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An Existence Result of Positive Solutions for the Bending Elastic Beam Equations

Yongxiang Li * and Dan Wang

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
* Correspondence: liyx@nwnu.edu.cn

Abstract: This paper is concerned with the existence of positive solutions to the fourth-order boundary value problem $u^{(4)}(x) = f(x, u(x), u''(x))$ on the interval $[0, 1]$ with the boundary condition $u(0) = u(1) = u''(0) = u''(1) = 0$, which models a statically bending elastic beam whose two ends are simply supported. Without assuming that the nonlinearity $f(x, u, v)$ is nonnegative, an existence result of positive solutions is obtained under the inequality conditions that $|(u, v)|$ is small or large enough. The discussion is based on the method of lower and upper solutions.

Keywords: bending elastic beam equations; lower and upper solutions; positive solution; existence

MSC: 34B18; 47H10

1. Introduction

In this paper, we discuss the existence of a positive solution for the fourth-order boundary value problem (BVP)

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u''(x)), & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $I = [0, 1]$, $f : I \times [0, \infty) \times (-\infty, 0] \to \mathbb{R}$ is continuous. BVP(1) models the deformations of an elastic beam whose two ends are simply supported in an equilibrium state, and u represents the deformation of the beam, u'' in f is the bending moment term which represents bending effect, see [1–5].

Since 1986, many researchers have studied the existence of solutions to this problem, see [1–12] and reference therein. Firstly, Aftabizadeh [1] showed the existence of solutions that f is a bounded function. Yang [2] extended Aftabizadeh's result and showed BVP(1) has a solution under f that satisfies a linear growth condition. Del Pino and Manasevich [5] further extended Yang's result, and they obtained existence and uniqueness theorems under a non-resonance condition involving a two-parameter linear eigenvalue problem and a linear growth condition on f. Later, De Coster et al. [6] and Li [9] extended the two-parameter non-resonance conditions in [5]. Agarwal [3] and Kaufmann [12] obtained the existence results by using the Schauder fixed point theorem under f satisfies certain growth conditions. Korman [4], Ma et al. [7], Cabada [8] and Li [10] discussed the existence of solutions by using monotone iterative technique assumed that BVP(1) has a pair of ordered lower and upper solutions and f satisfies certain monotone conditions between the lower and upper solutions. Recently, Li and Gao [11] obtained existence and uniqueness results under certain inequality conditions of f, and the inequality conditions allow f to grow superlinearly on u and u''.

Generally, for BVP(1) in statically elastic beams, only its positive solution is practical significance, see Figure 1.

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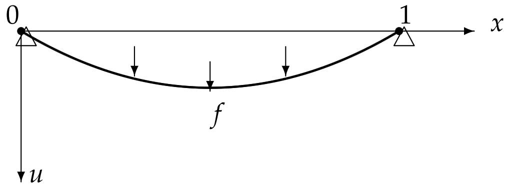

Figure 1. Simply Supported Beam.

Under the acting of the load f, the bean is deformed down, and the displacement $u(x)$ of the beam at x is positive for $x \in (0, 1)$. Hence, the solutions of the simply supported beam equations are usually positive. For the case that f is nonnegative, some authors have researched the existence of positive solutions, see [13–19]. Early, Ma and Wang [13] considered the special case of BVP(1) that f does not contain u'': the simple fourth-order boundary value problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \qquad x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{2}$$

They using the fixed point theorems of cone mapping obtained the existence of positive solutions of BVP(2) under f is nonnegative and $f(x, u)$ is superlinear or sublinear growth on u at 0 and $+\infty$. Later, Bai and Wang [15], Li [16], Liu [18] and Yao [19] improved and extended these results by choosing a cone in $C(I)$ and using the fixed-point index theory in cones. For general BVP(1), Li [17] obtained the existence and no existence results of positive solutions under f that are nonnegative and satisfy some inequality conditions involving the first eigenvalue-line of the corresponding two-parameter linear eigenvalue problem by counting the fixed-point index of the corresponding integral operator in a cone of $C^2(I)$. Ma and Xu [14] obtained the existence of positive solutions under f is nonnegative and $f(x, u, v)$ satisfies asymptotically linear conditions as $|(u, v)| \to 0$ and $|(u, v)| \to \infty$ by using Krein–Rutman theorem and the global bifurcation theory of positive operators obtained the existence of positive solutions.

