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Poly-Cauchy Numbers with Higher Level

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Abstract: In this article, mainly from the analytical aspect, we introduce poly-Cauchy numbers with higher levels (level s) as a kind of extensions of poly-Cauchy numbers with level 2 and the original poly-Cauchy numbers and investigate their properties. Such poly-Cauchy numbers with higher levels are yielded from the inverse relationship with an s-step function of the exponential function. We show such a function with recurrence relations and give the expressions of poly-Cauchy numbers with higher levels. Poly-Cauchy numbers with higher levels can be also expressed in terms of iterated integrals and a combinatorial summation. Poly-Cauchy numbers with higher levels for negative indices have a double summation formula. In addition, Cauchy numbers with higher levels can be also expressed in terms of determinants.

Keywords: poly-Cauchy numbers; Cauchy numbers; poly-Bernoulli numbers

MSC: 11B75; 11B37; 05A15; 05A19

1. Introduction

The Stirling numbers with higher level (level s) were first studied by Tweedie [1] in 1918. Namely, those of the first kind $\left[\!\left[{n \atop k} \right]\!\right]_s$ and the second kind $\left\{\!\left\{ {n \atop k} \right\}\!\right\}_s$ appeared as

$$x(x+1^s)(x+2^s)\cdots(x+(n-1)^s)=\sum_{k=0}^n \left[\!\left[{n \atop k} \right]\!\right]_s x^k$$

and

$$x^n=\sum_{k=0}^n \left\{\!\left\{ {n \atop k} \right\}\!\right\}_s x(x-1^s)(x-2^s)\cdots\bigl(x-(k-1)^s\bigr)\,,$$

respectively. They satisfy the recurrence relations

$$\left[\!\left[{n \atop k} \right]\!\right]_s=\left[\!\left[{n-1 \atop k-1} \right]\!\right]_s+(n-1)^s\left[\!\left[{n-1 \atop k} \right]\!\right]_s$$

and

$$\left\{\!\left\{ {n \atop k} \right\}\!\right\}_s=\left\{\!\left\{ {n-1 \atop k-1} \right\}\!\right\}_s+k\left\{\!\left\{ {n-1 \atop k} \right\}\!\right\}_s$$

with $\left[\!\left[{0 \atop 0} \right]\!\right]_s=\left\{\!\left\{ {0 \atop 0} \right\}\!\right\}_s=1$ and $\left[\!\left[{n \atop 0} \right]\!\right]_s=\left\{\!\left\{ {n \atop 0} \right\}\!\right\}_s=0$ $(n\ge 1)$. When $s=1$, they are the original Stirling numbers of both kinds. When $s=2$, they have been often studied as central factorial numbers of both kinds (see, e.g., [2]). The concept introduced by Tweedie This concept was used by Bell [3] to show a generalization of Lagrange and Wilson theorems. However, such generalized Stirling numbers have been forgotten or ignored for a long time.

Recently in [4,5], the Stirling numbers with higher levels have been rediscovered and studied more deeply, in particular, from the aspects of combinatorics. On the other hand, in [6], by using the Stirling numbers of the first kind with level 2, poly-Cauchy

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numbers with level 2 are introduced as a kind of generalizations of the original poly-Cauchy numbers, which may be interpreted as a kind of generalizations of the classical Cauchy numbers. In [7], by using the Stirling numbers of the second kind with level 2, poly-Bernoulli numbers with level 2 are introduced as a kind of generalizations of the original poly-Bernoulli numbers [8]. In [9], other poly-generalized numbers, which are called polycosecant numbers, are introduced and studied. This result leads to a variant of multiple zeta values of level 2 [10], which forms a subspace of the space of alternating multiple zeta values. However, no generalized Stirling number is considered in [9].

Another of the most famous generalized Stirling numbers is the r-Stirling number [11], which has meaningful relations with harmonic numbers from the summation formulas [12–14]. By using r-Stirling numbers, so-called various r-numbers are introduced.

It is remarkable to see that the original poly-Cauchy numbers (with level 1, ref. [15]), which may be also yielded by the logarithm function (an 1-step function) with the inverse relation of the exponential function. This can be said to be an analytical definition. Then, poly-Cauchy numbers with level 2 may be yielded or defined from the inverse relation about the hyperbolic sine function, which is a 2-step function of the exponential function [6]. Then, it would be a natural question how the poly-Cauchy numbers with level 3, 4, and generally level s can be defined by any functions (3, 4 and generally s-step functions, respectively) in a natural way.

In combinatorial ways, just as poly-Cauchy number with level 2 arises from the relationship with the Stirling numbers with level 2, poly-Cauchy number with level 3, 4 and generally level s could be hoped to arise from the Stirling numbers with level 3, 4 and generally level s, respectively. However, in the case of 3 or higher level, it is not easy to define and describe most of the properties including both combinatorial and analytical meanings naturally as well as those with levels 1 and 2. For example,

$$\left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_s = \frac{s}{(sk)!}\sum_{j=1}^{k}(-1)^{k-j}\binom{sk}{k-j}j^{sn}$$

holds for $s = 1, 2$ and does not for $s \ge 3$ ([5]).

