symmetry

MDPI

Article

Singularities for Timelike Developable Surfaces in Minkowski 3-Space

Yanlin Li [1,*], Zhizhi Chen [2], Sahar H. Nazra [3] and Rashad A. Abdel-Baky [4]

1 School of Mathematics, Hangzhou Normal University, Hangzhou 311120, China
2 School of Statistics, Jilin University of Finance and Economics, Changchun 130117, China
3 Department of Mathematical Sciences, College of Applied Sciences, Umm Al-Qura University, Mecca 24382, Saudi Arabia
4 Department of Mathematics, Faculty of Science, University of Assiut, Assiut 71516, Egypt
* Correspondence: liyl@hznu.edu.cn

Abstract: In this paper, we consider the singularities and geometrical properties of timelike developable surfaces with Bishop frame in Minkowski 3-space. Taking advantage of the singularity theory, we give the classification of generic singularities of these developable surfaces. Furthermore, an example of application is given to illustrate the applications of the results.

Keywords: timelike developable surfaces; singularities; developable surfaces

check for updates

Citation: Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singularities for Timelike Developable Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 277. https://doi.org/10.3390/sym15020277

Academic Editor: Abraham A. Ungar

Received: 21 October 2022
Revised: 9 January 2023
Accepted: 16 January 2023
Published: 19 January 2023

1. Introduction

A channel or canal surface is a surface traced by a one-parameter family of spheres, whose center is on a smooth space curve, its directrix or spine. If the radii of the traced spheres are stationary then the canal surface is named a tubular or sweeping surface. There are several famous examples: circular cylinder, right circular cone, torus, and rotation surface. This visualization is a generalization of the classical registration of a partner of a planar curve [1–6]. One of the eminent facts about sweeping surface is that the sweeping surface can be a developable surface, that is, a smooth surface can be flattened onto a plane without distortion. Therefore, the developable surface can be made out of sheet metal, since such a surface must be attainable by modification from a plane, and every point on such a surface lies on at least one straight line. Therefore, sweeping surfaces and developable surfaces have been paid interest to in engineering, architecture, and design, etc. [7–18].

In Euclidean 3-space $\mathbb{E}^3$, in spite of the fact that the Serret–Frenet frame can simply be calculated, it is not continuously defined for a C^1-continuous space curve, and even for a C^2-continuous space curve the Serret–Frenet frame becomes undefined at an inflection point (i.e., curvature $\kappa = 0$), thus causing unacceptable discontinuity when utilized for surface modeling. Thus, in Ref. [19], Bishop defined a novel frame named the alternative frame or Bishop frame, which could yield the desired means to move smoothly on a space curve. We know that the Bishop frame in Euclidean 3-space is a sharp tool used to investigate the topological and geometrical properties of curves in Euclidean 3-space. Even though the second derivative of the curves are vanishing, the Bishop frame can still work for them. The idea can be expanded to the Minkowski space. Therefore, we give the Minkowski version moving frame, which is the Minkowski Bishop frame. We can find many articles related with the Minkowski Bishop frames, for example, Refs. [20–23].

The present work studies the singularities and geometrical properties of timelike developable surfaces according to the Minkowski Bishop frame. The field, as far as we are aware, is not covered by any articles, though it has been carefully and ably worked by many researchers. Thus, the present paper hopes to help such a need, and it is inspired by the works of Izumiya, and Haiming et al. [3,23–26]. In this article, we construct the Bishop frame of speed timelike curve and take advantage of the singularity theory to study the singularities and geometrical properties of timelike developable.

2. Preliminaries

In this section, we present some definitions and employ notions that we will use in this paper (see for instance Refs. [1,24]). Let $\mathbb{R}^3 = \{(p_1, p_2, p_3) \mid, p_i \in \mathbb{R}$ (i=1, 2, 3)$\}$ be a 3-dimensional Cartesian space. For any $\mathbf{q} = (q_1, q_2, q_3)$, and $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$, the pseudo scalar product of $\mathbf{q}$ and $\mathbf{p}$ is defined by

$$< \mathbf{q}, \mathbf{p} > = -q_1 p_1 + q_2 p_2 + q_3 p_3.$$

We name $(\mathbb{R}^3, <,>)$ Minkowski 3-space. We write $\mathbb{E}_1^3$ instead of $(\mathbb{R}^3, <,>)$. We say that a non-zero vector $\mathbf{q} \in \mathbb{E}_1^3$ is spacelike, lightlike, or timelike if $<\mathbf{q},\mathbf{q}>>0$, $<\mathbf{q},\mathbf{q}> = 0$ or $<\mathbf{q},\mathbf{q}><0$, respectively. The norm of the vector $\mathbf{q} \in \mathbb{E}_1^3$ is defined to be $\|\mathbf{q}\| = \sqrt{|< \mathbf{q}, \mathbf{q} >|}$. For any two vectors $\mathbf{q}$, $\mathbf{p} \in \mathbb{E}_1^3$, we define the cross product $\mathbf{q} \times \mathbf{p}$ by

$$\mathbf{q} \times \mathbf{p} = \begin{vmatrix} -\mathbf{i} & \mathbf{j} & \mathbf{k} \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = (-(q_2 p_3 - q_3 p_2), (q_3 p_1 - q_1 p_3), (q_1 p_2 - q_2 p_1)),$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is the canonical basis of $\mathbb{E}_1^3$. The hyperbolic $(\mathbb{H}_+^2)$ and Lorentzian unit spheres $(\mathbb{S}_1^2)$, respectively, are

$$\mathbb{H}_+^2 = \left\{ \mathbf{q} \in \mathbb{E}_1^3 \mid \|\mathbf{q}\|^2 = -q_1^2 + q_2^2 + q_3^2 = -1 \right\},$$

and

$$\mathbb{S}_1^2 = \left\{ \mathbf{q} \in \mathbb{E}_1^3 \mid \|\mathbf{q}\|^2 = -q_1^2 + q_2^2 + q_3^2 = 1 \right\}.$$

Let $\gamma : I \to \mathbb{E}_1^3$ be a smooth curve in $\mathbb{E}_1^3$, where I is an interval of $\mathbb{R}$. We call γ is *spacelike*, *timelike*, *lightlike* if $\gamma'(t) = \frac{d\gamma}{dt}(t)$ is spacelike, timelike, lightlike for any $t \in \mathbb{R}$, respectively. A surface in the Minkowski 3-space $\mathbb{E}_1^3$ is named a timelike surface if the induced metric on the surface is a Lorentzian metric and is named a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on spacelike (timelike) surface is a timelike (spacelike) vector. Let $\gamma = \gamma(s)$ be a unit speed timelike curve in $\mathbb{E}_1^3$; by $\kappa(s)$ and $\tau(s)$ we indicate the natural curvature and torsion, respectively. $\gamma(s)$ is called the Serret–Frenet curve if $\kappa > 0$, and $\tau \neq 0$. Consider the Serret–Frenet frame $\{\chi_1(s), \chi_2(s), \chi_3(s)\}$ correlating with curve $\gamma(s)$, then the Serret–Frenet formulae reads:

$$\begin{pmatrix} \chi_1'(s) \\ \chi_2'(s) \\ \chi_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \chi_1(s) \\ \chi_2(s) \\ \chi_3(s) \end{pmatrix}, \tag{1}$$

where $\chi_1(s) = \gamma'(s)$, $\chi_2(s) = \gamma''(s)/\left\|\gamma''(s)\right\|$ and $\chi_3(s) = \chi_1(s) \times \chi_2(s)$ are named the unit tangent vector, the principal normal vector and the binormal vector, respectively. Here "prime" indicates the derivative with respect to the parameter s. The Serret–Frenet vector fields satisfy the relations:

$$\chi_1 \times \chi_2 = \chi_3, \chi_1 \times \chi_3 = -\chi_2, \chi_2 \times \chi_3 = -\chi_1. \tag{2}$$

