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Review

The Genus of a Graph: A Survey

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Abstract: The problem of determining the genus for a graph can be dated to the Map Color Conjecture proposed by Heawood in 1890. This was implied to be a Thread Problem by Hilbert and Cohn-Vossen. The conjecture was finally established by Ringel, Youngs, and many other mathematicians. Subsequently, the genera of some special graphs with symmetry were determined. The study of genus embeddings of graphs is closely related to other invariants of a graph. Specifically, the computational complexity is dependent on the genus of the underlying graph for certain well-known NP-hard problems. In this survey, main construction techniques and certain criteria are stated in the topic of the genus of a graph. Most graphs with a known genus are listed. A new theorem is shown that the method of joint trees of a graph is reasonable. Moreover, a formal set is introduced, and related results are obtained. Although a cubic graph of Hamilton cycle is asymmetric, it is interesting that a set of associate surfaces of all its joint trees is a formal set with symmetry.

Keywords: genus; embedding; orientable surface; formal set

1. Introduction

In this paper, we always consider a 2-cell embedding of a connected graph on an orientable closed surface without specific explanations. Its *genus* is the minimum genus of a surface on which a graph is embedded. The problem of determining the genus for a graph can be dated to the Map Color Conjecture proposed by Heawood in 1890 [1].

Conjecture 1. *Set S to be an orientable (or nonorientable) surface. Let $E(S)$, $\chi(S)$, and $\lceil z \rceil$ denote the Euler's characteristic, chromatic number, and the maximum integer that is not more than z, respectively. Then*

$$\chi(S) = \left\lceil \frac{1}{2}(7 + \sqrt{49 - 24E(S)}) \right\rceil, \quad E(S) \neq 2.$$

If $E(S) = 2$, then it is the famous Four Color Conjecture, which was established by Appel and Haken in 1976 [2]. The Map Color Conjecture was implied to be a Thread Problem by Hilbert and Cohn-Vossen (Chapter VI of [3]). This is the genus (or nonorientable genus) problem of each complete graph K_n for $n \geq 3$. It is the following problem for an orientable surface.

Conjecture 2. *Let $\gamma(n)$ denote the genus of K_n. Set $\lfloor x \rfloor$ to be the smallest integer that is not less than x. Then*

$$\gamma(n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor, \quad n \geq 3.$$

The Map Color Conjecture was finally established by Ringel, Youngs, and many other mathematicians in 1968 [4]. The proof of the conjecture was important progress in the field of topological graph theory. New methods were proposed to determine the genera of graphs, especially graphs with the high symmetry, during and after this period. Moreover, Thomassen proved that determining the genus of a graph is NP-complete [5]. The study of genus embeddings of graphs is closely related to the study of geometric realizations

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Citation: Wan, L. The Genus of a Graph: A Survey. *Symmetry* **2023**, *15*, 322. https://doi.org/10.3390/sym15020322

Academic Editor:Alice Miller

Received: 30 October 2022
Revised: 15 January 2023
Accepted: 18 January 2023
Published: 23 January 2023

of set systems. Especially, the computational complexity is dependent on the genus of the underlying graph for the well-known NP-hard problems, which are INDEPENDENT SET, VERTEX COVER, and DOMINATING SET. This paper mainly reviews some methods of constructing the genus embeddings of some special graphs with symmetry. Classical criteria and certain generalizations are presented. Most graphs with known genera are listed, which updates the table provided by Sthal [6]. Furthermore, a new conclusion is proved, which implies that joint trees of a graph are reasonable. A formal set is introduced, and related results are provided. Specifically, although a cubic graph of Hamilton cycle is asymmetric, the set of associate surfaces of all its joint trees is a special formal set with symmetry.

We organize this rest of paper as follows. In Section 2, we sketch some concepts and conclusions. In Section 3, we review main methods to determine the genus of a graph. Moreover, a new result is provided that implies that joint trees of a graph are reasonable. We give a review of classical planarity criteria and a generalization in Section 4. In Section 5, a formal set is proposed and related results are provided. Moreover, we prove that a set of all associate surfaces for a Hamiltonian cubic graph is a special formal set. In addition, most of the graphs with known genera are listed in Appendix A.

2. Preliminaries

In this section, basic concepts are sketched. These are mainly taken from [7]. The other undefined terms can be found in [8].

For a simple graph $G = (V, E)$, the graph $\overline{G} = (V, \overline{E})$ is called the *complement graph* of G such that for each $uv \in \overline{E}$ if and only if $uv \notin E$. Given a set V of intervals on a line, regard each interval as a vertex. If $u \cap v \neq \emptyset$ and neither $u \subseteq v$ nor $v \subseteq u$ for each $uv \in E$, then $G = (V, E)$ is called an *overlap graph* where $\emptyset$ is a set that does not contain any element.

Given two graphs G_1 and G_2, the *union* $G_1 \bigcup G_2$ is a graph. Here $V(G_1 \bigcup G_2) = V(G_1) \bigcup V(G_2)$, $E(G_1 \bigcup G_2) = E(G_1) \bigcup E(G_2)$. The concept of union can be generalized to a finite collection of graphs. If all the graphs are the same, we have the notation $nG = \bigcup_{i=1}^{n}(G)$. The *join* $G = G_1 + G_2$ is a graph where $V(G) = V(G_1) \bigcup V(G_2)$ and $E(G) = E(G_1) \bigcup E(G_2) \bigcup \{uv | \forall u \in V(G_1), \forall v \in V(G_2)\}$. If G_1 and G_2 are empty graphs, then G is a complete bipartite graph. Therefore, $\overline{K_m} + \overline{K_n} = K_{m,n}$, where K_n denotes a complete graph of order n. The *Cartesian product* $G_1 \times G_2$ is a graph where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) |$ either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)\}$. The *composition* (or lexicographic product), denoted by $G = G_1[G_2]$ is also a graph where $V(G) = V(G_1) \times V(G_2)$ and $E(G) = \{(v_1, w_1)(v_2, w_2) | v_1 = v_2$ and $w_1 w_2 \in E(G_2)$ or $v_1 v_2 \in E(G_1)\}$. If H is a subgraph both of G_1 and of G_2, then an *amalgamation* $G_1 \vee_H G_2$ of G_1 and G_2 along H is the graph by identifying a copy of H contained in G_1 with a copy of H contained in G_2. The *tensor product* (Kronecker product or conjunction) $G_1 \bigotimes G_2$ is the graph where $V(G_1 \bigotimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \bigotimes G_2) = \{(v_1, w_1)(v_2, w_2) | v_1 v_2 \in E(G_1)$ and $w_1 w_2 \in E(G_2)\}$.

For a given graph $G = (V, E)$ and a vertex $v \in V(G)$, a *rotation* $\sigma(v)$ is a cyclic permutation of edges that are incident with v. Assign a rotation to each vertex of $V(G)$, then obtain a *rotation system* of G. Edmonds found the following result [9]. Youngs provided the result of the first proof published [10]. The idea of the bijection can be found in earlier work about embeddings [11,12].

