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New Results on Integral Operator for a Subclass of Analytic Functions Using Differential Subordinations and Superordinations

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Abstract: In this paper, we discuss and introduce a new study using an integral operator $w_{k,\mu}^{m}$ in geometric function theory, especially sandwich theorems. We obtained some conclusions for differential subordination and superordination for a new formula generalized integral operator. In addition, certain sandwich theorems were found. The differential subordination theory's features and outcomes are symmetric to those derived using the differential subordination theory.

Keywords: analytic function; subordination; superordination; dominant; subordinant; sandwich theorem

MSC: 30C45

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1. Introduction

Let $\mathbb{G}(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer j and $a \in \mathbb{C}$, let $\mathbb{G}\,[a, j]$ be the subclass of $\mathbb{G}(U)$ of the form:

$$f(z) = z + a_j z^j + a_{j+1} z^{j+1} + \ldots \ (a \in \mathbb{C},\ j \in N = \{1, 2, \ldots\}).$$

Assume that A is a subclass of $\mathbb{G}(U)$ of functions f of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1}$$

If $f \in A$ is given by (1) and $g \in A$ is given by $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, the Hadamard product (or convolution) for the functions f and g is defined by:

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z).$$

The above was defined in [1].

Assuming that both f and g are analytically defined in U, f is called subordinate to g in U and denoted as $f \prec g$. If there is a function, w, which is Schwarz analytic in U, and $w(0) = 0, |w(z)| < 1, (z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. Moreover, if the function g is univalent in U, we have the following equivalence: $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$ (see [2–5]).

Definition 1 [6,7]**.** *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h(z) be analytic function is in U. If $p(z)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ are univalent in U and if $p(z)$ satisfies the second-order differential superordination*

$$h(z) \prec \psi\Big(p(z), zp'(z), z^2 p''(z); z\Big), (z \in U), \tag{2}$$

then, $p(z)$ is called a solution of the differential superordination (2). An analytic function $q(z)$ which is called a subordinant of the solutions of the differential superordination (2), or more simply a subordinant, if $q \prec p$ for all the functions $p(z)$ satisfying (2). A univalent subordinant $\widetilde{q}(z)$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants $q(z)$ of (2) is called the best subordinant.

Definition 2 [4]. *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be univalent function in U. If $p(z)$ is analytic in U and satisfies the second-order differential subordination:*

$$\psi\Big(p(z), zp'(z), z^2 p''(z); z\Big) \prec h(z), (z \in U), \tag{3}$$

then, p is called a solution of the differential subordination (3). The univalent function $q(z)$ is called a dominant of the solution of the differential subordination (3), or more simply dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q} \prec q$ for all dominant $q(z)$ of (3) is called the best dominant of (3).

Sufficient requirements for the functions h, q, and ψ that satisfy the following condition, were obtained by many authors (see [8–20]).

$$h(z) \prec \psi\Big(p(z), zp'(z), z^2 p''(z); z\Big) \Rightarrow p(z) \prec q(z), (z \in U). \tag{4}$$

By using the results (see [9–14,18,21] and also [19,22–29]), we obtain sufficient conditions for normalized analytic functions satisfying:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. In addition, many authors (see [9–15] and also [3,16–18,23,30]) derived some differential subordination and superordination results with some sandwich theorems. Our subject has some applications (see [8,31–38]).

Raina and Poonam Sharma [39] defined an integral operator for $\mu > -1$, $k > 0$

$$I_{k,\mu} f(z) = \frac{\mu+1}{k} z^{2-\frac{\mu+1}{k}} \int_0^Z t^{\frac{\mu+1}{k}-2} f(t)dt,$$

By using the function f of the form (1). We get:

$$I_{k,\mu} f(z) = z + \sum_{j=2}^{\infty} \frac{\mu+1}{\mu+1+k(j-1)} a_j z^j. \tag{5}$$

Now, we will generalize this operator as follows:

$$w^m_{k,\mu} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{\mu+1}{\mu+1+k(j-1)}\right)^m a_j z^j. \tag{6}$$

We observe that: $w^{m+1}_{k,\mu} : \mathbb{G}(U) \to \mathbb{G}(U)$ integral operator follows that:
From (6), we note that:

$$w^m_{0,0} f(z) = f(z)$$

$$z\Big(w^{m+1}_{k,\mu} f(z)\Big)' = \frac{(\mu+1)}{k}\Big(w^m_{k,\mu} f(z)\Big) - \left(\frac{\mu+1}{k} - 1\right)\Big(w^{m+1}_{k,\mu} f(z)\Big). \tag{7}$$

In this paper, we will establish our differential subordination and superordination results by the operator $w_{k,\mu}^{m} f(z)$.