The Bishop frame or rotation minimizing frame (RMF) of $\gamma(s)$ is defined by

$$\begin{pmatrix} \xi' \\ \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} 0 & \mu_1 & -\mu_2 \\ -\mu_1 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \\ \xi_2 \end{pmatrix}, \tag{3}$$

Here, Bishop functions of $\gamma(s)$ are defined by $\mu_1(s) = \kappa \cos\varphi$, $\mu_2(s) = -\kappa \sin\varphi$. The relation matrix can be expressed as

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\phi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}_1 \\ \boldsymbol{\chi}_2 \\ \boldsymbol{\chi}_3 \end{pmatrix}, \tag{4}$$

where $\varphi(s) \geq 0$ is a spacelike angle. One can show that

$$\left. \begin{array}{l} \mu_1^2 + \mu_2^2 = \kappa^2, \text{ and } \vartheta = -\tan^{-1}\left(\frac{\mu_2}{\mu_1}\right);\ \mu_1 \neq 0, \\ \varphi(s) = -\int\limits_{s_0}^{s} \tau ds + \varphi_0,\ \varphi_0 = \varphi(s_0). \end{array} \right\} \tag{5}$$

A ruled surface in $\mathbb{E}_1^3$ is locally the map $\mathfrak{D}_{(\gamma,x)}$: $I \times \mathbb{R} \to \mathbb{E}_1^3$ defined by

$$\mathfrak{D}_{(\gamma,x)}(s,t) = \boldsymbol{\gamma}(s) + t\mathbf{x}(s), \text{ t} \in \mathbb{R}, \tag{6}$$

where $\boldsymbol{\gamma}(s)$ is named the base curve, and $\mathbf{x}(s)$ the director curve. The straight lines $t \to \boldsymbol{\gamma}(s) + t\mathbf{x}(s)$ are named rulings. It is well known that $\mathfrak{D}_{(\gamma,x)}$ is a developable surface if and only if $\det(\boldsymbol{\gamma}'(s), \mathbf{x}(s), \mathbf{x}'(s)) = 0$.

3. Singularities for Timelike Developable Surfaces

In this section, we will investigate the singularities of two timelike developable ruled surfaces according to Bishop frame of regular unit speed timelike curve in Minkowski 3-space $\mathbb{E}_1^3$. Next we deduce Legendrian dualities between spherical indicatrixes of curves by using the theory of Legendrian duality as follows: By the theory of Legendrian duality we define one-forms

$$< \boldsymbol{dv}, \boldsymbol{\omega} >= -\omega_1 dv_1 + \omega_2 dv_2 + \omega_3 dv_3,$$

and

$$< \boldsymbol{d\omega}, \boldsymbol{v} >= -v_1 d\omega_1 + v_2 d\omega_2 + v_3 d\omega_3.$$

Let $\mathbb{S}_1^2$ be the Lorentzian (de Sitter space) unit sphere in $\mathbb{E}_1^3$. Then we have the following double fibration:

$$\left. \begin{array}{l} (a)\ \mathbb{S}_1^2 \times \mathbb{S}_1^2 \supseteq \triangle = \{(\boldsymbol{v}, \boldsymbol{\omega}) \mid < \boldsymbol{\omega}, \boldsymbol{v} >\} = 0, \\ (b)\ \pi_{11} :\ \triangle \to \mathbb{S}_1^2,\ \pi_{12} :\ \triangle \to \mathbb{S}_1^2, \\ (c)\ \theta_{11} = < \boldsymbol{dv}, \boldsymbol{\omega} > |_{\triangle},\ \theta_{12} = < \mathbf{v}, \boldsymbol{d\omega} > |_{\triangle}. \end{array} \right\}$$

Here $\pi_{11}(\boldsymbol{v}, \boldsymbol{\omega}) = \mathbf{v}$, $\pi_{12}(\boldsymbol{v}, \boldsymbol{\omega}) = \boldsymbol{\omega}$. $\theta_{11}^{-1}(0)$, and $\theta_{12}^{-1}(0)$ define the same tangent plane on $\triangle$, which is denoted by K, and indicates that $(\triangle, K)$ is a contact manifold and each of π_{1j} (j = 1, 2) is Legendrian fibration. If there exists an isotropic mapping i: $L \to \triangle$, which means that $i \times \theta_{11} = 0$, we say that $\pi_{11}(i(L))$ and $\pi_{12}(i(L))$ are $\triangle$-dual to each other. It is uncomplicated to see that the condition $i \times \theta_{11} = 0$ is equivalent to $i \times \theta_{12} = 0$. Then we have:

Corollary 1. *Let γ: $I \to \mathbb{E}_1^3$ be a unit speed timelike curve, with $\mu_i \neq 0$ (i=1, 2), we have:*
(1) $\boldsymbol{\chi}_1(s)$ and $\boldsymbol{\chi}_3(s)$ is Δ-dual to each other;
(2) $\boldsymbol{\xi}_1(s)$ and $\boldsymbol{\xi}_2$ is Δ-dual to each other.

Proof. (1) Consider the curve $L(s) = (\boldsymbol{\chi}_1(s), \boldsymbol{\chi}_3(s)) \subset \mathbb{H}_+^2 \times \mathbb{S}_1^2$. Then we have $< \boldsymbol{\chi}_1(s), \boldsymbol{\chi}_3(s) >= 0$, and $< \boldsymbol{\chi}_1'(s), \boldsymbol{\chi}_3(s) >=< \kappa\boldsymbol{\chi}_2, \boldsymbol{\chi}_3(s) >= 0$. The affirmation (1) holds.

(2) Consider the curve $L(s) = (\boldsymbol{\xi}_1(s), \boldsymbol{\xi}_2(s)) \subset \mathbb{S}_1^2 \times \mathbb{S}_1^2$. Then, we have

$<\ \boldsymbol{\xi}_1(s), \boldsymbol{\zeta}_2(s)\ >=\ 0$, and $<\ \boldsymbol{\xi}_1'(s), \boldsymbol{\zeta}_2(s)\ >=<\ \mu_1\boldsymbol{\xi}, \boldsymbol{\zeta}_2(s)\ >=\ 0$. The affirmation (2) holds. □

Let $\mathbf{y}(s,t)$ be a parametric sweeping given by:

$$\mathbf{y}(s,t) = \gamma(s) + r_1(t)\boldsymbol{\xi}_1(s) + r_2(t)\boldsymbol{\xi}_2(s). \tag{7}$$

Here $\gamma(s)$ is a timelike unit speed spine curve, $0 \leq s \leq T$, s is the arc length parameter, and $\mathbf{r}(t) = (0, r_1(t),\ r_2(t))$ is the characteristic circle, with another parameter $t \in \mathbb{R}$. It can be seen that $\mathbf{y}(s,t)$ is a timelike surface. From Equation (7) it follows that the expression of the two timelike developable surfaces is

$$\mathfrak{D}_i(s,t) = \gamma(s) + t\boldsymbol{\xi}_i(s), \text{ (i=1, 2).}$$

It is clear that $\mathfrak{D}_2(s,0) = \gamma(s)$ (resp. $\mathfrak{D}_1(s,0) = \gamma(s)$), $0 \leq s \leq L$, that is, the surface $\mathfrak{D}_2$ (resp. $\mathfrak{D}_1$) interpolate the curve $\gamma(s)$. Now, it seems natural to pose the following two questions. Under what condition does $\mathfrak{D}_i$ have singularities, and how do we recognize their different types? The answer can be stated as: To discuss the singularities of $\mathfrak{D}_i(s,t)$, the cross product can be obtained as

$$\frac{\partial \mathfrak{D}_1}{\partial s} \times \frac{\partial \mathfrak{D}_1}{\partial t} = (1 - t\mu_1)\boldsymbol{\xi}_2(s),$$

and

$$\frac{\partial \mathfrak{D}_2}{\partial s} \times \frac{\partial \mathfrak{D}_2}{\partial t} = -(1 - t\mu_2)\boldsymbol{\xi}_1(s).$$

It follows that $\mathfrak{D}_1$ (resp. $\mathfrak{D}_2$) is non-singular at (s_0, t_0) if and only if $1 - t_0\mu_1(s_0) \neq 0$ (resp. $(1 - t_0\mu_2(s_0) \neq 0)$. Hence, according to Theorem 3.2 in Ref. [10], we can give the following corollary:

Corollary 2. *For the developable ruled surfaces $\mathfrak{D}_i(s,t) = \gamma(s) + t\boldsymbol{\xi}_i(s)$, $s \in I$, $t \in \mathbb{R}$, we have:*

(1) $\mathfrak{D}_i(s,t)$ is locally diffeomorphic to the cuspidal edge (CE) C(2,3)$\times\mathbb{R}$ at (s_0, t_0) if and only if $t_0 = \mu_i^{-1}(s_0) \neq 0$, and $\mu_i'(s_0) \neq 0$.

