symmetry

MDPI

Article

Quasi-Monomiality Principle and Certain Properties of Degenerate Hybrid Special Polynomials

Rabab Alyusof

Department of Mathematics, College of Science, King Saud University, Riyadh 11421, Saudi Arabia; ralyusof@ksu.edu.sa

Abstract: This article aims to introduce degenerate hybrid type Appell polynomials ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ and establishes their quasi-monomial characteristics. Additionally, a number of features of these polynomials are established, including symmetric identities, implicit summation formulae, differential equations, series definition and operational formalism.

Keywords: degenerate hybrid special polynomials; appell polynomials; monomiality principle; operational rules

MSC: 33E20; 33B10; 33E30; 11T23

check for updates

Citation: Rabab, A. Quasi-Monomiality Principle and Certain Properties of Degenerate Hybrid Special Polynomials. *Symmetry* **2023**, *15*, 407. https://doi.org/10.3390/sym15020407

Academic Editors: Juan Luis García Guirao, Sergei D. Odintsov and Calogero Vetro

Received: 3 January 2023
Revised: 21 January 2023
Accepted: 25 January 2023
Published: 3 February 2023

Copyright: © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction and Preliminaries

To propose and pose solutions to a variety of differential and integro-differential equations in the areas of mathematics, biomedical sciences, engineering physics and approximation theory, numerous sequences of polynomials came into play as a vital role such as Bessel polynomials, tangent polynomials, Laguerre and Chebyshev polynomials, the Hermite, Jacobi and Legendre polynomials. A vast and the largest family of sequences of polynomials is, of course, the Appell polynomial family [1]. Extensive research is being carried out on the Appell family due to its significance in numerous areas of mathematical, biological and engineering sciences. These polynomials form a commutative group under the operation of composition and, thus, increase their significance in the areas of linear and abstract algebra.

Appell [1], denoted by $Q_m(u)$, was introduced in the 18th century, designated by the expression:

$$m\, Q_{m-1}(u) = \frac{d}{du} Q_m(u), \qquad m \in \mathbb{N}_0 \tag{1}$$

and specified by the designated generating expression as:

$$\sum_{k=0}^{\infty} Q_k(u) \frac{t^k}{k!} = Q(t) \exp(ut), \tag{2}$$

where, $Q(t)$ is a convergent power series, whose expansion in Taylors' form is listed as:

$$\sum_{k=0}^{\infty} Q_k \frac{t^k}{k!} = Q(t), \qquad Q_0 \neq 0. \tag{3}$$

A notable evolution was seen in recent years with the induction of multi-variable and index functions in polynomial families of special functions. To answer the challenges arising in a variety of mathematical fields, such as mathematical physics, engineering mathematics, approximation and automata theory and abstract algebra, multi-variate and indices of special functions are required. Over the past few years, a number of generalizations of mathematical physics, including special functions, have evolved considerably. Hermite [2]

himself first devised the notion of multiple-index, multiple-variable Hermite polynomials and degenerate hybrid polynomials [3–7].

Several writers, including S. A. Wani et al. [8–14], made an effort to introduce a hybrid family connected to Appell sequences of special polynomials. The authors developed the hybrid form of Appell polynomials and investigated a number of their characteristics, including their generating function, the definition of series, summation formulae, determinant forms, approximation features, etc.

The first effort to introduce 3-variable degenerate Hermite Kampé de Fériet polynomials and study their various properties was undertaken by Kyung-Won Hwang et al. [15]. The following generating function defines the $\mathcal{H}m(u,v,w;\eta)$, as listed below:

$$(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}} = \sum_{m=0}^{\infty}\mathcal{H}_m(u,v,w;\eta)\frac{t^m}{m!}. \tag{4}$$

The 3-variable degenerate Hermite polynomials $\mathcal{H}_m(u,v,w;\eta)$, given in (4), are the solution of the expression listed below:

$$\begin{aligned} \tfrac{\partial}{\partial v}\mathcal{H}_m(u,v,w;\eta) &= \left(\tfrac{\eta}{log(1+\eta)}\right)\tfrac{\partial^2}{\partial u^2}\{\mathcal{H}_m(u,v,w;\eta)\}, \\ \tfrac{\partial}{\partial w}\mathcal{H}_m(u,v,w;\eta) &= \left(\tfrac{\eta}{log(1+\eta)}\right)^2\tfrac{\partial^3}{\partial u^3}\{\mathcal{H}_m(u,v,w;\eta)\} \\ \textit{and} & \\ \mathcal{H}_m(u,0,0;\eta) &= \left(\tfrac{log(1+\eta)}{\eta}\right)^m\{u^m\}. \end{aligned} \tag{5}$$