The purpose of this paper is to define poly-Cauchy numbers with higher level (level s) from the analytical implications and investigate their properties. Such poly-Cauchy numbers with higher levels are yielded from the inverse relationship with an s-step function of the exponential function. We show such a function with recurrence relations and give the expressions of poly-Cauchy numbers with higher levels. Poly-Cauchy numbers with higher levels can be also expressed in terms of iterated integrals and a combinatorial summation. Poly-Cauchy numbers with higher levels for negative indices have a double summation formula. In addition, Cauchy numbers with higher levels can be also expressed in terms of determinants.

2. Definitions

For integers n and k with $n \ge 0$, *poly-Cauchy numbers* $C_{n,s}^{(k)}$ *with level* s ($s \ge 1$) are defined by

$$\mathrm{Lif}_{s,k}\big(\mathfrak{AF}_s(t)\big) = \sum_{n=0}^{\infty} C_{n,s}^{(k)}\frac{t^n}{n!}\,, \tag{1}$$

where

$$\mathrm{Lif}_{s,k}(z) = \sum_{m=0}^{\infty}\frac{z^{sm}}{(sm)!(sm+1)^k}\,.$$

The function $\mathfrak{AF}_s(t)$ is the inverse function of

$$\mathfrak{F}_s(t) = \sum_{m=0}^{\infty}\frac{t^{sm+1}}{(sm+1)!}\,.$$

When $s = 1$, $\mathcal{C}_{n,1}^{(k)} = c_n^{(k)}$ are the original poly-Cauchy numbers [15,16], defined by

$$\mathrm{Lif}_k\big(\mathfrak{AF}_1(t)\big) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!}\,,$$

where $\mathrm{Lif}_{1,k}(z) = \mathrm{Lif}_k(z)$ is the polylogarithm factorial function (or polyfactorial function) and $\mathfrak{AF}_1(t) = \log(t+1)$ is the inverse function of

$$\sum_{m=0}^{\infty} \frac{t^{m+1}}{(m+1)!} = e^t - 1\,.$$

When $k = 1$, $c_n = c_n^{(1)}$ are the original Cauchy numbers defined by

$$\mathrm{Lif}_1\big(\log(t+1)\big) = \frac{t}{\log(t+1)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}\,.$$

When $s = 2$, $\mathcal{C}_{n,2}^{(k)} = \mathbb{C}_n^{(k)}$ are poly-Cauchy numbers with level 2 [6], defined by

$$\mathrm{Lif}_{2,k}\big(\mathfrak{AF}_2(t)\big) = \sum_{n=0}^{\infty} \mathbb{C}_n^{(k)} \frac{t^n}{n!}\,,$$

where $\mathfrak{AF}_2(t) = \mathrm{arcsinh} t$ is the inverse function of

$$\sum_{m=0}^{\infty} \frac{t^{2m+1}}{(2m+1)!} = \sinh t$$

When $k = 1$, $\mathbb{C}_n = \mathbb{C}_n^{(1)}$ are Cauchy numbers with level 2, defined by

$$\mathrm{Lif}_{2,1}\big(\mathrm{arcsinh} t\big) = \frac{t}{\mathrm{arcsinh} t} = \sum_{n=0}^{\infty} \mathbb{C}_n \frac{t^n}{n!}\,.$$

When $k = 1$ and $s = 3$,

$$\mathrm{Lif}_{3,1}(z) = \frac{e^z + \omega^2 e^{\omega z} + \omega e^{\omega^2 z}}{3z} = \frac{\mathfrak{F}_3(z)}{z}\,,$$

where

$$\mathfrak{F}_3(z) = \sum_{m=0}^{\infty} \frac{z^{3m+1}}{(3m+1)!}\,.$$

and $\omega = (1+\sqrt{-3})/2$, satisfying $\omega^3 = 1$. Note that a similar function to $1/\mathfrak{F}_3(z)$ is studied in [17].

For an arbitrary $s \ge 1$ and $k = 1$, we have

$$\mathrm{Lif}_{s,1}(z) = \sum_{m=0}^{\infty} \frac{z^{sm}}{(sm+1)!} = \frac{1}{sz}\prod_{j=0}^{s-1} \zeta^j e^{\zeta^{s-j} z} = \frac{\mathfrak{F}_s(z)}{z}\,,$$

where $\zeta = e^{2\pi i/s}$, is the s-th root of the identity. The function $\mathrm{Lif}_{s,1}(z)$ becomes the *s-step exponential function* .

3. Basic Results

When $s = 3$,

$$\mathfrak{AF}_3(z) = z - \frac{1}{24}z^4 + \frac{17}{2520}z^7 - \frac{389}{259200}z^{10} + \frac{85897}{222393600}z^{13} - \frac{887731}{8211456000}z^{16}$$

$$+\frac{762918737}{23870702592000}z^{19}-\frac{16283723339}{1658385653760000}z^{22}+\cdots.$$

When $s=4$,

$$\begin{aligned}\mathfrak{A}\mathfrak{F}_4(z)=z-\frac{1}{120}z^5+\frac{25}{72576}z^9-\frac{1655}{83026944}z^{13}+\frac{32633}{24320507904}z^{17}\\ -\frac{4046837}{41098797121536}z^{21}+\frac{95346434209}{1247759480609832960 0}z^{25}\\ -\frac{13496484991405}{2188470308231198225203 2}z^{29}+\frac{7594510992880985}{1482243393315651829660385 28}z^{33}\\ -\frac{4010591254856244071}{9213624932850091773168954900 48}z^{37}\\ +\frac{1168313532343019269 49}{3103746517920095780021023077826 56}z^{41}-\cdots.\end{aligned}$$

In general, for the inverse function of $\mathfrak{F}_s(z)$, we have the following.

