, $\beta \leq \widetilde{\beta}$, $\gamma \geq \widetilde{\gamma}$, $\delta \geq \widetilde{\delta}$, $\varepsilon \geq \widetilde{\varepsilon}$, and $\zeta \leq \widetilde{\zeta}$. Therefore, using the score function defined in Definition 10 and 11, we have

$$sc\left(\text{CCFFWA}\left(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p\right)\right)=\frac{\alpha^3+\beta^3-\gamma^3-\delta^3}{2}+\left(\zeta^3-\varepsilon^3\right)$$
$$\leq\frac{\widetilde{\alpha}^3+\widetilde{\beta}^3-\widetilde{\gamma}^3-\widetilde{\delta}^3}{2}+\left(\widetilde{\zeta}^3-\widetilde{\varepsilon}^3\right)=sc\left(\text{CCFFWA}\left(\widetilde{\mathcal{L}}_1,\ \widetilde{\mathcal{L}}_2,\ldots,\widetilde{\mathcal{L}}_p\right)\right).$$

Thus, $\text{CCFFWA}\left(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p\right)\leq\text{CCFFWA}\left(\widetilde{\mathcal{L}}_1,\ \widetilde{\mathcal{L}}_2,\ldots,\widetilde{\mathcal{L}}_p\right)$. □

Property 3. *For any group of CCFFNs $\mathcal{L}_i$ $(i=1,\ 2,\ldots,\ p)$. If*

$$\mathcal{L}^-=\left(\left\langle\begin{bmatrix}min_i\left(\varphi_{\mathcal{L}_i}^{L}\right),\\ min_i\left(\varphi_{\mathcal{L}_i}^{U}\right)\end{bmatrix},\begin{bmatrix}max_i\left(\psi_{\mathcal{L}_i}^{L}\right),\\ max_i\left(\psi_{\mathcal{L}_i}^{U}\right)\end{bmatrix}\right\rangle,\left\langle\begin{matrix}max_i\left(\varphi_{\mathcal{L}_i}\right),\\ min_i\left(\psi_{\mathcal{L}_i}\right)\end{matrix}\right\rangle\right)\text{ and}$$
$$\mathcal{L}^+=\left(\left\langle\begin{bmatrix}max_i\left(\varphi_{\mathcal{L}_i}^{L}\right),\\ max_i\left(\varphi_{\mathcal{L}_i}^{U}\right)\end{bmatrix},\begin{bmatrix}min_i\left(\psi_{\mathcal{L}_i}^{L}\right),\\ min_i\left(\psi_{\mathcal{L}_i}^{U}\right)\end{bmatrix}\right\rangle,\left\langle\begin{matrix}min_i\left(\varphi_{\mathcal{L}_i}\right),\\ max_i\left(\psi_{\mathcal{L}_i}\right)\end{matrix}\right\rangle\right)$$

then $\mathcal{L}^-\leq\text{CCFFWA}\left(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n\right)\leq\mathcal{L}^+$. *This property is called Boundedness.*

Proof. As $min_i\left(\varphi_{\mathcal{L}_i}^{L}\right)\leq\varphi_{\mathcal{L}_i}^{L}\leq max_i\left(\varphi_{\mathcal{L}_i}^{L}\right)$, $min_i\left(\varphi_{\mathcal{L}_i}^{U}\right)\leq\varphi_{\mathcal{L}_i}^{U}\leq max_i\left(\varphi_{\mathcal{L}_i}^{U}\right)$, $min_i\left(\psi_{\mathcal{L}_i}^{L}\right)\leq\psi_{\mathcal{L}_i}^{L}\leq max_i\left(\psi_{\mathcal{L}_i}^{L}\right)$, $min_i\left(\psi_{\mathcal{L}_i}^{U}\right)\leq\psi_{\mathcal{L}_i}^{U}\leq max_i\left(\psi_{\mathcal{L}_i}^{U}\right)$, $min_i\left(\varphi_{\mathcal{L}_i}\right)\leq\varphi_{\mathcal{L}_i}\leq max_i\left(\varphi_{\mathcal{L}_i}\right)$, and $min_i\left(\psi_{\mathcal{L}_i}\right)\leq\psi_{\mathcal{L}_i}\leq max_i\left(\psi_{\mathcal{L}_i}\right)$ it follows that

$$\sqrt[3]{1-\prod_{i=1}^{n}\left(1-min_i\left(\varphi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-max_i\left(\varphi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}};$$
$$\sqrt[3]{1-\prod_{i=1}^{n}\left(1-min_i\left(\varphi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-max_i\left(\varphi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}};$$
$$\prod_{i=1}^{n}max_i\left(\psi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}\left(\psi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}min_i\left(\psi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i};$$
$$\prod_{i=1}^{n}max_i\left(\psi_{\mathcal{L}_i}^{U}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}\left(\psi_{\mathcal{L}_i}^{U}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}min_i\left(\psi_{\mathcal{L}_i}^{U}\right)^{\xi_i\sigma_i};$$
$$\prod_{i=1}^{n}max_i\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\leq\prod_{i=1}^{n}min_i\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i};$$
$$\sqrt[3]{1-\prod_{i=1}^{n}\left(1-min_i\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}\leq\sqrt[3]{1-\prod_{i=1}^{n}\left(1-max_i\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}$$

which implies that

$$
\begin{aligned}
min_i\left(\varphi^L_{\mathcal{L}_i}\right)^3 \leq \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi^L_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \leq max_i\left(\varphi^L_{\mathcal{L}_i}\right)^3;\\
min_i\left(\varphi^U_{\mathcal{L}_i}\right)^3 \leq \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi^U_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \leq max_i\left(\varphi^U_{\mathcal{L}_i}\right)^3;\\
max_i\left(\psi^L_{\mathcal{L}_i}\right) \leq \prod_{i=1}^{n}\left(\psi^L_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \leq min_i\left(\psi^L_{\mathcal{L}_i}\right);\\
max_i\left(\psi^U_{\mathcal{L}_i}\right) \leq \prod_{i=1}^{n}\left(\psi^U_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \leq min_i\left(\psi^U_{\mathcal{L}_i}\right);\\
max_i\left(\varphi_{\mathcal{L}_i}\right) \leq \prod_{i=1}^{n}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i} \leq min_i\left(\varphi_{\mathcal{L}_i}\right);\\
min_i\left(\psi_{\mathcal{L}_i}\right)^3 \leq \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\psi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}} \leq max_i\left(\psi_{\mathcal{L}_i}\right)^3.
\end{aligned}
$$

Thus, $\mathcal{L}^- \leq \mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n) \leq \mathcal{L}^+$. □

Property 4. *For the CCFFNs* $\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\ \mathcal{L}_p$ *and* $\widetilde{\mathcal{L}} = \left(\langle\left[\widetilde{\varphi}^L_{\widetilde{\mathcal{L}}},\ \widetilde{\varphi}^U_{\widetilde{\mathcal{L}}}\right],\ \left[\widetilde{\psi}^L_{\widetilde{\mathcal{L}}},\widetilde{\psi}^U_{\widetilde{\mathcal{L}}}\right]\rangle,\ \langle\widetilde{\varphi}_{\widetilde{\mathcal{L}}},\ \widetilde{\psi}_{\widetilde{\mathcal{L}}}\rangle\right)$, *we have*

$$\mathrm{CCFFWA}\left(\mathcal{L}_1\widetilde{\mathcal{L}} \oplus \mathcal{L}_2\widetilde{\mathcal{L}} \oplus \ldots \oplus \mathcal{L}_p\widetilde{\mathcal{L}}\right) = \mathrm{CCFFWA}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_p) \oplus \widetilde{\mathcal{L}}.$$

Proof. Straightforward. □

Property. 5. *For a positive real number* ζ, *we have*

$$\mathrm{CCFFWA}(\zeta\mathcal{L}_1,\zeta\ \mathcal{L}_2,\ldots,\zeta\mathcal{L}_p) = \zeta\mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p).$$

Proof. Straightforward. □

3.2.2. Ordered weighted Averaging Operator

Definition. 17. *A CCFFOWA is a mapping defined as* CCFFOWA : $\Gamma^n \to \Gamma$ *on a collection of CPFNs* $\mathcal{L}_i$, $(i = 1,2,\ldots p)$ *as follows*

$$\mathrm{CCFFOWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\ \mathcal{L}_p) = \xi_1\sigma_1\mathcal{L}_{\delta(1)} \oplus \xi_2\sigma_2\beta_{\delta(2)} \oplus \ldots \oplus\ \xi_p\sigma_p\mathcal{L}_{\sigma(p)} \tag{17}$$

where δ *is a permutation of* $(1,\ 2,\ldots,n)$, *such that* $\mathcal{L}_{\delta(i-1)} \geq \mathcal{L}_i$ *for* $i = 1,2,\ldots,p$ *and* $\xi = (\xi_1,\xi_2\ldots,\xi_n)^T$ *is its weight vector, such that* $\xi \succ 0$ *and* $\sum_{i=1}^{p}\xi_i = 1$ *with confidence levels* $0 \leq \sigma_i \leq 1$. *Furthermore, the i*th *largest CFFN among* $\mathcal{L}_i's$ *is* $\mathcal{L}_{\delta(i)}$.