Proposition 1. *There exists a bijection between the rotation systems and its orientable embeddings for a graph.*

The polygon representation of a surface was found in the argument of the classification of closed surfaces. Let $\mathcal{S}$ denote the set of all surfaces. Each surface is equivalent to one and only one of canonical forms by using the equivalence "$\sim$" defined on $\mathcal{S}$ with the following operation properties [13]:

Op1. $A \sim Aaa^-$, where $A \in \mathcal{S}$ and $a \notin A$;
Op2. $AabBb^-a^- \sim AcBc^- = Ac^-Bc$, where $AB \in \mathcal{S}$ and $a, b, c \notin AB$ for linear sequences A and B;
Op3. $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$, where A, B, C, D and E are linear sequences and $AaBbCa^- Db^-E \in \mathcal{S}$.

Let $\mathcal{S}$ denote the set of all surfaces. Let T be a spanning tree of G, and let G_σ be an embedding. Given a rotation system σ, splitting each cotree edge b into two semi-edges b and b^-, we get a *joint tree* $\widetilde{T}_\sigma$ [13]. The *associate surface* (or embedding surface) S_σ of $\widetilde{T}_\sigma$ is a polyhegon containing all letters of semi-edges. It is obvious that there is one and only one joint tree $\widetilde{T}_\sigma$ for a given rotation system of G_σ. Is the associate surface S_σ the surface that G_σ embeds on? The answer is affirmative. A proof is given in the following section. Thus, determining the genus of G is transformed into determining the genus of the set of its associate surfaces.

3. Methods for Determining the Genus of a Graph

In this section, we review methods used successfully to determine the genera of some special graphs. The reader is referred to the survey in [6]. Furthermore, we provide a new result related to the joint tree of a graph.

There exists several main methods to settle the genus of a graph. The first approach is the direct construction using rotation systems. Heffter first used it to get the genera of some K_n, and then Ringel calculated the genera of $K_{m,n}$ and certain K_n [4,14]. Subsequently, Ringel and Youngs obtained the genus of $K_{n,n,n}$ [15]. White calculated the genera of several families of complete tripartite graphs, including $K_{mn,n,n}$ [16].

The second method is a current graph denoted by (H, Γ, η). Here, H is an embedding, the finite group Γ has an identity e, and $\eta : H \to \Gamma - e$ is a function such that $(\eta(x))^{-1} = \eta(x^{-1})$ for any $x \in E(G)$; $\eta(x)$ is regarded as a current. It was created by Gustin [17] and developed by Youngs. Ringel and Youngs used it to compute the genera of some K_n. Jacques gave a general exposition for Cayley graphs [18]. Finally, Gross and Alpert unified all previously definitions of current graphs into one [19,20].

The third method is a voltage graph. It is a triple (H, Γ, η) with an embedding H, a finite group Γ of identity e, and a function $\eta : H \to \Gamma$ such that $(\eta(x))^{-1} = \eta(x^{-1})$ for all $x \in H$. Here, $\eta(x)$ is a voltage. Its derived graph $H \times \Gamma$ has vertex set $V(H) \times \Gamma$. Two vertices (u_1, ξ_1) and (u_2, ξ_2) in $H \times \Gamma$ are adjacent if and only if u_1 and u_2 are adjacent in H such that $\eta(u_1u_2) = \xi_1^{-1}\xi_2$. A voltage graph was developed by Gross in 1974 [21]. It is interpreted in the context of branched covering spaces and regarded as a dual of a current graph [19,20]. Gross and Tucker extended it to nonregular coverings via the permutation voltage [22].

The fourth method is a *transition graph* denoted by $\mathcal{G} = (D, \mathcal{T}, \lambda, \alpha)$, which was introduced by Ellingham et al. in 2006 [23]. Here,

(1) D is a diagraph where both the indegree and the outdegree are equal to 2 for any $u \in V(D)$;
(2) $\mathcal{T} = \{C_1, \cdots, C_m\}$, where each C_i is a closed trails for $1 \le i \le m$ and where $E(D) = \bigcup_{i=1}^{m} C_i$ and where C_i and C_j don't have a common edge for $i \neq j$;
(3) an ordering $C_i \to C_j$ of the C_i and C_j incident with u for each $u \in V(D)$;
(4) a function λ: $V \to \{-1, +1\}$;
(5) a function α: $V \to \Gamma$, where (usually infinite group) Γ is a voltage group and α is a voltage assignment.

This is equivalent to a voltage graph. Vertices and edges of a voltage graph correspond to vertices and edges of its derived graph, respectively. In case of a transition graph and the derived graph, vertices correspond to edges, and edges correspond to consecutive pairs of edges in the local rotations. The number and sizes of the derived faces can be easily determined.

The fifth method is a surgery, which is a scissors-and-paste approach to obtain a genus embedding of a graph. It was used to calculate the genus of the n-cube Q_n by Ringel, and Beineke and Harary [24,25]. White generalized this approach to establish many genus formulae for Cartesian, lexicographic, and strong tensor product graphs [16,26]. Alpert used it to obtain some results for amalgamations [27]. Ma and Ren provided the genus of $C_m \times K_n$ by two surgical constructions where $n = 4$ and $m \geq 12$, or $n \geq 5$ and $m \geq 6n - 13$. Lv and Chen provided the genus of $K_{n,n,1}$ by a handle-inserting operation for odd n.

Bouchet introduced a special surgical technique called a *diamond sum* to give a new proof of the genera of complete bipartite graphs in 1978 [28]. A primal version was applied by Mohar et al. [29,30]. Mohar and Thomassen gave a primal form of Bouchet's proof and used the definition [31]. A general form was given by Kawarabayashi et al. [32]. Let $\Psi_1 : G \to S_1$ and $\Psi_2 : H \to S_2$ be two embeddings of G and H into the surfaces S_1 and S_2, respectively. Set $x \in V(G)$, $y \in V(H)$, adjacent to m vertices $x_1, x_2, \cdots, x_m \in V(G)$ and $y_m, y_{m-1}, \cdots, y_1 \in V(H)$, respectively, in the clockwise rotation. Set D_1 to be a closed disk which contains in a small neighborhood of the star $st(x) = \{xx_1, xx_2, \cdots, xx_m\}$. Here, it contains $st(x)$ and intersects G only at $x_1, x_2, \cdots, x_m$. Similarly, choose D_2 in a small neighborhood of the star $\{y\} \bigcup \{yy_1, yy_2, \cdots, yy_m\}$. Delete the interior of D_i from S_i and S_2 for $1 \leq i \leq 2$, identify their boundaries of $S_i \backslash D_i$, and then get an embedding Ψ of a new graph U in the surface $S_1 \circ S_2$. Here, $S_1 \circ S_2$ is the disk sum of S_1 and S_2; U is obtained from $G \backslash \{x\}$ and $H \backslash \{y\}$ by identifying x_i with y_i for $i = 1, 2, \cdots, m$. The operations on the graph and the embedding are, respectively, called the diamond sum of graphs and the diamond sums of embeddings, denoted by

$$(G, u) \diamondsuit (H, v)$$

and

$$\Psi_1(G, u) \diamondsuit \Psi_2(H, v).$$

The sixth method is generative n-valuations introduced by Bouchet in 1976 [33]. Let H be a Eulerian graph and H_σ is an embedding of H. If the boundary of each face for H_σ is a triangle, then H_σ is called an even triangulation. Assign to each triangle of H_σ an element of Z_n and obtain a map η called an n-valuation of an even triangulation. Set $u \in V(H)$ and suppose $\sigma(u)$ is a rotation at u conformal with H_σ. The rotation induces a cyclic order of all the triangles of H_σ incident to u, denoted by $\Delta_1, \Delta_2, \cdots, \Delta_{2l}$. Set

$$\lambda(u) = \sum_{i=1}^{2l} (-1)^i \eta(\Delta_i).$$

If η is an n-valuation such that $\lambda(u)$ is a generator for any $u \in V(H)$, then η is called a *generative n-valuation* of H_σ. This approach was applied in [33,34].