The target of this paper is to find sufficient conditions for normalized analytic functions to get:

$$q_1(z) \prec \left(\frac{w_{k,\mu}^{m+1} f(z)}{z} \right)^{\delta} \prec q_2(z),$$

and

$$q_1(z) \prec \left(\frac{\propto w_{k,\mu}^{m} f(z) + (1- \propto) w_{k,\mu}^{m+1} f(z)}{z} \right)^{\delta} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries

In order to establish our subordination and superordination results, we need the following lemmas and definitions:

Definition 3 [3]. *Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $\overline{U} = U \bigcup \{z \in \partial U\}$, and $E(q) = \{\zeta \in \partial U : q(z) = \infty\}$ and are such that $q'(\zeta) \neq 0$ such that for $\zeta \in \partial U \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) = Q_0$, and $Q(1) = Q_1 = \{q \in Q,\ q(0) = 1\}$.*

Lemma 1 [3]. *Let $q(z)$ be a convex univalent function in U and let $\propto \in \mathbb{C}$, $\zeta \in \mathbb{C} \setminus \{0\}$, and suppose that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\frac{\propto}{\zeta}\right)\right\}.$$

If $p(z)$ is analytic in U, and

$$\propto p(z) + \zeta z p'(z) \prec \propto q(z) + \zeta z q'(z), \tag{8}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2 [4]. *Let q be a univalent function in U and let Φ and θ be analytic in the domain D containing $q(U)$ with $\Phi(w) \neq 0$, when $w \in q(U)$. Put $Q(z) = zq'(z)\Phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that,*

(i) *Q is starlike univalent in U.*

(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)), \tag{9}$$

then $p \prec q$ and q is the best dominant.

Lemma 3 [4]. *Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\zeta \in \mathbb{C}$, that $Re\,(\zeta) > 0$. If $p(z) \in \mathbb{G}\,[q(0), 1] \cap Q$ and $p(z) + \zeta zp'(z)$ is univalent in U, then $q(z) + \zeta zq'(z) \prec p(z) + \zeta zp'(z)$, which implies that $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.*

Lemma 4 [6]. *Let $q(z)$ be convex univalent in the unit disk U and let θ and Φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $Re\left\{\frac{\theta'(q(z))}{\Phi(q(z))}\right\} > 0$ *for* $z \in U$,

(ii) $zq'(z)\Phi(q(z))$ *is starlike univalent in* $z \in U$

If $p \in \mathbb{G}[q(0),1] \cap Q$ with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\Phi(p(z))$ is univalent in U, and

$$\theta q(z) + zq'(z)\Phi(q(z)) \prec \theta\, p(z) + zp'(z)\Phi(p(z)), \tag{10}$$

then $q \prec p$ and q is the best subordinant.

3. Differential Subordination Results

Here, some differential subordination results are introduced using the operator $w^{m}_{k,\mu}f(z)$.

Theorem 1. *Let* $q(z)$ *be univalent convex in the unit disk* U *and let* $\mu,\ \delta \in \mathbb{C}$, $k \in \mathbb{C} \setminus \{0\}$. *Suppose that:*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\delta\frac{\mu+1}{k}\right)\right\}.$$

If

$$\tau(m,k,\mu,\delta) = \left(\frac{w^{m+1}_{k,\mu}f(z)}{z}\right)^{\delta}\left(\frac{w^{m}_{k,\mu}f(z)}{w^{m+1}_{k,\mu}f(z)}\right), \tag{11}$$

hold the following subordination:

$$\tau(m,k,\mu,\delta) \prec q(z) + \frac{k}{\delta(\mu+1)}zq'(z), \tag{12}$$

then $\left(\frac{w^{m+1}_{k,\mu}f(z)}{z}\right)^{\delta} \prec q(z)$ and q is the best dominant.