Theorem. 4. *The value obtained by using the CCFFOWA operator for CFFNs* $\mathcal{L}_i$ $(i = 1,2,\ldots,p)$ *is again a CFFN and given by*

$$\mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n) = \left(\begin{array}{c}\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^L_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^U_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right],\left[\begin{array}{c}\prod_{i=1}^{p}\left(\psi^L_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{p}\left(\psi^U_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i}\end{array}\right]\rangle,\\ \langle\prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i},\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}}\rangle\end{array}\right). \tag{18}$$

Proof. Similar proof as Theorem 3. □

Example 2. *Let* $\mathcal{L}_1 = (\langle[0.3, 0.4], [0.2.0.3], (0.2, 0.6)\rangle; 0.5)$, $\mathcal{L}_2 = (\langle[0.4, 0.5], [0.3.0.4], (0.6, 0.2)\rangle; 0.4)$, *and* $\mathcal{L}_3 = (\langle[0.6, 0.7], [0.5.0.6], (0.4, 0.3)\rangle; 0.7)$ *be three CFFNs with confidence levels, and* $\xi = (0.5, 0.3, 0.2)$ *be their corresponding weight vector. By using Equations (10) and (11) to calculate the score values of each CFFN it follows that*

$$\begin{aligned} sc(\mathcal{L}_1) &= \tfrac{(0.3)^3+(0.4)^3-(0.2)^3-(0.3)^3}{2} + \left((0.6)^3 - (0.2)^3\right) = 0.2360; \\ sc(\mathcal{L}_2) &= \tfrac{(0.4)^3+(0.5)^3-(0.3)^3-(0.4)^3}{2} + \left((0.2)^3 - (0.6)^3\right) = -0.1590; \\ sc(\mathcal{L}_3) &= \tfrac{(0.6)^3+(0.7)^3-(0.5)^3-(0.6)^3}{2} + \left((0.3)^3 - (0.4)^3\right) = 0.0720. \end{aligned}$$

The order of these CFFNs with respect to score values is $\mathcal{L}_1 \succ \mathcal{L}_3 \succ \mathcal{L}_2$.
Arrange these CFFNs with respect to score values, i.e.,

$$\mathcal{L}_1 = (\langle[0.3, 0.4], [0.2.0.3], (0.2, 0.6)\rangle; 0.5), \mathcal{L}_3 = (\langle[0.6, 0.7], [0.5.0.6], (0.4, 0.3)\rangle; 0.7);$$

and

$$\mathcal{L}_2 = (\langle[0.4, 0.5], [0.3.0.4], (0.6, 0.2)\rangle; 0.4).$$

Therefore, $\mathcal{L}_{\delta(1)} = \mathcal{L}_1$, $\mathcal{L}_{\delta(2)} = \mathcal{L}_3$, and $\mathcal{L}_{\delta(3)} = \mathcal{L}_2$.
Now, we have

$$\begin{aligned} \sqrt[3]{1 - \prod_{i=1}^{3}\left(1 - \left(\varphi^L_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}} &= \left(1 - \left(\begin{array}{c}\left(1-(0.3)^3\right)^{(0.5)(0.5)} \times \left(1-(0.6)^3\right)^{(0.3)(0.7)} \\ \times\left(1-(0.4)^3\right)^{(0.2)(0.4)}\end{array}\right)\right)^{1/3} = 0.3942; \\ \sqrt[3]{1 - \prod_{i=1}^{3}\left(1 - \left(\varphi^U_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}} &= \left(1 - \left(\begin{array}{c}\left(1-(0.4)^3\right)^{(0.5)(0.5)} \times \left(1-(0.7)^3\right)^{(0.3)(0.7)} \\ \times\left(1-(0.5)^3\right)^{(0.2)(0.4)}\end{array}\right)\right)^{1/3} = 0.4777; \\ \prod_{i=1}^{3}\left(\psi^L_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} &= (0.2)^{(0.5)(0.5)} \times (0.5)^{(0.3)(0.7)} \times (0.3)^{(0.2)(0.4)} = 0.5251; \\ \prod_{i=1}^{3}\left(\psi^L_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} &= (0.3)^{(0.5)(0.5)} \times (0.6)^{(0.3)(0.7)} \times (0.4)^{(0.2)(0.4)} = 0.6178; \\ \prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} &= (0.2)^{(0.5)(0.5)} \times (0.4)^{(0.3)(0.7)} \times (0.6)^{(0.2)(0.4)} = 0.5296; \\ \sqrt[3]{1 - \prod_{i=1}^{p}\left(1 - \left(\psi_{\mathcal{L}_{\delta(i)}}\right)^3\right)^{\xi_i\sigma_i}} &= \left(1 - \left(\begin{array}{c}\left(1-(0.6)^3\right)^{(0.5)(0.5)} \times \left(1-(0.3)^3\right)^{(0.3)(0.7)} \\ \times\left(1-(0.2)^3\right)^{(0.2)(0.4)}\end{array}\right)\right)^{1/3} = 0.4021. \end{aligned}$$

Hence,

$$\text{CCFFOWA}(\mathcal{L}_1,\ \mathcal{L}_2, \mathcal{L}_3) = \left(\begin{array}{c}\langle[0.3942, 0.4777], [0.5251.0.6178]\rangle, \\ \langle 0.5296,\ 0.4021\rangle\end{array}\right).$$

3.2.3. Geometric Operator

Definition 18. *A CCFFWG operator is a mapping* $\text{CCFFWG} : \Gamma^n \to \Gamma$ *defined as*

$$\text{CCFFWG}\left(\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_p\right) = \sqsupseteq_1\mathcal{L}_1 \otimes \sqsupseteq_2\mathcal{L}_2 \otimes \ldots \otimes \sqsupseteq_p\mathcal{L}_p \tag{19}$$

where Ω *is the collection of CPFNs* $\mathcal{L}_i (i = 1, 2, \ldots, p)$, *and* $\xi = \left(\xi_1,\ \xi_2, \ldots, \xi_p\right)^T$ *is the weight vector of* $\mathcal{L}_i$ *such that* $\xi_i \succ 0$ *and* $\sum_{i=1}^{n} \xi_i = 1$. *We also set,* σ_p *be the confidence levels of CFFN* $\mathcal{L}_p$.