The seventh method is a joint tree, which was introduced by Liu [13]. Wan et al. also verified the genera of complete bipartite graphs by using joint trees, the paper for which was posted on Science Online in 2012 [35]. Shao and Liu used this technique to obtain genus embeddings of complete bipartite graphs $K_{n,n,l}$ for $l \geq n \geq 2$ [36]. Shao et al. obtained the genera of other graphs [37,38].

We prove the following result. The result explains that the associate surface of a joint tree $\widetilde{T}_\sigma$ is the surface that it is embedded on for any embedding G_σ of G. The result holds for any nonorientable embedding by applying a similar argument.

Theorem 1. *Set T to be a spanning tree of a connected graph G. Let G_σ and $\widetilde{T}_\sigma$ be the embedding and the joint tree for any rotation system σ of G, respectively. Set S_σ to be the associate surface of $\widetilde{T}_\sigma$. Then S_σ is the surface that G_σ embeds on.*

Proof. If $G = T$, then S_σ is a sphere. The result obviously holds. Suppose that G has m cotree-edges denoted by $x_1, x_2, \cdots, x_m$, where $m \geq 1$. We verify the result by induction on the edge number n of T.

Now consider the case $n = 0$. Suppose that $\sigma = (x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_{2m}})$, where each x_j occurs twice in σ for $1 \leq j \leq m$. Let S be the surface of G_σ embeds on. Then S is formed by identifying edges with the same letters of the polygon with the symbol $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3} \cdots x_{i_{2m}}^{\varepsilon_{2m}}$. Here, x_j and x_j^- occur once on the polygon for $1 \leq j \leq m$. The expression $(x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} x_{i_3}^{\varepsilon_3} \cdots x_{i_{2m}}^{\varepsilon_{2m}})$ is in fact the associate surface S_σ of the joint tree $\widetilde{T}_\sigma$. The result therefore holds.

Assume that the result holds for any integers less than n $(n \geq 1)$. Next we verify the case for n. Because T is a tree, there exists a vertex u such that u is incident with one and only one edge in T. Without loss of generality, set $e_1 = uv$, $\sigma_u = (e_1, x_{i_1}, x_{i_2}, \cdots, x_{i_r})$ and $\sigma_v = (e_1, L_1, e_2, L_2, \cdots, e_k, L_k)$, where each L_j is either empty or L_j consists of certain cotree edges for $k \geq 1$, $1 \leq j \leq k$, $1 \leq i_l \leq m$ for each $1 \leq l \leq r$ and some $1 \leq r \leq 2m$. Here, each e_j is a tree edge for $1 \leq j \leq k$.

Set $H = G \cdot e_1$. Therefore, $T' = T \cdot e_1$ is a spanning tree of H. Denote the new vertex with w by contracting the edge e_1 and keep the labels of all other verties. Let τ be the rotation system of H by setting $\tau_w = (x_{i_1}, x_{i_2}, \cdots, x_{i_r}, L_1, e_2, L_2, \cdots, e_k, L_k)$ and $\tau_a = \sigma_a$ for $a \neq w$ and $a \in V(H)$. According to the definition of the associate surface of a joint tree,

$$S_\tau = S_\sigma \tag{1}$$

where S_τ and S_σ are the associate surfaces of the joint tree $\widetilde{T}'_\tau$ of H and the joint tree $\widetilde{T}_\sigma$ of G, respectively. It is clear that

$$V(H_\tau) = V(G_\sigma) - 1, E(H_\tau) = E(G_\sigma) - 1 \text{ and } \phi(G_\sigma) = \phi(H_\tau).$$

Using the Euler–Ponincaré formula,

$$V(G_\sigma) - E(G_\sigma) + \phi(G_\sigma) = 2 - 2\gamma(G_\sigma)$$

and

$$V(H_\tau) - E(H_\tau) + \phi(H_\tau) = 2 - 2\gamma(H_\tau).$$

Thus

$$\gamma(G_\tau) = \gamma(H_\tau).$$

By induction assumption, S_σ is the surface that G_σ embeds on. Therefore, S_σ is the surface that G_σ embeds on by applying Equation (1).

Hence the result holds by induction. □

Moreover, Conder and Stokes introduced methods that are the subgroup orbit, the independence number, and the use of integer linear programming [39]. They used these techniques to get the genera of $C_3 \times C_3 \times C_3$ and the Gray graph, etc. In addition, Mohar et al. and Brin and Squier obtained the genus of $C_3 \times C_3 \times C_3$ [40,41]. Marušič et al. supplied the genus of the Gray graph [42].

In addition, some of these methods can be combined together to calculate the genus of a graph. Surgery can be used to augment embeddings derived from other methods, as in the additional adjacency constructions of Ringel and Youngs [4]. White combined the voltage graph theory and a surgery. If a voltage–current group is Abelian, then Archdeacon supplied an approach for simultaneously assigning a voltage and a current on an embedded graph [43]. He used it to obtain the genera of $K_{n,n,n}$ for $n \geq 2$. Most of graphs with known genera are listed in Appendix A, where Stahl's table is updated.

4. Planarity Criteria and Generalizations

It is a very important problem to characterize graphs embedded on surfaces, which can be used to determine the genus of a graph. In this section, we review several classical planarity criteria and certain generalizations.

Kuratowski proved the following famous result to characterize those graphs that embed on the plane in 1930 [44].

Proposition 2. *It is planar if and only if a graph does not contain a subgraph homomorphic to* K_5 *and* $K_{3,3}$*.*

If G can be formed from a subgraph of H by contracting edges, G is a *minor* of H. Wagner rephrased the result above by using the minor as follows [45].

Proposition 3. *It is planar if and only if a graph does not contain* K_5 *and* $K_{3,3}$ *as a minor.*

A *cocycle* is a minimal edge cut. If there exists a function $\lambda : E(H) \to E(H^*)$ such that each cycle of H is a cocycle of H^*, then H^* is an *algebraic dual* of H. Whitney found the following result [46].

Proposition 4. *A graph is planar if and only if it has an algebraic dual.*

If H^* is an algebraic dual of H, then there is a planar embedding of H such that H^* is the geometric dual. MacLane characterized the planarity by a certain abstract property of its cycle space [47,48].