Since

$$(1+\eta)^{\frac{t}{\eta}} \ \to e^t, \ \ as \ \ \eta \ \to \ 0. \tag{6}$$

Thus, in view of the above, expression (4) reduces to 3-variable Hermite polynomials [16] given by expression:

$$\sum_{m=0}^{\infty}H_m(u,v,w)\,\frac{t^m}{m!} = \exp\left(ut+vt^2+wt^3\right). \tag{7}$$

Dattoli [17] refined the monomiality principle, which was proposed in 1941 by Steffensen [18], by two remarkable operators, which satisfy the following expressions:

$$c_{m+1}(u) = \hat{\mathcal{M}}\{c_m(u)\} \tag{8}$$

and

$$m\,c_{m-1}(u) = \hat{\mathcal{D}}\{c_m(u)\} \tag{9}$$

and, therefore, were given the name of multiplicative and derivative operators for a polynomial set $\{c_m(u)\}_{m\in\mathbb{N}}$.

Thus, polynomials $\{c_m(u)\}_{m\in\mathbb{N}}$ are referred to as a quasi-monomial if they obey the formula:

$$\hat{1} = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = [\hat{\mathcal{D}},\hat{\mathcal{M}}], \tag{10}$$

and is, therefore, designated as a Weyl group structure.

The $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ expressions are significant from the point of view to determine the properties of $\{c_m(u)\}_{m\in\mathbb{N}}$ when it is quasi-monomial. Therefore, the succeeding axioms hold:

(i) If $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ possesses differential realizations, then the polynomial $c_m(u)$ satisfies the differential equation by the expression:

$$m\,c_m(u) = \hat{\mathcal{M}}\hat{\mathcal{D}}\{c_m(u)\}. \tag{11}$$

(ii) The expression:

$$\hat{\mathcal{M}}^m\,\{1\} = c_m(u), \tag{12}$$

is said to be an explicit form of the underlying polynomial, while taking, $c_0(u) = 1$.

(iii) Further, in view of (12), the expression:

$$\sum_{m=0}^{\infty} c_m(u)\frac{t^m}{m!} = e^{t\hat{\mathcal{M}}}\{1\}, \quad |t| < \infty \ , \tag{13}$$

derived is called the exponential generating relation.

The techniques or approaches of operational formalism are very much significant in numerous areas varying from mathematical physics to classical and quantum optics, engineering mathematics to lie-algebras. Therefore, due to such importance and significance, these rules prove beneficial for research; see for example [8,19,20].

Thus, in view of Equations (8) and (9), we derived the multiplicative and derivative operators by differentiating the expression (4) w.r.t. t and u, respectively, and they are thus given by the expressions:

$$\begin{aligned} \mathcal{H}_{m+1}(u,v,w;\eta) &= \hat{\mathcal{M}}_{\mathcal{H}}\{\mathcal{H}_m(u,v,w;\eta)\} \\ &= \left(u\frac{log(1+\eta)}{\eta} + 2v\frac{\partial}{\partial u} + 3w\left(\frac{\eta}{log(1+\eta)}\right)\frac{\partial^2}{\partial u^2}\right)\{\mathcal{H}_m(u,v,w;\eta)\} \end{aligned} \tag{14}$$

and

$$m\ \mathcal{H}_{m-1}(u,v,w;\eta) = \hat{\mathcal{D}_{\mathcal{H}}}\{\mathcal{H}_m(u,v,w;\eta)\} = \frac{\eta}{log(1+\eta)}\frac{\partial}{\partial u}\{\mathcal{H}_m(u,v,w;\eta)\}. \tag{15}$$

Additionally, in view of (11), we derive the expression for the differential equation by making use of expressions (14) and (15) as:

$$\left(u\frac{\partial}{\partial u} + 2\ v\ \frac{\eta}{log(1+\eta)}\frac{\partial^2}{\partial u^2} + 3\ w\ (\frac{\eta}{log(1+\eta)})^2\ \frac{\partial^3}{\partial u^3} - m\right)\{\mathcal{H}_m(u,v,w;\eta)\} = 0. \tag{16}$$

Concerning the significance of these results, regenerated and motivated by them, here, 3-variable degenerate Hermite-based Appell polynomials ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$, which are given by the relation:

$$\mathcal{Z}(t,u,v,w;\eta) = \sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = Q(t)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}, \tag{17}$$

are constructed by proving the above result and their several properties such as quasi-monomiality are deduced. The article is presented in the form of sections as listed: in Section 2, we construct ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ polynomials and obtain some of their significant and basic properties by taking the help of multiplicative and derivative operators derived in Section 1. In Section 3, symmetry identities for ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ polynomials are obtained. In Section 4, the operational rule for ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ polynomials are constructed, and then a few examples are given in the last section.