Theorem 5. *For* $\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_n$, *the value obtained by CCFFWG is a CFFN, which is determined by*

$$\mathrm{CCFFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n)=\begin{pmatrix}\left\langle\begin{bmatrix}\prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{p}\left(\varphi_{\mathcal{L}_i}^{U}\right)^{\xi_i\sigma_i}\end{bmatrix},\begin{bmatrix}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}\end{bmatrix}\right\rangle,\\ \left\langle\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}},\prod_{i=1}^{p}\left(\psi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\right\rangle\end{pmatrix}. \tag{20}$$

Proof. Similar to Theorem 3, therefore omitted here. □

Example 3. *Let* $\mathcal{L}_1 = (\langle[0.4, 0.6], [0.3.0.7], (0.3, 0.5)\rangle; 0.8)$, $\mathcal{L}_2 = (\langle[0.5, 0.6], [0.4.0.5], (0.2, 0.4)\rangle; 0.7)$, *and* $\mathcal{L}_3 = (\langle[0.2, 0.3], [0.4.0.5], (0.7, 0.2)\rangle); 0.6)$ *be three CFFNs with confidence levels and* $\xi = (0.25, 0.35, 0.4)$ *be their corresponding weight vector then*

$$\prod_{i=1}^{3}\left(\varphi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}=(0.4)^{(0.25)(0.8)}\times(0.5)^{(0.35)(0.7)}\times(0.2)^{(0.4)(0.6)}=0.4774;$$
$$\prod_{i=1}^{3}\left(\varphi_{\mathcal{L}_i}^{L}\right)^{\xi_i\sigma_i}=(0.6)^{(0.25)(0.8)}\times(0.6)^{(0.35)(0.7)}\times(0.3)^{(0.4)(0.6)}=0.5967;$$
$$\sqrt[3]{1-\prod_{i=1}^{3}\left(1-\left(\psi_{\mathcal{L}_i}^{L}\right)^3\right)^{\xi_i\sigma_i}}=\left(1-\begin{pmatrix}\left(1-(0.3)^3\right)^{(0.25)(0.8)}\times\left(1-(0.4)^3\right)^{(0.35)(0.7)}\\ \times\left(1-(0.4)^3\right)^{(0.4)(0.6)}\end{pmatrix}\right)^{\frac{1}{3}}=0.3328;$$
$$\sqrt[3]{1-\prod_{i=1}^{3}\left(1-\left(\psi_{\mathcal{L}_i}^{U}\right)^3\right)^{\xi_i\sigma_i}}=\left(1-\begin{pmatrix}\left(1-(0.7)^3\right)^{(0.25)(0.8)}\times\left(1-(0.5)^3\right)^{(0.35)(0.7)}\\ \times\left(1-(0.5)^3\right)^{(0.4)(0.6)}\end{pmatrix}\right)^{\frac{1}{3}}=0.5171;$$
$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_i}\right)^3\right)^{\xi_i\sigma_i}}=\left(1-\begin{pmatrix}\left(1-(0.3)^3\right)^{(0.25)(0.8)}\times\left(1-(0.2)^3\right)^{(0.35)(0.7)}\\ \times\left(1-(0.7)^3\right)^{(0.4)(0.6)}\end{pmatrix}\right)^{\frac{1}{3}}=0.4682:$$
$$\prod_{i=1}^{p}\left(\psi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}=(0.5)^{(0.25)(0.8)}\times(0.4)^{(0.35)(0.7)}\times(0.2)^{(0.4)(0.6)}=0.4727.$$

Hence, we have

$$\mathrm{CCFFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\mathcal{L}_3)=\begin{pmatrix}\langle[0.4774, 0.5967], [0.3328,\ 0.5171]\rangle,\\ \langle 0.4682,\ 0.4727\rangle\end{pmatrix}.$$

3.2.4. Ordered Weighted Geometric Operator

Definition 19. *A CPFOWG is a mapping* $\mathrm{CPFOWG}:\Gamma^n\to\Gamma$ *defined over a collection of CCFFNs* $\mathcal{L}_i$ *with confidence levels* σ_i $(i = 1, 2, \ldots p)$ *as follows*

$$\mathrm{CCFFOWG}\left(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\ \mathcal{L}_p\right)=\sigma_I\xi_1\mathcal{L}_{\sigma(1)}\otimes\sigma_2\xi_2\mathcal{L}_{\sigma(2)}\otimes\ldots\otimes\sigma_p\xi_p\mathcal{L}_{\delta(p)} \tag{21}$$

where δ *is a permutation of* $(1, 2, \ldots, p)$, *such that* $\mathcal{L}_{\delta(i-1)} \geq \mathcal{L}_i$ *for* $i = 1, 2, \ldots, n$ *and* $\xi = \left(\xi_1, \xi_2 \ldots, \xi_p\right)^T$ *is its weight vector, such that* $\xi \succ 0$ *and* $\sum_{i=1}^{n}\xi_i = 1$. *Moreover, the* ith *largest CFFN among* $\mathcal{L}_i$*s is* $\mathcal{L}_{\delta(i)}$.

Theorem 6. *The value obtained by using the CPFOWG operator for CFFNs* $\mathcal{L}_i$ $(i = 1, 2, \ldots, p)$ *is again a CFFN and given by*

$$\mathrm{CCFFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n)=\left(\begin{array}{c}\langle\begin{bmatrix}\prod_{i=1}^{p}\left(\varphi^{L}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{p}\left(\varphi^{U}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i}\end{bmatrix},\begin{bmatrix}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^{L}_{\mathcal{L}_{\delta(i)}}\right)^{3}\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi^{U}_{\mathcal{L}_{\delta(i)}}\right)^{3}\right)^{\xi_i\sigma_i}}\end{bmatrix}\rangle,\\ \langle\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_{\delta(i)}}\right)^{3}\right)^{\xi_i\sigma_i}},\prod_{i=1}^{p}\left(\psi_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i}\rangle\end{array}\right). \quad (22)$$

Theorem 7. *Let* $\mathcal{L}_i(i=1,2,\ldots,p)$, *and* $\xi=(\xi_1,\ \xi_2,\ldots,\xi_p)^T$ *be the weight vector of* $\mathcal{L}_i$ *such that* $\xi_i\succ 0$ *and* $\sum_{i=1}^{p}\xi_i=1$, then we *have*

1. $\mathrm{CCFFWA}\left(\mathcal{L}_1^c,\ \mathcal{L}_2^c,\ldots,\mathcal{L}_p^c\right)=(\mathrm{CPFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p))^c$;
2. $\mathrm{CCFFWG}\left(\mathcal{L}_1^c,\ \mathcal{L}_2^c,\ldots,\mathcal{L}_p^c\right)=(\mathrm{CPFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p))^c$.

Proof. Since $\mathcal{L}_\mathrm{i}=\left(\langle\left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right],\ \left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)$ and $\mathcal{L}_\mathrm{i}^c=\left(\langle\left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right],\ \left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right]\rangle,\ \langle\psi_{\mathcal{L}_i},\ \varphi_{\mathcal{L}_i}\rangle\right)$, then using Equation (17), we have

$$\mathrm{CCFFWA}\left(\mathcal{L}_1^c,\ \mathcal{L}_2^c,\ldots,\mathcal{L}_p^c\right)=\left(\begin{array}{c}\langle\begin{bmatrix}\prod_{i=1}^{n}\left(\psi^{L}_{\mathcal{L}_i}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{n}\left(\psi^{L}_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\end{bmatrix},\begin{bmatrix}\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi^{L}_{\mathcal{L}_i}\right)^{2}\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\varphi^{L}_{\mathcal{L}_i}\right)^{2}\right)^{\xi_i\sigma_i}}\end{bmatrix}\rangle,\\ \langle\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\psi_{\mathcal{L}_i}\right)^{2}\right)^{\xi_i\sigma_i}},\prod_{i=1}^{n}\left(\varphi_{\mathcal{L}_i}\right)^{\xi_i\sigma_i}\rangle\end{array}\right)$$
$$=(\mathrm{CCFFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p))^c.$$

Similarly, we can prove that $\mathrm{CCFFWG}\left(\mathcal{L}_1^c,\ \mathcal{L}_2^c,\ldots,\mathcal{L}_p^c\right)=(\mathrm{CCPFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p))^c$. □

Theorem 8. *Let* $\mathcal{L}_i(i=1,2,\ldots,p)$, and $\xi=(\xi_1,\ \xi_2,\ldots,\xi_p)^T$ *be the weight vector of* $\mathcal{L}_i$ *such that* $\xi_i\succ 0$ *and* $\sum_{i=1}^{p}\xi_i=1$, *then we have*

$$\mathrm{CCFFWG}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p)\leq\mathrm{CCFFWA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p) \quad (23)$$

