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Compound-Combination Synchronization for Fractional Hyperchaotic Models with Different Orders

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Abstract: In this paper, we introduce a new type of synchronization for the fractional order (FO) hyperchaotic models with different orders called compound-combination synchronization (CCS). Using the tracking control method, a theorem to calculate the analytical controllers which achieve our proposed synchronization is described and proved. We introduce, also, the FO hyperchaotic complex Lü, Chen, and Lorenz models with complex periodic forcing. The symmetry property is found in the FO hyperchaotic complex Lü, Chen, and Lorenz models. These hyperchaotic models are found in many areas of applied sciences, such as physics and secure communication. These FO hyperchaotic models are used as an example for our proposed synchronization. The numerical simulations show a good agreement with the analytical results. The complexity and existence of additional variables mean that it is safer and interesting to transmit and receive signals in communication theory. The proposed scheme of synchronization is considered a generalization of many types in the literature and other examples can be found in similar studies.

Keywords: compound-combination synchronization; fractional order; tracking control; hyperchaotic; symmetry

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1. Introduction

During recent decades, fractional calculus has been used in a broad area of applications, including chaotic models [1–5], signal processing [6,7], fluid mechanics [8,9], and biological population models [10,11]. FO derivatives provide an excellent instrument to describe memory and the inherited properties of various materials and processes compared to integer-order derivatives [12]. Therefore, modeling with FO derivatives may be more accurate than modeling with integer-order derivatives. Many models have chaotic and hyperchaotic solutions in fractional calculus, such as chaotic neural networks models [13], hyperchaotic complex Duffing–van der Pol models [14] and chaotic generalized fractional Lü and Lorenz models [15], etc. For other models see [16–20].

Chaos synchronization has begun to receive increasing attention and has become an interesting problem due to its potential applications in secure communication and control processing. Many control methods, including the adaptive control scheme [21], adaptive back-stepping technique [22], active control method [23], sliding mode control scheme [24], and tracking control method [25], have been developed for chaos synchronization for FO calculus. Furthermore, many types of synchronization for FO calculus, such as combination synchronization between three hyperchaotic FO models, have been investigated [26]. Mahmoud et al. introduced combination–combination synchronization among four chaotic FO models [27], while Sun et al. illustrated compound synchronization among four chaotic FO models [28]. Different kinds of modulus–modulus synchronization were investigated by Mahmoud et al. [29]. In this paper, we introduce a new type of synchronization (synchronization between three master models and two slave models) for the FO hyperchaotic models, which has possible applications in modeling FO hyperchaotic circuits, such as the hyperchaotic models in [30–32].

There also exist interesting cases of complex dynamical models [33–37], which have many applications in many important fields of physics and engineering. Many hyperchaotic complex Lü models with complex periodic forcing are introduced in [35], for examples the following three models:

$$\begin{aligned} \dot{x_1} &= a_1(y_1 - x_1) + k_1(1+i)cosw_1 t, \\ \dot{y_1} &= c_1 y_1 - x_1 z_1, \\ \dot{z_1} &= \frac{1}{2}(x_1\bar{y}_1 + \bar{x}_1 y_1) - b_1 z_1, \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{x_2} &= a_2(y_2 - x_2), \\ \dot{y_2} &= c_2 y_2 - x_2 z_2 + k_2(1+i)sinw_2 t, \\ \dot{z_2} &= \frac{1}{2}(x_2\bar{y}_2 + \bar{x}_2 y_2) - b_2 z_2, \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{x_3} &= a_3(y_3 - x_3) + k_3 exp(jw_3 t), \\ \dot{y_3} &= c_3 y_3 - x_3 z_3, \\ \dot{z_3} &= \frac{1}{2}(x_3\bar{y}_3 + \bar{x}_3 y_3) - b_3 z_3, \end{aligned} \tag{3}$$

where $a_i, b_i,, c_i, w_i$, and k_i; $i = 1,2,3$ are positive parameters, $x_i = x_{i1} + jx_{i2}$, $y_i = x_{i3} + jx_{i4}$, $z_i = x_{i5}$; $j = \sqrt{-1}$ are the state variables for models (1)–(3), respectively.

Mahmoud et al. introduced the complex Chen model with complex periodic forcing [36] and the complex Lorenz model with complex periodic forcing [37] as:

$$\begin{aligned} \dot{x_4} &= a_4(y_4 - x_4) + k_4(1+i)cosw_4 t, \\ \dot{y_4} &= (c_4 - a_4)x_4 + c_4 y_4 - x_4 z_4, \\ \dot{z_4} &= \frac{1}{2}(x_4\bar{y}_4 + \bar{x}_4 y_4) - b_4 z_4, \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{x_5} &= a_5(y_5 - x_5) + k_5 exp(jw_5 t), \\ \dot{y_5} &= c_5 x_5 - y_5 - x_5 z_5, \\ \dot{z_5} &= \frac{1}{2}(x_5\bar{y}_5 + \bar{x}_5 y_5) - b_5 z_5, \end{aligned} \tag{5}$$

where $a_l, b_l,, c_l, w_l$, and k_l; $l = 4,5$ are positive parameters, $x_l = y_{(l-3)1} + jy_{(l-3)2}$, $y_l = y_{(l-3)3} + jy_{(l-3)4}$, $z_l = y_{(l-3)5}$, are the state variables for models (4) and (5), respectively. The models (1)–(5) without periodic forcing are symmetric.