2. 3-Variable Degenerate Hermite-Based Appell Polynomials

To generate and frame the 3-variable degenerate Hermite-based Appell polynomials represented by the notation ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ in the context of the monomiality principle, the succeeding results are proved:

Theorem 1. *For the 3-variable degenerate Hermite-based Appell polynomials, represented by the notation* ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$*, the succeeding relation holds:*

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = Q(t)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}. \tag{18}$$

Proof. In view of expression (12), we have

$${}_{\mathcal{H}}Q_m(u,v,w;\eta) = \left(u\frac{log(1+\eta)}{\eta} + 2v\frac{\partial}{\partial u} + 3w\Big(\frac{\eta}{log(1+\eta)}\Big)\frac{\partial^2}{\partial u^2}\right)^m\{1\}, \tag{19}$$

and in view of expression (13), we have

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = \exp\left(t\left(u\frac{log(1+\eta)}{\eta} + 2v\frac{\partial}{\partial u} + 3w\Big(\frac{\eta}{log(1+\eta)}\Big)\frac{\partial^2}{\partial u^2}\right)\right)\{1\}, \tag{20}$$

thus, the result (18). □

Theorem 2. *For the polynomials* ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$*, the succeeding generating expression holds true:*

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = Q(t)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}, \tag{21}$$

or

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = Q(t)(1+\eta)^{\frac{ut+vt^2+wt^3}{\eta}}, \tag{22}$$

respectively.

Proof. Changing u in Equation (2) by $\hat{\mathcal{M}}_{\mathcal{H}}$, that is, the multiplicative operator of the polynomials $\mathcal{H}_m(u,v,w;\eta)$, we have

$$Q(t)\exp\big(\hat{\mathcal{M}}_{\mathcal{H}}\, t\big) = \sum_{m=0}^{\infty} Q_m(\hat{\mathcal{M}}_{\mathcal{H}})\frac{t^m}{m!}. \tag{23}$$

Making use of expression $\hat{\mathcal{M}}_{\mathcal{H}}$ represented by expression (14), we obtain

$$Q(t)\exp\left(\left(\frac{(u+2vt+3wt^2)log(1+\eta)}{\eta}\right)t\right) = \sum_{m=0}^{\infty} Q_m\left(\frac{(u+2vt+3wt^2)log(1+\eta)}{\eta}\right)\frac{t^m}{m!}. \tag{24}$$

Denoting the 3-variable degenerate Hermite-based Appell polynomials in the r.h.s. of expression (24) by ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$,

$$Q_m\left(\frac{(u+2vt+3wt^2)log(1+\eta)}{\eta}\right) = {}_{\mathcal{H}}Q_m(u,v,w;\eta) \tag{25}$$

and the ist exponential in the l.h.s. of expression (24) is expanded on usage of expression (4), assertion (21) or assertion (22) is established. □

Theorem 3. *The succeeding multiplicative and derivative operators for the polynomials* ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$*, holds:*

$$\hat{\mathcal{M}_{\mathcal{H}}Q} = \left(\frac{Q'(t)}{Q(t)} + u\frac{log(1+\eta)}{\eta} + 2v\frac{\partial}{\partial u} + 3w\Big(\frac{\eta}{log(1+\eta)}\Big)\frac{\partial^2}{\partial u^2}\right) \tag{26}$$

and

$$\mathcal{D}_{\hat{\mathcal{H}}}Q = \frac{\eta}{log(1+\eta)}\, D_u, \tag{27}$$

respectively.