Proof. Easy to prove. □

Definition. 20. *For the CFFNs* $\mathcal{L}_i\ (i=1,\ 2,\ldots,p)$ *the operator* $\mathrm{CCFFHA}:\ \Gamma^n\to\Gamma$ *is given as*

$$\mathrm{CCFFHA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_p)=\sigma_1\xi_1\dot{\mathcal{L}}_{\sigma(1)}\oplus\sigma_2\xi_2\dot{\mathcal{L}}_{\sigma(2)}\oplus\ldots\oplus\sigma_p\xi_p\dot{\mathcal{L}}_{\sigma(p)} \quad (24)$$

where, $\xi=(\xi_1,\ \xi_2,\ldots,\xi_p)^T$ *be the weight vector, such that* $\xi_i\succ 0$ *and* $\sum_{i=1}^{n}\xi_i=1$ *and* $\dot{\mathcal{L}}_i{}'$s $\left(\dot{\mathcal{L}}_i=n\zeta_i\mathcal{L}_i\right)$ is $\dot{\mathcal{L}}_{\sigma(i)}$, *where n is the number of CPFNs and* $\eta=(\eta_1,\ \eta_2,\ldots,\eta_p)^T$ *is the vector corresponding to* $\mathcal{L}_i$ *with* $\zeta_i\succ 0$ *and* $\sum_{i=1}^{p}\zeta_i=1$.

Theorem. 9. *The value obtained using the CCFFHA operator for the CFFNs* $\mathcal{L}_i\ (i=1,2,\ldots,p)$ *is again a CFFN and given by*

$$\text{CCFFHA}(\mathcal{L}_1,\ \mathcal{L}_2,\ldots,\mathcal{L}_n)=\left(\begin{array}{c}\left\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\dot{\varphi}^{L}_{\mathcal{L}_{\delta(i)}}\right)^2\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\dot{\mu}^{U}_{\mathcal{L}_{\delta(i)}}\right)^2\right)^{\xi_i\sigma_i}}\end{array}\right],\left[\begin{array}{c}\prod_{i=1}^{n}\left(\dot{\psi}^{L}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{n}\left(\dot{\psi}^{U}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i}\end{array}\right]\right\rangle\\ \left\langle\prod_{i=1}^{p}\left(\dot{\varphi}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i},\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\dot{\psi}_{\mathcal{L}_{\delta(i)}}\right)^2\right)^{\xi_i\sigma_i}}\right\rangle\end{array}\right). \tag{25}$$

Proof. Easy to prove. □

Example 4. *Consider three CPFNs* β_i $(i=1,\ 2,\ 3)$, *such that* $\mathcal{L}_1=(\langle[0.30,\ 0.50],\ [0.60,\ 0.70]\rangle,\ \langle0.50,\ 0.40\rangle,0.3)$, $\mathcal{L}_2=(\langle[0.60,\ 0.70],\ [0.40,\ 0.50]\rangle,\ \langle0.50,\ 0.60\rangle,0.4)$, *and* $\mathcal{L}_3=(\langle[0.70,\ 0.80],\ [0.20,\ 0.40]\rangle,\ \langle0.50,\ 0.40\rangle,0.5)$. *Additionally, if* $\eta=(0.25,\ 0.35,\ 0.40)^T$ *is the weight vector of* $\mathcal{L}_i$ *then* $\dot{\mathcal{L}}_i=3\eta_i\mathcal{L}_i=\left(\langle\left[\dot{\varphi}^{L}_{\mathcal{L}_{\delta(i)}},\ \dot{\varphi}^{U}_{\mathcal{L}_{\delta(i)}}\right],\ \left[\dot{\psi}^{L}_{\mathcal{L}_{\delta(i)}},\ \dot{\psi}^{U}_{\mathcal{L}_{\delta(i)}}\right]\rangle,\langle\dot{\varphi}_{\mathcal{L}_{\delta(i)}},\ \dot{\psi}_{\mathcal{L}_{\delta(i)}}\rangle\right)$ $(i=1,\ 2,\ 3)$ *is computed for each CFFN as*

$$\begin{aligned}\dot{\mathcal{L}}_1&=\left(\left\langle\left[\begin{array}{c}\sqrt[3]{1-\left(1-(0.30)^3\right)^{3\times0.25}},\\ \sqrt[3]{1-(1-(0.50)^3)^{3\times0.2}5}\end{array}\right],\left[\begin{array}{c}(0.60)^{3\times0.25},\\ (0.70)^{3\times0.25}\end{array}\right]\right\rangle,\ \left\langle\begin{array}{c}(0.50)^{3\times0.25},\\ \sqrt[3]{1-(1-(0.40)^3)^{3\times0.25}}\end{array}\right\rangle\right)\\&=\left(\left\langle\left[\begin{array}{c}0.2729,\\ 0.4568\end{array}\right],\left[\begin{array}{c}0.6817,\\ 0.7653\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.5946,\\ 0.3644\end{array}\right\rangle\right);\\ \dot{\mathcal{L}}_2&=\left(\left\langle\left[\begin{array}{c}\sqrt[3]{1-\left(1-(0.60)^3\right)^{3\times0.35}},\\ \sqrt[3]{1-(1-(0.70)^3)^{3\times0.35}}\end{array}\right],\left[\begin{array}{c}(0.40)^{3\times0.35},\\ (0.50)^{3\times0.35}\end{array}\right]\right\rangle,\ \left\langle\begin{array}{c}(0.50)^{3\times0.35},\\ \sqrt[3]{1-(1-(0.60)^3)^{3\times0.35}}\end{array}\right\rangle\right)\\&=\left(\left\langle\left[\begin{array}{c}0.6087,\\ 0.7092\end{array}\right],\left[\begin{array}{c}0.3821,\\ 0.4830\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.4830,\\ 0.6087\end{array}\right\rangle\right);\\ \dot{\mathcal{L}}_3&=\left(\left\langle\left[\begin{array}{c}\sqrt[3]{1-\left(1-(0.70)^3\right)^{3\times0.4}},\\ \sqrt[3]{1-(1-(0.80)^3)^{3\times0.4}}\end{array}\right],\left[\begin{array}{c}(0.20)^{3\times0.4},\\ (0.40)^{3\times0.4}\end{array}\right]\right\rangle,\ \left\langle\begin{array}{c}(0.50)^{3\times0.4},\\ \sqrt[3]{1-(1-(0.40)^3)^{3\times0.4}}\end{array}\right\rangle\right)\\&=\left(\left\langle\left[\begin{array}{c}0.7343,\\ 0.8326\end{array}\right],\left[\begin{array}{c}0.1450,\\ 0.3330\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.4353,\\ 0.4241\end{array}\right\rangle\right).\end{aligned}$$

The score values of these numbers are calculated as

$$\begin{aligned}sc\left(\dot{\mathcal{L}}_1\right)&=\tfrac{(0.2729)^3+(0.4568)^3-(0.6817)^3-(0.7653)^3}{2}-\left((0.5946)^3-(0.3644)^3\right)=-0.4865;\\ sc\left(\dot{\mathcal{L}}_2\right)&=\tfrac{(0.6087)^3+(0.7092)^3-(0.3821)^3-(0.4830)^3}{2}-\left((0.4830)^3-(0.6087)^3\right)=0.0948;\\ sc\left(\dot{\mathcal{L}}_3\right)&=\tfrac{(0.7343)^3+(0.8326)^3-(0.1450)^3-(0.3330)^3}{2}-\left((0.4353)^3-(0.4241)^3\right)=0.3841.\end{aligned}$$