Proposition 5. *A* 2*-connected graph* G *is planar if and only if there is a family* $\mathcal{C}$ *of* $\beta(G)$ *cycles such that each* $e \in E(G)$ *occurs either once or twice in* $\mathcal{C}$*.*

Set H to be a graph of $\delta(H) \geq 3$ and set S to be a surface. If H is not embedded on S and $G - x$ is embedded on S for any $x \in E(H)$, then H is a *topological obstruction* for S. If H with each edge x contracted can be embedded on S for a topological obstruction H, then it is a *minor order obstruction* of S. Vollmerhaus, Bodendiek and Wagner, and Robertson and Seymour independently proved the following result, which is a generalization of the Kuratowski theorem, to an arbitrary surface [49–51].

Proposition 6. *There is a finite set of obstructions (topological or minor order) for every surface of fixed genus.*

However, it is very difficult to find the complete list of obstructions for a surface of genus $g \geq 1$. Gagarina et al. obtained the following result [52].

Proposition 7. *Suppose that a graph* G *does not contain* $K_{3,3}$ *as a minor.* G *is toroidal if and only if it has no* G_i *as a minor in Figure 1 for* $1 \leq i \leq 4$*.*

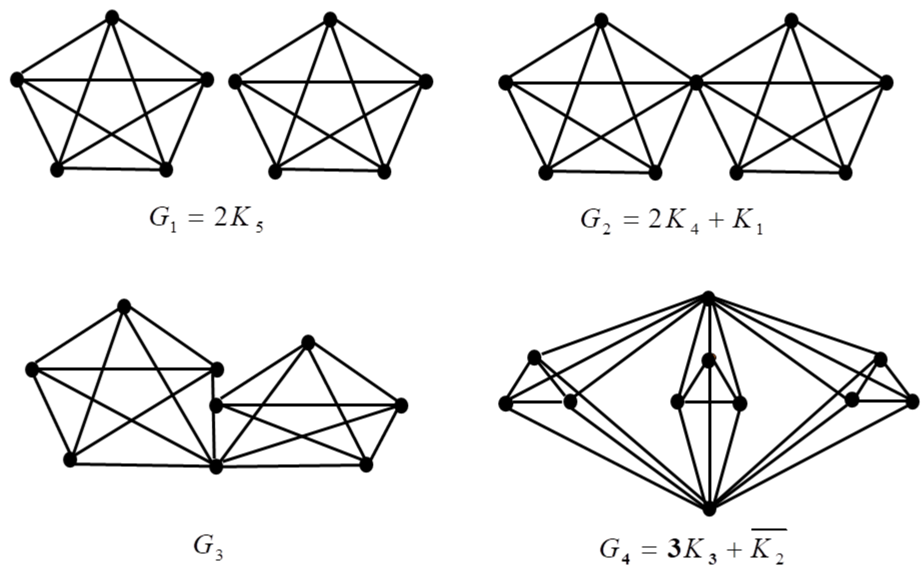

Figure 1. The minor–minimal non-toroidal graphs G_1, G_2, G_3, and G_4 containing no $K_{3,3}$ as a minor.

5. Formal Sets

In this section, we introduce a formal set that is useful to compute the genus of a graph. Specifically, if G is a cubic graph with a Hamilton cycle, then the set of all its associate surfaces is a special formal set.

A *formal sequence* $\mathcal{A}$ is a sequence consisting of lowercase letters and uppercase letters. Let n and m_j be positive integers and let x_l be distinct letters for $1 \leq l \leq n$ and $1 \leq j \leq 2n$. A *formal set* $\mathcal{F}$ of *size* n is a formal sequence consisting of x_l^{ϵ}, X_{l,s_l}, and $Y_{l,s_{n+l}}$ for $\epsilon \in \{+,-\}$, $1 \leq l \leq n$, $1 \leq s_l \leq m_l$ and $1 \leq s_{l+n} \leq m_{l+n}$ satisfying the following conditions:

(1) If $m_l = 1$, then x_l occurs once on $\mathcal{F}$, otherwise X_{l,s_l} occurs once for each $1 \leq s_l \leq m_l$;
(2) If $m_{l+n} = 1$, then x_l^- occurs once on $\mathcal{F}$, otherwise $Y_{l,s_{l+n}}$ occurs once for each $1 \leq s_{l+n} \leq m_{l+n}$

where
$$\begin{cases} X_{l,s_l} \in \{x_l, \emptyset\}, \bigcup\limits_{s_l=1}^{m_l} X_{l,s_l} = x_l, X_{l,p} \bigcap X_{l,q} = \emptyset, \\ \qquad\qquad \text{if } 1 \leq s_l \leq m_l \text{ and any } 1 \leq p \neq q \leq m_l; \\ Y_{l,s_{n+l}} \in \{x_l^-, \emptyset\}, \bigcup\limits_{s_{n+l}=1}^{m_{n+l}} Y_{l,s_{n+l}} = x_l^-, Y_{l,p} \bigcap Y_{l,q} = \emptyset, \\ \qquad\qquad \text{if } 1 \leq s_{n+l} \leq m_{n+l} \text{ and any } 1 \leq p \neq q \leq m_{n+l}. \end{cases}$$

Here, m_l and m_{n+l} are called *valencies* of x_l and x_l^-, denoted by $d(x_l)$ and $d(x_l^-)$, respectively. Specifically, if $m_l = 2$ for $1 \leq l \leq n$, then let $X_{l,1} = X_l$ and let $X_{l,2} = \bar{X}_l$. If $m_{l+n} = 2$ for $1 \leq l \leq n$, then let $Y_{l,1} = Y_l$ and let $Y_{l,2} = \bar{Y}_l$.

Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be formal sequences. A formal set has no difference if one uses the following operations:

(1) Exchange $\mathcal{A}$ and $\mathcal{B}$ for a formal set $\mathcal{AB}$;
(2) Exchange $X_{l,p}$ and $X_{l,q}$ or exchange $Y_{l,p}$ and $Y_{l,q}$ for $1 \leq l \leq n$ and $p \neq q$;
(3) Replace each x_l by z^{ϵ} and simultaneously replace each x_l^- by $z^{-\epsilon}$ for $z \notin \mathcal{F}$ and $\epsilon \in \{+,-\}$;
(4) Replace $X_{l,p}$ (or $Y_{l,p}$) by Z for $Z \notin \mathcal{F}$, $Z \in \{x_l, \emptyset\}$ (or $Z \in \{x_l^-, \emptyset\}$), $1 \leq l \leq n$ and $1 \leq p \leq m_l$ (or $1 \leq p \leq m_{n+l}$).

If a formal set $\mathcal{F} = xZ_1Z_2 \cdots Z_{2n}x^-\bar{Z}_{2n} \cdots \bar{Z}_2\bar{Z}_1$, where $Z_1, Z_2, \cdots, Z_{2n}$ is a permutation of X_i and Y_i for $1 \leq i \leq n$, then $\mathcal{F}$ is called a *cubic basic* set.