Proof. Set

$$p(z) = \left(\frac{w^{m+1}_{k,\mu}f(z)}{z}\right)^{\delta}.$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, if we differentiate $p(z)$ with respect to z and by (7), in the last equation, it follows that:

$$\frac{zp'(z)}{p(z)} = \left(\delta\frac{\mu+1}{k}\right)\left(\frac{w^{m}_{k,\mu}f(z)}{w^{m+1}_{k,\mu}f(z)} - 1\right), \tag{13}$$

then

$$zp'(z) = \left(\frac{w^{m+1}_{k,\mu}f(z)}{z}\right)^{\delta}\left(\delta\frac{\mu+1}{k}\right)\left(\frac{w^{m}_{k,\mu}f(z)}{w^{m+1}_{k,\mu}f(z)} - 1\right). \tag{14}$$

From the hypothesis the subordination (12) follows and becomes

$$p(z) + \frac{k}{\delta(\mu+1)}zp'^{(z)} \prec q(z) + \frac{k}{\delta(\mu+1)}zq'(z). \tag{15}$$

Then by apply Lemma 1, we obtain:

$$\left(\frac{w^{m+1}_{k,\mu}f(z)}{z}\right)^{\delta} \prec q(z).$$

The proof is complete. □

Now, in the above theorem, if we taking the convex function $q(z) = \frac{1+Dz}{1+Ez}$, we get the following corollary:

Corollary 1. *Let $D, E \in \mathbb{C},\ D \neq E,\ |E| < 1$ and $\delta > 0$, with $f \in A$. Suppose that:*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -Re\left(\delta\frac{\mu+1}{k}\right)\right\}.$$

If

$$\tau(m,k,\mu,\delta) = \left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}\left(\frac{w_{k,\mu}^{m}f(z)}{w_{k,\mu}^{m+1}f(z)}\right),$$

hold the following subordination:

$$\tau(m,k,\mu,\delta) \prec \frac{1+Dz}{1+Ez} + \frac{k}{\delta(\mu+1)}\frac{(D-E)z}{(1+Ez)^2}.$$

Then

$$\left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta} \prec \frac{1+Dz}{1+Ez}$$

and $\frac{1+Dz}{1+Ez}$ is the best dominant.

Theorem 2. *Let $q(z)$ be univalent convex in the unit disk U with $q(0) = 1,\ q'(z) \neq 0,\ z \in U$ and let $\xi, \mu,\ \delta,\ \propto \in \mathbb{C}, \rho,\ k \in\ \mathbb{C} \setminus \{0\}$. Suppose that:*

$$Re\left\{\frac{zq''(z)}{q'(z)} - \frac{3\xi}{\rho}q^2(z) + 1\right\} > 0.$$

If $f \in A$ satisfies:

$$N(\xi,\rho,k,\mu,\propto,\delta) \prec\ \xi q^3(z) - \rho z q'(z), \tag{16}$$

where

$$\begin{aligned} &N(\xi,\rho,k,\mu,\propto,\delta) \\ &= \left(\frac{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}\left(\xi\left(\frac{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{2\delta}\right. \\ &\left.-\rho\delta\left(\frac{\mu+1}{k}\right)\left(\frac{w_{k,\mu}^{m-1}f(z)}{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m}f(z)} - 1\right)\right) \end{aligned} \tag{17}$$

then

$$\left(\frac{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta} \prec\ q(z), \tag{18}$$

and q is the best dominant.

Proof. Consider a function p by:

$$p = \left(\frac{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}$$

is analytic in U and $p(0) = 1$, differentiating (18) with respect to z, and using the identity (7), we get:

$$\frac{zp'(z)}{p(z)} = \delta\left[\frac{\propto\left(w_{k,\mu}^{m}f(z)\right)' + (1-\propto)\left(w_{k,\mu}^{m+1}f(z)\right)'}{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)} + 1\right] \tag{19}$$

by setting $\theta(w) = \xi w^3$ and $\Phi(w) = -\rho$, where θ is analytic in $\mathbb{C}$ and Φ is analytic in $\mathbb{C}\backslash\{0\}$.