Thus, $sc\left(\dot{\mathcal{L}}_3\right)\succcurlyeq sc\left(\dot{\mathcal{L}}_2\right)\succcurlyeq sc\left(\dot{\mathcal{L}}_1\right)$, *which gives*

$$\begin{aligned}\dot{\beta}_{\sigma(1)}&=\left(\left\langle\left[\begin{array}{c}0.7343,\\ 0.8326\end{array}\right],\left[\begin{array}{c}0.1450,\\ 0.3330\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.4353,\\ 0.4241\end{array}\right\rangle\right),\\ \dot{\beta}_{\sigma(2)}&=\left(\left\langle\left[\begin{array}{c}0.6087,\\ 0.7092\end{array}\right],\left[\begin{array}{c}0.3821,\\ 0.4830\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.4830,\\ 0.6087\end{array}\right\rangle\right),\end{aligned}$$

and

$$\dot{\beta}_{\delta(3)}=\left(\left\langle\left[\begin{array}{c}0.2729,\\ 0.4568\end{array}\right],\left[\begin{array}{c}0.6817,\\ 0.7653\end{array}\right]\right\rangle,\left\langle\begin{array}{c}0.5946,\\ 0.3644\end{array}\right\rangle\right)$$

Let $\xi=(0.35,\ 0.4,\ 0.25)$ *be the position vector, then by using Equation (25), we have*

$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\dot{\varphi}^{L}_{\mathcal{L}_{\delta(i)}}\right)^{3}\right)^{\xi_i\sigma_i}} = \sqrt[3]{1-\left(\begin{array}{c}\left(1-(0.7343)^3\right)^{0.35\times 0.5}\times\left(1-(0.6087)^3\right)^{0.4\times 0.4}\\ \times\left(1-(0.2729)^3\right)^{0.25\times 0.3}\end{array}\right)} = 0.4966;$$

$$\sqrt[3]{1-\prod_{i=1}^{n}\left(1-\left(\dot{\varphi}^{U}_{\sigma(i)}\right)^{3}\right)^{\xi_i\sigma_i}} = \sqrt[3]{1-\left(\begin{array}{c}\left(1-(0.8326)^3\right)^{0.35\times 0.5}\times\left(1-(0.7092)^3\right)^{0.4\times 0.4}\\ \times\left(1-(0.4568)^3\right)^{0.25\times 0.3}\end{array}\right)} = 0.5891;$$

$$\prod_{i=1}^{p}\left(\dot{\psi}^{L}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} = (0.1450)^{0.35\times 0.5}(0.3821)^{0.4\times 0.4}(0.6817)^{0.25\times 0.3} = 0.5942;$$

$$\prod_{i=1}^{p}\left(\dot{\psi}^{U}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} = (0.3330)^{0.35\times 0.5}(0.4830)^{0.4\times 0.4}(0.7653)^{0.25\times 0.3} = 0.7197;$$

$$\prod_{i=1}^{p}\left(\dot{\psi}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i} = (0.4353)^{0.35\times 0.5}(0.4830)^{0.4\times 0.4}(0.5946)^{0.25\times 0.3} = 0.7401;$$

$$\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\dot{\psi}_{\mathcal{L}_{\delta(i)}}\right)^{3}\right)^{\xi_i\sigma_i}} = \sqrt[3]{1-\left(\begin{array}{c}\left(1-(0.4241)^3\right)^{0.35\times 0.5}\times\left(1-(0.6087)^3\right)^{0.4\times 0.4}\\ \times\left(1-(0.3644)^3\right)^{0.25\times 0.3}\end{array}\right)} = 0.3844.$$

Therefore,

$$\mathrm{CCFFHA}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3) = \left(\left\langle\begin{bmatrix}0.4966,\\ 0.5891\end{bmatrix},\begin{bmatrix}0.5942,\\ 0.7197\end{bmatrix}\right\rangle,\left\langle\begin{matrix}0.7401,\\ 0.3844\end{matrix}\right\rangle\right).$$

4. Decision-Making Approach under Cubic Fermatean Fuzzy Sets with Confidence Levels

This section presents an MCDM approach to deal with MCDM problems by using the proposed aggregation operators under a cubic Fermatean fuzzy environment. The MCDM problem is presented for evaluation with a cubic Fermatean fuzzy environment with the following presumptions and abbreviations. Let $X = \{X_1, X_2, \ldots, X_p\}$ be the set of m different alternatives which have to be analyzed under the set of q different criteria $C = \{C_1, C_2, \ldots, C_q\}$. Suppose that all these possibilities are examined by experts, which provide their choices for each X_i $(i = 1, 2, \ldots, p)$, under a cubic Fermatean fuzzy environment, and that these values can be considered as CFFNs $\mathcal{D} = \left[\lceil_{ij}\right]_{p\times q}$ where $\lceil_{ij} = \left(\left\langle\left[\varphi^{L}_{\mathcal{L}_{ij}}, \varphi^{U}_{\mathcal{L}_{ij}}\right], \left[\psi^{L}_{\mathcal{L}_{ij}}, \psi^{U}_{\mathcal{L}_{ij}}\right]\right\rangle, \left\langle\varphi_{\mathcal{L}_{ij}}, \psi_{\mathcal{L}_{ij}}\right\rangle\right)$ characterizes the importance values of alternative X_i given by the decision-maker such that $0 \le \varphi^{L}_{\mathcal{L}_{ij}}, \varphi^{U}_{\mathcal{L}_{ij}}, \psi^{L}_{\mathcal{L}_{ij}}, \psi^{U}_{\mathcal{L}_{ij}}, \varphi_{\mathcal{L}_{ij}}, \psi_{\mathcal{L}_{ij}} \le 1$, $\left[\varphi^{L}_{\mathcal{L}_{ij}}, \varphi^{U}_{\mathcal{L}_{ij}}\right], \left[\psi^{L}_{\mathcal{L}_{ij}}, \psi^{U}_{\mathcal{L}_{ij}}\right] \subseteq [0, 1]$, which satisfy the conditions that $\left(\varphi^{U}_{ij}\right)^3 + \left(\psi^{U}_{ij}\right)^3 \le 1$ and $\left(\varphi_{\mathcal{L}_{ij}}\right)^3 + \left(\psi_{\mathcal{L}_{ij}}\right)^3 \le 1$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ be the weight vector of the criteria such that $\xi_i \succ 0$ and $\sum_{i=1}^{p}\xi_i = 1$. Additionally, let σ_i be the confidence levels of the CFFNs $\mathcal{L}_i$ such that $0 \le \sigma_i \le 1$. In order to identify the optimal alternative(s), the presented approach is divided into the following steps.

Step 1. Arrange the confidence and capability for each alternative X_i in the form of $\lceil_{ij} = \left(\left\langle\left[\varphi^{L}_{\mathcal{L}_{ij}}, \varphi^{U}_{\mathcal{L}_{ij}}\right], \left[\psi^{L}_{\mathcal{L}_{ij}}, \psi^{U}_{\mathcal{L}_{ij}}\right]\right\rangle, \left\langle\varphi_{\mathcal{L}_{ij}}, \psi_{\mathcal{L}_{ij}}\right\rangle\right)$. These rating values are expressed as a decision matrix $\mathcal{D}$ as:

$$\mathcal{D} = \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_p \end{array}\begin{pmatrix} \lceil_{11} & \lceil_{12} & \cdots & \lceil_{1p} \\ \lceil_{21} & \lceil_{22} & \cdots & \lceil_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \lceil_{q1} & \lceil_{q2} & \cdots & d_{pq} \end{pmatrix}. \quad (26)$$

Step 2. Convert the cost-type criteria into benefit-type criteria by using the normalization formula as given below:

$$\mathcal{r}_{ij}=\begin{cases}\left(\langle\left[\varphi^{L}_{\mathcal{L}_{ij}},\varphi^{U}_{\mathcal{L}_{ij}}\right],\ \left[\psi^{L}_{\mathcal{L}_{ij}},\psi^{U}_{\mathcal{L}_{ij}}\right]\rangle,\langle\varphi_{\mathcal{L}_{ij}},\psi_{\mathcal{L}_{ij}}\rangle\right);\ \text{if the benefit}-\text{type conditions are met}\\ \left(\left[\psi^{L}_{\mathcal{L}_{ij}},\psi^{U}_{\mathcal{L}_{ij}}\right],\left[\varphi^{L}_{\mathcal{L}_{ij}},\varphi^{U}_{\mathcal{L}_{ij}}\right],\langle\nu_{\mathcal{L}_{ij}},\mu_{ij}\rangle\right);\ \text{if the cost}-\text{type conditions are met}\end{cases} \tag{27}$$

Step 3. Calculate the aggregated value $\mathcal{r}_i$ $(i=1,2,\ldots,p)$ of the alternative X_i, using CCFFWA, CCFFOWA, CCFFHA, CCFFWG, or CCFFOWG operators.