In fact, a formal set $\mathcal{F}$ is a set of orientable surfaces. A surface in $\mathcal{F}$ is obtained by replacing an uppercase letter by the associated letter according to its definition. Then $\mathcal{F}$ contains $\prod_{j=1}^{2n} m_j$ orientable surfaces; $\dot{U}_{x_l} = \{x_l, X_{l,s_l} | 1 \leq s_l \leq m_l\}$ and $\dot{U}_{x_l^-} = \{x_l^-, Y_{l,s_{n+l}} | 1 \leq s_{n+l} \leq m_{n+l}\}$ are called *siblings* of x_l and x_l^-, respectively. For convenience, let Ω and Ω_n denote the family of all formal sets and of all formal sets of size n, respectively.

Example 1. *Let* $\mathcal{F}= X_{2,1}x_1Y_{1,1}X_{2,2}x_2^-X_{2,3}Y_{1,2} \in \Omega$ *where* $Y_{1,i} \in \{x_1^-, \emptyset\}$ *for* $1 \leq i \leq 2$, $X_{2,j} \in \{x_2, \emptyset\}$ *for* $1 \leq j \leq 3$, $Y_{1,1} \bigcup Y_{1,2} = x_1^-$, $Y_{1,1} \bigcap Y_{1,2} = \emptyset$, $\bigcup_{j=1}^{3} X_{1,j} = x_2$, $\bigcap_{j=1}^{3} X_{1,j} = \emptyset$. *Then* $\mathcal{F}$ *consists of the following surfaces:*

$$x_2x_1x_1^-x_2^- \quad x_1x_1^-x_2x_2^- \quad x_1x_1^-x_2^-x_2 \quad x_2x_1x_2^-x_1^- \quad x_1x_2x_2^-x_1^- \quad x_1x_2^-x_2x_1^-.$$

Given a formal set $\mathcal{F}$, its *genus* denoted by $\gamma(\mathcal{F})$ is the minimum genus of surfaces contained in $\mathcal{F}$. If it contains a surface of genus 0, then it is *planar*. In order to study the planarity, we construct its *accompanying graph* $Acc(\mathcal{F})$. Put each letter on the real line according to the order in $\mathcal{F}$. We regard $[x_l, x_l^-]$ (or $[x_l^-, x_l]$), $[x_l, Y_{l,s_{n+l}}]$ (or $[Y_{l,s_{n+l}}, x_l]$), $[X_{l,s_l}, x_l^-]$ (or $[x_l^-, X_{l,s_l}]$), and $[X_{l,s_l}, Y_{l,s_{n+l}}]$ (or $[Y_{l,s_{n+l}}, X_{l,s_l}]$) as intervals, denoted by $u_{l,0,0}$, $u_{l,0,s_{n+l}}$, $u_{l,s_l,0}$, and $u_{l,s_l,s_{n+l}}$ for $1 \leq l \leq n$, $1 \leq s_l \leq m_l$, and $1 \leq s_{n+l} \leq m_{n+l}$, respectively; $Acc(\mathcal{A})$ is the associated overlap graph with vertices $u_{l,j,k}$ and edges such that $u_{p,j,k}$ is adjacent to u_{q,j_1,k_1} if and only if $u_{p,j,k} \cap u_{q,j_1,k_1} \neq \emptyset$ (empty set) and neither $u_{p,j,k} \subseteq u_{q,j_1,k_1}$ nor $u_{q,j_1,k_1} \subseteq u_{p,j,k}$ for $1 \leq l, p, q \leq n$, $0 \leq j \leq m_p$, $0 \leq j_1 \leq m_q$, $0 \leq k \leq m_{n+p}$ and $0 \leq k_1 \leq m_{n+q}$.

Theorem 2. *For* $\mathcal{A} \in \Omega_n$, $\mathcal{F}$ *is planar if and only if the accompanying graph* $Acc(\mathcal{F})$ *contains an independent set* I *such that* $u_{l,j,k} \in I$ *for each* $1 \leq l \leq n$ *and some* j, k *with* $j = 0$ *for* $d(x_l) = 1$, $k = 0$ *for* $d(x_l^-) = 1$, $1 \leq j \leq m_l$, *and* $1 \leq k \leq m_{n+l}$ *for others where* $d(x_l) = m_l$ *and* $d(x_l^-) = m_{n+l}$.

Proof. Suppose that $\mathcal{F}$ is planar. Then there exists a surface $S \in \mathcal{F}$ such that $\gamma(S) = 0$, where S consists of x_l and x_l^- for $1 \leq l \leq n$. Consider the intervals $[x_l, x_l^-]$, denoted by u_l for notational convenience, which are determined by their orders on S. It is obvious that $u_p \cap u_q = \emptyset$, $u_p \subseteq u_q$ or $u_q \subseteq u_p$ for $p \neq q$. and $1 \leq p, q \leq n$. Since $S \in \mathcal{F}$, there exists u_{l,j_l,k_l} such that $u_{l,j_l,k_l} = u_l$ for each $1 \leq l \leq n$ and some j_l, k_l with $j_l = 0$ for $d(x_l) = 1$, $k_l = 0$ for $d(x_l^-) = 1$, $1 \leq j_l \leq m_l$ and $1 \leq k_l \leq m_{n+l}$ for others where $d(x_l) = m_l$ and $d(x_l^-) = m_{n+l}$. Thus, $I = \{u_{l,j_l,k_l} | 1 \leq l \leq n\}$ is an independent set of $Acc(\mathcal{F})$.

Conversely, suppose that I is an independent set of $Acc(\mathcal{F})$ such that $u_{l,j_l,k_l} = u_l$ for each $1 \leq l \leq n$ and some j_l, k_l with $j_l = 0$ for $d(x_l) = 1$, $k_l = 0$ for $d(x_l^-) = 1$, $1 \leq j_l \leq m_l$ and $1 \leq k_l \leq m_{n+l}$ for others where $d(x_l) = m_l$ and $d(x_l^-) = m_{n+l}$. Let S be the the associated surface of $\mathcal{F}$ such that

$$\begin{cases} X_{l,j_l} = x_l, & \text{if } j_l \geq 1; \\ Y_{l,k_l} = x_l^-, & \text{if } k_l \geq 1; \\ x \in S, & \text{if } d(x) = 1 \text{ for } x \in \{x_l, x_l^- | 1 \leq l \leq n\}. \end{cases}$$

It is obvious that $\gamma(S) = 0$. □

If xy, xZ, or $XZ \in \mathcal{A}$, for $y, Z \in \dot{U}_{x^-}$ and $X \in \dot{U}_x$, then x is an *eliminable letter*.

Lemma 1. *Let* $\mathcal{A} \in \Omega_n$, *let* x_p *be an eliminable letter, and let* $\dot{U}_{x_p,x_p^-} = \dot{U}_{x_p} \bigcup \dot{U}_{x_p^-}$ *for some* $1 \leq p \leq n$. *Then* $\gamma(\mathcal{A})$=$\gamma(\mathcal{B})$ *where* $\mathcal{B} = \mathcal{A} \setminus \dot{U}_{x_p,x_p^-}$.