The aims of this paper are: (1) propose a scheme for the CCS between three master and two slave FO hyperchaotic models with different orders using the tracking control method. (2) A theorem to calculate the analytical controllers which achieve our proposed synchronization is stated and proved. (3) We stated special cases of our synchronization which give other kinds of synchronization [26–28]. (4) The FO hyperchaotic complex Lü, Chen and Lorenz models with complex periodic forcing are introduced. (5) As an example for our proposed synchronization, we used the FO hyperchaotic complex Lü, Chen, and Lorenz models. (6) The numerical simulation results verify the feasibility of the proposed CCS scheme.

The rest of this paper is organized as follows. Section 2 defines CCS for five FO hyperchaotic models with different orders. In Section 3, the CCS for five identical FO hyperchaotic models is investigated by tracking control method and the FO stability theory.

In Section 4, we introduce the FO hyperchaotic complex Lü, Chen and Lorenz models with complex periodic forcing. Using Lyapunov exponents via a modified technique of Wolf algorithm [38], these models have hyperchaotic solutions. These models are used as an example for proposed CCS. In Section 5, the numerical treatments of our example are used to test the analytical formula of the controller forces to achieve the CCS. Finally, Section 6 concludes the results of this paper.

2. Compound-Combination Synchronization Definition

The CCS among three master and two slave FO models with different orders is designed in this section.

First, the three master FO models are given as:

$$^{c}D^{\alpha_1}x_1(t) = f_1(t, x_1(t)), \tag{6}$$

$$^{c}D^{\alpha_2}x_2(t) = f_2(t, x_2(t)), \tag{7}$$

$$^{c}D^{\alpha_3}x_3(t) = f_3(t, x_3(t)). \tag{8}$$

second, the two slave FO models are defined as:

$$^{c}D^{\beta}y_1(t) = g_1(t, y_1(t)) + u_1, \tag{9}$$

$$^{c}D^{\beta}y_2(t) = g_2(t, y_2(t)) + u_2, \tag{10}$$

where $^{c}D^{\alpha_i}$ and $^{c}D^{\beta}$ are the Caputo derivatives for fractional orders α_i and β, respectively, α_i, $\beta \in (0, 1]$ $(i = 1, 2, 3)$ [39], $x_i = diag(x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in})$ and $y_j = diag(y_{j1}, y_{j2}, y_{j3}, \ldots, y_{jn})$ are the state variables of models (6)–(10), $f_i(t, x_i) = diag(f_{i1}(t, x_i), f_{i2}(t, x_i), \ldots, f_{in}(t, x_i))$, $g_j(y_j) = diag(g_{j1}(t, y_j), g_{j2}(t, y_j), \ldots, g_{jn}(t, y_j))$ are continuous diagonal matrices functions, and $u_j(t, x_1, x_2, x_3, y_1, y_2) = diag(u_{j1}, u_{j2}, \ldots, u_{jn})$, $i = 1, 2, 3$, $j = 1, 2$, are controllers of the slave models (9) and (10).

Definition 1. *If there exist five constant diagonal matrices $A_1, A_2, A_3, B_1, B_2 \in (\mathbb{R}^n \times \mathbb{R}^n)$ and $B_1 \neq 0$ or $B_2 \neq 0$, such that*

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|B_1y_1 + B_2y_2 - A_1x_1(A_2x_2 + A_3x_3)\| = 0, \tag{11}$$

the compound-combination synchronization of the master FO models (6)–(8) and the slave FO models (9) and (10) is hold. Where $\|.\|$ expresses the matrix norm.

Remark 1. *A compound synchronization of four FO hyperchaotic models [28] can be obtained, if $B_1 = 0$ or $B_2 = 0$ in the above definition.*

Remark 2. *The combination–combination synchronization of four FO hyperchaotic models [27] is given from Definition 1 for the case of x_1 as a constant matrix.*

Remark 3. *For the choice x_1 as a constant matrix and either B_1 or B_2 as zero, then the combination synchronization of three FO hyperchaotic models is deduced [26].*

3. The Compound-Combination Synchronization Planner

This section introduces the planner of the CCS of models with three master FO models (6)–(8) and two slave FO models (9) and (10). We suppose the controller $U = B_1u_1 + B_2u_2$ as follows:

$$U(t, x_1, x_2, x_3, y) = \rho(t, x_1, x_2, x_3) + \tau(t, x_1, x_2, x_3, y_1, y_2), \tag{12}$$

where $\rho(t,x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n$ is a compensation control and given by:

$$\rho(t,x_1,x_2,x_3) = {}^cD^\beta(A_1x_1(A_2x_2 + A_3x_3)) - g_1(t,A_1x_1A_2x_2) - g_2(t,A_1x_1A_3x_3), \quad (13)$$

and $\tau : \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^n)$ is a matrix function.