(a) Using a CCFFWA operator

$$\begin{aligned}\mathcal{r}_i&=\left(\langle\left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right],\ \left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)=\text{CCFFWA}(\mathcal{L}_{i1},\ \mathcal{L}_{i2},\ldots,\mathcal{L}_{ip})\\&=\left(\begin{array}{c}\left[\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\varphi^{L}_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\varphi^{U}_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right],\left[\begin{array}{c}\prod_{j=1}^{p}\left(\psi^{L}_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i},\\ \prod_{j=1}^{p}\left(\psi^{L}_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i}\end{array}\right],\\ \prod_{j=1}^{p}\left(\varphi_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i},\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\psi_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right);\end{aligned} \tag{28}$$

(b) using a CCFFOWA operator

$$\begin{aligned}\mathcal{r}_i&=\left(\langle\left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right],\ \left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)=\text{CCFFOWA}\left(\mathcal{r}_{\delta(i1)},\ \mathcal{r}_{\delta(i2)},\ldots,\mathcal{r}_{\delta(ip)}\right)\\&=\left(\begin{array}{c}\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\varphi^{L}_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\varphi^{U}_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right],\left[\begin{array}{c}\prod_{j=1}^{p}\left(\psi^{L}_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i},\\ \prod_{j=1}^{p}\left(\psi^{U}_{\mathcal{L}_{\delta(i)}}\right)^{\xi_i\sigma_i}\end{array}\right]\rangle,\\ \langle\prod_{j=1}^{p}\left(\varphi_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i},\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\psi_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}\rangle\end{array}\right);\end{aligned} \tag{29}$$

(c) using a CCFFHA operator

$$\begin{aligned}\mathcal{r}_i&=\left(\langle\left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right],\ \left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right)=\text{CPFFHA}\left(\dot{\mathcal{r}}_{\delta(i1)},\ \dot{\mathcal{r}}_{\delta(i2)},\ldots,\dot{\mathcal{r}}_{\delta(ip)}\right)\\&=\left(\begin{array}{c}\langle\left[\begin{array}{c}\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\dot{\varphi}^{L}_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\left(\dot{\mu}^{U}_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right],\left[\begin{array}{c}\prod_{j=1}^{p}\left(\dot{\psi}^{L}_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i},\\ \prod_{j=1}^{p}\left(\dot{\psi}^{U}_{\mathcal{L}_{\delta(iJ)}}\right)^{\xi_i\sigma_i}\end{array}\right]\rangle,\\ \langle\prod_{j=1}^{p}\left(\dot{\varphi}_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i},\sqrt[3]{1-\prod_{j=1}^{p}\left(1-\left(\dot{\psi}_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}\rangle\end{array}\right);\end{aligned} \tag{30}$$

(d) using a CCFFWG operator

$$\begin{aligned}\mathcal{r}_i&=\left(\langle\left[\varphi^{L}_{\mathcal{L}_i},\ \varphi^{U}_{\mathcal{L}_i}\right],\ \left[\psi^{L}_{\mathcal{L}_i},\psi^{U}_{\mathcal{L}_i}\right]\rangle,\ \varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\right)=\text{CCFFWG}(\mathcal{r}_{i1},\ \mathcal{r}_{i2},\ldots,\mathcal{r}_{ip})\\&=\left(\begin{array}{c}\langle\left[\begin{array}{c}\prod_{i=1}^{p}\left(\varphi^{L}_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i},\\ \prod_{i=1}^{p}\left(\varphi^{U}_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i}\end{array}\right],\left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi^{L}_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}},\\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi^{U}_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right]\rangle,\\ \langle\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_{ij}}\right)^3\right)^{\xi_i\sigma_i}},\prod_{i=1}^{p}\left(\psi_{\mathcal{L}_{ij}}\right)^{\xi_i\sigma_i}\rangle\end{array}\right);\end{aligned} \tag{31}$$

(e) using a CCFFOWG operator

$$\mathscr{r}_i = \left(\langle\left[\varphi^L_{\mathcal{L}_i},\ \varphi^U_{\mathcal{L}_i}\right],\ \left[\psi^L_{\mathcal{L}_i}, \psi^U_{\mathcal{L}_i}\right]\rangle,\ \langle\varphi_{\mathcal{L}_i},\ \psi_{\mathcal{L}_i}\rangle\right) = \mathrm{CCFFOWG}(\mathscr{r}_{i1},\ \mathscr{r}_{i2}, \ldots, \mathscr{r}_{ip})$$
$$= \left(\begin{array}{c} \langle\left[\begin{array}{c}\prod_{i=1}^{p}\left(\varphi^L_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i}, \\ \prod_{i=1}^{p}\left(\varphi^U_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i}\end{array}\right], \left[\begin{array}{c}\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi^L_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}, \\ \sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\psi^U_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}\end{array}\right]\rangle, \\ \langle\sqrt[3]{1-\prod_{i=1}^{p}\left(1-\left(\varphi_{\mathcal{L}_{\delta(ij)}}\right)^3\right)^{\xi_i\sigma_i}}, \prod_{i=1}^{p}\left(\psi_{\mathcal{L}_{\delta(ij)}}\right)^{\xi_i\sigma_i}\rangle \end{array}\right). \tag{32}$$

Step 4. Compute the collected score values of each alternative as follows:

$$sc(\mathscr{r}_i) = \frac{\left(\varphi^L_{\mathcal{L}_{ij}}\right)^3 + \left(\varphi^U_{\mathcal{L}_{ij}}\right)^3 - \left(\psi^L_{\mathcal{L}_{ij}}\right)^3 - \left(\psi^U_{\mathcal{L}_{ij}}\right)^3}{2} + \left(\psi^3_{\mathcal{L}_{ij}} - \varphi^3_{\mathcal{L}_{ij}}\right). \tag{33}$$

If $sc(\mathscr{r}_{i_1}) = sc(\mathscr{r}_{i_2})$ for any two indices i_1 and i_2, then compute accuracy values as

$$ac(\mathscr{r}_i) = \frac{\left(\varphi^L_{\mathcal{L}_{ij}}\right)^3 + \left(\varphi^U_{\mathcal{L}_{ij}}\right)^3 + \left(\psi^L_{\mathcal{L}_{ij}}\right)^3 + \left(\psi^U_{\mathcal{L}_{ij}}\right)^3}{2} + \left(\varphi^3_{\mathcal{L}_{ij}} + \psi^3_{\mathcal{L}_{ij}}\right). \tag{34}$$

Step 5. By rating all of the alternatives in order of importance of the score values choose the best alternative.

4.1. Case Study

Inventory management is a major subject these days. From an industrial standpoint, a corporation cannot achieve targeted levels of manufacturing unless its inventory is adequately maintained. Therefore, appropriate inventory management is the first stage of the ladder of suitable levels of production. Any scarcity of raw materials in stock might cause a disruption of the entire manufacturing process, which would result in a significant loss for the industry. Suppose a food corporation wishes to monitor different inventory products. The corporation primarily manufactures four different types of food: drinks (X_1), palm oil (X_2), pickles (X_3), and sweets (X_4). Three factors namely cost price (C_1), storage facilities (C_2), and staleness level (C_3) must be taken into consideration while deciding whether to reorder ingredients for making these food products such that $\xi = (0.25, 0.35, 0.4)$ is the weight vector of these factors. The presented alternatives are examined under these three factors and their values are scored in terms of CFFNs. In each CFFN, the interval-valued FFNs (IVFFNs) indicate the current stock level in the inventory, and the FFNs represent the estimate of agreement and disagreement towards the present stock level for a coming week. Since the corporation does not sacrifice product quality, reducing staleness levels is given top attention. The main objective is then to determine the food products for which the ingredient stock must be reordered frequently. The following steps of the proposed approach were carried out for it.

Step 1. As described in Table 1, the desired data for each alternative is presented in CFFNs, and the collection evaluation is provided in a decision matrix.