Proof. Without loss of generality, suppose that $\mathcal{A} = \mathcal{A}_1 X_{p,j} Y_{p,k} \mathcal{A}_2$ for $1 \le j \le m_p$ and $1 \le k \le m_{n+p}$ where $d(x_p) = m_p$ and $d(x_p^-) = m_{n+p}$. Let $\mathcal{B}_{j,k}$ be the formal set by letting $X_{p,j} = x_p$ and $Y_{p,k} = x_p^-$ and deleting elements of U_{x_p,x_p^-} from $\mathcal{A}$ where $U_{x_p,x_p^-} = \dot{U} \setminus \{x_p, x_p^-\}$.

Because $S \in \mathcal{A}$ for each surface $S \in \mathcal{B}_{j,k}$,

$$\gamma(\mathcal{A}) \le \gamma(\mathcal{B}_{j,k}). \quad (2)$$

Since for each surface $S_1 \in \mathcal{B}$ there exists one and only one surface $S \in \mathcal{B}_{j,k}$ such that $S = x_p x_p^- S_1$, by Op1

$$S \sim S_1.$$

This concludes that

$$\gamma(\mathcal{B}) = \gamma(\mathcal{B}_{j,k}). \quad (3)$$

By the definition of $\mathcal{A}$ and $\mathcal{B}$, S is formed from a surface $S_1 \in \mathcal{B}$ by adding letters x_p and x_p^- for each $S \in \mathcal{A}$ and $S \notin \mathcal{B}_{j,k}$. This concludes that

$$\gamma(S_1) \le \gamma(S). \quad (4)$$

Hence the result is implied by applying (2)–(4). □

Now introduce an operation "$\lesssim$" defined on Ω with the following properties:

Op. set $\mathcal{F} = \mathcal{A} X_1 \mathcal{B} X_2 \mathcal{C} X_3 \mathcal{D} X_4 \mathcal{E} \in \Omega_n$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ are formal sequences. If $X_1 \in \dot{U}_{x_p^r}, X_3 \in \dot{U}_{x_p^{-r}}, X_2 \in \dot{U}_{x_q^\epsilon}, X_4 \in \dot{U}_{x_q^{-\epsilon}}$ with $r, \epsilon \in \{+,-\}$, and $x_p \ne x_q$ for $1 \le p \ne q \le n$, then

$$\mathcal{F} \lesssim \mathcal{F}_1 x_p x_q x_p^- x_q^-$$

where $W = \dot{U}_{x_p} \bigcup \dot{U}_{x_p^-} \bigcup \dot{U}_{x_q} \bigcup \dot{U}_{x_q^-}$ and $\mathcal{F}_1 = \mathcal{A}\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{E} \setminus W$; $[X_1, X_3]$ and $[X_2, X_4]$ are called *poles* of $\mathcal{F}_1$.

Lemma 2. *Let* $\mathcal{F} \in \Omega_n$ *and let* $\mathfrak{A} = \{\mathcal{A} | \mathcal{F} \lesssim \mathcal{A} x_p x_q x_p^- x_q^-, 1 \le p \ne q \le n\}$. *If* $\mathcal{F}$ *is non-planar, then*

$$\gamma(\mathcal{F}) = \min_{\mathcal{A} \in \mathfrak{A}} \{\gamma(\mathcal{A})\} + 1.$$

Proof. Because $\mathcal{F}$ is non-planar, there exists $1 \le p \ne q \le n$ such that S has one of the following forms for each surface $S \in \mathcal{F}$:

$$A_1 x_p A_2 x_q^- A_3 x_p^- A_4 x_q A_5 \qquad A_1 x_p^- A_2 x_q^- A_3 x_p A_4 x_q A_5$$
$$A_1 x_p^- A_2 x_q A_3 x_p A_4 x_q^- A_5 \qquad A_1 x_p A_2 x_q A_3 x_p^- A_4 x_q^- A_5$$

where A_i is a sequence of lowercase letters containing x_j^ϵ for $1 \le i \le 5$, $1 \le j \le n$, $j \ne p$, $j \ne q$, and $\epsilon \in \{+,-\}$. By Op2 and Op3,

$$\begin{aligned} A_1 x_p A_2 x_q^- A_3 x_p^- A_4 x_q A_5 &= A_1 x_p^- A_2 x_q^- A_3 x_p A_4 x_q A_5 \\ &= A_1 x_p^- A_2 x_q A_3 x_p A_4 x_q^- A_5 \\ &= A_1 x_p A_2 x_q A_3 x_p^- A_4 x_q^- A_5 \\ &\sim A_1 A_4 A_3 A_2 A_5 x_p x_q x_p^- x_q^-. \end{aligned}$$

Therefore,

$$\gamma(S) = \gamma(A_1 A_4 A_3 A_2 A_5) + 1$$

where $A_1 A_4 A_3 A_2 A_5 \in \mathcal{A}$ for some $\mathcal{A} \in \mathfrak{A}$.

For each surface $S_1 \in \mathfrak{A}$, there exists $\mathcal{A} \in \mathfrak{A}$ such that $S_1 \in \mathcal{A}$ and that $\mathcal{F} \lesssim \mathcal{A}x_p x_q x_p^- x_q^-$ with $1 \leq p, q \leq n$ and $p \neq q$. This concludes that there is a surface $S \in \mathcal{F}$ such that $S \sim S_1 x_p x_q x_p^- x_q^-$.

Hence the result is clear. □

It is obvious that the following result holds by Lemma 2.

Theorem 3. *Let $\mathcal{F}$ be a non-planar formal set. Then $\gamma(\mathcal{F}) = g$ if and only if there exists a positive integer g such that $\mathcal{F}_g$ is planar and $\mathcal{F}_k$ is non-planar for $k \leq g-1$, where $\mathcal{F}_i$ is obtained by applying Op i times from $\mathcal{F}$ for $i \leq g$.*

Suppose that G is a cubic graph with a Hamilton cycle and that G has $n+1$ cotree edges with $n \geq 1$. Then G has $2n$ vertices. Without loss of generality, set $C = v_1 v_2 \cdots v_{2n} v_1$ to be a Hamilton cycle of G and set $x = v_1 v_{2n}$. Choose $T = v_1 v_2 \cdots v_{2n}$ as a spanning tree of G and denote other cotree edges with $x_1, x_2, \cdots, x_n$. A joint tree $\widetilde{T}$ is obtained by splitting cotree edges a into two semi-edges with labels a and a^- for $a \in \{x, x_i | 1 \leq i \leq n\}$. Because each vertex has two rotations, set $\sigma_1(v_1) = (x, x_{l_1}, v_1 v_2)$, $\sigma_2(v_1) = (x, v_1 v_2, x_{l_1})$, $\sigma_1(v_{2n}) = (v_{2n-1} v_{2n}, x_{l_{2n}}, x)$, $\sigma_2(v_{2n}) = (v_{2n-1} v_{2n}, x, x_{l_{2n}})$, $\sigma_1(v_j) = (v_{j-1} v_j, x_{l_j}, v_j v_{j+1})$, $\sigma_2(v_j) = (v_{j-1} v_j, v_j v_{j+1}, x_{l_j})$ for $2 \leq j \leq 2n-1$, $1 \leq l_j \leq n$. Set the formal set $\mathcal{F} = x Z_1 Z_2 \cdots Z_{2n} x^- \bar{Z}_{2n} \cdots \bar{Z}_2 \bar{Z}_1$. Here, if x_k are adjacent to v_i and v_j with $i < j$, then $Z_i = X_k$ and $Z_j = Y_k$, where $X_k \in \{x_k, \varnothing\}$, $Y_k \in \{x_k^-, \varnothing\}$ for $1 \leq k \leq n$, $1 \leq i, j \leq 2n$. Let each vertex have a clockwise rotation in any joint tree. There is a bijection between $\mathcal{F}$ and the set of associate surfaces of all joint trees of G. The bijection is obtained by letting $Z_j \neq \varnothing$ if and only if $\sigma(v_j) = \sigma_1(v_j)$, $\bar{Z}_j \neq \varnothing$ if and only if $\sigma(v_j) = \sigma_2(v_j)$ for $1 \leq j \leq 2n$. Therefore, we have the following result.