Using Equations (9) and (10), we have

$$B_1{}^cD^\beta y_1 + B_2{}^cD^\beta y_2 = B_1g_1(t,y_1) + B_2g_2(t,y_2) + U. \quad (14)$$

due to Equations (12)–(14), the model of error for CCS is:

$${}^cD^\beta e = B_1g_1(t,y_1) + B_2g_2(t,y_2) - g_1(t,A_1x_1A_2x_2) - g_2(t,A_1x_1A_3x_3) + \tau(t,x_1,x_2,x_3,y_1,y_2). \quad (15)$$

it is clear that CCS can be achieved if the error model (15) is asymptotically stable. So, the following theory is presented to obtain the analytical formula of the matrix τ,

Theorem 1. *If the matrix function $\tau(t,x_1,x_2,x_3,y_1,y_2)$ takes the form:*

$$\tau(t,x_1,x_2,x_3,y_1,y_2) = g_1(t,A_1x_1A_2x_2) + g_2(t,A_1x_1A_3x_3) - B_1g_1(t,y_1) - B_2g_2(t,y_2) - Ke, \quad (16)$$

the CCS for the three master FO models (6)–(8) and the two slave FO models will be achieved

Proof. Since τ is given by Equation (16), the error of model (15) can be written as:

$${}^cD^\beta e = -Ke. \quad (17)$$

one defines a Lyapunov function as:

$$V(t) = \frac{1}{2}e^Te, \quad (18)$$

the FO derivative of $V(t)$ is given by,

$${}^cD^\beta V(t) = {}^cD^\beta(\frac{1}{2}e^Te) \leq e^{T\,c}D^\beta e, \quad (19)$$

using Equation (17), then we have

$${}^cD^\beta V(t) \leq e^T(-Ke) = -K\|e\|^2 \leq -\mu_{\min}\|e\|^2 < 0. \quad (20)$$

where $\mu_{\min} = \min(\mu_1,\mu_2,\ldots,\mu_n)$ is the minimum value of the eigenvalues of K. Since $V(t)$ is positive definite function and its FO derivative is negative definite, then the error $e(t) \to 0$ as $t \to \infty$, and, hence, the CCS among three master FO models (6)–(8) and the two slave FO models (9) and (10) can be achieved. □

Corollary 1. *(i) The master FO models (6)–(8) will be in compound synchronization with the slave FO model (9) if $B_2 = 0$. So the controllers are written as:*

$$U = B_1u_1 = \rho(t,x_1,x_2,x_3) + \tau(t,x_1,x_2,x_3,y_1), \quad (21)$$

where $\rho(t,x_1,x_2,x_3) = {}^cD^\beta(A_1x_1(A_2x_2 + A_3x_3)) - g_1(t,A_1x_1(A_2x_2 + A_3x_3))$, and $\tau(t,x_1,x_2,x_3,y_1) = -Ke - B_1g_1(y_1) + g_1(t,A_1x_1(A_2x_2 + A_3x_3))$.

(ii) The master models (6)–(8) will be in compound synchronization with the slave model (10) if $B_1 = 0$. Then, the controllers are:

$$U = B_2u_2 = \rho(t,x_1,x_2,x_3) + \tau(t,x_1,x_2,x_3,y_2), \quad (22)$$

where $\rho(t, x_1, x_2, x_3) = {}^cD^{\beta}(A_1x_1(A_2x_2 + A_3x_3)) - g_2(A_1x_1(A_2x_2 + A_3x_3))$, *and* $\tau(t, x_1, x_2, x_3, y_2) = -Ke - B_2g_2(y_2) + g_2(A_1x_1(A_2x_2 + A_3x_3))$.

Corollary 2. *For the choice* $x_1 = N$ *as a constant matrix, the master models* (7) *and* (8) *will be in combination–combination synchronization with the slave models* (9) *and* (10) *under the following controllers:*

$$U = B_1u_1 + B_2u_2 = \rho(t, x_2, x_3) + \tau(t, x_2, x_3, y_1, y_2), \tag{23}$$

where $\rho(t, x_2, x_3) = {}^cD^{\beta}(A_1N(A_2x_2 + A_3x_3)) - g_1(A_1NA_2x_2) - g_2(A_1NA_3x_3)$, *and* $\tau(t, x_2, x_3, y_1, y_2) = -Ke - B_1g_1(y_1) - B_2g_2(y_2) + g_1(A_1NA_2x_2) + g_2(A_1NA_3x_3)$.

Corollary 3. *(i) The master models* (7) *and* (8) *will be in combination synchronization with the slave model (9) if* $B_2 = 0$ *and* $x_1 = N$ *is a constant matrix. Therefore, the controllers are given as:*

$$U = B_1u_1 = \rho(t, x_2, x_3) + \tau(t, x_2, x_3, y_1), \tag{24}$$

where $\rho(t, x_2, x_3) = {}^cD^{\beta}(A_1N(A_2x_2 + A_3x_3)) - g_1(A_1N(A_2x_2 + A_3x_3))$, *and* $\tau(t, x_2, x_3, y_1) = -Ke - B_1g_1(y_1) + g_1(A_1N(A_2x_2 + A_3x_3))$.