By using Lemma 2, we obtain $Q(z) = zq'(z)\Phi\,(q(z)) = -\rho zq'(z)$ and $h(z) = \theta(q(z)) + Q(z) = \xi q^3(z) - \rho zq'(z)$, where $Q(z)$ is a starlike function in U.

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{zq''(z)}{q'(z)} - \frac{3\xi}{\rho}q^2(z) + 1\right\} > 0.$$

By a straightforward computation, we obtain:

$$N(\xi,\rho,k,\mu,\propto,\delta) = \xi p^3(z) - \rho zp'(z). \tag{20}$$

By making use of (17), we obtain:

$$\xi p^3(z) - \rho Zp'(z) \prec \xi q^3(z) - \rho Zq'(z).$$

Therefore, by Lemma 2, we get:

$$\left(\frac{\propto w_{k,\mu}^{m}f(z) + (1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta} \prec q.$$

Thus, the proof is complete. □

4. Differential Superordination Results

Theorem 3. *Let $q(z)$ be a convex univalent function in U and $q(0) = 1$. Let $\mu,\delta \in \mathbb{C}, k \in \mathbb{C}\setminus\{0\}$ such that $Re\left\{\delta\frac{\mu+1}{k}\right\} > 0$. If $f \in A$ satisfies:*

$$0 \neq \left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta} \in \mathbb{G}[q(0),1] \cap Q$$

and τ that is defined as Equation (11) is univalent in U, then $q\,(z) + \frac{k}{\delta(\mu+1)}zq\prime(z) \prec \tau(m,k,\mu,\delta)$, which implies that

$$q(z) \prec \left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta} \tag{21}$$

and $q(z)$ is the best subordinant.

Proof. If, we put

$$p = \left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}. \tag{22}$$

Differentiating (22) with respect to z, we get

$$\frac{zp(z)'}{p(z)} = \delta\left[\frac{z\left(w_{k,\mu}^{m+1}f(z)\right)'}{\left(w_{k,\mu}^{m+1}f(z)\right)} - 1\right]. \tag{23}$$

After some computations and using (10), from (23), we obtain:

$$p+\left(\delta\frac{\mu+1}{k}\right)zp(z)'=\left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}\left(1+\left(\delta\frac{\mu+1}{k}\right)\left(\frac{w_{k,\mu}^{m}f(z)}{w_{k,\mu}^{m+1}f(z)}-1\right)\right) \quad (24)$$

and by using Lemma 3 we get:

$$q(z)\prec\left(\frac{w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta},$$

where $q(z)$ is the best subordinant. □

Theorem 4. *Let $q(z)$ be a convex univalent function in the unit disk U. Let $\xi,\ \propto,\ \mu,\ \delta\in\mathbb{C}$, $k,\ \rho\in\mathbb{C}\backslash\{0\}$ such that $Re\left\{\delta\frac{\mu+1}{k}\right\}>0$ and $f\in A$. Suppose that:*

$$Re\left\{-3\frac{\xi(q(z))^{2}}{\rho}q'(z)\right\}>0,\ for\ z\ \in U. \quad (25)$$

If

$$0\neq\left(\frac{\propto w_{k,\mu}^{m}f(z)+(1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}\in\mathbb{G}[q(0),1]\ \cap Q,$$

and $\left(\frac{\propto w_{k,\mu}^{m}f(z)+(1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}$ is univalent in U, and

$$\xi q^{3}(z)-\rho zq'(z)\prec N(\xi,\rho,k,\mu,\propto,\delta), \quad (26)$$

where N is defined in Equation (17), then

$$q\prec\left(\frac{\propto w_{k,\mu}^{m}f(z)+(1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}$$

and q is the best subordinant.