Proposition 1.

$$\mathfrak{A}\mathfrak{F}_s(z)=d_0z-d_1z^{s+1}+d_2z^{2s+1}-\cdots+(-1)^n d_n z^{sn+1}+\cdots,$$

where the coefficients d_i satisfy the recurrence relation

$$d_n=\sum_{m=0}^{n-1}(-1)^{n-m-1}d_m\sum_{\substack{i_1+\cdots+i_{sm+1}=n-m\\ i_1,\dots,i_{sm+1}\ge 0}}\frac{1}{(si_1+1)!\cdots(si_{sm+1}+1)!}\quad(n\ge 1)\tag{2}$$

with $d_0=1$.

Proof. The expression can be obtained by the following process. First, put $\mathfrak{F}_s^{-1}(z):=\mathfrak{A}\mathfrak{F}_s(z)$ as

$$\mathfrak{F}_s^{-1}(z)=d_0z-d_1z^{s+1}+d_2z^{2s+1}-\cdots+(-1)^n d_n z^{sn+1}+\cdots.\tag{3}$$

Then we can find $d_0=1,d_1,d_2,\dots$ as follows. For convenience, put

$$H_{s,n}(j):=\sum_{\substack{i_1+\cdots+i_{sn+1}=j\\ i_1,\dots,i_{sn+1}\ge 0}}\frac{1}{(si_1+1)!\cdots(si_{sn+1}+1)!}.$$

Since $\mathfrak{F}_s^{-1}\big(\mathfrak{F}_s(z)\big)=\mathfrak{F}_s\big(\mathfrak{F}_s^{-1}(z)\big)=z$, we see that

$$\begin{aligned}z&=\sum_{n=0}^{\infty}(-1)^n d_n\left(\sum_{m=0}^{\infty}\frac{z^{sm+1}}{(sm+1)!}\right)^{sn+1}\\ &=\sum_{n=0}^{\infty}(-1)^n d_n\sum_{m=0}^{\infty}H_{s,n}(m)z^{sn+sm+1}\\ &=\sum_{n=0}^{\infty}(-1)^n d_n\sum_{l=n}^{\infty}H_{s,n}(l-n)z^{sl+1}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^m d_m H_{s,m}(n-m)\right)z^{sn+1}.\end{aligned}$$

Hence, for $n\ge 1$

$$\sum_{m=0}^{n}(-1)^m d_m H_{s,m}(n-m)=0$$

with $d_0 = 1$. The exact values of $d_0, d_1, d_2, \dots$ can be obtained by the recurrence relation (2). Some values of $H_{s,n}(j)$ for smaller j can be given as follows.

$$
\begin{aligned}
H_{s,n}(0) &= 1\,,\\
H_{s,n}(1) &= \frac{sn+1}{(s+1)!}\,,\\
H_{s,n}(2) &= \frac{sn+1}{(2s+1)!} + \frac{1}{\big((s+1)!\big)^2}\binom{sn+1}{2}\,,\\
H_{s,n}(3) &= \frac{sn+1}{(3s+1)!} + \frac{(sn+1)(sn)}{(2s+1)!(s+1)!} + \frac{1}{\big((s+1)!\big)^3}\binom{sn+1}{3}\,,\\
H_{s,n}(4) &= \frac{sn+1}{(4s+1)!} + \frac{(sn+1)(sn)}{(3s+1)!(s+1)!} + \frac{1}{\big((2s+1)!\big)^2}\binom{sn+1}{2}\\
&\quad + \frac{sn+1}{(2s+1)!\big((s+1)!\big)^2}\binom{sn}{2} + \frac{1}{\big((s+1)!\big)^4}\binom{sn+1}{4}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d_1 &= H_{s,0}(1) = \frac{1}{(s+1)!},\\
d_2 &= -H_{s,0}(2) + d_1 H_{s,1}(1) = \frac{1}{(s+1)!r!} - \frac{1}{(2s+1)!}\,,\\
d_3 &= H_{s,0}(3) - d_1 H_{s,1}(2) + d_2 H_{s,2}(1)\\
&= \frac{3r+2}{2\big((s+1)!\big)^2 s!} - \frac{3s+2}{(2s+1)!(s+1)!} + \frac{1}{(3s+1)!}\,,\\
d_4 &= -H_{s,0}(4) + d_1 H_{s,1}(3) - d_2 H_{s,2}(2) + d_3 H_{s,3}(1)\\
&= \frac{(4s+3)(2s+1)}{3\big((s+1)!\big)^3 s!} + \frac{1}{(2s+1)!(2s)!} - \frac{4s+3}{(2s)!\big((s+1)!\big)^2}\\
&\quad + \frac{2(2s+1)}{(3s+1)!(s+1)!} - \frac{1}{(4s+1)!}\,,
\end{aligned}
\tag{4}
$$