Table 1. Assessment values of alternatives in terms of CFFNs with confidence levels.

Alternatives	C_1	C_2	C_3
X_1	$\begin{pmatrix} \langle \begin{bmatrix}0.55,\\0.65\end{bmatrix}, \begin{bmatrix}0.15,\\0.25\end{bmatrix} \rangle;0.4 \\ (0.45,0.25)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.45,\\0.55\end{bmatrix} \rangle;0.6 \\ (0.35,0.25)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.65\end{bmatrix}, \begin{bmatrix}0.25,\\0.35\end{bmatrix} \rangle;0.5 \\ (0.25,0.40)\end{pmatrix}$
X_2	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.45,\\0.55\end{bmatrix} \rangle;0.5 \\ (0.45,0.65)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.20,\\0.30\end{bmatrix}, \begin{bmatrix}0.35,\\0.40\end{bmatrix} \rangle;0.4 \\ (0.25,0.35)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.45\end{bmatrix}, \begin{bmatrix}0.15,\\0.25\end{bmatrix} \rangle;0.6 \\ (0.45,0.55)\end{pmatrix}$
X_3	$\begin{pmatrix} \langle \begin{bmatrix}0.55,\\0.65\end{bmatrix}, \begin{bmatrix}0.25,\\0.35\end{bmatrix} \rangle;0.3 \\ (0.25,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.65\end{bmatrix}, \begin{bmatrix}0.30,\\0.40\end{bmatrix} \rangle;0.3 \\ (0.35,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.55,\\0.75\end{bmatrix}, \begin{bmatrix}0.15,\\0.20\end{bmatrix} \rangle;0.7 \\ (0.35,0.55)\end{pmatrix}$
X_4	$\begin{pmatrix} \langle \begin{bmatrix}0.35,\\0.55\end{bmatrix}, \begin{bmatrix}0.15,\\0.35\end{bmatrix} \rangle;0.5 \\ (0.15,0.35)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.60\end{bmatrix}, \begin{bmatrix}0.20,\\0.25\end{bmatrix} \rangle;0.7 \\ (0.35,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.55\end{bmatrix}, \begin{bmatrix}0.25,\\0.35\end{bmatrix} \rangle;0.8 \\ (0.50,0.40)\end{pmatrix}$

Step 2. Using Equation (26), it is possible to derive a normalized decision matrix which is summarized in Table 2.

Table 2. Normalized decision matrix.

Alternatives	C_1	C_2	C_3
X_1	$\begin{pmatrix} \langle \begin{bmatrix}0.15,\\0.25\end{bmatrix}, \begin{bmatrix}0.55,\\0.65\end{bmatrix} \rangle;0.4 \\ (0.25,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.45,\\0.55\end{bmatrix} \rangle;0.6 \\ (0.35,0.25)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.45,\\0.65\end{bmatrix} \rangle;0.5 \\ (0.40,0.25)\end{pmatrix}$
X_2	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.55\end{bmatrix}, \begin{bmatrix}0.25,\\0.35\end{bmatrix} \rangle;0.5 \\ (0.65,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.20,\\0.30\end{bmatrix}, \begin{bmatrix}0.35,\\0.40\end{bmatrix} \rangle;0.4 \\ (0.25,0.35)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.15,\\0.25\end{bmatrix}, \begin{bmatrix}0.25,\\0.45\end{bmatrix} \rangle;0.6 \\ (0.55,0.45)\end{pmatrix}$
X_3	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.55,\\0.65\end{bmatrix} \rangle;0.3 \\ (0.45,0.25)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.65\end{bmatrix}, \begin{bmatrix}0.30,\\0.40\end{bmatrix} \rangle;0.3 \\ (0.35,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.15,\\0.20\end{bmatrix}, \begin{bmatrix}0.55,\\0.75\end{bmatrix} \rangle;0.7 \\ (0.55,0.35)\end{pmatrix}$
X_4	$\begin{pmatrix} \langle \begin{bmatrix}0.15,\\0.35\end{bmatrix}, \begin{bmatrix}0.35,\\0.55\end{bmatrix} \rangle;0.5 \\ (0.35,0.15)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.45,\\0.60\end{bmatrix}, \begin{bmatrix}0.20,\\0.25\end{bmatrix} \rangle;0.7 \\ (0.35,0.45)\end{pmatrix}$	$\begin{pmatrix} \langle \begin{bmatrix}0.25,\\0.35\end{bmatrix}, \begin{bmatrix}0.45\\,0.55\end{bmatrix} \rangle;0.8 \\ (0.40,0.50)\end{pmatrix}$

Step 3. Aggregate the values of Table 2 with the proposed operators:

(a) Using Equation (28) we get

$$\begin{aligned}
r_1 &= (\langle[0.1892, 0.2685], [0.6790, 0.7751]\rangle, (0.5814, 0.2514));\\
r_2 &= (\langle[0.2398, 0.3104], [0.5205, 0.6369]\rangle, (0.6761, 0.3425));\\
r_3 &= (\langle[0.2295, 0.3376], [0.5715, 0.7032]\rangle, (0.7135, 0.5335));\\
r_4 &= (\langle[0.3053, 0.4237], [0.4579, 0.5457]\rangle, (0.5058, 0.4008)).
\end{aligned}$$

(b) Using Equation (29) we get

$$\begin{aligned}
r_1 &= (\langle[0.1757, 0.2536], [0.6941, 0.7895]\rangle, (0.5603, 0.2904));\\
r_2 &= (\langle[0.2773, 0.3507], [0.5100, 0.6239]\rangle, (0.7056, 0.3493));\\
r_3 &= (\langle[0.2855, 0.4200], [0.5999, 0.6988]\rangle, (0.6441, 0.4218));\\
r_4 &= (\langle[0.2766, 0.3923], [0.5368, 0.6297]\rangle, (0.5933, 0.3195)).
\end{aligned}$$

(c) Using Equation (30) we get

$$\begin{aligned}
r_1 &= (\langle[0.4724, 0.5803], [0.4016, 0.5234]\rangle, (0.2971, 0.5310));\\
r_2 &= (\langle[0.4612, 0.5723], [0.2154, 0.3146]\rangle, (0.4333, 0.6352));\\
r_3 &= (\langle[0.4321, 0.5276], [0.2311, 0.3214]\rangle, (0.4523, 0.5876));\\
r_4 &= (\langle[0.5889, 0.6794], [0.4023, 0.3367]\rangle, (0.4429, 0.6054)).
\end{aligned}$$

(d) Using Equation (31) we get

$$\begin{aligned}
\mathcal{r}_1 &= (\langle[0.4686, 0.5661], [0.3820, 0.5012]\rangle, (0.2873, 0.5230));\\
\mathcal{r}_2 &= (\langle[0.4582, 0.5621], [0.2277, 0.3331]\rangle, (0.4352, 0.6450));\\
\mathcal{r}_3 &= (\langle[0.4872, 0.5629], [0.2746, 0.3829]\rangle, (0.3932, 0.7646));\\
\mathcal{r}_4 &= (\langle[0.4163, 0.5530], [0.3341, 0.4333]\rangle, (0.3323, 0.5197)).
\end{aligned}$$

(e) Using Equation (32) we get

$$\begin{aligned}
\mathcal{r}_1 &= (\langle[0.3390, 0.4463], [0.4369, 0.5695]\rangle, (0.3109, 0.4437));\\
\mathcal{r}_2 &= (\langle[0.3665, 0.4768], [0.2426, 0.3644]\rangle, (0.5061, 0.5586));\\
\mathcal{r}_3 &= (\langle[0.3768, 0.4715], [0.3237, 0.4366]\rangle, (0.4268, 0.6303));\\
\mathcal{r}_4 &= (\langle[0.4092, 0.5759], [0.2921, 0.4118]\rangle, (0.3191, 0.4462)).
\end{aligned}$$

Step 4. Compute the score values by using Equation (33); the results are listed in Table 3.

Table 3. Score values and ranking order of alternatives with different operators.