Proof. Differentiating (22) with respect to t partially and simplifying, we have

$$\left[\frac{Q'(t)}{Q(t)} + \hat{\mathcal{M}}_{\mathcal{H}}\right] Q(t) \exp(\hat{\mathcal{M}}_{\mathcal{H}}\, t) = \sum_{m=0}^{\infty} m\, Q_m(\hat{\mathcal{M}}_{\mathcal{H}}) \frac{t^{m-1}}{m!}. \tag{28}$$

Replacing m by $m+1$ in the r.h.s. of the above expression and simplifying, we find

$$\left[\frac{Q'(t)}{Q(t)} + \hat{\mathcal{M}}_{\mathcal{H}}\right] Q(t) \exp(\hat{\mathcal{M}}_{\mathcal{H}}\, t) = \sum_{m=0}^{\infty} Q_{m+1}(\hat{\mathcal{M}}_{\mathcal{H}}) \frac{t^{m}}{m!}. \tag{29}$$

Inserting expressions (14), (25) on both sides of the above expression (29), it follows that

$$\begin{array}{c} \left[\frac{Q'(t)}{Q(t)} + u\frac{log(1+\eta)}{\eta} + 2v\frac{\partial}{\partial u} + 3w\left(\frac{\eta}{log(1+\eta)}\right)\frac{\partial^2}{\partial u^2}\right] \\ \times \sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = \sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_{m+1}(u,v,w;\eta)\frac{t^m}{m!}. \end{array} \tag{30}$$

In view of Equation (5), equating the coefficients of like exponents of t on both sides of the above expression, we are led to the result (26).

Again, differentiating expression (21) w.r.t. u, it follows that

$$D_u\left\{Q(t)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}\right\} = \frac{t}{\eta}\, log(1+\eta)\left\{Q(t)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}\right\}. \tag{31}$$

Using the generating expression (21) on both sides of the above expression and comparing the coefficients of like exponents of t on both sides of the above expression, we are led to the result (27). □

Remark 1. *Using expressions* (26) *and* (27) *in Equation* (11)*, we find the following differential equation for the 3-variable degenerate Hermite-based Appell polynomials* ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$*:*

$$\begin{array}{c} \left[\left(u + \frac{Q'(t)}{Q(t)}\frac{\eta}{log(1+\eta)}\right)\frac{\partial}{\partial u} + 2v\left(\frac{\eta}{log(1+\eta)}\right)\frac{\partial^2}{\partial u^2} + 3w\left(\frac{\eta}{log(1+\eta)}\right)^2\frac{\partial^3}{\partial u^3} - m\right] \\ \times {}_{\mathcal{H}}Q_m(u,v,w;\eta) = 0. \end{array} \tag{32}$$

Next, a series representation of the polynomials ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ is derived:

Theorem 4. *The following series expansion:*

$${}_{\mathcal{H}}Q_m(u,v,w;\eta) = \sum_{k=0}^{m} \binom{m}{k} \mathcal{H}_{m-k}(u,v,w;\eta)\, Q_k, \tag{33}$$

for the polynomials ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ *holds true.*

Proof. Using Equations (2) with $u = 0$ and (4) in the l.h.s. of the expression (21), we find

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}Q_m(u,v,w;\eta)\frac{t^m}{m!} = \sum_{k=0}^{\infty} Q_k \frac{t^k}{k!} \sum_{m=0}^{\infty} \mathcal{H}_m(u,v,w;\eta)\frac{t^m}{m!} \tag{34}$$

and thus on the usage of the c.P. rule in the r.h.s. of the above expression, assertion (33) is proved. □

3. Symmetric Identities

In this section, we give symmetric identities for polynomials ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$.

Theorem 5. *For, $\alpha,\beta>0$ and $\alpha\neq\beta$, it follows that*

$$\alpha^m\,{}_{\mathcal{H}}Q_m(\beta u,\beta^2 v,\beta^3 w;\eta)=\beta^m\,{}_{\mathcal{H}}Q_m(\alpha u,\alpha^2 v,\alpha^3 w;\eta). \tag{35}$$

Proof. Since, $\alpha,\beta>0$ and $\alpha\neq\beta$, we can start by writing:

$$\mathcal{R}(t;u,v,w;\eta)=Q(t)\,(1+\eta)^{\frac{\alpha\beta ut}{\eta}}(1+\eta)^{\frac{\alpha^2\beta^2 vt^2}{\eta}}(1+\eta)^{\frac{\alpha^3\beta^3 wt^3}{\eta}}. \tag{36}$$

Therefore, the above expression $\mathcal{R}(t;u,v,w;\eta)$ is symmetric in α and β.