Theorem 4. *Let G be a Hamiltonian cubic graph, then its set of all associate surfaces is a cubic basic set.*

Now we show a simple example to determine the genus of a cubic graph with a Hamilton cycle.

Example 2. *Let H be the cubic graph in Figure 2.*

Obviously, H is non-planar. Choose the Hamilton path $u_1 u_2 \cdots u_{18}$ as the spanning tree T. Then the set of all associate surfaces of H is the formal set

$$\begin{aligned} \mathcal{F} \quad = \quad & x X_1 X_2 X_3 X_4 X_5 X_6 X_7 Y_3 X_8 Y_5 X_9 Y_7 Y_1 Y_2 Y_8 Y_4 Y_9 Y_6 x^- \\ & \bar{Y}_6 \bar{Y}_9 \bar{Y}_4 \bar{Y}_8 \bar{Y}_2 \bar{Y}_1 \bar{Y}_7 \bar{X}_9 \bar{Y}_5 \bar{X}_8 \bar{Y}_3 \bar{X}_7 \bar{X}_6 \bar{X}_5 \bar{X}_4 \bar{X}_3 \bar{X}_2 \bar{X}_1 \end{aligned}$$

Let $[X_1, \bar{Y}_1]$ and $[\bar{Y}_4, \bar{X}_4]$ be poles. By applying Op

$$\mathcal{F} \lesssim \mathcal{F}_1 x_1 x_4 x_1^- x_4^-$$

where $\mathcal{F}_1 = x \bar{Y}_7 \bar{X}_9 \bar{Y}_5 \bar{X}_8 \bar{Y}_3 \bar{X}_7 \bar{X}_6 \bar{X}_5 \bar{Y}_8 \bar{Y}_2 X_2 X_3 X_5 X_6 X_7 Y_3 X_8 Y_5 X_9 Y_7 Y_2 Y_8 Y_9 Y_6 x^- \bar{Y}_6 \bar{Y}_9 \bar{X}_3 \bar{X}_2$.

Because $\mathcal{A}_1 = \bar{Y}_7 \bar{X}_9 \bar{Y}_5 \bar{X}_8 \bar{Y}_3 \bar{X}_7 \bar{X}_6 \bar{X}_5 \bar{Y}_8 \bar{Y}_2 X_2 X_3 X_5 X_6 X_7 Y_3 X_8 Y_5 X_9 Y_7 Y_2 Y_8 Y_9 Y_6$ contains an eliminable letter x_2, by Lemma 1

$$\gamma(\mathcal{A}_1) = \gamma(\mathcal{A})$$

where

$$\mathcal{A} = \bar{Y}_7 \bar{X}_9 \bar{Y}_5 \bar{X}_8 \bar{Y}_3 \bar{X}_7 \bar{X}_6 \bar{X}_5 \bar{Y}_8 X_3 X_5 X_6 X_7 Y_3 X_8 Y_5 X_9 Y_7 Y_8 Y_9 Y_6.$$

Because $Acc(\mathcal{A})$ contains an independent set $I = \{u_{2,1,2}, u_{3,1,1}, u_{5,2,1}, u_{6,2,1}, u_{7,2,2}, u_{8,1,2}, u_{9,1,1}\}$ where $u_{3,1,1} = [X_3, Y_3]$, $u_{5,1,1} = [\bar{X}_5, Y_5]$, $u_{6,1,1} = [\bar{X}_6, Y_6]$, $u_{7,1,2} = [\bar{Y}_7, \bar{X}_7]$, $u_{8,1,2} = [\bar{Y}_8, X_8]$, $u_{9,1,1} = [X_9, Y_9]$, it implies that $\mathcal{F}_1$ is planar. By applying Theorem 3

$$\gamma(\mathcal{F}) = 1.$$

Therefore

$$\gamma(H) = 1.$$

Let each vertex have a clockwise rotation in $\widetilde{T}$ in Figure 2. Then the genus of the associate surface of $\widetilde{T}$ is equal to 1.

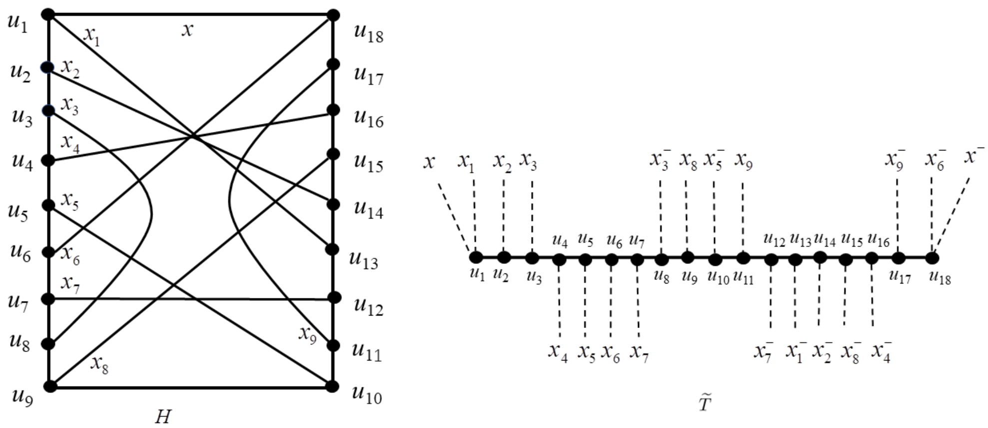

Figure 2. A cubic graph H and a joint tree $\widetilde{T}$.

In the rest of this section, several problems are given for further study.

Problem 1. *Characterize these graphs of genus $g \geq 1$.*

There exist several classical criteria to characterize planar graphs. If a graph contain no K3,3 as a minor, then Gagarin et al. obtained the complete list of obstructions for the torus [52]. How to generalize these results for a graph of a genus $g \geq 1$?

Problem 2. *Is there a polynomial algorithm to determine the genus of a cubic graph with a given Hamilton cycle?*

Thomassen proved that determining the genus of a cubic graph is NP-complete [53]. Its set of all associate surfaces is a cubic basic formal set for a cubic graph with a given Hamilton cycle. Gavril provided an algorithm to find the maximum independent set of an overlap graph [54]. Because the accompanying graph of a formal set is an overlap graph, there is an algorithm to determine whether a formal set is planar in a polynomial time. Is there an algorithm to determine the genus of a cubic basic formal set in a polynomial time?