(ii) The master models (7) *and* (8) *will be in combination synchronization with the slave model (10) if* $B_1 = 0$ *and* $x_1 = N$ *is a constant matrix. Then, the controllers are written as:*

$$U = B_2u_2 = \rho(t, x_2, x_3) + \tau(t, x_2, x_3, y_2), \tag{25}$$

where $\rho(t, x_2, x_3) = {}^cD^{\beta}(A_1N(A_2x_2 + A_3x_3)) - g_2(A_1N(A_2x_2 + A_3x_3))$, *and* $\tau(t, x_2, x_3, y_2) = -Ke - B_2g_2(y_2) + g_2(A_1N(A_2x_2 + A_3x_3))$.

4. An Example

We study the CCS for five hyperchaotic fractional models with different orders as an example using the scheme of Section 3. We consider the fractional versions of models (1)–(3) in real forms, respectively, as:

$$\begin{aligned}
{}^cD^{\alpha_1}x_{11} &= a_1(x_{13} - x_{11}) + k_1cosw_1t,\\
{}^cD^{\alpha_1}x_{12} &= a_1(x_{14} - x_{12}) + k_1cosw_1t,\\
{}^cD^{\alpha_1}x_{13} &= c_1x_{13} - x_{11}x_{15},\\
{}^cD^{\alpha_1}x_{14} &= c_1x_{14} - x_{12}x_{15},\\
{}^cD^{\alpha_1}x_{15} &= x_{11}x_{13} + x_{12}x_{14} - b_1x_{15},
\end{aligned} \tag{26}$$

$$\begin{aligned}
{}^cD^{\alpha_2}x_{21} &= a_2(x_{23} - x_{21}),\\
{}^cD^{\alpha_2}x_{22} &= a_2(x_{24} - x_{22}),\\
{}^cD^{\alpha_2}x_{23} &= c_2x_{23} - x_{21}x_{25} + k_2sinw_2t,\\
{}^cD^{\alpha_2}x_{24} &= c_2x_{24} - x_{22}x_{25} + k_2sinw_2t,\\
{}^cD^{\alpha_2}x_{25} &= x_{21}x_{23} + x_{22}x_{24} - b_2x_{25},
\end{aligned} \tag{27}$$

$$\begin{aligned}
{}^cD^{\alpha_3}x_{31} &= a_3(x_{33} - x_{31}) + k_3cosw_3t,\\
{}^cD^{\alpha_3}x_{32} &= a_3(x_{34} - x_{32}) + k_3sinw_3t,\\
{}^cD^{\alpha_3}x_{33} &= c_3x_{33} - x_{31}x_{35},\\
{}^cD^{\alpha_3}x_{34} &= c_3x_{34} - x_{32}x_{35},\\
{}^cD^{\alpha_3}x_{35} &= x_{31}x_{33} + x_{32}x_{34} - b_3x_{35},
\end{aligned} \tag{28}$$

where ${}^cD^{\alpha_i}$ are the Caputo fractional derivatives with order $0 < \alpha_i \leq 1$; $i = 1, 2, 3$. For the choice $a_1 = 35$, $b_1 = 4$, $c_1 = 25$, $k_1 = 10$, $w_1 = 5$, $\alpha_1 = 0.95$ for the model (26) and the initial values $x_{10} = diag(-4.2595, -4.3055, -6.1868, -6.2533, 27.5840)$, we used a modified technique of the Wolf algorithm to calculate the Lyapunov exponents of the model, and the results are: $\lambda_1 = 12.4479$, $\lambda_2 = 2.9483$, $\lambda_3 = 0.5194$, $\lambda_4 = -6.6970$, and $\lambda_5 = -19.6768$. These Lyapunov exponent results show that model (26) has hyperchaotic solution of order 3, as shown in Figure 1 in (x_{12}, x_{14}, x_{15}) space. Model (27) has a hyperchaotic solution for the values of the parameters $a_2 = 34$, $b_2 = 4$, $c_2 = 25$, $k_2 = 10$, $w_2 = 5$, $\alpha_2 = 0.96$ and the initial values $x_{20} = diag(0.1, 0.2, 0.14, 0.2, 0.4)$ as depicted in Figure 2 for (x_{23}, x_{21}, x_{25}) space. By similar way, if we choose $a_3 = 35$, $b_3 = 4$, $c_3 = 25$, $k_3 = 10$, $w_3 = 5$, $\alpha_3 = 0.97$ and the initial values $x_{30} = diag(0.1, 0.2, 0.14, 0.2, 0.4)$, model (28) has hyperchaotic solution as drown in Figure 3 in (x_{31}, x_{32}, x_{35}) space.