□

Thus, by the definition (1), explicit expressions of $\mathcal{C}_{n,s}^{(k)}$ for each concrete s and small n can be achieved. For $s = 3$, we have

$$
\begin{aligned}
\mathcal{C}_{0,3}^{(k)} &= 1\,,\\
\mathcal{C}_{3,3}^{(k)} &= \frac{1}{4^k}\,,\\
\mathcal{C}_{6,3}^{(k)} &= -\binom{6}{4}\frac{1}{4^k} + \frac{1}{7^k}\,,\\
\mathcal{C}_{9,3}^{(k)} &= \frac{3(35\cdot 1+79)}{8}\binom{9}{7}\frac{1}{4^k} - \binom{9}{4}\frac{1}{7^k} + \frac{1}{10^k}\,,\\
\mathcal{C}_{12,3}^{(k)} &= -\frac{9\cdot 22\cdot 153}{4}\binom{12}{10}\frac{1}{4^k} + \frac{3(35\cdot 2+79)}{8}\binom{12}{7}\frac{1}{7^k} - \binom{12}{4}\frac{1}{10^k} + \frac{1}{13^k}\,.
\end{aligned}
$$

For $s = 4$, since

$$
\begin{aligned}
&\frac{\big(\mathfrak{A}\mathfrak{F}_4(x)\big)^{4m}}{(4m)!}\\
&= \frac{x^{4m}}{(4m)!} - \binom{4m+4}{5}\frac{x^{4m+4}}{(4m+4)!} + \frac{2(126m+281)}{5}\binom{4m+8}{9}\frac{x^{4m+8}}{(4m+8)!}
\end{aligned}
$$

$$
- \frac{8(6006m^2+40183m+67157)}{5}\binom{4m+12}{13}\frac{x^{4m+12}}{(4m+12)!}
$$
$$
+ \frac{16(12864852m^3+172143972m^2+767355367m+1139488217)}{45}\binom{4m+16}{17}\frac{x^{4m+16}}{(4m+16)!} - \cdots\,,
$$

we have

$$
\begin{aligned}
\mathcal{C}_{0,4}^{(k)} &= 1\,,\\
\mathcal{C}_{4,4}^{(k)} &= \frac{1}{5^k}\,,\\
\mathcal{C}_{8,4}^{(k)} &= -\binom{8}{5}\frac{1}{5^k} + \frac{1}{9^k}\,,\\
\mathcal{C}_{12,4}^{(k)} &= \frac{2(126\cdot 1+281)}{5}\binom{12}{9}\frac{1}{5^k} - \binom{12}{5}\frac{1}{9^k} + \frac{1}{13^k}\,,\\
\mathcal{C}_{16,4}^{(k)} &= -\frac{8(6006\cdot 1^2+40183\cdot 1+67157)}{5}\binom{16}{13}\frac{1}{5^k}\\
&\quad + \frac{2(126\cdot 2+281)}{5}\binom{16}{9}\frac{1}{9^k} - \binom{16}{5}\frac{1}{13^k} + \frac{1}{17^k}\,.
\end{aligned}
$$

4. Iterated Integrals

Similarly to the cases of the poly-Cauchy numbers with levels 1 and 2 ([6,15]), Cauchy numbers with higher levels have an expression in terms of iterated integrals.

Since

$$
\frac{d}{dz}(z\,\mathrm{Lif}_{s,k}(z)) = \mathrm{Lif}_{s,k-1}(z)\,,
$$

we have

$$
\frac{d}{dz}(z\,\mathrm{Lif}_{s,k}(z)) = \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{z^{3n+1}}{(sn)!(sn+1)^k}\right) = \sum_{n=0}^{\infty}\frac{z^{sn}}{(sn)!(sn+1)^{k-1}} = \mathrm{Lif}_{s,k-1}(z)\,.
$$

Therefore,

$$
\mathrm{Lif}_{s,k-1}(z) = \frac{1}{z}\int_0^z \mathrm{Lif}_{s,k-1}(z)\,dz\,.
$$

By iteration, we get

$$
\mathrm{Lif}_{s,k}(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \mathrm{Lif}_{s,1}(z)\,\underbrace{dz\cdots dz}_{k-1}\,.
$$

Putting $z = \mathfrak{A}\mathfrak{F}_s(t)$, we get

$$
\mathrm{Lif}_{s,k}(\mathfrak{A}\mathfrak{F}_s(t)) = \frac{1}{\mathfrak{A}\mathfrak{F}_s(t)}\underbrace{\int_0^t \frac{\mathfrak{G}_s(t)}{\mathfrak{A}\mathfrak{F}_s(t)}\cdots\int_0^t}_{k-1}\frac{t\mathfrak{G}_s(t)}{\mathfrak{A}\mathfrak{F}_s(t)}\,\underbrace{dt\cdots dt}_{k-1}\,,
$$