Problem 3. *Find a new algebraic or combinatorial construction to study the genus of a graph, especially* 3*-connected cubic graphs.*

Conjecture 3. *For a complete tripartite graph $K_{m,n,l}$ with $m \geq n \geq l \geq 1$*

$$\gamma(K_{m,n,l}) = \left\lceil \frac{(m-2)(n+l-2)}{4} \right\rceil.$$

White proposed the conjecture above [16]. Although many cases have been obtained, which are listed in Appendix A, this conjecture is still open.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Graphs with Known Genera

Graph		Reference
K_n		[50]
$K_{m,n}$		[28,55]
		[14,35]
$K_{3(n)}$		[15,56]
$K_{mn,n,n}$		[16]
$K_{n,n,n-2}$	n even	[56]
$K_{2n,2n,n}$		[56]
$K_{2n,2n,n-m}$	$m+1$ generates Z_{2n}	Stahl [unpublished work]
$K_{2m,n,l}$	$n+l$ even, $m \geq n+l-2$	[57]
$K_{m,n,n-2}$, $K_{m,n,\frac{n}{2}}$	$m \geq n, n$ even	[58]
$K_{n,n,l}$	$l \geq n \geq 1$	[36,58]
$K_{n,n,1}$	n even	[59]
$K_{n,n,1}$	n odd	[60]
$K_{m,n,l}$	$n+l \leq 6, m \geq n, l$	[16]
$K_{m,n,l}$	l, m, n even	[61]
$K_{m,n,l}$	$n+l$ even, $m \geq 2(n+l-2)$	[32]
$K_{m,n+2,n}$	$m \geq n+2 \geq 2$, n even	[32]
$K_{m,2n,n}$	$m \geq 2n \geq 2$	[32]
$K_{n(m)}$	$n \equiv 3 \pmod{12}$, $m = 2^k$	[62]
$K_{n(m)}$	$n \equiv 7 \pmod{12}$, $m = 3^k, 2 \cdot 3^k$	[63]
$K_{p^r(3^k)}$	p prime, $p \equiv 3 \pmod 4$	[63]
$K_{n(m)}$	$n \equiv 0,1,4,6,9,10 \pmod{12}$	[33]
	$m \equiv 0,2,4 \pmod 6$	
	$n \equiv 3,7 \pmod{12}$;	
	$n \equiv 11 \pmod{12}$, $m \equiv 0,3 \pmod{12}$	
$K_{n(m)}$	$n \equiv 3 \pmod 4$	[34]
	$m \equiv 0 \pmod 3$, $m \geq 3$	
$K_{4(n)}$		[62,64,65]
$K_{2n,n,n,n}$		[66,67]
$K_{t,n,n,n}$	$n \geq 1, t \geq 2n$	[66]
$K_{n(2)}$	$n \neq 11$	[68,69]
$K((i-2)n_1, n_2, \cdots, n_i)$	$n_j = n$ for $1 \leq j \leq i$	[67]
$K_{n_1,n_2,\cdots,n_k}$	n_i even	[61]
$(K_{s,s})^n$, $K_{2m,2m} \times K_{2n,2n}$		[26]
$K_{2m,2m} \times K_{r,s}$	$r, s \leq 2m$	[26]
$K_n \times K_2$	$n \not\equiv 5,9 \pmod{12}$	[70]
$\prod_{i=1}^{n} P_{m_i}$	m_1, m_2, m_3 are even	[26]
$\prod_{i=1}^{n} P_i$	$i \geq 6$	[71]
$\prod_{i=1}^{n} C_{2m_i}$	$n \geq 2$	[26]
$K_{m,n} \vee_{K_2} K_{p,q}$		[72]
$K_m \vee_{K_q} K_n$	$q = 2,3,4,5$	[27]
Q_n		[24,25,68]

Graph		Reference
$G \leq K_8$		[73]
$\bigotimes_{i=1}^{n} Q_s, \bigotimes_{i=1}^{n} C_{2m}, \bigotimes_{i=1}^{n} K_{2r,2q}$	$s \geq 2, m \geq 2, r \geq 1, q \geq 1$	[74]
$(C_r \times C_s) \otimes Q_n$	$r, s \geq 4$	[75]
$(C_r \times C_s) \otimes K_m$	$r, s \geq 4, m = 2^n, n \geq 1$	[76]
$P_3 \otimes K_m$	$m = 2^n, n \geq 2$	[76]
$C_m[C_{2n}], Q_m[Q_n], K_{m,n}[K_{p,q}]$	$m, n \geq 2, p + q = 2k$	[57]
$C_m + P_l, C_m + C_n, P_l + P_r$	$m, n \geq 3$	[58]
$C_m + K_n$		
	$n = 4, m \geq 12;$	[77]
	$n \geq 5, m \geq 6n - 13$	[77]
$\overline{K_m} + lK_{n_1}, \overline{K_m} + \overline{K_{l(n_1)}}, \overline{K_m} + \bigcup_{j=1}^{k} 2K_{n_j}$	$m \geq 2n_1, n_1 \geq n_2 \geq \cdots \geq n_k$	[58]
$K_{2n} + \overline{K_m}$	$m \geq 4(n-1)$	[58]
$K_n + \overline{K_m}$	n even $, m \geq n$	[78]
	$n = 2^p + 2, p \geq 3, m \geq n - 1$	
	$n = 2^p + 1, p \geq 3, m \geq n + 1$	
$K_n + \overline{K_m}$	$m \geq n - 1, n = 3^q(2^p + \frac{1}{2}) + \frac{3}{2}, p \geq 3, q \geq 0$	[66]
$K_{12p} + \overline{K_{2q}}$	$p \geq 3, q = 1 + 3h, 1 \leq h \leq p - 2$	[79]
	$p \geq 2, q = 2 + 3h, 0 \leq h \leq p - 2$	[79]
$K_{12p+2} + \overline{K_{1+6q}}$	$p \geq 2, 1 \leq q \leq p - 1$	[79]
$K_{12p+3} + \overline{K_m}$	$p \geq 2, m = 4, 7, 8, \cdots, 4p - 2$	[79]
	$p = 2, 3, m = 6$	
$K_{12p+4} + \overline{K_{6q}}$	$p \geq 2, 1 \leq q \leq p - 1$	[79]
$K_{12p+6} + \overline{K_{1+2q}}$	$p \geq 2, q = 3h, 1 \leq h \leq p - 1$	[79]
	$p \geq 3, q = 3h + 2, 1 \leq h \leq p - 2$	[79]
	$p \geq 1, 3 \leq q \leq 2p + 1$	[79]
$K_{12p+8} + \overline{K_{4+6q}}$	$p \geq 2, 1 \leq q \leq p - 2$	[79]
$K_{12p+10} + \overline{K_{3+6q}}$	$p \geq 2, 1 \leq q \leq p - 1$	[79]
$K_{36p+6} + \overline{K_{1+2q}}$	$p \geq 1, 0 \leq q \leq 12p$	[80]
$K_{36p+18} + \overline{K_{1+2q}}$	$p \geq 0, 0 \leq q \leq 12p + 4$	[80]

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