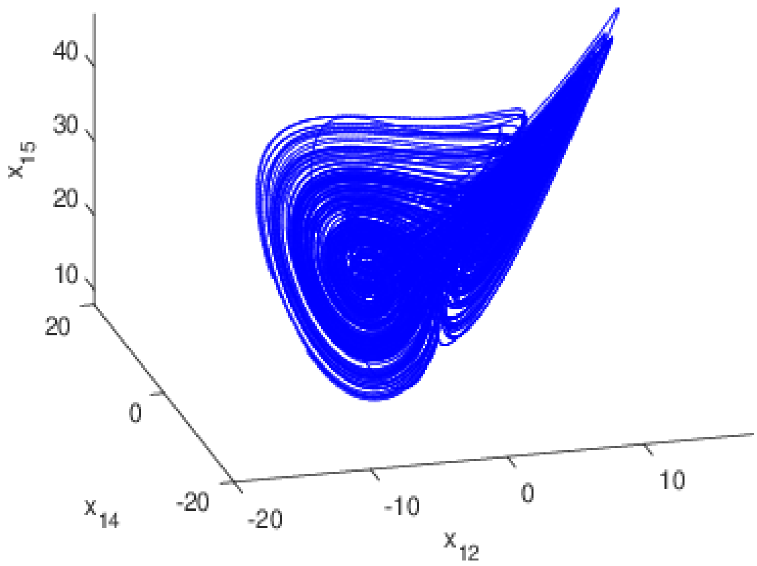

Figure 1. Hyperchaotic solution for the fractional complex Lü model with complex periodic forcing (26).

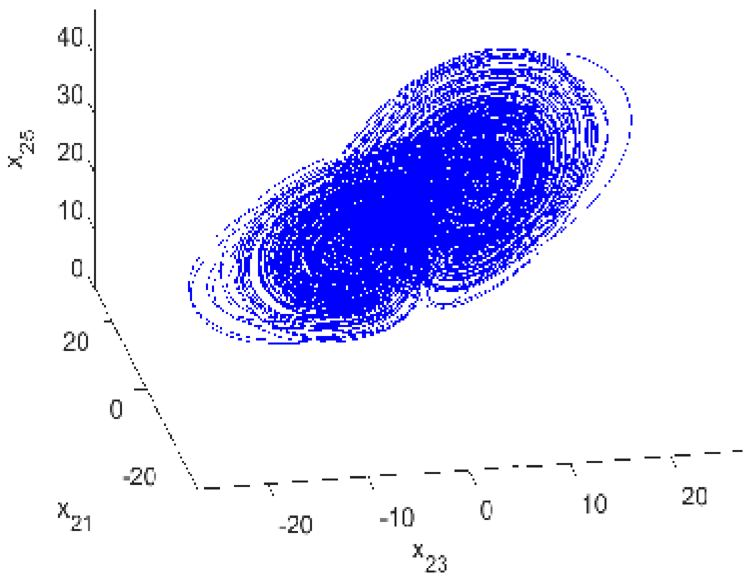

Figure 2. Hyperchaotic solution for the fractional complex Lü model with complex periodic forcing (27).

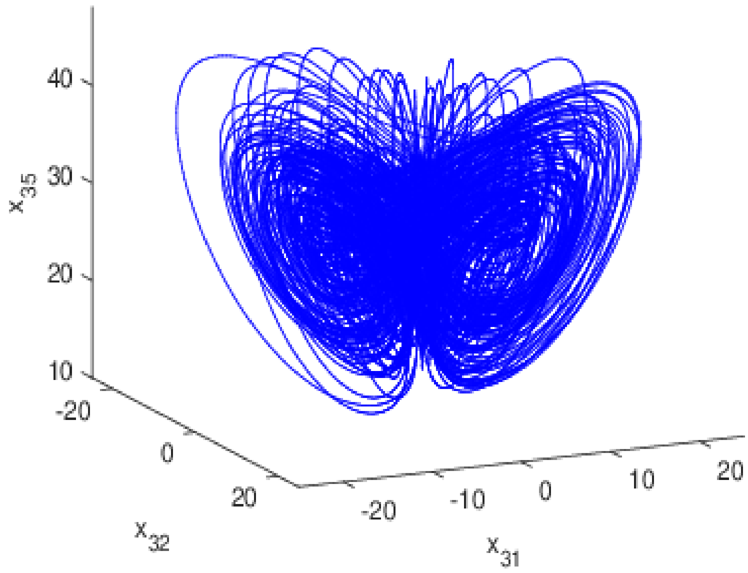

Figure 3. Hyperchaotic solution for the fractional complex Lü model with complex periodic forcing (28).

The FO of models (4) and (5) can be written in the real forms, respectively, as:

$$\begin{aligned}
{}^cD^\beta y_{11} &= a_4(y_{13} - y_{11}) + k_4 cos w_4 t, \\
{}^cD^\beta y_{12} &= a_4(y_{14} - y_{12}) + k_4 cos w_4 t, \\
{}^cD^\beta y_{13} &= (c_4 - a_4)y_{11} + c_4 y_{13} - y_{11}y_{15}, \\
{}^cD^\beta y_{14} &= (c_4 - a_4)y_{12} + c_4 y_{14} - y_{12}y_{15}, \\
{}^cD^\beta y_{15} &= y_{11}y_{13} + y_{12}y_{14} - b_4 y_{15},
\end{aligned} \tag{29}$$

$$\begin{aligned}
{}^cD^\beta y_{21} &= a_5(y_{23} - y_{21}) + k_5 cos w_5 t, \\
{}^cD^\beta y_{22} &= a_5(y_{24} - y_{22}) + k_5 sin w_5 t, \\
{}^cD^\beta y_{23} &= c_5 y_{21} - y_{23} - y_{21}y_{25}, \\
{}^cD^\beta y_{24} &= c_5 y_{22} - y_{24} - y_{22}y_{25}, \\
{}^cD^\beta y_{25} &= y_{21}y_{23} + y_{22}y_{24} - b_5 y_{25}.
\end{aligned} \tag{30}$$

