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Theory of Quantum Mechanical Scattering in Hyperbolic Space

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Abstract: The theory of quantum mechanical scattering in hyperbolic space is developed. General formulas based on usage of asymptotic form of the solution of the Shrödinger equation in hyperbolic space are derived. The concept of scattering length in hyperbolic space, a convenient measurable in describing low-energy nuclear interactions is introduced. It is shown that, in the limit of the flat space, i.e., when $\rho \to \infty$, the obtained expressions for quantum mechanical scattering in hyperbolic space transform to corresponding formulas in three-dimensional Euclidean space.

Keywords: low-energy scattering; hyperbolic space; scattering length; cross section

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1. Introduction

The problems of mechanics in spaces of constant curvature were dealt with by many prominent mathematicians and mechanics in the 19th century (for more information, see books [1,2]).

In the beginning of the 20th century, the general theory of relativity and quantum mechanics were created. These greatest achievements in physics raised the interest in the problems of both classical and quantum mechanics in Riemannian spaces. The important role of the tetrad formalism, which is systematically presented in [3,4] for the formulation of such problems, should be emphasized.

In the work [5], the quantum mechanical problem of a hydrogen atom on a three-dimensional sphere was first solved by E. Schrödinger. A similar problem in hyperbolic space was considered in Refs. [6–8]. These authors showed that the energy spectrum of the hydrogen atom in spaces of constant curvature has a degeneracy analogous to the degeneracy in the planar case. Subsequently, it was demonstrated that the reason for the degeneration is in the additional conserved operators, which are analogues of the Runge–Lenz vector found in [9–11] in the case of positive constant curvature space, and in [12] for hyperbolic space.

Non-Euclidean geometry is used for the description of various physical problems [13–23]. For example, hyperbolic geometry finds application in the theory of relativistic nuclear collisions [13–16]. In [14], the connection between geometric relations in the hyperbolic space and kinematic characteristics determined from the experiment (transverse momentum, longitudinal velocity, etc.) was considered. It was shown that accounting for the properties of hyperbolic space—in particular, the absence of geometric similarity (in contrast to Euclidean geometry)—is very important in the analysis of experimental data and construction of adequate models for describing the multiple production of particles. Hyperbolic geometry was used for the description of particle production on the basis of experimental data obtained at bubble chambers in $n - p$ and other reactions in an energy range from a few to tens of GeV in [15]. The model based on the Coulomb interaction on the sphere has been used for description of the spectrum of quarkonium [17] and the excited states of excitons in quantum dots [18–20]. In addition, the usage of hyperbolic geometry to solve the problems of relativistic kinematics [24–26] can be highlighted. This is due to the fact that the group of motions of the Lobachevsky space is isomorphic to the Lorentz group. The geometric approach to the kinematics of relativistic particles is essentially based on the connection between the vector parametrization of the Lorentz group established in [25] and the quaternion calculus.

In the scattering theory, in hyperbolic space, two directions can be highlighted. One is the abstract scattering theory on the hyperbolic plane using the theory of automorphic functions, which is a spectral harmonic analysis on the group SL (2.R). The group SL(2.R) is locally isomorphic to the group O(2.1), the three-dimensional Lorentz group, and the group of motions of the Lobachevsky plane [27]. Well-known scientists, such as D. Lax, R. Phillips, L.D. Faddeev, B.S. Pavlov, and other experts in the field of mathematical physics, contributed to this approach, in which physics is the starting point for setting new mathematical problems.

In a series of papers by the present authors and their colleagues [28–30], a scattering theory in three-dimensional hyperbolic space similar to the scattering theory in three-dimensional Euclidean space was developed. The problems solved in those works were aimed at modeling physical problems in the spirit of Refs. [14,15,17–21]. In the flat space limit, they reduce to the corresponding problems in the Euclidean space.

The present paper focuses on the case of low-energy particle scattering, which is typical of nuclear physics. The dependences of the scattering length and effective scattering radius on the radius of the curvature of the hyperbolic space were studied.

The symmetry of the spaces used by physicists is crucial for the formulation and solution of the inherent problems of classical and quantum mechanics. Our approach is based on group-theoretical methods, the mathematical framework of symmetry.

2. The Scattering Problem in Hyperbolic Space

In Ref. [28], the formulation of the quantum-mechanical scattering problem in the hyperbolic space using Shapiro plane wave solutions of Schrödinger's equation [31] as an expression for the incident wave was considered.