(2) $\mathfrak{D}_i(s,t)$ is locally diffeomorphic to the Swallowtail (SW) at (s_0, t_0) if and only if $t_0 = \mu_i^{-1}(s_0) \neq 0$, $\mu_i'(s_0) = 0$, and $\left(\mu_i^{-1}(s_0)\right)'' \neq 0$.

Now, we should address the next concept regarding the contact of curves with some surfaces. Let F: $\mathbb{E}_1^3 \to \mathbb{R}$ be a regular function, and $\gamma:\ \mathbb{R} \to \mathbb{E}_1^3$ be a smooth unit speed timelike Frenet–Serret curve. We say that $\gamma(s)$ and $F^{-1}(0)$ have k-point contact for $s = s_0$ if the function $g(s_0) = F \circ \gamma$ satisfies $g(s_0) = g'(s_0) = g''(s_0) = \ldots = g^{(k-1)}(s_0) = 0$, and $g^{(k)}(s_0) \neq 0$. Further, for any fixed $\mathbf{x}_0 \in \mathbb{E}_1^3$, we define the set

$$\mathfrak{BR}_j(\mathbf{x}_0) = \{<\boldsymbol{\xi}_i, \mathbf{x}_0 - \gamma(s)>= \mathbf{0}, j = 1,\ 2\}.$$

We call it the first (resp. second) Bishop rectifying bundle of $\gamma(s)$ through $\mathbf{x}_0$ when j = 1 (resp. j = 2).

The major aim of this work is in the following theorem:

Theorem 1. *Let γ: $I \to \mathbb{E}_1^3$ be a unit speed timelike curve, with $\mu_i \neq 0$; i = 1, 2. Then we:*
A- For $\mathbf{x}_0 = \mathfrak{D}_1(s_0, t_0)$ and the second Bishop rectifying bundle $\mathfrak{BR}_2(\mathbf{x}_0)$ of $\gamma(s)$. One has the following:

(a) The curve $\gamma(s)$ and $\mathfrak{BR}_2(\mathbf{x}_0)$ have at least a 2-point contact at s_0;

(b) The curve $\gamma(s)$ and $\mathfrak{BR}_2(\mathbf{x}_0)$ have at least a 3-point contact at s_0 if and only if

$$\mathbf{x}_0 = \gamma(s_0) - \frac{1}{\mu_1(s_0)}\boldsymbol{\xi}_1(s_0),\ \mu_1'(s_0) \neq 0.$$

Under this situation $\mathfrak{D}_1(s,t)$ at (s_0,t_0) is locally diffeomorphic to CE C(2,3)×$\mathbb{R}$, and $\mathfrak{D}_1(s_0, \frac{1}{\mu_1(s_0)})$ is locally diffeomorphic to a line;

(c) The curve $\gamma(s)$ and $\mathfrak{BR}_2(x_0)$ has at least a 4-point contact at s_0 if and only if

$$\mathbf{x}_0 = \gamma(s_0) - \frac{1}{\mu_1(s_0)}\xi_1(s_0),\ \mu_1'(s_0) = 0,\ \mu_1''(s_0) \neq 0.$$

Under this situation $\mathfrak{D}_1(s,t)$ at (s_0,t_0) is locally diffeomorphic to SW, and $\mathfrak{D}_1(s_0, \frac{1}{\mu_1(s_0)})$ is locally diffeomorphic to the (2,3,4)-cusp.

For $\mathbf{x}_0 = \mathfrak{D}_2(s_0,t_0)$ and the first Bishop rectifying bundle $\mathfrak{BR}_1(\mathbf{x}_0)$ of $\gamma(s)$. One has:

(a) The curve $\gamma(s)$ and $\mathfrak{BR}_1(\mathbf{x}_0)$ have at least a 2-point contact at s_0;

(b) The curve $\gamma(s)$ and $\mathfrak{BR}_1(\mathbf{x}_0)$ have at least a 3-point contact at s_0 if and only if

$$\mathbf{x}_0 = \gamma(s_0) + \frac{1}{\mu_2(s_0)}\xi_2(s_0),\ \mu_2'(s_0) \neq 0.$$

Under this situation $\mathfrak{D}_2(s,t)$ at (s_0,t_0) is locally diffeomorphic to CE C(2,3)×$\mathbb{R}$, and $\mathfrak{D}_2(s_0, \frac{1}{\mu_2(s_0)})$ is locally diffeomorphic to a line; (c) The curve $\gamma(s)$ and $\mathfrak{BR}_1(\mathbf{x}_0)$ have at least a 4-point contact at s_0 if and only if

$$\mathbf{x}_0 = \gamma(s_0) + \frac{1}{\mu_2(s_0)}\xi_2(s_0),\ \mu_2'(s_0) = 0,\ \mu_2''(s_0) \neq 0.$$

Under this situation $\mathfrak{D}_2(s,t)$ at (s_0,t_0) is locally diffeomorphic to SW, and $\mathfrak{D}_2(s_0, \frac{1}{\mu_2(s_0)})$ is locally diffeomorphic to the (2,3,4)-cusp.

Here, C(2,3)×$\mathbb{R} = \{(x,y)|x^2 = y^3\} \times \mathbb{R}$, C(2, 3, 4) = $\{(x,y,z)|x = t^2, y = t^3, z = t^4\}$, and $SW = \{(x,y,z)|x = 3u^4 + u^2v,\ y = 4u^3 + 2uv,\ z = v\}$. The graphs of cuspidal edge and swallowtail can be seen in Figures 1 and 2.

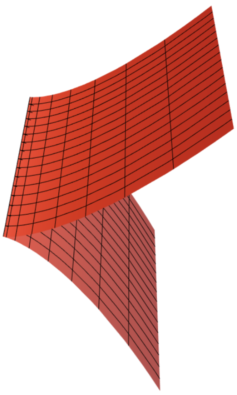

Figure 1. Cuspidal edge.

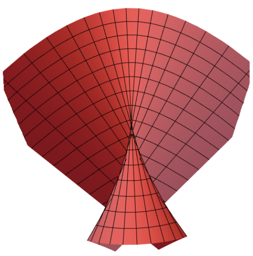

Figure 2. Swallowtail.