where

$$
\begin{aligned}
\mathfrak{G}_s(z) &= \frac{d}{dz}\mathfrak{A}\mathfrak{F}_s(z) = \sum_{n=0}^{\infty}(-1)^n(sn+1)d_n z^{sn}\\
&= 1 - \frac{1}{s!}z^s + \frac{1}{(2s)!}\left(\binom{2s+1}{s} - 1\right)z^{2s}\\
&\quad - \frac{1}{(3s)!}\left(\frac{1}{2}\binom{3s+2}{s+1,s+1,s} - \binom{3s+2}{s+1} + 1\right)z^{3s}
\end{aligned}
$$

$$+\frac{1}{(4s)!}\left(\frac{1}{6}\binom{4s+3}{s+1,s+1,s+1,s}+\binom{4s+1}{2s}\right.$$
$$\left.-\frac{1}{2}\binom{4s+3}{2s+1,s+1,s+1}+\binom{4s+2}{s+1}-1\right)z^{4s}-\cdots,$$

where $\binom{n}{s_1,\ldots,s_m}=\frac{n!}{(s_1)!\cdots(s_m)!}$ denotes the multinomial coefficient with $n=s_1+\cdots+s_m$.

Moreover we can express the Laurent series of $\mathfrak{G}_s(t)/\mathfrak{AF}_s(t)$, in fact,

$$\frac{\mathfrak{G}_s(t)}{\mathfrak{AF}_s(t)}=\frac{\mathfrak{AF}_s'(t)}{\mathfrak{AF}_s(t)}$$

with $\mathfrak{AF}_s(t)=t\mathfrak{D}_s(t)$ and $\mathfrak{D}_s(0)\neq 0$. Hence

$$\frac{\mathfrak{G}_s(t)}{\mathfrak{AF}_s(t)}=\frac{\mathfrak{D}_s(t)+t\mathfrak{D}_s'(t)}{t\mathfrak{D}_s(t)}=\frac{1}{t}+\frac{\mathfrak{D}_s'(t)}{\mathfrak{D}_s(t)}=\frac{1}{t}+\frac{d}{dt}\log\mathfrak{D}_s(t)\,.$$

From (3), we have

$$\mathfrak{D}_s(t)=d_0-d_1t^s+d_2t^{2s}-d_3t^{3s}+\cdots=d_0+u(t)\,,$$

with $d_0=1$ and $u(t)=-d_1t^s+d_2t^{2s}-d_3t^{3s}+\cdots$. So,

$$\log\mathfrak{D}_s(t)=\log(1+u(t))=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{u(t)^k}{k}$$

Therefore,

$$\log\mathfrak{D}_s(t)=(-d_1t^s+d_2t^{2s}-d_3t^{3s}+\cdots)-\frac{1}{2}(-d_1t^s+d_2t^{2s}-d_3t^{3s}+\cdots)^2+$$
$$+\frac{1}{3}(-d_1t^s+d_2t^{2s}-d_3t^{3s}+\cdots)^3+\cdots.$$

So, it follows that

$$\log\mathfrak{D}_s(t)=-d_1t^s+\left(d_2-\frac{d_1^2}{2}\right)t^{2s}+\left(-d_3+d_1d_2-\frac{d_1^3}{3}\right)t^{3s}$$
$$+\left(d_4-d_1d_3-\frac{d_2^2}{2}+d_1^2d_2-\frac{d_1^4}{4}\right)t^{4s}+\cdots,$$

yielding the expression

$$\frac{d}{dt}\log\mathfrak{D}_s(t)=-sd_1t^{s-1}+2s\left(d_2-\frac{d_1^2}{2}\right)t^{2s-1}+3s\left(-d_3+d_1d_2-\frac{d_1^3}{3}\right)t^{3s-1}$$
$$+4s\left(d_4-d_1d_3-\frac{d_2^2}{2}+d_1^2d_2-\frac{d_1^4}{4}\right)t^{4s-1}+\cdots.$$

After substituting the vales of d_n, we have

$$\frac{\mathfrak{G}_s(t)}{\mathfrak{AF}_s(t)}=\frac{1}{t}-\frac{s}{(s+1)!}t^{s-1}+2s\left(\frac{2s+1}{2\big((s+1)!\big)^2}-\frac{1}{(2s+1)!}\right)t^{2s-1}$$
$$+3s\left(\frac{3s+1}{(2s+1)!(s+1)!}-\frac{(3s+2)(3s+1)}{6\big((s+1)!\big)^3}-\frac{1}{(3s+1)!}\right)t^{3s-1}$$

$$+4s\left(\frac{4s+1}{(3s+1)!(s+1)!}-\frac{4s+1}{(2s)!\big((s+1)!\big)^2}+\frac{4s+1}{2\big((2s+1)!\big)^2}\right.$$
$$\left.+\frac{(4s+3)(4s+1)(2s+1)}{12\big((s+1)!\big)^4}-\frac{1}{(4s+1)!}\right)t^{4s-1}+\cdots. \quad (5)$$

Proposition 2. *We have*

$$\sum_{n=0}^{\infty}\mathcal{C}_{n,s}^{(k)}\frac{t^n}{n!}=\frac{1}{\mathfrak{A}\mathfrak{F}_s(t)}\underbrace{\int_0^t\frac{\mathfrak{G}_s(t)}{\mathfrak{A}\mathfrak{F}_s(t)}\cdots\int_0^t}_{k-1}\frac{t\mathfrak{G}_s(t)}{\mathfrak{A}\mathfrak{F}_s(t)}\underbrace{dt\cdots dt}_{k-1},$$

where $\mathfrak{G}_s(z)=\frac{d}{dz}\mathfrak{A}\mathfrak{F}_s(z)$ and a more precise expression of $\mathfrak{G}_s(t)/\mathfrak{A}\mathfrak{F}_s(t)$ is given in (5).