The above authors who studied the existence of positive solutions of BVP(1.1) all required that the nonlinear term f is nonnegative. When f is not nonnegative, the corresponding integral operator of BVP(1) is not a positive operator and the method in references [13–19] is not applicable. The purpose of this paper is to obtain the existence of positive solutions for general BVP(1) without assuming that f is nonnegative. Under the inequality conditions of f when $|(u, v)|$ is small or large enough, we obtained an existence result of positive. Our main result is as follows:

Theorem 1. *Let $f : I \times [0, \infty) \times (-\infty, 0] \to \mathbb{R}$ be continuous and satisfy the following conditions*

(F1) *For every $x \in I$ and $v \in (-\infty, 0]$, $f(x, u, v)$ is increasing on u in $[0, \infty)$;*

(F2) *there exist constant $\alpha, \beta \geq 0$ satisfying $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} \geq 1$ and $\delta > 0$ such that*

$$f(x, u, v) \geq \alpha u - \beta v, \quad \textit{for} \ \ u \geq 0, \ \ v \leq 0 \ \ \textit{and} \ \ |(u, v)| \leq \delta;$$

(F3) *there exist constant $\alpha_1, \beta_1 \geq 0$ satisfying $\frac{\alpha_1}{\pi^4} + \frac{\beta_1}{\pi^2} < 1$ and $H > 0$ such that*

$$f(x, u, v) \leq \alpha_1 u - \beta_1 v, \quad \textit{for} \ \ u \geq 0, \ \ v \leq 0 \ \ \textit{and} \ \ |(u, v)| \geq H.$$

Then BVP(1) has at least one positive solution.

Note that the straight line

$$\ell_1 = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \;\middle|\; \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} = 1 \right\} \tag{3}$$

on α and β is the first eigenvalue-line of the two-parameter linear eigenvalue problem corresponding to BVP(1)

$$\begin{cases} u^{(4)}(x) = \alpha u(x) - \beta u''(x), \qquad x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{4}$$

(see [5], Proposition 2.1), the coefficient conditions of the inequalities in (F2) and (F3) are optimal.

The proofs of Theorem 1 are based on the method of lower and upper solutions. A lower solution v of BVP(1.1) means that $v \in C^4(I)$ and satisfies

$$\begin{cases} v^{(4)}(x) \le f(x, v(x), v''(x)), \qquad x \in I, \\ v(0) \le 0, \quad v(1) \le 0, \quad v''(0) \ge 0, \quad v''(1) \ge 0, \end{cases}$$

and an upper solution w of BVP(1.1) means that $w \in C^4(I)$ and satisfies

$$\begin{cases} w^{(4)}(x) \ge f(x, w(x), w''(x)), \qquad x \in I, \\ w(0) \ge 0, \quad w(1) \ge 0, \quad w''(0) \le 0, \quad w''(1) \le 0. \end{cases}$$

The method of lower and upper solutions for BVP(1) is, by finding a pair of lower solution v_0 and upper w_0 with $v_0 \le w_0$ and $v_0'' \ge w_0''$, to obtain a solution u_0 satisfied $v_0 \le u_0 \le w_0$ and $v_0'' \ge u_0'' \ge v_0''$, see [7,10]. The advantages of this method are that it is no any restriction for the growth of $f(x, u, v)$ with respect to u and v, and that ones can find the solution u_0 with monotone iteration technique starting from v_0 and w_0 under some monotonicity conditions of f. The disadvantage is that it is not easy to find the required pair of upper and lower solutions. In Theorem 1, we give the concrete conditions (F2) and (F3) for finding the lower solution v_0 and the upper solution w_0.

In [20], Minhós, Gyulov and Santos established a theorem of upper and lower solutions for the more general fourth-order boundary value problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \qquad x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

provided a pair of lower and upper solutions, see [20] (Theorem 1). However, the definition of upper and lower solutions in [20] is different from ours, and our upper and lower solutions do not meet the conditions of the pair of upper and lower solutions in [20]. Hence, Theorem 1 is not covered by [20] (Theorem 1).