3.1. Bishop Height Functions

Now we will consider two various sets of Bishop height functions, which will be helpful for studying the singularities of $\mathfrak{D}_i(s,t)$ as follows [27,28]: $H_i : I \times \mathbb{E}^3_1 \to \mathbb{R}$, by $H_i(s,\mathbf{x}) = <\xi_i, \mathbf{x}-\gamma>$; $i = 1,2$. We name it the first (resp. second) Bishop height function for $i = 1$ (resp. 2). We use the notation $h_{i\mathbf{x}}(s) = H_i(s,\mathbf{x})$ for any fixed $\mathbf{x} \in \mathbb{E}^3_1$. From now on, we shall not often write the parameter s. Then, we have the following proposition:

Proposition 1. *Let γ: $I \to \mathbb{E}^3_1$ be a unit speed timelike curve, with $\mu_1 \neq 0$, and $\mu_2 \neq 0$. Then:*

(A)

(1) $h_{1\mathbf{x}}(s) = 0$ *if and only if there exist* $a, b \in \mathbb{R}$ *such that* $\boldsymbol{x} - \gamma = a\boldsymbol{\xi} + b\boldsymbol{\xi}_2$,

(2) $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) + t\boldsymbol{\xi}_2(s)$.

(3) $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = h''_{1x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) - \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$.

(4) $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = h''_{1x}(s) = h'''_{1x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) - \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$, *and* ${\mu_2}'(s) = 0$.

(5) $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = h''_{1x}(s) = h'''_{1x}(s) = h^{(4)}_{1x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) - \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$, *and* ${\mu_2}'(s) = {\mu_2}''(s) = 0$.

(B)

(1) $h_{2\mathbf{x}}(s) = 0$ *if and only if there exist* $a, b \in \mathbb{R}$ *such that* $\boldsymbol{x} - \gamma = a\boldsymbol{\xi} + b\boldsymbol{\xi}_1$,

(2) $h_{2x}(s) = h'_{2\mathbf{x}}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) + t\boldsymbol{\xi}_1(s)$.

(3) $h_{2x}(s) = h'_{2x}(s) = h''_{2x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) + \frac{1}{\mu_1(s)}\boldsymbol{\xi}_1(s)$.

(4) $h_{2x}(s) = h'_{2\mathbf{x}}(s) = h''_{2\mathbf{x}}(s) = h'''_{2x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) + \frac{1}{\mu_1(s)}\boldsymbol{\xi}_2(s)$, *and* $\mu'_1(s) = 0$.

(5) $h_{2x}(s) = h'_{2\mathbf{x}}(s) = h''_{2\mathbf{x}}(s) = h'''_{2x}(s) = h^{(4)}_{2x}(s) = 0$ *if and only if* $\boldsymbol{x} = \gamma(s) + \frac{1}{\mu_1(s)}\boldsymbol{\xi}_2(s)$, *and* $\mu'_1(s) = \mu''_1(s) = 0$.

Proof. (A). (1) Since $h_{1\mathbf{x}}(s) =< \boldsymbol{x}-\gamma, \boldsymbol{\xi}_1 >= 0$, and $\{\boldsymbol{\xi}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ is RMF along $\gamma(s)$, then there exists $a, b \in \mathbb{R}$ such that $\boldsymbol{x} - \gamma = a\boldsymbol{\xi} + b\boldsymbol{\xi}_2$.

(2) When $h_{1\mathbf{x}}(s) = 0$, the affirmation (2) follows from the fact that $h'_{1\mathbf{x}}(s) =< \boldsymbol{\xi}'_1, \boldsymbol{x}-\gamma >=< \mu_1\boldsymbol{\xi}, \boldsymbol{x}-\gamma >= 0$. Thus, we have that $< \mu_1\boldsymbol{\xi}, \boldsymbol{x}-\gamma >= 0$. It follows from the fact $\mu_1 \neq 0$ that $< \boldsymbol{\xi}, \boldsymbol{x}-\gamma >$, and $\boldsymbol{x}-\gamma = b\boldsymbol{\xi}_2$. Thus, we get that $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = 0$ if and only if $\boldsymbol{x}-\gamma = b\boldsymbol{\xi}_2$.

(3) When $h_{1x}(s) = h'_{1\mathbf{x}}(s) = 0$, the affirmation (3) follows from the fact that

$$
\begin{aligned}
h''_{1\mathbf{x}}(s) &= -\mu_1 + < \boldsymbol{x}-\gamma, \mu'_1\boldsymbol{\xi} + \mu_1^2\boldsymbol{\xi}_1 - \mu_1\mu_2\boldsymbol{\xi}_2 >= 0 \\
&= -\mu_1(1 + b\mu_2),
\end{aligned}
$$

Since $\mu_1 \neq 0$, we get that $h_{1x}(s) = h'_{1\mathbf{x}}(s) = h''_{\mathbf{x}}(s) = 0$ if and only if $\boldsymbol{x} = \gamma(s) - \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$.

(4) Under the hypothesis that $h_{1x}(s) = h'_{1\mathbf{x}}(s) = h''_{1\mathbf{x}}(s) = 0$, this derivative is calculated as follows:

$$
\left.
\begin{array}{l}
h^{(3)}_{1\mathbf{x}}(s) = -2\mu'_1 + < \left(\mu''_1 - \mu_1^3 + \mu_1\mu_2^2\right)\boldsymbol{\xi} + 3\mu_1\mu'_1\boldsymbol{\xi}_1 \\
-\left(\mu_1\mu'_2 + 2\mu_2\mu'_1\right)\boldsymbol{\xi}_2, \boldsymbol{x}-\gamma > \\
= \frac{\mu_1(s)\mu'_2(s)}{\mu_2(s)} \\
= 0.
\end{array}
\right\}
$$

Since $\mu_2(s) \neq 0$, we get $h^{(3)}_{1\mathbf{x}}(s) = 0$, which is equivalent to the assumption $\mu'_2(s) = 0$. The affirmation (4) follows.

(5) Under the hypothesis that $h_{1\mathbf{x}}(s) = h'_{1\mathbf{x}}(s) = h''_{1\mathbf{x}}(s) = \overset{(3)}{h}_{1\mathbf{x}}(s) = 0$, this derivative is calculated as:

$$\overset{(4)}{h}_{1\mathbf{x}}(s) = -3\mu_1'' + \mu_1^3 - \mu_1\mu_2^2 + < a\boldsymbol{\xi} + a_1\boldsymbol{\xi}_1 + a_2\boldsymbol{\xi}_2, , \boldsymbol{x} - \boldsymbol{\gamma} >,$$

where

$$\left.\begin{array}{l} a(s) = \mu_1'' - 6\mu_1'\mu_1^2 + 3\mu_1'\mu_2^2 + 3\mu_1'\mu_1\mu_2', \\ a_1(s) = 3(\mu_1')^2 + 3\mu_1\mu_1''' + \mu_1^4 + \mu_1^2\mu_1^2, \\ a_2(s) = -3\mu_1''\mu_2 - 3\mu_1'\mu_2' + \mu_1^3\mu_2 - \mu_1\mu_2^3 - \mu_1\mu_2''. \end{array}\right\}$$

By $\boldsymbol{x} = \boldsymbol{\gamma}(s) - \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$ in the above, we have that:

$$\begin{aligned} \overset{(4)}{h}_{1\mathbf{x}}(s) &= -3\mu_1'' + \mu_1^3 - \mu_1\mu_2^2 - \frac{1}{\mu_2}\left[-3\mu_1'''\mu_2 - 3\mu_1'\mu_2' + \mu_1^3\mu_2 - \mu_1\mu_2^3 - \mu_1\mu_2''\right] \\ &= \frac{3\mu_1'\mu_2' + \mu_1\mu_2''}{\mu_2}. \end{aligned}$$

Since $\mu_1(s) \neq 0$, $\mu_2'(s) = 0$, we get that $\overset{(4)}{h}_{1\mathbf{x}}(s) = 0$ is synonymous to the assumption $\mu_2''(s) = 0$. The affirmation (5) follows. Using the identical computation as the proof of (A), we can get (B). $\square$

By simple calculations, we have:

Proposition 2. *Let γ: $I \to \mathbb{E}_1^3$ be a unit speed timelike curve, with $\mu_1 \neq 0$, and $\mu_2 \neq 0$. Then:*

(1) $\mu_2' = 0$ if and only if $\boldsymbol{x} = \boldsymbol{\gamma}(s) + \frac{1}{\mu_2(s)}\boldsymbol{\xi}_2(s)$ is a constant vector.