5. An Explicit Expression

If we know the coefficients d_n ($n\ge 0$) appeared in $\mathfrak{A}\mathfrak{F}_s(t)$ in Proposition 1, we can get an expression of $\mathcal{C}_{n,s}^{(k)}$.

Theorem 1. *For integers n and k with $n\ge 0$,*

$$\mathcal{C}_{sn,s}^{(k)}=\sum_{m=0}^{n}\frac{(-1)^{n-m}(sn)!}{(sm)!(sm+1)^k}\sum_{\substack{i_1+\cdots+i_{sm}=n-m\\ i_1,\dots,i_{sm}\ge 0}}d_{i_1}\cdots d_{i_{sm}}t^{sn}.$$

Proof. By the definition in (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{C}_{n,s}^{(k)}\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\mathcal{C}_{sn,s}^{(k)}\frac{t^{sn}}{(sn)!}\\&=\sum_{m=0}^{\infty}\frac{1}{(sm)!(sm+1)^k}\left(\sum_{l=0}^{\infty}(-1)^l d_l t^{sl+1}\right)^{sm}\\&=\sum_{m=0}^{\infty}\frac{1}{(sm)!(sm+1)^k}\\&\quad\times\sum_{n=m}^{\infty}\sum_{\substack{i_1+\cdots+i_{sm}=n-m\\ i_1,\dots,i_{sm}\ge 0}}(-1)^{i_1+\cdots+i_{sm}}d_{i_1}\cdots d_{i_{sm}}t^{(si_1+1)+\cdots+(si_{sm}+1)}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(sm)!(sm+1)^k}\sum_{\substack{i_1+\cdots+i_{sm}=n-m\\ i_1,\dots,i_{sm}\ge 0}}(-1)^{n-m}d_{i_1}\cdots d_{i_{sm}}t^{sn}.\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result. □

6. Some Expressions of Poly-Cauchy Numbers with Higher Levels for Negative Indices

The poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ [8], defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}\mathbb{B}_n^{(-k)}\frac{t^n}{n!},$$

where

$$\mathrm{Li}_k(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^k}$$

is the polylogarithm function, satisfy the duality formula $\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}$ for $n, k > 0$, because of the symmetric formula

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{B}_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}\,.$$

Though the corresponding duality formula does not hold for the original poly-Cauchy numbers (ref. [16], Proposition 1) and poly-Cauchy numbers with level 2 (ref. [6], Theorem 4.1), we still have the double summation formula of poly-Cauchy numbers with higher level.

Theorem 2. *For nonnegative integers n and k,*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{C}_{sn,s}^{(-sk)}\frac{x^{sn}}{(sn)!}\frac{y^{sk}}{(sk)!} = \frac{1}{s^2}\sum_{j=0}^{s-1}\sum_{h=0}^{s-1}e^{\zeta^j y}\big(\mathfrak{B}\mathfrak{F}_s(x)\big)^{\zeta^h e^{\zeta^j y}}\,,$$

where $\mathfrak{B}\mathfrak{F}_s(x) = e^{\mathfrak{A}\mathfrak{F}_s(x)}$ and ζ is the s-th root of unity as $\zeta = e^{2\pi i/s} = \cos(2\pi/s) + i\sin(2\pi/s)$.

Proof. From the definition in (1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{C}_{sn,s}^{(-sk)}\frac{x^{sn}}{(sn)!}\frac{y^{sk}}{(sk)!} = \sum_{k=0}^{\infty}\mathrm{Lif}_{s,k}\big(\mathfrak{A}\mathfrak{F}_s(x)\big)\frac{y^{sk}}{(sk)!}\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\big(\mathfrak{A}\mathfrak{F}_s(x)\big)^{sm}}{(sm)!}(sm+1)^{sk}\frac{y^{sk}}{(sk)!}\\
&= \sum_{m=0}^{\infty}\frac{\big(\mathfrak{A}\mathfrak{F}_s(x)\big)^{sm}}{(sm)!}\frac{1}{s}\sum_{j=0}^{s-1}e^{\zeta^j(sm+1)y}\\
&= \frac{1}{s}\sum_{j=0}^{s-1}e^{\zeta^j y}\sum_{m=0}^{\infty}\frac{e^{\zeta^j y}\mathfrak{A}\mathfrak{F}_s(x)}{(sm)!}\\
&= \frac{1}{s^2}\sum_{j=0}^{s-1}\sum_{h=0}^{s-1}e^{\zeta^j y}e^{\zeta^h e^{\zeta^j y}\mathfrak{A}\mathfrak{F}_s(x)}\,,
\end{aligned}$$

yielding the desired result. □

7. Cauchy Numbers with Higher Level

When $k = 1$ in (1), $\mathcal{C}_{n,s} = \mathcal{C}_{n,s}^{(1)}$ are the Cauchy numbers with higher level, defined by

$$\frac{t}{\mathfrak{A}\mathfrak{F}_s(t)} = \sum_{n=0}^{\infty}\mathcal{C}_{n,s}\frac{t^n}{n!}\,. \tag{6}$$

In this section, we shall show some properties of $\mathcal{C}_{n,s} = \mathcal{C}_{n,s}^{(1)}$. First, we give its determinant expression. A similar expression for the hypergeometric Cauchy numbers is given in [18].