The stationary problem for the Schrödinger equation in Cartesian coordinates x_μ has the form ($\hbar = m = 1$)

$$H\Psi = E\Psi, \quad H = \frac{1}{4\rho^2} M_{\mu\nu} M_{\mu\nu} + U, \quad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad \mu,\nu = 1,2,3,4, \tag{1}$$

where $x_\mu x_\mu = \mathbf{x}^2 + x_4^2 = \mathbf{x}^2 - x_0^2 = -\rho^2$, $\mathbf{x} = (x_1, x_2, x_3)$, $x_4 = ix_0$, ρ is the radius of curvature of the space, and U is the potential energy.

The solution of Schrödinger equation far beyond the region of scattering is a superposition of the incident and scattered (spherical) waves. However, in hyperbolic space, contrary to the flat space, simple plane wave solutions do not exist and it is required to construct analogues of such waves. The closest to a plane wave in its properties is the solution of the free Schrödinger equation known as Shapiro's plane waves (see [31]), which can be written as

$$\xi(x, \mathbf{n}) = \left(\frac{x_0 - \mathbf{x}\mathbf{n}}{\rho}\right)^{-1-i\eta}, \qquad \eta = \sqrt{2E\rho^2 - 1}, \tag{2}$$

where $\mathbf{n}$ is a unit vector and E is the energy of the system.

In the hyperbolic space, Schrödinger equation has solutions in the form of a spherical diverging wave by analogy with flat space.

Let us introduce the spherical coordinate system in hyperbolic space as

$$x_0 = \rho ch\beta, \quad x_1 = \rho sh\beta \sin\theta \cos\phi,$$

$$x_2 = \rho sh\beta \sin\theta \sin\phi, \quad x_3 = \rho sh\beta \cos\theta,$$
$$0 \le \beta < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi.$$

The solution of Schrödinger equation with potential $V(\beta)$ depending only on the distance can be written in factorized form $\Psi = R_l(\beta)Y_l^m(\theta,\phi)$, where the radial equation is

$$\Big[\frac{1}{2\rho^2}\Big(-\frac{1}{sh^2\beta}\frac{d}{d\beta}\Big(sh^2\beta\frac{d}{d\beta}\Big)+\frac{l(l+1)}{sh^2\beta}\Big)+V(\beta)-E\Big]R_l(\beta)=0. \tag{3}$$

The solution of Equation (3), which is regular at $\beta = 0$, in the absence of interaction has the form

$$R_{\eta l}(\beta) = \sqrt{\frac{\pi}{2sh\beta}}\frac{\Gamma(i\eta+l+1)}{\Gamma(i\eta+1)}P^{-\frac{1}{2}-l}_{-\frac{1}{2}+i\eta}(ch\beta). \tag{4}$$

where $P^{-\frac{1}{2}-l}_{-\frac{1}{2}+i\eta}(ch\beta)$ is the Legendre functions of the first kind.

The asymptotic form of the solution $R_{\eta l}$ at $\beta \longrightarrow \infty$ for a rapidly decreasing potential $V(\beta) \sim sh^{-n}\beta, n \ge 2$ is given by an expression in a form close to the standard one, namely

$$R_{\eta l}(\beta) \approx \frac{1}{2i\eta sh\beta}\Big(e^{i(\eta\beta-\frac{\pi}{2}+\delta_l)} - e^{-i(\eta\beta-\frac{\pi}{2}+\delta_l)}\Big), \qquad \eta = \sqrt{2E\rho^2-1}. \tag{5}$$

The solution of Equation (3), which is a diverging spherical wave in the hyperbolic space, has the form

$$R^+_{\eta l}(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{2sh\beta}}\Big[\frac{\Gamma(i\eta+l+1)}{\Gamma(i\eta+1)}P^{-\frac{1}{2}-l}_{-\frac{1}{2}+i\eta}(ch\beta) + \frac{\Gamma(i\eta-l)}{\Gamma(i\eta+1)}P^{\frac{1}{2}+l}_{-\frac{1}{2}+i\eta}(ch\beta)\Big]. \tag{6}$$

As $\beta \longrightarrow \infty$, we have

$$R^+_{\eta l}(\beta) \approx \frac{1}{2i\eta sh\beta}e^{i\eta\beta} \approx \frac{1}{i\eta e^{\beta}}e^{i\eta\beta}. \tag{7}$$