(2) $\mu_1' = 0$ if and only if $\boldsymbol{x} = \boldsymbol{\gamma}(s) - \frac{1}{\mu_1(s)}\boldsymbol{\xi}_1(s)$ is a constant vector.

3.2. Unfolding of Functions by One-Variable

Now we employ some public results on the singularity theory for families of function germs [27,28]. Let F: $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a smooth function, and $f(s) = F_{x_0}(s, \mathbf{x}_0)$. Then F is called an r-parameter unfolding of $f(s)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has $\mathrm{A}_{\geqslant k}$-singularity ($k \geqslant 1$) at s_0. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 be $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)\right)(s_0) = \Sigma_{j=0}^{k-1} L_{ji}(s - s_0)^j$ (without the constant term), for $i = 1, ..., r$. Then $F(s)$ is called an p-versal unfolding if the $k \times r$ matrix of coefficients $\left(L_{ji}\right)$ has rank k ($k \leq r$). Therefore, we write serious sets on the unfolding regarding to the above notations. The discriminant set of order t of F is the set

$$\mathfrak{D}_F^t = \left\{\mathbf{x} \in \mathbb{R}^r \,|\, \exists\, s \in \mathbb{R}, F(s, \mathbf{x}) =, \frac{\partial F}{\partial s}(s, \mathbf{x}) = \ldots = \frac{\partial^t F}{\partial s^t}(s, \mathbf{x}) = 0 \text{ at } (s, \mathbf{x})\right\}. \tag{8}$$

Then $\mathfrak{D}_F^1 = \mathfrak{D}_F$, and $\mathfrak{D}_F^2$ is the set of singular points of $\mathfrak{D}_F$.

We state the following theorem:

Theorem 2. *Let F: $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$, which has the A_k singularity at s_0.*

Suppose that F is a p-versal unfolding.

(a) If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-1}$, and $\mathfrak{D}_F^2 = \varnothing$;

(b) If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C(2,3) \times \mathbb{R}^{r-2}$, $\mathfrak{D}_F^2$ is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-2}$, and $\mathfrak{D}_F^3 = \varnothing$.

(c) If $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, $\mathfrak{D}_F^3$ is locally diffeomorphic to $C(2,3,4) \times \mathbb{R}^{r-3}$, and $\mathfrak{D}_F^3$ is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-3}$, and $\mathfrak{D}_F^4 = \varnothing$.

Then, the following proposition can be obtained.

Proposition 3. *Under the situations of Proposition 1, if $h_{1\mathbf{x}_0}(s)$ has A_k-singularity $(k=2,\ 3)$ at $s_0 \in \mathbb{R}$, then $H_1(s,\mathbf{x})$ is a $p-$versal unfolding of $h_{1\mathbf{x}_0}(s_0)$. (2). If $h_{2\mathbf{x}_0}(s)$ has A_k-singularity $(k=2,\ 3)$ at $s_0 \in \mathbb{R}$, then $H_2(s,\mathbf{x})$ is a $p-$versal unfolding of $h_{2\mathbf{x}_0}(s_0)$.*

Proof. (1) Let $\gamma(s) = (\gamma_1(s),\gamma_2(s),\gamma_3(s))$, $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{E}^3_1$, and $\boldsymbol{\xi}_1(s) = (\xi_{11}(s), \xi_{12}(s), \xi_{13}(s)) \in \mathbb{S}^2_1$. Then,

$$H_1(s,\boldsymbol{x}) = -(\gamma_1(s) - x_1)\xi_{11}(s) + (\gamma_2(s) - x_2)\xi_{12}(s) + (\gamma_3(s) - x_3)\xi_{13}(s).$$

Thus, we have

$$\left.\begin{array}{l} \frac{\partial H_1}{\partial x_1} = \xi_{11}(s),\ \frac{\partial H_1}{\partial x_2} = -\xi_{12}(s),\ \frac{\partial H_1}{\partial x_3} = -\xi_{13}(s) \\ \frac{\partial^2 H_1}{\partial s \partial x_1} = \xi'_{11}(s),\ \frac{\partial^2 H_1}{\partial s \partial x_2} = -\xi'_{12}(s),\ \frac{\partial^2 H_1}{\partial s \partial x_3} = -\xi'_{13}(s). \end{array}\right\}$$

Therefore, the 2-jets of $\frac{\partial H}{\partial x_i}$ at s_0 (i=0, 1) are:

$$\left.\begin{array}{r} j^1\left(\frac{\partial H_1}{\partial x_1}(s,\mathbf{x}_0)\right) = \xi'_{11}(s - s_0), \\ j^1\left(\frac{\partial H_1}{\partial x_2}(s,\mathbf{x}_0)\right) = -\xi'_{12}(s)(s - s_0), \\ j^1\left(\frac{\partial H_1}{\partial x_3}(s,\mathbf{x}_0)\right) = -\xi'_{13}(s - s_0), \end{array}\right\}$$

and

$$\left.\begin{array}{r} j^2\left(\frac{\partial H_1}{\partial x_1}(s,\mathbf{x}_0)\right) = \xi'_{11}(s - s_0) + \frac{1}{2}\xi''_{11}(s - s_0)^2, \\ j^2\left(\frac{\partial H_1}{\partial x_2}(s,\mathbf{x}_0)\right) = -\xi'_{12}(s)(s - s_0) - \frac{1}{2}\xi''_{12}(s - s_0)^2, \\ j^2\left(\frac{\partial H_1}{\partial x_2}(s,\mathbf{x}_0)\right) = -\xi'_{13}(s)(s - s_0) - \frac{1}{2}\xi''_{13}(s - s_0)^2. \end{array}\right\}$$

(i) If $h_{1\mathbf{x}_0}(s_0)$ has the A_2-singularity at s_0, then $h'_{1\mathbf{x}_0}(s_0) = 0$. So the 1×3 matrix of coefficients (L_{ji}) is:

$$A = \begin{pmatrix} \xi'_{11} & -\xi'_{12} & -\xi'_{13} \end{pmatrix}.$$

Suppose that the rank of the matrix A is zero, then we have:

$$\xi'_{11} = \xi'_{12} = \xi'_{13} = 0.$$

Since $\left\|\gamma'(s_0)\right\| = \|\boldsymbol{\xi}(s_0)\|^2 = -1$, we have $-\left(\xi'_{11}\right)^2 + \left(\xi'_{11}\right)^2 + \left(\xi'_{11}\right)^2 = -\mu_1^2 \neq 0$, so that we have a contradiction . Therefore rank$(A) = 1$, and H is the (p) versal unfolding of $h_{1\mathbf{x}_0}$ at s_0.
(ii) If $h_{1\mathbf{x}_0}(s_0)$ has the A_3-singularity at $s_0 \in \mathbb{R}$, then $h'_{1\mathbf{x}_0}(s_0) = h''_{1\mathbf{x}_0}(s_0) = 0$. Therefore, the 3×3 matrix of the coefficients (L_{ji}) is

$$B(s_0) = \begin{pmatrix} \xi_{11} & -\xi_{12} & -\xi_{13} \\ \xi'_{11} & -\xi'_{12} & -\xi'_{13} \\ \frac{1}{2}\xi''_{11} & -\frac{1}{2}\xi''_{12} & -\frac{1}{2}\xi''_{13} \end{pmatrix}.$$