for the choice $a_4 = 42$, $b_4 = 4$, $c_4 = 26$, $k_4 = 85$, $w_4 = 5$, $\beta = 0.99$ for the model (29) and the initial values $y_{10} = diag(5.441, 5.5045, 4.3299, 4.3645, 16.6665)$, model (29) has hyperchaotic solution, as shown in Figure 4 in (y_{11}, y_{13}, y_{15}) space. Model (27) also has a hyperchaotic solution for the values of the parameters $a_5 = 15$, $b_5 = 5$, $c_5 = 45$, $k_5 = 10$, $w_5 = 13$, $\beta = 0.99$ and the initial values $y_{20} = diag(1, 2, 3, 4, 5)$ as depicted in Figure 5 for (y_{21}, y_{22}, y_{24}) space.

We consider the three hyperchaotic fractional complex Lü models (26)–(28) as the master models and the hyperchaotic fractional complex Chen model (29) and the hyperchaotic fractional complex Lorenz model (30) as the slave models as an example to achieve the CCS. The slave models after adding the controllers are:

$$\begin{aligned}
{}^cD^\beta y_{11} &= a_4(y_{13} - y_{11}) + k_4 cos w_4 t + u_1, \\
{}^cD^\beta y_{12} &= a_4(y_{14} - y_{12}) + k_4 cos w_4 t + u_2, \\
{}^cD^\beta y_{13} &= (c_4 - a_4)y_{11} + c_4 y_{13} - y_{11}y_{15} + u_3, \\
{}^cD^\beta y_{14} &= (c_4 - a_4)y_{12} + c_4 y_{14} - y_{12}y_{15} + u_4, \\
{}^cD^\beta y_{15} &= y_{11}y_{13} + y_{12}y_{14} - b_4 y_{15} + u_5,
\end{aligned} \tag{31}$$

$$
\begin{aligned}
{}^{c}D^{\beta}y_{21} &= a_5(y_{23} - y_{21}) + k_5 cos w_5 t + v_1, \\
{}^{c}D^{\beta}y_{22} &= a_5(y_{24} - y_{22}) + k_5 sin w_5 t + v_2, \\
{}^{c}D^{\beta}y_{23} &= c_5 y_{21} - y_{23} - y_{21}y_{25} + v_3, \\
{}^{c}D^{\beta}y_{24} &= c_5 y_{22} - y_{24} - y_{22}y_{25} + v_4, \\
{}^{c}D^{\beta}y_{25} &= y_{21}y_{23} + y_{22}y_{24} - b_5 y_{25} + v_5.
\end{aligned} \tag{32}
$$

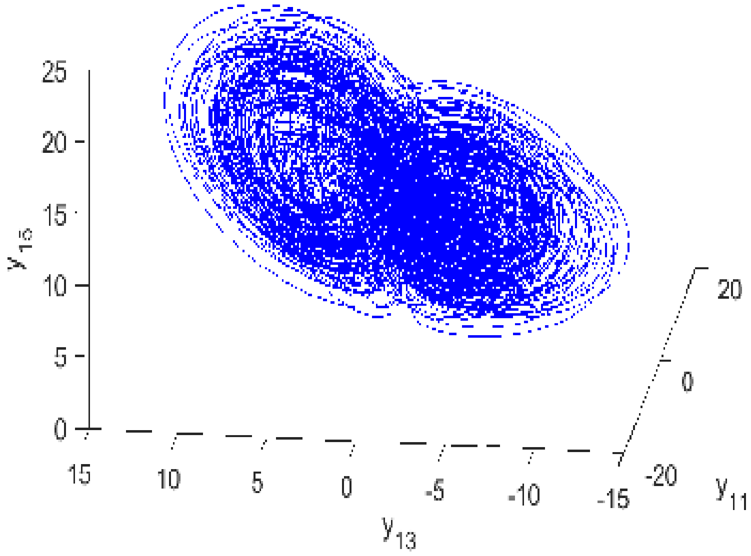

Figure 4. Hyperchaotic solution for the fractional complex Chen model with complex periodic forcing (29).

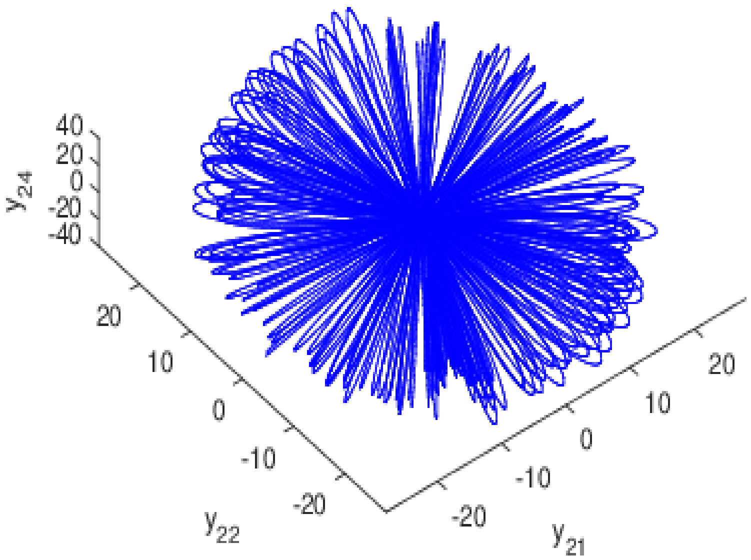

Figure 5. Hyperchaotic solution for the fractional complex Lorenz model with complex periodic forcing (30).