In Section 3, we will use the method of lower and upper solutions and a truncation function technique to prove Theorem 1. Some preliminaries to discuss BVP(1) are presented in Section 2.

2. Preliminaries

As usual, we use $C(I)$ to denote the Banach space of all continuous function u on I with maximum norm $\|u\| = \max_{x \in I} |u(x)|$. For $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all nth-order continuous differentiable function u with the norm $\|u\|_{C^n} = \max\{\|u\|, \|u'\|, \dots, \|u^{(n)}\|\}$.

To prove Theorem 1, we first consider the linear fourth-order boundary value problem (LBVP)

$$\begin{cases} u^{(4)}(x) + \beta\, u''(x) - \alpha\, u(x) = h(x), & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{5}$$

where α, $\beta \in \mathbb{R}$ and $h \in C(I)$ are given.

Lemma 1. *Let α, $\beta \geq 0$ and $\frac{\alpha}{\pi^4} + \frac{\beta^2}{\pi^2} < 1$. Then for every $h \in C(I)$, LBVP(5) has a unique solution $u \in C^4(I)$. Moreover, when $h \geq 0$, the solution u satisfies: $u \geq 0$, $u'' \leq 0$.*

Proof. Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$, that is

$$\lambda_1,\, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}.$$

By the assumption we easy to obtain that: $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2$. Let $G_i(x, y)(i = 1, 2)$ be the Green's function of the linear boundary value problem

$$-u''(x) + \lambda_i\, u(x) = 0, \qquad u(0) = u(1) = 0.$$

By Lemma 2.1 of [16], $G_i(x, y) \geq 0$ for every x, $y \in I$. Since

$$u^{(4)}(x) + \beta u''(x) - \alpha u(x) = \Big(-\frac{d^2}{dx^2} + \lambda_1\Big)\Big(-\frac{d^2}{dx^2} + \lambda_2\Big)\, u,$$

setting $v = -u'' + \lambda_2 u$, then LBVP(5) becomes to the second-order boundary value problem

$$\begin{cases} -v''(x) + \lambda_1\, v(x) = h(x), & x \in I, \\ v(0) = v(1) = 0. \end{cases} \tag{6}$$

Obviously, BVP(6) has a unique solution $v \in C^2(I)$ given by

$$v(x) = \int_0^1 G_1(x, z)\, h(z) dz, \qquad x \in I. \tag{7}$$

Hence, solving the the second-order boundary value problem

$$\begin{cases} -u''(x) + \lambda_2\, u(x) = v(x), & x \in I, \\ v(0) = v(1) = 0, \end{cases} \tag{8}$$

it follows that LBVP(5) has a unique solution $u \in C^4(I)$ give by

$$u(x) = \int_0^1 G_2(x, y) v(y) dy = \int_0^1 \int_0^1 G_2(x, y) G_1(y, z) h(z) dz dy. \tag{9}$$

When $h \geq 0$, by (9) and the nonnegativity of the Green functions G_1 and G_2, $u \geq 0$. By (7), $v \geq 0$. Since $v = -u'' + \lambda_2 u$ and $\lambda_2 \leq 0$, we obtain that $u'' = -v + \lambda_2 u \leq 0$. □

Let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then there exists a constant $M > 0$ such that

$$|f(x, u, v)| \leq M, \qquad (x, u, v) \in I \times \mathbb{R} \times \mathbb{R}. \tag{10}$$

We consider the nonlinear boundary value problem

$$\begin{cases} u^{(4)}(x) + \beta\, u''(x) - \alpha\, u(x) = f(x, u(x), u''(x)), & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{11}$$

Lemma 2. *Let* α, $\beta \geq 0$ *and* $\frac{\alpha}{\pi^4} + \frac{\beta^2}{\pi^2} < 1$ *and* $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be continuous and bounded. Then BVP(11) has at least one solution* $u \in C^4(I)$.