For the intention, we also require the 3×3 matrix B to be non-singular, which it constantly is. Furthermore, the determinate of this matrix at s_0 is

$$\begin{aligned}\det(B) &= \frac{1}{2}\begin{vmatrix} \xi_{11} & -\xi_{12} & -\xi_{13} \\ \xi'_{11} & -\xi'_{12} & -\xi'_{13} \\ \xi''_{11} & -\xi''_{12} & -\xi''_{13} \end{vmatrix} \\ &= \frac{1}{2} < \boldsymbol{\xi}_1 \times \boldsymbol{\xi}'_1, \boldsymbol{\xi}''_1 > .\end{aligned}$$

Since $\boldsymbol{\xi}'_1 = -\mu_1\boldsymbol{\xi}$, we have $\boldsymbol{\xi}''_1 = -\mu'_1\boldsymbol{\xi} - \mu_1^2\boldsymbol{\xi}_1 + \mu_1\mu_2\boldsymbol{\xi}_2$. Substituting these equations to the above equality, we obtain

$$\det(B) = \frac{1}{2}\mu_1^2(s_0)\mu_2(s_0) \neq 0.$$

Thus $\operatorname{rank}(B) = 3$, if we consider the matrix which consists of the first and the second row of the matrix B, so that $\operatorname{rank}(B) = 2$. (2) Using the same computation as the proof of (1), we can get (2). □

Proof of Theorem 1. (1) Let γ: $I \to \mathbb{E}^3_1$ be a unit speed timelike curve, with $\mu_1 \neq 0$, and $\mu_2 \neq 0$. For $\mathbf{x}_0 = \gamma(s_0) + t_0\boldsymbol{\xi}_i(s_0)$, we define a function $\mathfrak{H}_i(\mathbf{p}) = < \boldsymbol{\xi}_i, \mathbf{x}_0 - \mathbf{p} >$ (i = 1, 2), then we have $h_{i\mathbf{x}_0}(s) = \mathfrak{N}_i(\gamma(s))$.

First, we consider the affirmation (1). For $\mathbf{x}_0 = \mathfrak{D}_1(s_0, t_0)$, since $\mathfrak{B}\mathfrak{R}_2(\mathbf{x}_0) = \mathfrak{H}_2^{-1}(0)$, where 0 is a regular value of $\mathfrak{H}_2$; $h_{2\mathbf{x}_0}(s)$ has the A$_k$-singularity at s$_0$ if and only if $\gamma(s_0)$ and $\mathfrak{B}\mathfrak{R}_2(\mathbf{x}_0)$ have (k+1)-point contact for s_0. On the other hand, the discriminant set $\mathfrak{D}_{H_2}$ of H_2 is

$$\mathfrak{D}_{H_2} = \{\mathbf{x} = \gamma(s) + t\boldsymbol{\xi}_1(s) | s \in I, t \in \mathbb{R}\}.$$

The affirmation (1) follows from Theorem 2 and Proposition 1. Since the trajectory of the singularities of CE is locally diffeomorphic to the line, the affirmation (b) holds. Since the trajectory of singularities of SW is C(2,3,4), the affirmation (c) holds.

For the affirmation (2), the discriminant set of $\mathfrak{D}_{H_1}$ of H_1 is

$$\mathfrak{D}_{H_1} = \{\mathbf{x} = \gamma(s) + t\boldsymbol{\xi}_2(s) | s \in I, t \in \mathbb{R}\}.$$

By a similar argument, we can also prove the affirmation (2). This completes the proof. □

Example 1. *Given the timelike helix:*

$$\gamma(s) = (\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sqrt{2}s), \ -1 \leq s \leq 1,$$

It is simple to have that

$$\left.\begin{aligned} \chi_1(s) &= (\sqrt{3}\cosh s, \sqrt{3}\sinh s, \sqrt{2}), \\ \chi_2(s) &= (\sinh s, \cosh s, 0), \\ \chi_3(s) &= (\sqrt{2}\cosh s, \sqrt{2}\sinh s, \sqrt{3}), \\ \kappa(s) &= \sqrt{3}, \textit{and } \tau(s) = \sqrt{2}. \end{aligned}\right\}$$

Taking $\varphi_0 = 0$ *we have* $\varphi(s) = -\sqrt{2}s$. *Using Equations (3)–(5), we obtain* $\mu_1(s) = \sqrt{3}\cos\sqrt{2}s$, *and* $\mu_2(s) = -\sqrt{3}\sin\sqrt{2}s$. *Hence:*

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\sqrt{2}s\right)s & -\sin\left(\sqrt{2}s\right) \\ 0 & \sin\left(\sqrt{2}s\right) & \cos\left(\sqrt{2}s\right) \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}.$$

Thus, we get

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} = \begin{pmatrix} \sinh s \cos\left(\sqrt{2}s\right) - \sqrt{2}\cosh s \sin\left(\sqrt{2}s\right) \\ \cosh s \cos\left(\sqrt{2}s\right) - \sqrt{2}\sinh s \sin\left(\sqrt{2}s\right) \\ -\sqrt{3}\sin\left(\sqrt{2}s\right) \end{pmatrix},$$

$$\boldsymbol{\xi}_2 = \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix} = \begin{pmatrix} \sinh s \sin\left(\sqrt{2}s\right) + \sqrt{2}\cosh s \cos\left(\sqrt{2}s\right) \\ \sin\left(\sqrt{2}s\right)\cosh s + \sqrt{2}\cos\left(\sqrt{2}s\right)\sinh\ s \\ \sqrt{3}\cos\left(\sqrt{2}s\right) \end{pmatrix}.$$

So, the timelike developable surface is

$$\mathbf{y}(s,t) = \left(\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sqrt{2}s\right) + r_1(t)\begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix} + r_2(t)\begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix}.$$

(1) If we take $r_1(t) = \cos t$, *and* $r_2(t) = \sin t$ *, then we instantly obtain the surface (see Figure 3).*

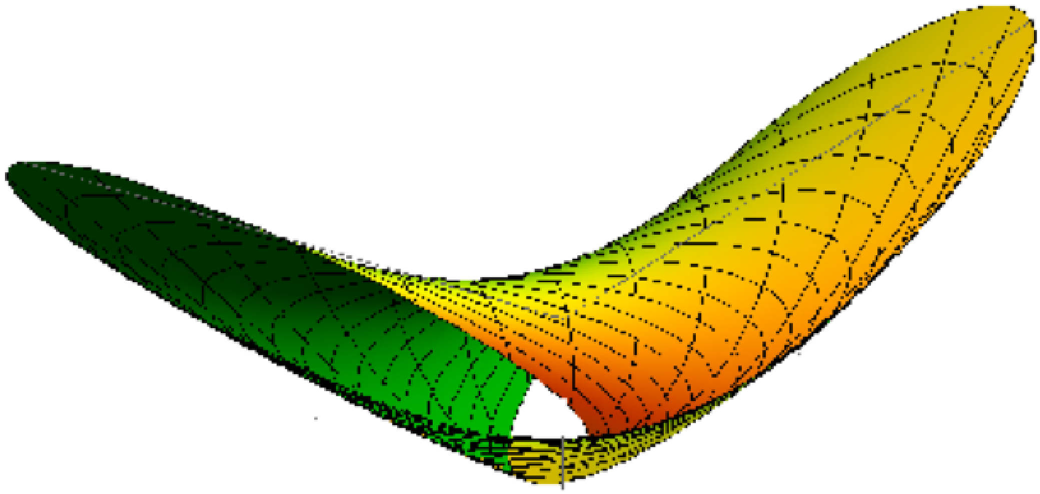

Figure 3. Timelike developable surface.