Proof. Define the function p by:

$$p(z)=\left(\frac{\propto w_{k,\mu}^{m}f(z)+(1-\propto)w_{k,\mu}^{m+1}f(z)}{z}\right)^{\delta}. \quad (27)$$

Differentiating (27) with respect to z, we get

$$\frac{zp'(z)}{p(z)}=\delta\left[\frac{\propto\left(w_{k,\mu}^{m}f(z)\right)'+(1-\propto)\left(w_{k,\mu}^{m+1}f(z)\right)'}{\propto w_{k,\mu}^{m}f(z)+(1-\propto)w_{k,\mu}^{m+1}f(z)}+1\right]. \quad (28)$$

By setting

$$\theta(w)=\xi w^{3}\text{ and }\Phi(w)=-\rho,$$

we see that $\theta(w)$ and $\Phi(w)$ are analytic in $\mathbb{C}$ and $\Phi(w)\neq 0,\ w\in\mathbb{C}\backslash\{0\}$. In addition, we obtain:

$$Q(z)=\ zq'(z)\Phi(q(z))=-\rho Zq'(z).$$

It is clear that $Q(z)$ is a starlike univalent function in U,

$$Re\left\{\frac{\theta'(q)}{\Phi(q)}\right\} = Re\left\{-3\frac{\xi(q(z))^2}{\rho}q'(z)\right\} > 0.$$

By straightforward computation, we get:

$$N(\xi,\rho,k,\mu,\propto,\delta) = \xi q^3(z) - \rho z q'(z), \quad (29)$$

where $N(\xi,\rho,k,\mu,\propto,\delta)$ is given by (17). From (26) and (29), we have

$$\xi q^3(z) - pzq'(z) \prec \xi p^3(z) - pzp'(z)$$

Therefore, by Lemma 4, we get:

$$q \prec \left(\frac{\propto w_{k,\mu}^{m} f(z) + (1-\propto) w_{k,\mu}^{m+1} f(z)}{z}\right)^{\delta},$$

and q is the best subordinant. □

5. Sandwich Results

If we set Theorem 1 against Theorem 3, we will get the following sandwich result:

Theorem 5. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$, where q_1 satisfies Theorem 1 and q_2 satisfies Theorem 3 with*

$$0 \neq \left(\frac{w_{k,\mu}^{m+1} f(z)}{z}\right)^{\delta} \in G[q(0),1] \cap Q,$$

and $\tau(z)$ is defined by (11) such that:

$$q_1(z) + \frac{k}{\delta(\mu+1)} z q_1'(z) \prec \tau(m,k,\mu,\delta) \prec q_2(z) + \frac{k}{\delta(\mu+1)} z q_2'(z).$$

Then

$$q_1(z) \prec \left(\frac{w_{k,\mu}^{m+1} f(z)}{z}\right)^{\delta} \prec q_2(z),$$

where q_1 is the best subordinant and q_2 is the best dominant.

Theorem 6. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$, where q_1 satisfies Theorem 2 and q_2 satisfies Theorem 4 with*

$$0 \neq \left(\frac{\propto w_{k,\mu}^{m} f(z) + (1-\propto) w_{k,\mu}^{m+1} f(z)}{z}\right)^{\delta} \in G[q(0),1] \cap Q,$$

and $N(z)$ is defined by relation (17), and suppose $Re\left\{-3\frac{\xi(q(z))^2}{\rho}q'(z)\right\} > 0$ such that:

$$\xi q_1^3(z) - \rho z q_1'(z) \prec N(\xi,\rho,k,\mu,\propto,\delta) \prec \xi q_2^3(z) - \rho z q_2'(z).$$

Then

$$q_1 \prec \left(\frac{\propto w_{k,\mu}^{m} f(z) + (1-\propto) w_{k,\mu}^{m+1} f(z)}{z}\right)^{\delta} \prec q_2,$$

where q_1 and q_2 are the best subordinant and the best dominant, respectively.

6. Conclusions and Future Work

We aimed to give some new results for an integral operator $w^m_{k,\mu} f(z)$ for a subclass of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ using differential subordinations and superordinations. The theorems and corollaries were derived by investigating relevant lemmas of second-order differential subordinations. Some new outcomes on differential subordination and superordination with some sandwich theorems were expressed. Moreover, several particular cases were also noted. The properties and outcomes of the differential subordination are symmetry to the properties of the differential superordination to form the sandwich theorems. The outcomes included in this current paper revealed new ideas for continuing the study, and we opened some windows for researchers to generalize the classes to establish new results in univalent and multivalent function theory.

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