5. Numerical Simulation

In this section, we tested and demonstrated the validity of the CCS for our example. Using Theorem 1 and the same values of the parameter values of the master models (26)–(28) and the slave models (29) and (30) which are used in Figures 1–5, the gain matrix $K = diag(1,2,3,4,5)$ and the scaling matrices are chosen as $A_1 = A_2 = A_3 = B_1 = B_2 = I$, where I is (5×5) identity matrix, then the matrix τ takes the form:

$$\tau = -diag(1,2,3,4,5)diag(e_1,e_2,e_3,e_4,e_5) + diag(a_4(x_{13}x_{23} - x_{11}x_{21}) + k_4cosw_4t, a_4(x_{14}x_{24} - x_{12}x_{22}) + k_4cosw_4t,$$
$$(c_4 - a_4)x_{11}x_{21} + c_4x_{13}x_{23} - x_{11}x_{21}x_{15}x_{25}, (c_4 - a_4)x_{12}x_{22} + c_4x_{14}x_{24} - x_{12}x_{22}x_{15}x_{25}, x_{11}x_{21}x_{13}x_{23}$$
$$+ x_{12}x_{22}x_{14}x_{24} - b_4x_{15}x_{25}) + diag(a_5(x_{13}x_{33} - x_{11}x_{31}) + k_5cosw_5t, a_5(x_{14}x_{34} - x_{12}x_{32}) + k_5sinw_5t,$$
$$c_5x_{11}x_{31} - x_{13}x_{33} - x_{11}x_{31}x_{15}x_{35}, c_5x_{12}x_{32} - x_{14}x_{34} - x_{12}x_{32}x_{15}x_{35}, x_{11}x_{31}x_{13}x_{33} + x_{12}x_{32}x_{14}x_{34} - b_5x_{15}x_{35}) \quad (33)$$
$$- diag(a_4(y_{13} - y_{11}) + k_4cosw_4t, a_4(y_{14} - y_{12}) + k_4cosw_4t, (c_4 - a_4)y_{11} + c_4y_{13} - y_{11}y_{15}, (c_4 - a_4)y_{12} + c_4y_{14}$$
$$- y_{12}y_{15}, y_{11}y_{13} + y_{12}y_{14} - b_4y_{15}) - diag(a_5(y_{23} - y_{21}) + k_5cosw_5t, a_5(y_{24} - y_{22}) + k_5sinw_5t, c_5y_{21} - y_{23} - y_{21}y_{25},$$
$$c_5y_{22} - y_{24} - y_{22}y_{25}, y_{21}y_{23} + y_{22}y_{24} - b_5y_{25}),$$

where $e = diag(e_1, e_2, e_3, e_4, e_5)$, the matrix of error, $e_i = r_i - d_i = y_{1i} + y_{2i} - x_{1i}(x_{2i} + x_{3i})$; $i = 1, 2, \ldots, 5$.

For our example, the error of model (17) is:

$$^{c}D^{\beta}e = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} e. \quad (34)$$

According to Theorem 1, the CCS between the three master models (26)–(28) and the two slave models (31) and (32) is hold. In the numerical results, we used the PECE (Predict–Evaluate–Correct–Evaluate) method [40]. The results of the CSS are described in Figures 6–8. Figure 6 shows the same hyperchaotic solution for the three master models (26)–(28) in (d_1, d_2, d_5) space and the two slave model (31) and (32) in (r_1, r_2, r_5) space. The state variables between the three master models (26)–(28) and the two slave models (31) and (32) are depicted in Figure 7. Figure 8 gives the time slave of the synchronization errors; it is clear that CCS is achieved, as indicated by the convergence of the error state variables to zero.

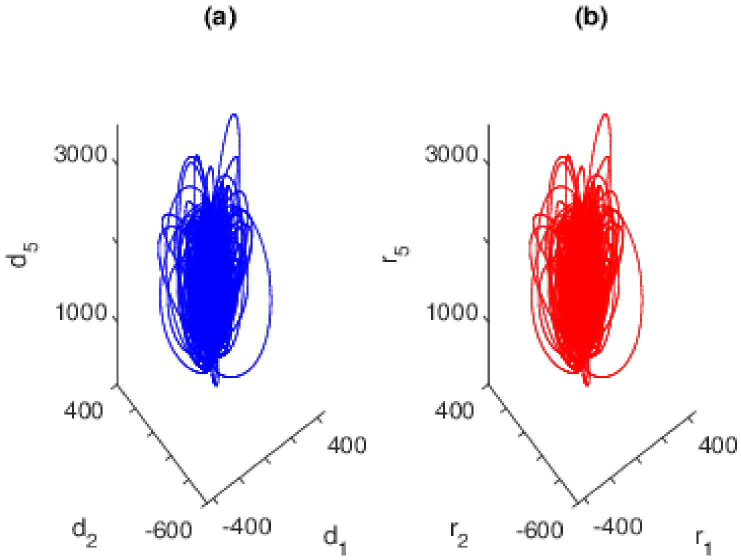

Figure 6. Hyperchaotic solution for (**a**) the three master models (26)–(28) in (d_1, d_2, d_5) space, (**b**) the two slave model (31) and (32) in (r_1, r_2, r_5) space.