(2) If $r_1(t) = r_2(t) = t$, *then* $\mathfrak{D}_1(s,t)$, *and* $\mathfrak{D}_2(s,t)$, *respectively, are:*

$$\mathfrak{D}_1(s,t) = \left(\sqrt{3}\sinh s + t\xi_{11}, \sqrt{3}\cosh s + t\xi_{13}, \sqrt{2}s + t\xi_{12}\right),$$

and

$$\mathfrak{D}_2(s,t) = \left(\sqrt{3}\sinh s + t\xi_{21}, \sqrt{3}\cosh s + t\xi_{23}, \sqrt{2}s + t\xi_{22}\right).$$

In addition, the singular locus of $\mathfrak{D}_1(s,t)$, *and* $\mathfrak{D}_2(s,t)$, *respectively, are:*

$$\mathfrak{D}_1(s) = \left(\sqrt{3}\sinh s + \frac{1}{\sqrt{3}\cos\sqrt{2}s}\xi_{11}, \sqrt{3}\cosh s + \frac{1}{\sqrt{3}\cos\sqrt{2}s}\xi_{13}, \sqrt{2}s + \frac{1}{\sqrt{3}\cos\sqrt{2}s}\xi_{12}\right),$$

and

$$\mathfrak{D}_2(s) = \left(\sqrt{3}\sinh s + \frac{1}{\sqrt{3}\sin\sqrt{2}s}\xi_{21}, \sqrt{3}\cosh s + \frac{1}{\sqrt{3}\sin\sqrt{2}s}\xi_{23}, \sqrt{2}s + \frac{1}{\sqrt{3}\sin\sqrt{2}s}\xi_{22}\right).$$

We consider a local part of this curve when $\frac{\pi}{6\sqrt{2}} \leq s \leq \frac{\pi}{3\sqrt{2}}$. *We see that* $\mu_1^{-1}(s) = \frac{1}{\sqrt{3}\cos\sqrt{2}s} \neq 0$, *and* $\mu_1'(s) = -\sqrt{6}\sin\left(\sqrt{2}s\right) \neq 0$ *for* $\frac{\pi}{6\sqrt{2}} \leq s \leq \frac{\pi}{3\sqrt{2}}$. *This means that* $\mathfrak{D}_1(s,t)$ *is locally diffeomorphic to a CE and its singular locus is locally diffeomorphic to a line (the red line), see Figure 4. For* $\mathfrak{D}_2(s,t)$, *when* $\frac{\pi}{6\sqrt{2}} \leq s \leq \frac{5\pi}{6\sqrt{2}}$. *We see that* $\mu_2^{-1}(s) = \frac{1}{\sqrt{3}\cos\sqrt{2}s} \neq 0$, $\mu_2'(s) = \sqrt{6}\cos\left(\sqrt{2}s\right) = 0$ *gives one real root* $s = \frac{\pi}{2\sqrt{2}}$. *We can also get that* $\left(\mu_2^{-1}(\frac{\pi}{2\sqrt{2}})\right)'' = 0$, *that is,* $\mathfrak{D}_2(s,t)$ *fails to be locally diffeomorphic to SW and its singular locus is not locally diffeomorphic to the C(2,3,4)-cusp at* $s = \frac{\pi}{2\sqrt{2}}$. *Hence* $H_2(s,\mathbf{x})$ *fails to be a versal unfolding of the* $h_{2x}(s)$ *at* $s = \frac{\pi}{2\sqrt{2}}$; *see Figure 5.*

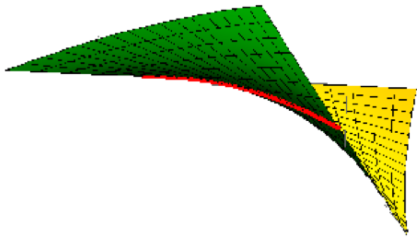

Figure 4. $\mathfrak{D}_1(s,t)$.

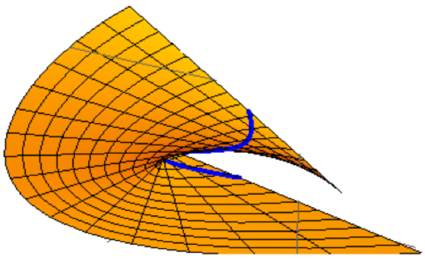

Figure 5. $\mathfrak{D}_2(s,t)$.

4. Conclusions

In this paper, we considered the notion of timelike developable surfaces with rotation minimizing frames in Minkowski 3–Space $\mathbb{E}_1^3$. By applying singularities theory, we classified the generic properties of the cuspidal edge and swallowtail. Finally, an example of application is offered to demonstrate the theoretical results. Furthermore, recently the application of singularity theory and submanifolds theory and so forth, presented in Refs. [29–57], has attracted great interest. In the following work, we will connect the results of this paper with the techniques and methods in Refs. [29–57] to explore more results and theorems related to symmetric properties about this topic.

Author Contributions: Conceptualization, Y.L., Z.C., S.H.N., R.A.A.-B.; methodology, Y.L., Z.C., S.H.N., R.A.A.-B.; investigation, Y.L., Z.C., S.H.N., R.A.A.-B.; writing—original draft preparation, Y.L., Z.C., S.H.N., R.A.A.-B.; writing—review and editing, Y.L., Z.C., S.H.N., R.A.A.-B. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (grant no. 12101168) and the Zhejiang Provincial Natural Science Foundation of China (grant no. LQ22A010014).