We further can write

$$\mathcal{R}(t;u,v,w;\eta)={}_{\mathcal{H}}Q_m(\alpha u,\alpha^2 v,\alpha^3 w;\eta)\frac{(\beta t)^m}{m!}=\beta^m\,{}_{\mathcal{H}}Q_m(\alpha u,\alpha^2 v,\alpha^3 w;\eta)\frac{t^m}{m!}. \tag{37}$$

Thus, it follows that

$$\mathcal{R}(t;u,v,w;\eta)={}_{\mathcal{H}}Q_m(\beta u,\beta^2 v,\beta^3 w;\eta)\frac{(\alpha t)^m}{m!}=\alpha^m\,{}_{\mathcal{H}}Q_m(\beta u,\beta^2 v,\beta^3 w;\eta)\frac{t^m}{m!}. \tag{38}$$

Equating the coefficients of like term of t in the last two equations, we obtain the assertion (30). □

Theorem 6. *For $\alpha,\beta>0$ and $\alpha\neq\beta$, it follows that*

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\alpha^i\beta^{m+1-i}\mathcal{Q}_n(\eta)$$

$${}_{\mathcal{H}}Q_{i-n}(\beta u,\beta^2 v,\beta^3 w;\eta)\mathcal{P}_{m-i}(\alpha-1;\eta)=$$

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\beta^i\alpha^{m+1-i}\mathcal{Q}_n(\eta)\,{}_{\mathcal{H}}Q_{i-n}(\alpha u,\alpha^2 v,\alpha^3 w;\eta)\mathcal{P}_{m-i}(\beta-1;\eta). \tag{39}$$

Proof. Since $\alpha,\beta>0$ and $\alpha\neq\beta$, we can start by writing:

$$\mathcal{S}(t;u,v,w;\eta)=Q(t)\,\frac{\alpha\beta t(1+\eta)^{\frac{\alpha\beta ut}{\eta}}(1+\eta)^{\frac{\alpha^2\beta^2 vt^2}{\eta}}(1+\eta)^{\frac{\alpha^3\beta^3 wt^3}{\eta}}\left((1+\eta)^{\frac{\alpha\beta t}{\eta}-1}\right)}{\left((1+\eta)^{\frac{\alpha t}{\eta}}-1\right)\left((1+\eta)^{\frac{\beta t}{\eta}}-1\right)}. \tag{40}$$

Using the same fashion as in the above theorem, we obtain assertion (34). □

Similarly, we can establish other symmetric identities by taking different functions.

4. Operational Formalism

As we are well aware, operational techniques are extensively used to generate new polynomial families of the doped type so that these are connected to the regular and generalized special functions easily. These techniques are utilized to form new hybrid special polynomials whose properties lie within the parental polynomial.

Differentiating successively (21) w.r.t. u,v,w, we find

$$D_u\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\}=\frac{log(1+\eta)}{\eta}\,m\,\Big\{{}_{\mathcal{H}}Q_{m-1}(u,v,w;\eta)\Big\}, \tag{41}$$

$$D_u^2\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} = \Big(\frac{log(1+\eta)}{\eta}\Big)^2 m(m-1)\Big\{{}_{\mathcal{H}}Q_{m-1}(u,v,w;\eta)\Big\}, \tag{42}$$

$$D_v\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} = \frac{log(1+\eta)}{\eta} m(m-1)\Big\{{}_{\mathcal{H}}Q_{m-1}(u,v,w;\eta)\Big\}, \tag{43}$$

and

$$D_w\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} = \frac{log(1+\eta)}{\eta} m(m-1)(m-2)\Big\{{}_{\mathcal{H}}Q_{m-1}(u,v,w;\eta)\Big\}, \tag{44}$$

respectively.

In view of expressions (41)–(44), we observe that ${}_{\mathcal{H}}Q_m(u,v,w;\eta)$ are the solutions of the expressions:

$$\frac{\eta}{log(1+\eta)} D_u^2\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} = D_v\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} \tag{45}$$

and

$$\Big(\frac{\eta}{log(1+\eta)}\Big)^2 D_u^3\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\} = D_w\Big\{{}_{\mathcal{H}}Q_m(u,v,w;\eta)\Big\}, \tag{46}$$

respectively, under the following initial condition

$${}_{\mathcal{H}}Q_m(u,0,0;\eta) = Q_m(u;\eta). \tag{47}$$

Thus, from expressions (45)–(47), it follows that

$${}_{\mathcal{H}}Q_m(u,v,w;\eta) = \exp\left(\frac{v\,\eta}{log(1+\eta)} D_u^2 + w\Big(\frac{\eta}{log(1+\eta)}\Big)^2 D_u^3\right)\{Q_m(u)\}. \tag{48}$$

In light of the aforementioned viewpoint, the polynomials ${}_{\mathcal{H}}Q_m(u,v,w:\eta)$ can be constructed from the Appell type degenerate polynomials $Q_m(u;\eta)$ by employing the operational rule (48).