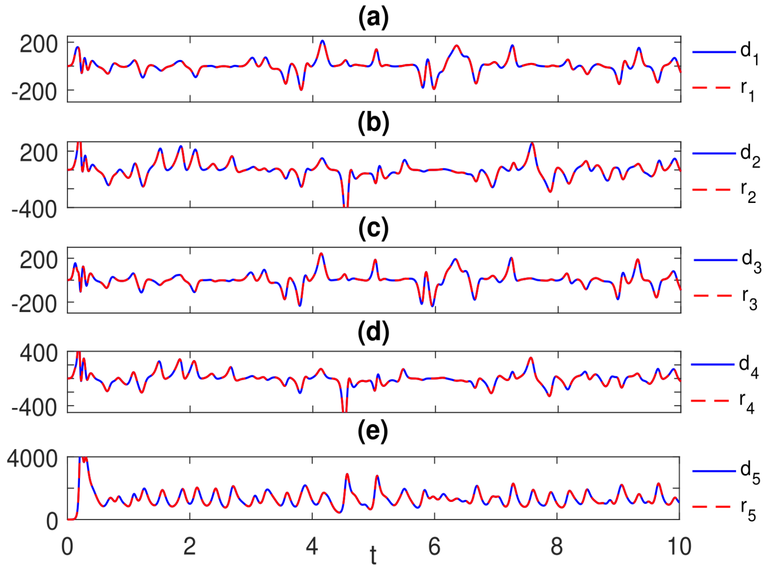

Figure 7. The state variables after the CCS between the three master models (26)–(28) (solid curves) and the two slave models (31) and (32) (dashed curves): (**a**) d_1 and r_1 versus t, (**b**) d_2 and r_2 versus t, (**c**) d_3 and r_3 versus t, (**d**) d_4 and r_4 versus t, (**e**) d_5 and r_5 versus t.

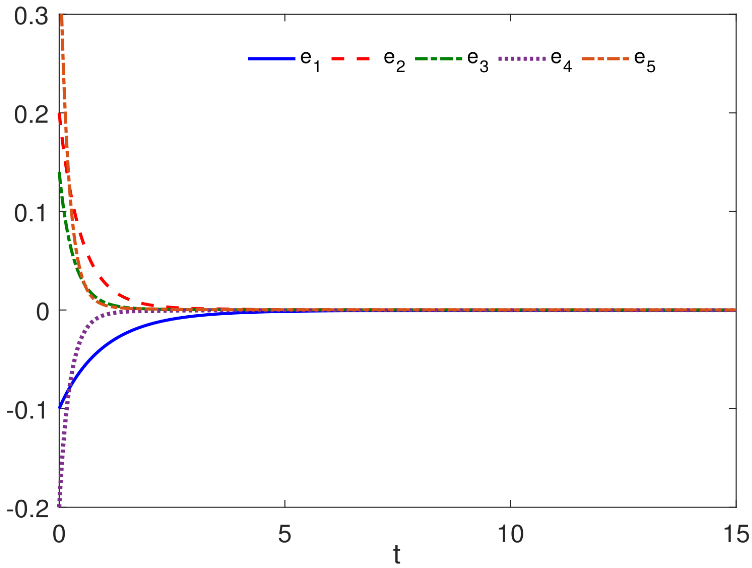

Figure 8. The synchronization errors of the three master models (26)–(28) and the two slave models (31) and (32).

6. Conclusions

In this paper, we proposed a new type of synchronization for three master fractional models and two slave fractional models with different orders, called CCS. According to Remarks 1–3, this kind of synchronization can be considered as a generalization of other synchronization types. The proposed CCS is achieved using stability theory and tracking control. We stated and proved Theorem 1 to derive the analytical controller which used to achieve the CCS. The FO hyperchaotic complex Lü (1)–(3), Chen (4), and Lorenz (5) models with complex periodic forcing are presented. These models have hyperchaotic solutions as shown in Figures 1–5, respectively. To test our proposed CCS, we used these models as an example. Numerical simulations used to test the correction and validity of the CCS. Because the CCS has more dimensions, our results increase the security of signal transmission and reception in secure communications. The results of the CCS are depicted in Figures 6–8.

Author Contributions: G.M.M.: Conceptualization; Supervision; Writing—review & editing; A.A.: Methodology; Validation; Writing—original draft; T.M.A.-E.: Formal analysis; Software; Writing—original draft; and A.A.F.: Formal analysis; Methodology; Writing—original draft. All authors have read and agreed to the published version of the manuscript.

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