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. do Carmo, M.P. *Differential Geometry of Curves and Surface*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1976.
2. Xu, Z.; Feng, R. Analytic and algebraic properties of canal Surfaces. *J. Comput. Appl. Math.* **2006**, *195*, 220–228. [CrossRef]
3. Izumiya, S.; Saji, K.; Takeuchi, N. Circular surfaces. *Adv. Geom.* **2007**, *7*, 295–313. [CrossRef]
4. Cui, L.; Wang, D.; Dai, J. Kinematic geometry of circular surfaces with a fixed radius based on Euclidean invariants. *ASME J. Mech.* **2009**, *131*, 101009. [CrossRef]
5. Qian, J.; Kim, Y. Some characterizations of canal surfaces. *Bull. Korean Math. Soc.* **2016**, *53*, 461–477.
6. Qian, J.; Kim, Y. Some Classification of canal surfaces with the Gauss map. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 261–3272. [CrossRef]
7. Abdel-Baky, R.A. Developable surfaces through sweeping surfaces. *Bull. Iran. Math. Soc.* **2019**, *45*, 951–963. [CrossRef]
8. Mofarreh, F.; Abdel-Baky, R.; Naghi, M. Developable surfaces through spacelike sweeping surfaces in Minkowski 3–Space. *Appl. Math. Inf. Sci.* **2021**, *15*, 263–270.
9. Mofarreh, F.; Abdel-Baky, R. Timelike sweeping surfaces and Singularities. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 2150006. [CrossRef]
10. Abdel-Baky, R.; Mofarreh, F.; Alluhaii, N. Spacelike sweeping surfaces and singularities in Minkowski 3-Space. *Math. Probl. Eng.* **2021**, *2021*, 5130941.
11. Pottmann, H.; Wallner, M. Approximation Algorithms for Developable Surfaces. *Comput.-Aided Geom. Des.* **1999**, *16*, 539–556. [CrossRef]
12. Kose, O. A method of the determination of a developable ruled surface. *Mech. Mach. Theory.* **1999**, *34*, 1187–1193. [CrossRef]
13. Zhao, H.; Wang, G. A new method for designing a developable surface utilizing the surface pencil through a given curve. *Prog. Nat. Sci.* **2008**, *18*, 105–110. [CrossRef]
14. Yıldız, O.G.; Karakus, S.O.; Hacısalihoglu, H.H. On the determination of a developable spherical orthotomic ruled surface. *Bull. Math. Sci.* **2015**, *5*, 137–146. [CrossRef]
15. Izumiya, S.; Takeuchi, N. New Special Curves and Developable Surfaces. *Turk. J. Math.* **2004**, *28*, 153–163.
16. Li, Y.; Nazra, S.H.; Abdel-Baky, R.A. Singularity Properties of Timelike Sweeping Surface in Minkowski 3-Space. *Symmetry* **2022**, *14*, 1996. [CrossRef]
17. Li, Y,; Eren, K.; Ayvacı, K.H.; Ersoy, S. The developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in Euclidean 3-space. *AIMS Math.* **2023**, *8*, 2226–2239. [CrossRef]
18. Izumiya, S.; Takeuchi, N. Singularities of ruled surfaces in R^3. *Math. Proc. Camb. Philos. Soc.* **2001**, *130*, 1–11. [CrossRef]
19. Bishop, R.L. There is more than one way to frame a curve. *Am. Math. Mon.* **1975**, *82*, 246–251. [CrossRef]
20. Bükcu, B.; Karacan, M.K. On the slant helices according to Bishop frame of the timelike curve in Lorentzian space. *Tamkang J. Math.* **2008**, *39*, 255–262. [CrossRef]
21. Keskin, O.; Yayli, Y. An application of N-Bishop frame to spherical images for direction curves. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750162. [CrossRef]
22. Grbovic, M.; Nešovic, E. On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space. *J. Math. Anal. Appl.* **2018**, *461*, 219–233. [CrossRef]
23. Liu, H.; Pei, D. Legendrian dualities between spherical indicatrixes of curves and surfaces according to Bishop frame. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2875–2887. [CrossRef]
24. O'Neil, B. *Semi-Riemannian Geometry Geometry, with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
25. Walfare, J. Curves and Surfaces in Minkowski Space. Ph.D. Thesis, K.U. Leuven, Faculty of Science, Leuven, Belgium, 1995.
26. Nagai, T. The Gauss map of a hypersurface in Euclidean sphere and the spherical Legendrian duality. *Topol. Appl.* **2012**, *159*, 545–554. [CrossRef]
27. Bruce, J.; Giblin, P. Generic geometry. *Am. Math. Mon.* **1983**, *90*, 529–545. [CrossRef]
28. Bruce, J.W.; Giblin, P.J. *Curves and Singularities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1992.
29. Antic, M. Characterization ofWarped Product Lagrangian Submanifolds in Cn. *Results Math.* **2022**, *77*, 106. [CrossRef]
30. Antic, M.; Vrancken, L. Conformally flat, minimal, Lagrangian submanifolds in complex space forms. *Sci. China Math.* **2022**, *65*, 1641–1660. [CrossRef]
31. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike Circular Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 173. [CrossRef]
32. Li, Y.; Alluhaibi, N.; Abdel-Baky, R.A. One-Parameter Lorentzian Dual Spherical Movements and Invariants of the Axodes. *Symmetry* **2022**, *14*, 1930. [CrossRef]
33. Li, Y.; Mofarreh, F.; Abdel-Baky, R.A. Timelike Circular Surfaces and Singularities in Minkowski 3-Space. *Symmetry* **2022**, *14*, 1914. [CrossRef]
34. Li, Y.; Erdoğdu, M.; Yavuz, A. Differential Geometric Approach of Betchow-Da Rios Soliton Equation. *Hacet. J. Math. Stat.* **2022**, 1–12 . [CrossRef]
35. Li, Y.; Abdel-Salam, A.A.; Saad, M.K. Primitivoids of curves in Minkowski plane. *AIMS Math.* **2023**, *8*, 2386–2406. [CrossRef]
36. Li, Y.; Prasad, R.; Haseeb, A.; Kumar, S.; Kumar, S. A Study of Clairaut Semi-Invariant Riemannian Maps from Cosymplectic Manifolds. *Axioms* **2022**, *11*, 503. [CrossRef]
37. Li, Y.; Eren, K.; Ayvacı, K.H.; Ersoy, S. Simultaneous characterizations of partner ruled surfaces using Flc frame. *AIMS Math.* **2022**, *7*, 20213–20229. [CrossRef]

38. Li, Y.; Abolarinwa, A.; Alkhaldi, A.H.; Ali, A. Some Inequalities of Hardy Type Related to Witten–Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580. [CrossRef]
39. Li, Y.; Mofarreh, F.; Laurian-Ioan, P.; Ali, A. Geometric Inequalities on the Dirac Operator and the Frobenius Norms. *Adv. Math. Phys.* **2023**, *2023*, 1275374.
40. Gür, S.; Şenyurt, S.; Grilli, L. The Dual Expression of Parallel Equidistant Ruled Surfaces in Euclidean 3-Space. *Symmetry* **2022**, *14*, 1062.
41. Çalışkan, A.; Şenyurt, S. Curves and ruled surfaces according to alternative frame in dual space. *Commun. Fac. Sci. Univ.* **2020**, *69*, 684–698. [CrossRef]
42. Çalışkan, A.; Şenyurt, S. The dual spatial quaternionic expression of ruled surfaces. *Therm. Sci.* **2019**, *23*, 403–411. [CrossRef]
43. Şenyurt, S.; Çalışkan, A. The quaternionic expression of ruled surfaces. *Filomat* **2018**, *32*, 5753–5766. [CrossRef]
44. Şenyurt, S.; Gür, S. Spacelike surface geometry. *Int. J. Geom. Methods Mod. Phys.* **2017**, *14*, 1750118. [CrossRef]
45. As, E.; Şenyurt, S. Some Characteristic Properties of Parallel-Equidistant Ruled Surfaces. *Math. Probl. Eng.* **2013**, *2013*, 587289. [CrossRef]
46. Özcan, B.; Şenyurt S. On Some Characterizations of Ruled Surface of a Closed Timelike Curve in Dual Lorentzian Space. *Adv. Appl. Clifford Al.* **2012**, *22*, 939–953.
47. Ali, A.T. Non-lightlike constant angle ruled surfaces in Minkowski 3-space. *J. Geom. Phys.* **2020**, *157*, 103833. [CrossRef]
48. Ali, A.T. A constant angle ruled surfaces. *Int. J. Geom.* **2018**, *7*, 69–80.
49. Ali, A.T. Non-lightlike ruled surfaces with constant curvatures in Minkowski 3-space. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15*, 1850068. [CrossRef]
50. Ali, A.T.; Hamdoon, F.M. Surfaces foliated by ellipses with constant Gaussian curvature in Euclidean 3-space. *Korean J. Math.* **2017**, *25*, 537–554.
51. Ali, A.T.; Abdel Aziz, H.S.; Sorour, A.H. On some geometric properties of quadric surfaces in Euclidean space. *Honam Math. J.* **2016**, *38*, 593–611. [CrossRef]
52. Ali, A.T.; Abdel Aziz, H.S.; Sorour, A.H. On curvatures and points of the translation surfaces in Euclidean 3-space. *J. Egyptian Math. Soc.* **2015**, *23*, 167–172. [CrossRef]
53. Gulbahar, M.; Kilic, E.; Keles, S.; Tripathi, M.M. Some basic inequalities for submanifolds of nearly quasi-constant curvature manifolds. *Differ. Geom. Dyn. Sys.* **2014**, *16*, 156–167.
54. Tripathi, M.M.; G ülbahar, M.; Kiliç, E.; Keleş, S. Inequalities for scalar curvature of pseudo-Riemannian submanifolds. *J. Geom. Phys.* **2017**, *112*, 74–84. [CrossRef]
55. Gulbahar, M. Qualar curvatures of pseudo Riemannian manifolds and pseudo Riemannian submanifolds. *AIMS Math.* **2021**, *6*, 1366–1377. [CrossRef]
56. Kiliç, E.; Gulbahar, M.; Kavuk, E. Concurrent Vector Fields on Lightlike Hypersurfaces. *Mathematics* **2020**, *9*, 59. [CrossRef]
57. Gulbahar, M.; Kiliç, E.; Keles, S. A useful orthonormal basis on bi-slant submanifolds of almost Hermitian manifolds. *Tamkang J. Math.* **2016**, *47*, 143–161. [CrossRef]