Proof. For every $h \in C(I)$, by Lemma 1, LBVP(5) has a unique solution $u := Sh \in C^4(I)$ given by (9). This defines a linear bounded operator $S : C(I) \to C^4(I)$, and it is called the solution operator of LBVP(5). By the compactness of the embedding $C^4(I) \hookrightarrow C^2(I)$, $S : C(I) \to C^2(I)$ is a linear completely continuous operator. We denote the norm of the linear $S : C(I) \to C^2(I)$ by $\|S\|_{\mathfrak{B}(C(I),C^2(I))}$. Define a nonlinear mapping $F : C^2(I) \to C(I)$ by

$$F(u)(x) := f(x, u(x), u''(x)), \qquad x \in I, \quad u \in C^2(I).$$

Clearly, $F : C^2(I) \to C(I)$ is continuous and bounded. By (10), F satisfies

$$\|F(u)\| \leq M, \qquad u \in C^2(I). \tag{12}$$

Hence, the composite mapping $A = S \circ F : C^2(I) \to C^2(I)$ is completely continuous. Let $R \geq \|S\|_{\mathfrak{B}(C(I),C^2(I))} M$ and set $\Omega = \{u \in C^2(I) : \ \|u\|_{C^2 \leq \mathbb{R}}\}$. Clearly, Ω is bounded convex closed set of $C^2(I)$. For every $u \in \Omega$, by (12) we have

$$\|Au\|_{C^2} = \|S(F(u))\| \leq \|S\|_{\mathfrak{B}(C(I),C^2(I))} \|F\| \leq \|S\|_{\mathfrak{B}(C(I),C^2(I))} M \leq R.$$

Hence, $Au \in \Omega$. This means that $A(\Omega) \subset \Omega$. By the Schauder fixed-point theorem [21], A has a fixed-point $u_0 \in \Omega$. Since $u_0 = Au_0 = S(F(u_0))$, by the definition of S, u_0 is the unique solution of LBVP(5) for $h = F(u_0) \in C(I)$. Hence, $u_0 \in C^4(I)$ satisfies Equation (11), and it is a solution of BVP(11). □

3. Proofs of the Main Result

Proof of Theorem 1. We use the method of lower and upper solutions and a truncation function technique to prove Theorem 1.

Firstly, we construct a pair of positive lower solution v_0 and upper solution w_0 of BVP(1), such that $v_0 \leq w_0$ and $v_0'' \geq w_0''$.

Let α_1, β_1, H be the constant in Condition (F3). Set

$$C_0 = \max\{|f(x, u, v) - (\alpha_1 u - \beta_1 v)| \mid x \in I,\ u \geq 0,\ v \leq 0,\ |(u,v)| \leq H\} + 1,$$

then by Condition (F3),

$$f(x, u, v) \leq \alpha_1 u - \beta_1 v + C_0, \qquad x \in I,\ u \geq 0,\ v \leq 0. \tag{13}$$

By Lemma 1, the boundary value problem

$$\begin{cases} u^{(4)}(x) + \beta_1 u''(x) - \alpha_1 u(x) = C_0, & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{14}$$

has a unique solution $w_0 \in C^4(I)$, and it satisfies $w_0 \geq 0$ and $w_0' \leq 0$. By (13) and Equation (14), we easily see that w_0 is an upper solution of BVP(1).

Let δ be the constant in Condition (F2). Choose a constant by

$$\sigma = \min\left\{\frac{\delta}{\sqrt{1+\pi^4}}, \quad \frac{C_0}{\pi^4 - \beta_1 \pi^2 - \alpha_1}\right\} \tag{15}$$

and define a function by $v_0(x) = \sigma \sin \pi x$. We show that v_0 is a lower solution of BVP(1). For every $x \in I$, since

$$v_0(x) = \sigma \sin \pi x \geq 0, \qquad {v_0}''(x) = -\pi^2 \sigma \sin \pi x \leq 0,$$
$$|(v_0(x),\ {v_0}''(x))| = \sigma\sqrt{1+\pi^4}\sin \pi x \leq \delta,$$

form (F2) it follows that,

$$\begin{aligned} f(x,\ v_0(x), {v_0}''(x)) &\geq \alpha v_0(x) - \beta {v_0}''(x) \\ &= (\alpha + \beta\pi^2)\sigma \sin \pi x \geq \pi^4 \sigma \sin \pi x = {v_0}^{(4)}(x). \end{aligned}$$