Let us set a direction in (2) as $\mathbf{n} = (0,0,1)$. Consequently, a plane Shapiro wave in a spherical coordinate system will be written as

$$\xi(\beta,\theta) = (ch\beta - sh\beta\cos\theta)^{-1-i\eta}. \tag{8}$$

The expression (8) expanded in spherical waves is

$$\xi(\beta,\theta) = \sum_{l=0}^{\infty}(2l+1)R_{\eta l}(\beta)P_l(\cos\theta). \tag{9}$$

Accordingly, the asymptotic expression for the expansion of the incident wave, taking into account (5) (see [32]), is

$$\xi(\beta,\theta) = \frac{1}{2i\eta sh\beta}\sum_{l=0}^{\infty}(2l+1)\Big((-1)^l e^{-i\eta\beta} + e^{i\eta\beta}\Big)P_l(\cos\theta). \tag{10}$$

The exact solution of the stationary Schrödinger Equation (1) for the potential $V(\beta)$ while $\beta \longrightarrow \infty$ should have the form

$$\Psi(\beta,\theta) \approx (ch\beta - sh\beta\cos\theta)^{-1-i\eta} + \frac{f(\theta)}{\rho sh\beta}e^{i\eta\beta}, \tag{11}$$

where $f(\theta)$ is a scattering amplitude.

The solution (11) in terms of Legendre polynomials will take the form

$$\Psi(\beta,\theta) \approx \frac{1}{2i\eta sh\beta}\sum_{l=0}^{\infty}(2l+1)[(-1)^{l+1}e^{i\eta\beta}+S_l e^{-i\eta\beta}]P_l(\cos\theta), \tag{12}$$

where $S_l = e^{2i\delta_l}$ and δ_l are the scattering phases.

Let us expand the scattering amplitude in Legendre polynomials:

$$f(\theta) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta). \tag{13}$$

By subtracting the incident wave in the form Equation (10) from Equation (12) and comparing the result with (11), we obtain

$$f(\theta) = \frac{\rho}{2i\eta}\sum_{l=0}^{\infty}(2l+1)(S_l-1)P_l(\cos\theta). \tag{14}$$

Comparing Equations (13) and (14), we obtain the expression for the coefficients

$$A_l = \frac{\rho}{2i\eta}(2l+1)(e^{2i\delta_l}-1).$$

The quantity

$$f_l = \frac{\rho}{2i\eta}(e^{2i\delta_l}-1) \tag{15}$$

is called partial amplitude.

Since the expression for the scattering amplitude (15) coincides with the corresponding formula in three-dimensional Euclidean space, then, accounting for the isotropy of the hyperbolic space and the orthogonality of the Legendre polynomials, we obtain an expression for the total cross-section through partial scattering phases, which coincides with the analogous expression in the plane space and has the form

$$\sigma = 4\pi|f(\theta)|^2 = \frac{4\pi\rho^2}{\eta^2}\sum_{l=0}^{\infty}(2l+1)sin^2\delta_l. \tag{16}$$

As follows from Equation (16), the maximum total cross section is equal to

$$\sigma_{max} = 4\pi|f(\theta)|^2 = \frac{4\pi\rho^2}{\eta^2}\sum_{l=0}^{\infty}(2l+1). \tag{17}$$

In Equation (17), the curvature radius ρ is contained only in η, which allows us to explicitly express the dependence on the space curvature $1/\rho^2$:

$$\sigma_{max} = \frac{4\pi}{2E-\frac{1}{\rho^2}}\sum_{l=0}^{\infty}(2l+1), \tag{18}$$

or in the first approximation in curvature

$$\sigma_{max} = \frac{2\pi}{E}\left(1+\frac{1}{2E\rho^2}\right)\sum_{l=0}^{\infty}(2l+1). \tag{19}$$

It should be noted that partial phases generally depend on the energy and space curvature. However, for short-range forces and low energies of the scattered particle, the main contribution to the amplitude is given by the wave with $l = 0$:

$$f(\theta) = f_0 = \frac{\rho}{2i\eta}(e^{2i\delta_0} - 1). \quad (20)$$

In this case, instead of Equation (16), we obtain

$$\sigma = \frac{4\pi\rho^2}{\eta^2} sin^2\delta_0. \quad (21)$$

It is known [33] that, in the case of a flat space, at $\delta_0 = \pi/2$, as the energy tends to zero $E \longrightarrow 0$, the cross section tends to infinity. In the case of the problem under consideration, the maximum of the cross-section (21) is reached at an energy equal to the curvature, as follows from the formula below:

$$\sigma = \frac{4\pi}{2E - \frac{1}{\rho^2}}. \quad (22)$$

In the case of low-energy scattering, when δ_0 can be considered small and constant, by expanding the exponent in Formula (20) and limiting ourselves by two terms, we obtain

$$f_0 = L = \frac{\rho\delta_0}{\eta}. \quad (23)$$

The quantity L is similar to the scattering length in flat space [34], and reduces to it at $\rho \longrightarrow \infty$. Therefore, expression (23) will be considered as the scattering length in the hyperbolic space, and in terms of the small curvature, is written as

$$f_0 = L \approx \frac{\delta_0}{\sqrt{2E}}\Big(1 + \frac{1}{4E\rho^2}\Big). \quad (24)$$

The cross-section corresponding to the amplitude (24) is expressed by the formula

$$\sigma = 4\pi L^2. \quad (25)$$

Obviously, with vanishing curvature $\rho \longrightarrow \infty$, Equations (16)–(24) coincide with the corresponding formulas in a flat three-dimensional space. An interesting feature of scattering in the hyperbolic space is the fact that, when the energy is large, but ρ is not, the dependence on the curvature radius in Equations (18), (19), (22), and (24) also disappears. This feature is a consequence of Equation (2) and is essential when the mentioned formulas are considered as model ones and when ρ acts as some additional parameter, whose interpretation should be discussed separately in each particular case. These formulas may be useful for describing the processes of neutron scattering on nuclei and nano-objects.

Let us demonstrate the above general principles of the scattering theory in non-relativistic quantum mechanics in hyperbolic space by means of examples of scattering by a spherically symmetric potential well and Coulomb potential. We would also like to compare the results to those in flat (Euclidean) space.

3. Scattering Length in Euclidean and Hyperbolic Spaces

Let us first consider the scattering of s-waves for a low-energy particle with a short-range interaction. The interaction will be described by a spherically symmetric potential well of the form

$$V = -U_0, \quad r \leq a; \qquad V = 0, \quad r > a - in \quad Euclidean \quad space$$

and

$$V = -U_0, \quad \beta \leq a; \qquad V = 0, \quad \beta > a - in \quad hyperbolic \quad space$$

Here, a is the "width" of the well. It has the dimension of length in Euclidean space and is dimensionless in hyperbolic space.

As known [35], the scattering length on a spherically symmetric potential well in Euclidean space is given by the expression

$$L = a\left(1 - \frac{tg(K_0 a)}{K_0 a}\right), \tag{26}$$

where $K_0^2 = 2mU_0$.

The scattering length in the hyperbolic space in a potential well can be obtained from Formula (26) by replacing a with $a\rho$ as shown in [36]:

$$L = a\rho\left(1 - \frac{tg(K_0 a\rho)}{K_0 a\rho}\right). \tag{27}$$

The essential difference in scattering length expressions for both spaces should be discussed. In deriving (26) (see [35]), we consider the limit of low-energy $E \longrightarrow 0$ and, hence, $k \longrightarrow 0$, since, in the Euclidean space, $k^2 = 2mE$. Moreover, the dimension of k is length-inverse. On the other hand, in the hyperbolic space, the role of the wave number is played by the dimensionless quantity $\eta = \sqrt{2m\rho^2 E - 1}$, and the scattering length (27) is determined from the first expansion term of the series of the function $\frac{\eta}{\rho} ctg\delta_0$ at $\eta \longrightarrow 0$, i.e.,

$$\frac{\eta}{\rho} ctg\delta_0 \approx -\frac{1}{L}. \tag{28}$$

Thus, in hyperbolic space, the energy does not tend to zero, but $E \longrightarrow E_{min}$ and $E_{min} = 1/(2m\rho^2)$, which is small when the radius of curvature of the space is sufficiently large. When the energy of the incoming particle is lower than E_{min}, only bound states are possible.