Theorem 3. *For $n \ge 1$,*

$$\mathcal{C}_{sn,s} = (sn)!\begin{vmatrix} d_1 & 1 & 0 & & \\ d_2 & d_1 & 1 & & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & d_1 & 1 \\ d_n & \cdots & \cdots & d_2 & d_1 \end{vmatrix}\,,$$

where d_n is the coefficient of t^{sn+1} appeared in $\mathfrak{AF}_s(t)$ in Proposition 1.

Remark 1. *By using the values of d's in (4), Theorem 3 yields*

$$\begin{aligned}
&\mathcal{C}_{0,s}=1,\quad \mathcal{C}_{s,s}=\frac{1}{s+1},\quad \mathcal{C}_{2s,s}=\frac{1}{2s+1}-\frac{s(2s)!}{\big((s+1)!\big)^2},\\
&\mathcal{C}_{3s,s}=\frac{1}{3s+1}-\frac{3s(3s)!}{(2s+1)!(s+1)!}+\frac{s(3s+1)!}{2\big((s+1)!\big)^3},\\
&\mathcal{C}_{4s,s}=\frac{1}{4s+1}-\frac{4s(4s)!}{(3s+1)!(s+1)!}-\frac{(8s+3)(4s)!}{(2s+1)!\big((s+1)!\big)^2}\\
&\qquad -\frac{2s(4s)!}{(2s+1)!\big((s+1)!\big)^2}+\frac{(4s+3)(4s)!}{(2s)!\big(s+1)!\big)^2}-\frac{s(8s^2+6s+1)(4s)!}{\big((s+1)!\big)^4},\dots.
\end{aligned}$$

Proof of Theorem 3. From (6), we have

$$\begin{aligned}
1&=\left(\sum_{m=0}^\infty \mathcal{C}_{sm,s}\frac{t^{sm}}{(sm)!}\right)\left(\sum_{l=0}^\infty(-1)^l d_l t^{sl}\right)\\
&=\sum_{n=0}^\infty\sum_{l=0}^n \frac{\mathcal{C}_{sn-sl,s}}{(sn-sl)!}(-1)^l d_l t^{sn}.
\end{aligned}$$

where the coefficients $d_0, d_1, \dots$ are also given in (3) with (4). Comparing the coefficients on both sides,

$$\sum_{l=0}^n \frac{\mathcal{C}_{sn-sl,s}}{(sn-sl)!}(-1)^l d_l=0\quad(n\ge 1).$$

By the inversion relation

$$\sum_{k=0}^n(-1)^{n-k}\alpha_k R(n-k)=0\quad(n\ge 1)\quad\text{with}\quad \alpha_0=R(0)=1$$

$$\Longleftrightarrow$$

$$\alpha_n=\begin{vmatrix}R(1)&1&&0\\R(2)&\ddots&\ddots&\\\vdots&\ddots&\ddots&1\\R(n)&\cdots&R(2)&R(1)\end{vmatrix}\quad\Longleftrightarrow\quad R(n)=\begin{vmatrix}\alpha_1&1&&0\\\alpha_2&\ddots&\ddots&\\\vdots&\ddots&\ddots&1\\\alpha_n&\cdots&\alpha_2&\alpha_1\end{vmatrix}$$

(e.g., see [19]), we get the result as

$$\alpha_n=d_n\quad\text{and}\quad R(n)=\frac{\mathcal{C}_{sn,s}}{(sn)!}.$$

□

By the inversion formula shown in the above proof, we also have the following Corollary. Similar determinant expressions of Bernoulli, Cauchy and related numbers were found in [20]).

Corollary 1. *For $n \ge 1$,*

$$d_n=\begin{vmatrix}\frac{\mathcal{C}_{s,s}}{s!}&1&&0\\\frac{\mathcal{C}_{2s,s}}{(2s)!}&\frac{\mathcal{C}_{3s,s}}{(3s)!}&&\\\vdots&&\ddots&1\\\frac{\mathcal{C}_{sn,s}}{(sn)!}&\cdots&\frac{\mathcal{C}_{2s,s}}{(2s)!}&\frac{\mathcal{C}_{s,s}}{s!}\end{vmatrix}.$$

By Trudi's formula

$$\begin{vmatrix} a_1 & a_2 & \cdots & \cdots & a_m \\ a_0 & a_1 & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_1 & a_2 \\ 0 & & & a_0 & a_1 \end{vmatrix}$$
$$= \sum_{t_1+2t_2+\cdots+mt_m=m} \binom{t_1+\cdots+t_m}{t_1,\ldots,t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m}$$

(refs. [21,22]; ref. [23],Volume 3, pp. 208–209, p. 214), we have a different expression of $\mathcal{C}_{n,s}$.