Hence v_0 is a lower solution of BVP(1). We show that

$$v_0 \leq w_0, \quad {v_0}'' \geq {w_0}''. \tag{16}$$

Consider the function $u = w_0 - v_0$. Noting w_0 is a solution of BVP(14), we have

$$\begin{aligned} h(x) &:= u^{(4)}(x) + \beta_1\, u''(x) - \alpha_1\, u(x) \\ &= C_0 - ({v_0}^{(4)}(x) + \beta_1\, {v_0}''(x) - \alpha_1\, v_0(x)\,) \\ &= C_0 - (\pi^4 - \beta_1 \pi^2 - \alpha_1)\, \sigma \sin \pi x \\ &\geq C_0 - (\pi^4 - \beta_1 \pi^2 - \alpha_1)\, \sigma \\ &\geq 0, \qquad x \in I. \end{aligned}$$

This means that $h \geq 0$ and $u \in C^4(I)$ is a solution of LBVP(5). By Lemma 2, $u \geq 0$ and $u'' \leq 0$. Hence (16) holds.

Secondly, we make a bounded truncation function f^* of f through the lower solution v_0 and upper solution w_0.

Define functions $\xi,\ \eta : I \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \xi(x,\ u) &= \min\{\max\{v_0(x),\ u\},\ \ w_0(x)\}, \\ \eta(x,\ v) &= \min\{\max\{w_0''(x),\ v\},\ \ v_0''(x)\}, \end{aligned} \tag{17}$$

Then $\xi,\ \eta : I \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$\begin{aligned} v_0(x) \leq \xi(t,\ u) \leq w_0(x), &\quad (x,\ u) \in I \times \mathbb{R}, \\ w_0''(x) \leq \eta(x,\ v) \leq v_0''(x), &\quad (x,\ v) \in I \times \mathbb{R}. \end{aligned} \tag{18}$$

Define a truncating function of f by

$$f^*(x,\ u,\ v) = f(t,\ \xi(x,u), \eta(x,v)) + \frac{v - \eta(x,v)}{v^2+1}, \quad (x,\ u,\ v) \in I \times \mathbb{R}^2. \tag{19}$$

By (17) and (18), $f^* : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Next, we consider the boundary value problem

$$\begin{cases} u^{(4)}(x) = f^*(x,\ u(x),\ u''(x)), & t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{20}$$

and prove its solution is also the solution of BVP(1).

By Lemma 2, BVP(20) has a solution $u_0 \in C^4(I)$. We show that

$$w_0'' \leq u_0'' \leq v_0''. \tag{21}$$

In fact, if $w_0'' \not\leq u_0''$, then for the function

$$\phi(x) = u_0''(x) - w_0''(x), \quad x \in I, \tag{22}$$

$\min_{0 \leq x \leq 1} \phi(t) < 0$. Since $\phi(0)$, $\phi(1) \geq 0$, there exists $x_0 \in (0, 1)$ such that $\min_{0 \leq x \leq 1} \phi(x) = \phi(x_0)$. By the properties of ϕ at minimum points, we have

$$\phi(x_0) < 0, \quad \phi'(x_0) = 0, \quad \phi''(x_0) \geq 0.$$

from this and (22) it follows that

$$u_0''(x_0) < w_0''(x_0), \qquad u_0^{(4)}(x_0) \geq w_0^{(4)}(x_0). \tag{23}$$

Hence by definition (17), we have

$$\eta(x_0, u_0''(x_0)) = w_0''(x_0). \tag{24}$$

By Equations (20) and (18), (24), Condition (F1) and the definition of the upper solution w_0, we have

$$\begin{aligned}
u_0^{(4)}(x_0) &= f^*(x_0, u_0(x_0), u_0''(x_0)) \\
&= f(x_0,\ \xi(x_0, u_0(x_0)),\ \eta(x_0, u_0''(x_0))) + \frac{u_0''(x_0) - \eta(x_0, u_0''(x_0))}{u_0''^2(x_0) + 1} \\
&= f(x_0,\ \xi(x_0, u_0(x_0)),\ w_0''(x_0)) + \frac{u_0''(x_0) - w_0''(x_0)}{u_0''^2(x_0) + 1} \\
&< f(x_0,\ \xi(x_0, u_0(x_0)),\ w_0''(x_0)) \\
&\leq f(x_0, w_0(x_0), w_0''(x_0)) \\
&\leq w_0^{(4)}(x_0).
\end{aligned}$$

Namely, $u_0^{(4)}(x_0) < w_0^{(4)}(x_0)$, this contradict the second inequality of (23). Hence, $w_0'' \leq u_0''$.