Let us consider the case of the Coulomb potential of the form

$$U = \frac{\alpha}{r}, \qquad \alpha > 0 - \text{repulsion potential in Euclidean space.}$$

and

$$U = \frac{-\alpha x_0}{\rho|\vec{x}|}, \qquad \alpha < 0 - \text{repulsion potential in hyperbolic space.}$$

As known [35], the scattering amplitude in the Euclidean space is given by the formula (here, $\hbar = 1$)

$$f(\theta) = -\frac{\Gamma(1 + i\alpha/k)}{\Gamma(1 - i\alpha/k)} \frac{\alpha}{2k^2 \sin^2(\theta/2)} \exp(-\frac{2i\alpha}{k} \ln \sin(\theta/2)) \tag{29}$$

The partial amplitude f_0 is determined as

$$f_0 = \frac{1}{2} \int_{-1}^{1} f(\theta) d(\cos\theta). \tag{30}$$

Calculations yield

$$f_0 = \frac{1}{2ik} \frac{\Gamma(1 + i\alpha/k)}{\Gamma(1 - i\alpha/k)}, \tag{31}$$

which can be written as

$$f_0 = \frac{1}{2ik} e^{2i\delta_0}, \tag{32}$$

recalling that

$$e^{2i\delta_l} = \frac{\Gamma(1 + l + i\alpha/k)}{\Gamma(1 + l - i\alpha/k)}.$$

Hence, the partial cross-section for $l = 0$ is

$$\sigma_0 = 4\pi|f_0|^2 = \frac{\pi}{k^2}. \tag{33}$$

Determining the scattering length from the equation $\sigma_0 = 4\pi L^2$, we have

$$L = \frac{1}{2k}. \tag{34}$$

In [37], it is noted that the length of scattering in Coulomb potential is large but the formula is not given.

Since S-scattering ($l = 0$) occurs at low energy, $k \longrightarrow 0$ and $L \longrightarrow \infty$. Thus, in Euclidean space, the scattering length of Coulomb potential is infinite, confirming the long-range nature of the Coulomb force.

Let us derive the corresponding formulas for Coulomb scattering in hyperbolic space. As shown in [29], the scattering amplitude is determined by the expression

$$f(\theta) = \frac{\rho(\gamma_+ - \gamma_-)}{\gamma_+ + \gamma_-}\frac{\Gamma(1 - i\gamma_+ + i\gamma_-)}{\Gamma(1 + i\gamma_+ - i\gamma_-)}2^{-i(\gamma_+ - i\gamma_-)}(1 - \cos\theta)^{i\gamma_+ - i\gamma_- - 1}, \tag{35}$$

where

$$\gamma_\pm = \sqrt{\frac{E\rho^2 m \pm \alpha\rho}{2} - \frac{1}{4}}. \tag{36}$$

Performing a similar calculation, we obtain

$$f_0 = \frac{\rho}{2i(\gamma_+ + \gamma_-)}\frac{\Gamma(1 - i\gamma_+ + i\gamma_-)}{\Gamma(1 + i\gamma_+ - i\gamma_-)}, \tag{37}$$

and thus

$$\sigma_0 = \frac{\pi\rho^2}{(\gamma_+ + \gamma_-)^2}. \tag{38}$$

The low-energy limit corresponds to the condition $\gamma_+ \longrightarrow 0$ and $E \longrightarrow \frac{\hbar^2}{2m\rho^2}(1 - 2\alpha\rho)$. At this limit, $(\gamma_+ + \gamma_-)^2 \approx -\alpha\rho$ (here, $\alpha < 0$). Thus, we have

$$\sigma_0 = \frac{-\pi\rho}{\alpha}, \tag{39}$$

and, for the scattering length,

$$L = \sqrt{\frac{-\rho}{4\alpha}}. \tag{40}$$

As follows from (40), the scattering length in hyperbolic space is finite, i.e., the force is not of a long range any more. However, when $\rho \longrightarrow \infty$, the scattering length transforms to $L \longrightarrow \infty$, which is in agreement with (34).

4. Conclusions

In the introduction, a brief literature review of the practical applications of non-Euclidean geometry in physics is presented.

Let us summarize the main results of the paper:

1. As follows from the formula for the modulus of the wave vector Equation (2) and subsequent expressions obtained within the framework of the approach presented by Equations (16)–(22), the effect of a "flat" space limit can be achieved with a constant radius of curvature due to high-energy particles.

2. Studies of low-energy scattering—in particular, the use of Equation (22)—may help in the identification of curvature effects.

3. The formulae for the scattering length on the spherically symmetric potential will have the same form in the hyperbolic space as in the Euclidean space.

4. Whereas, in the Euclidean space, the scattering length in the case of the Coulomb potential is infinite, it is finite in the hyperbolic space, showing that the Coulomb potential does not have long-range potential in the hyperbolic space.

It will be interesting to investigate these properties in other spaces as well.

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