Theorem 4.

$$\mathcal{C}_{sn,s} = (sn)! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} (d_1)^{t_1} (d_2)^{t_2} \cdots (d_n)^{t_n}$$

and

$$d_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} \times \left(\frac{\mathcal{C}_{s,s}}{s!}\right)^{t_1} \left(\frac{\mathcal{C}_{2s,s}}{(2s)!}\right)^{t_2} \cdots \left(\frac{\mathcal{C}_{sn,s}}{(sn)!}\right)^{t_n}.$$

8. A Recurrence Relation for $\mathcal{C}_{n,s}^{(k)}$ in Terms of $\mathcal{C}_{n,s}$

We can show a recurrence formula for $\mathcal{C}_{n,s}^{(k)}$ in terms of $\mathcal{C}_{n,s}^{(k-1)}$ and $\mathcal{C}_{n,s}$.

Theorem 5. *For integers n and k with $n \geq 0$ and $k \geq 1$,*

$$\mathcal{C}_{sn,s}^{(k)} = (sn)! \sum_{\nu=0}^{n} \sum_{m=0}^{\nu} \frac{(-1)^{\nu-m}(s\nu - sm + 1) d_{\nu-m} \mathcal{C}_{sn-s\nu,s} \mathcal{C}_{sm,s}^{(k-1)}}{(sn - s\nu)!(sm)!(s\nu+1)},$$

where d_n is the coefficient of t^{sn+1} appeared in $\mathfrak{AF}_s(t)$ in Proposition 1.

Remark 2. *Poly-Cauchy numbers $c_n^{(k)}$ have a recurrence formula (ref. [16], Theorem 7)*

$$c_n^{(k)} = n! \sum_{\nu=0}^{n} \sum_{m=0}^{\nu} \frac{(-1)^{\nu-m} c_{n-\nu} c_m^{(k-1)}}{(n-\nu)! m! (\nu+1)}.$$

Poly-Cauchy numbers $\mathbb{C}_n^{(k)}$ with level 2 have a recurrence formula (ref. [6], Theorem 3.4)

$$\mathbb{C}_{2n}^{(k)} = (2n)! \sum_{\nu=0}^{n} \sum_{m=0}^{\nu} \left(-\frac{1}{4}\right)^{\nu-m} \binom{2\nu-2m}{\nu-m} \frac{\mathbb{C}_{2n-2\nu} \mathbb{C}_{2m}^{(k-1)}}{(2n-2\nu)!(2m)!(2\nu+1)}.$$

Proof of Theorem 5. Similarly to the description in Section 4, we obtain

$$\sum_{n=0}^{\infty} \mathcal{C}_{sn,s}^{(k)} \frac{x^{sn}}{(sn)!} = \mathrm{Lif}_{s,k}(\mathfrak{AF}_s(x))$$
$$= \frac{1}{\mathfrak{AF}_s(x)} \int_0^x \mathrm{Lif}_{s,k-1}(\mathfrak{AF}_s(\sigma)) \mathfrak{G}_s(\sigma) d\sigma$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{C}_{sn,s} \frac{x^{sn-1}}{(sn)!}\right) \int_0^x \left(\sum_{m=0}^{\infty} \mathcal{C}_{sm,s}^{(k-1)} \frac{\sigma^{sm}}{(sm)!}\right) \left(\sum_{j=0}^{\infty} (-1)^j (sj+1) d_j \sigma^{rj}\right) d\sigma$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{C}_{sn,s} \frac{x^{sn-1}}{(sn)!}\right) \int_0^x \left(\sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} (-1)^{\nu-m} (s\nu - sm + 1) d_{\nu-m} \frac{\mathcal{C}_{sm,s}^{(k-1)}}{(sm)!} \sigma^{s\nu}\right) d\sigma$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{C}_{sn,s} \frac{x^{sn-1}}{(sn)!}\right) \left(\sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} (-1)^{\nu-m} (s\nu - sm + 1) d_{\nu-m} \frac{\mathcal{C}_{sm,s}^{(k-1)}}{(sm)!} \frac{x^{s\nu+1}}{s\nu+1}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \sum_{m=0}^{\nu} \frac{(-1)^{\nu-m} (s\nu - sm + 1) d_{\nu-m} \mathcal{C}_{sn-s\nu,s} \mathcal{C}_{sm,s}^{(k-1)}}{(sn-s\nu)!(sm)!(s\nu+1)} x^{sn}.$$

Comparing the coefficients on both sides, we get the result. □

9. Conclusions

In this paper, we define poly-Cauchy numbers with higher level (level s) from the analytical implications, and study their properties. Such poly-Cauchy numbers with higher levels are yielded from the inverse relationship with an s-step function of the exponential function. When $s \geq 3$, the inverse function is not given using a known function, but it can be used to obtain the expressions and relations.

Poly-Bernoulli numbers with level 2 are defined and studied in [7]. Is it possible to introduce poly-Bernoulli numbers with higher levels? If so, is there any relation between them and poly-Cauchy numbers with higher levels?

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