With a similar argument, we can show that $u_0'' \leq v_0''$. Thus, (21) holds. Furthermore, from (21) we show that

$$v_0 \leq u_0 \leq w_0. \tag{25}$$

Consider the function $u = u_0 - v_0$. Since

$$-u''(x) = -(u_0''(x) - v_0''(x)) \geq 0, \quad x \in I; \qquad u(0),\ u(1) \geq 0,$$

by the maximum principle of second-order differential operators, $u \geq 0$. That is, $v_0 \leq u_0$. Similarly, $u_0 \leq w_0$. Hence, (25) holds.

Now, from (21), (25) and definition (17), it follows that

$$\xi(x, u_0(x)) = u_0(x), \quad \eta(x, u_0''(x)) = u_0''(x), \quad x \in I.$$

Hence by Equation (20), we have

$$
\begin{aligned}
u_0^{(4)}(x) &= f^*(x,\, u_0(x),\, {u_0}''(x)) \\
&= f(x,\; \xi(x,\, u_0(x)),\; \eta(x,\, {u_0}''(x))) + \frac{{u_0}''(x) - \eta(x,\, {u_0}''(x))}{{u_0}''^2(x) + 1} \\
&= f(x,\, u_0(x),\, {u_0}''(x)), \quad x \in I.
\end{aligned}
$$

That is, u_0 is a solution of BVP(1) and it is positive.

The proof of Theorem 1 is completed. □

Example 1. *Consider the following fourth-order boundary value problem*

$$
\begin{cases}
u^{(4)}(x) = 2\sqrt{u(x)} - 3u''(x) - 5u''^2(x), & x \in I, \\
u(0) = u(1) = u''(0) = u''(1) = 0\,.
\end{cases} \tag{26}
$$

Clearly, this problem has a trivial solution $u \equiv 0$. In addition, we can easily verify that the nonlinearity of BVP(26)

$$
f(x,\, u,\, v) = 2\sqrt{u} - 3v - 5v^2, \qquad u \geq 0, \;\; v \leq 0 \tag{27}
$$

satisfies the conditions (F1)–(F3). By Theorem 1, BVP(26) has at least one positive solution. Since the nonlinearity f defined by (27) is not nonnegative, this conclusion cannot be obtained from the known results in [13–19].

4. Conclusions

In this paper, we obtained the existing result of positive solutions for the bending elastic beam equation BVP(1) by applying the method of lower and upper solutions. The method of lower and upper solutions is an important technique to solve BVP(1). The key to applying this method is to find a pair of lower solution v_0 and upper w_0 satisfied $v_0 \leq w_0$ and ${v_0}'' \geq {w_0}''$. Some authors mentioned in Section 1 discussed the existence of solutions under the assumption that the equation has such a pair of lower and upper solutions, and they did not provide the search method or existence conditions for such a pair of lower and upper solutions. In Theorem 1, we give the concrete conditions to obtain such a pair of lower and upper solutions, these are (F2) and (F3). Condition (F2) implies that

$$
v_0(x) = \sigma \sin \pi x \;(\, 0 < \sigma < \delta / \sqrt{1 + \pi^4}\,)
$$

is a lower solution of BVP(1), where δ is the constant in (F2); and (F3) implies that the unique positive solution w_0 of LBVP(14) is a upper solution of BVP(1). By Lemma 1, we showed that when σ is small enough, v_0 and w_0 satisfy

$$
v_0 \leq w_0, \quad {v_0}'' \geq {w_0}''.
$$

Hence, in Section 3 we proved that when f also satisfies the condition (F1), BVP(1) has a positive solution u_0 satisfied

$$
v_0 \leq u_0 \leq w_0, \quad {v_0}'' \geq {u_0}'' \geq {w_0}''.
$$

This conclusion allows f with negative values, and the previous works on the existence of positive solutions only discussed the case that f is nonnegative. Our conclusion develops the study on the existence of positive solutions of the static simply supported beam equations.

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