Operators	$sc(\mathcal{r}_1)$	$sc(\mathcal{r}_2)$	$sc(\mathcal{r}_3)$	$sc(\mathcal{r}_4)$	Ranking
CCFFWA	−0.5569	−0.4467	−0.4533	−0.1420	$X_4 \succ X_2 \succ X_3 \succ X_1$
CCFFOWA	−0.5538	−0.4202	−0.4221	−0.3376	$X_4 \succ X_2 \succ X_3 \succ X_1$
CCFFHA	0.1698	0.1971	0.2014	0.3423	$X_4 \succ X_2 \succ X_3 \succ X_1$
CCFFWG	0.1650	0.2948	0.1707	0.4732	$X_4 \succ X_2 \succ X_3 \succ X_1$
CCFFOWG	−0.0128	0.1932	0.0922	0.2387	$X_4 \succ X_2 \succ X_3 \succ X_1$

Step 5. Rankings of all the alternatives based on the score values and ordering are listed in the last column of Table 3. From this analysis, it is seen that X_4 is the best one among the others.

4.2. Validity Tests

To illustrate the feasibility of the proposed strategy in a multitude of environments, we used testing procedures defined by Wang and Trianaphyllou [31] as follows:

Test 1. If we replace the rating values of the non-optimal alternatives with those of a worse alternative, the best alternative should remain stable as long as the relative weighted criteria remain fixed.

Test 2. The procedure should be transitive.

Test 3. When a specific problem is separated into smaller ones while the same decision-making approach is used, the aggregated ranking of the alternatives should be equivalent to the original ranking.

Validity test using criterion 1

The ranking order of alternatives obtained by the proposed approach is $X_4 \succ X_2 \succ X_3 \succ X_1$. To test the corresponding nature of the proposed approach by test criterion 1, the non-optimal alternative X_1 was replaced with the worst alternative X_1^* where rating values of X_1^* were assumed to be $([\langle 0.1, 0.2], [0.6, 0.7], (0.2, 0.6)\rangle)$, $(\langle[0.2, 0.3], [0.5, 0.6], (0.3, 0.5)\rangle)$, and $(\langle[0.25, 0.35], [0.3, 0.4], (0.1, 0.5)\rangle)$. Following the observations, the presented approach was used, and the aggregated score values of the alternatives were $sc(X_1) = 0.1606$, $sc(X_1^*) = 0.0031$, $sc(X_3) = 0.3676$, and $sc(X_4) = 0.3646$. As a result, the ranking order was $X_4 \succ X_3 \succ X_1 \succ X_1^*$ with the best alternative remaining the same as in the proposed approach. Thus, the presented approach yielded consistent findings in term of test criterion 1.

Validity test using criteria 2 and 3

For testing validity according to criteria 2 and 3, the fragmented decision-making subcases are taking as $\{X_1, X_2, X_4\}$, $\{X_2, X_3, X_4\}$, and $\{X_2, X_3, X_1\}$. Then, using the described process, their rank order is as follows: $X_4 \succ X_2 \succ X_1$, $X_4 \succ X_2 \succ X_3$ and $X_2 \succ X_3 \succ X_1$, for criterion 2 and 3, respectively. When all of the results are combined, the overall ranking is $X_4 \succ X_2 \succ X_3 \succ X_1$ which is the same as the outcomes of the original decision-making approach. Hence, our proposed approach is valid under test criteria 2 and 3.

4.3. Comparative Analysis

In the literature, there are numerous types of fuzzy sets that are used for specific situations based on their properties. The fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Fermatean fuzzy sets are some of the most popular sets of fuzzy set theory. Cubic Fermatean fuzzy sets with confidence levels are an innovative variation of the fuzzy set theory that we introduce in this study. Table 4 compares each of these fuzzy sets with respect to a number of attributes. Each of them has a graded membership value and the capacity to describe uncertainty across multiple attributes.

Table 4. Different type of fuzzy sets and their features.

Characteristics	Different Types of Fuzzy Sets					
	Fuzzy Set	**IFS**	**PFS**	**FFS**	**CFFS**	**CFFSCL**
Membership value	✓	✓	✓	✓	✓	✓
Describe ambiguity	MG	MG and NMG	MG and NMG	MG and NMG	MG and NMG	MG and NMG with confidence levels
Unknown parameters	×	×	×	×	×	✓
Ability of multi-attribute modeling	✓	✓	✓	✓	✓	✓
Modeling of increasing uncertainty	×	×	×	×	×	✓
Taking reluctance into account while making decisions	×	✓	×	✓	×	✓

Abbreviations: IFS: intuitionistic fuzzy set; PFS: Pythagorean fuzzy set; FFS: Fermatean fuzzy set; CFFS: Cubic Fermatean fuzzy set; CFFSCL: Cubic Fermatean fuzzy set with confidence levels; MG: membership grade; NMG: Non-membership grade.

4.4. Comparison with Some Existing Approaches

An evaluation was conducted to examine the performance of the new method compared to existing approaches [23,27,28,32] in the context of CPFSs and CIFSs. Sacrificing flexibility, we examined the situation by using the weight of decision-makers as $\xi = (0.25, 0.35, 40)$ which allows for the existing approaches to be used with the original dataset. The results obtained with different methods are summarized in Table 5 and we conclude that the ranking order of the given alternatives is $X_4 \succ X_2 \succ X_3 \succ X_1$, hence the best alternative is X_4 which coincides with the proposed approach results given in Table 3, which validates the stability of our approach. Furthermore, the structure of the relative score values follows the same pattern, demonstrating that the presented approach is conservative in nature.

Table 5. Comparison with existing studies.

Existing Approaches	$sc(X_1)$	$sc(X_2)$	$sc(X_3)$	$sc(X_4)$	Ranking Order
Garg and Kaur [32]	−0.6294	−0.4567	−0.5321	−0.2310	$X_4 \succ X_2 \succ X_3 \succ X_1$
Amin et al. [27]	−0.7312	−0.5438	−0.4877	−0.3421	$X_4 \succ X_2 \succ X_3 \succ X_1$
Rahim et al. [28]	−0.3643	−0.3241	−0.3575	−0.2783	$X_4 \succ X_2 \succ X_3 \succ X_1$
Kaur and Garg [23]	−0.5666	−0.4763	−0.5198	−0.3417	$X_4 \succ X_2 \succ X_3 \succ X_1$

According to the comparative study described above, the presented strategy for handling decision-making problems has significant improvements over existing ones.

(1) Cubic Fermatean fuzzy sets are a new development in fuzzy set theory, which can handle the uncertainty more accurately in real situations. Therefore, the proposed approach is more suitable than existing approaches to solve real-life and engineering decision problems.
(2) Furthermore, Table 4 demonstrates that the findings calculated using the different available methods are performed without taking the confidence levels of the attributes into account throughout the analysis. In other words, all of these techniques examined their theories on the premise that decision-makers are completely confident in the analyzed objects. However, in practice these sorts of prerequisites are only partially met.
(3) The existing aggregation operators are a special case of the presented operators. As a result, we conclude that the presented aggregation operators are more general in nature and more appropriate to solve real-world issues than the existing ones.

5. Conclusions

The main purpose of this research was to modify the existing operational laws of cubic Fermatean fuzzy sets, and propose a number of aggregation operators by taking into account the degree of confidence levels of each decision-maker during evaluation. Previously, all decision-makers were considered to express their opinions of numerous alternatives with a same level of certainty. However, this issue has been solved in the current article by factoring in the decision- maker's confidence levels. We introduced a number of aggregating operators under the cubic Fermatean fuzzy framework, including CCFFWA, CCFFOWA, CCFFWG, and CCFFHA, by taking confidence levels into account. A few significant traits of each were also described. Additionally, the standard cubic Fermatean fuzzy weighted averaging and cubic Fermatean fuzzy weighted geometric operators were transformed into the provided aggregation operators when $\sigma = 1$ for all preferences. A comparison with several existing operators was performed to show that the provided operators offer a reducible approach to the MCDM problem.

Future research may further develop the outlined technique to support a wider range of applications and address a variety of uncertain programming difficulties, such as K-mean clustering [33] and fuzzy controllers [34,35]. The application will also be expanded to include neural networks and convolutional networks [36–39].

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