5. Examples

Depending on the choice of function $Q(t)$, numerous members of the Appell family can be obtained:

The generating function for the Bernoulli polynomials $c_m(u)$ is given by ([21], p. 36)

$$\Big(\frac{t}{e^t-1}\Big)e^{ut} = \sum_{m=0}^{\infty} c_m(u)\frac{t^m}{m!}, \qquad |t| < 2\pi, \tag{49}$$

where $c_k := c_k(0)$ are called Bernoulli numbers.

The generating function for the Euler polynomials $E_m(u)$ is given by ([21], p. 40)

$$\Big(\frac{2}{e^t+1}\Big)e^{ut} = \sum_{m=0}^{\infty} E_m(u)\frac{t^m}{m!}, \qquad |t| < \pi, \tag{50}$$

where $E_k := 2^k E_k\Big(\frac{1}{2}\Big)$ are called Euler numbers.

The generating function for the Genocchi polynomials $G_m(u)$ is given by [22]

$$\Big(\frac{2t}{e^t+1}\Big)e^{ut} = \sum_{m=0}^{\infty} G_m(u)\frac{t^m}{m!}, \qquad |t| < \pi, \tag{51}$$

where $G_k := G_k(0)$ are called Genocchi numbers.

A large variety of applications in advanced number theory, biomedical sciences and engineering mathematics such as numerical analysis and actuarial mathematics make use of these above-mentioned polynomials extensively. The powers of natural numbers, the binomial expansion and the Taylor expansion are some of the few examples where the Bernoulli numbers can be found. In close coordination with the trigonometric and hyperbolic cotangent are a few examples, where the Euler numbers can be seen and enter the Taylor expansion, while Genocchi numbers and tangent numbers are significant in areas of graph theory and automata theory.

Thus, for suitable selection of $Q(t)$ in (21), the expressions that we called generating expressions for 3-variable degenerate Hermite-based Bernoulli, Euler and Genocchi polynomials hold:

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}c_m(u,v,w;\eta)\frac{t^m}{m!} = \left(\frac{t}{e^t-1}\right)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}, \tag{52}$$

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}E_m(u,v,w;\eta)\frac{t^m}{m!} = \left(\frac{2}{e^t+1}\right)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}, \tag{53}$$

and

$$\sum_{m=0}^{\infty} {}_{\mathcal{H}}G_m(u,v,w;\eta)\frac{t^m}{m!} = \left(\frac{2t}{e^t+1}\right)(1+\eta)^{\frac{ut}{\eta}}(1+\eta)^{\frac{vt^2}{\eta}}(1+\eta)^{\frac{wt^3}{\eta}}, \tag{54}$$

respectively.

Additionally, in view of Equation (33), these polynomials satisfy the following series form:

Remark 2. *The 3-variable degenerate Hermite-based Bernoulli, Euler and Genocchi polynomials satisfy the following explicit form:*

$${}_{\mathcal{H}}c_m(u,v,w;\eta) = \sum_{k=0}^{m}\binom{m}{k}\mathcal{H}_{m-k}(u,v,w;\eta)\,c_k, \tag{55}$$

$${}_{\mathcal{H}}E_m(u,v,w;\eta) = \sum_{k=0}^{m}\binom{m}{k}\mathcal{H}_{m-k}(u,v,w;\eta)\,E_k, \tag{56}$$

and

$${}_{\mathcal{H}}G_m(u,v,w;\eta) = \sum_{k=0}^{m}\binom{m}{k}\mathcal{H}_{m-k}(u,v,w;\eta)\,G_k, \tag{57}$$

respectively

Similarly, other corresponding results can be established for these polynomials.

6. Conclusions

In this paper, we study the general properties and identities of the 3-variable degenerate Hermite-based Appell polynomials by convoluting Appell and degenerate 3-variable Hermite polynomials. These presented results can be applied in any 3-variable degenerate Hermite-based Appell type polynomials, such as Bernoulli, Euler, Genocchi and tangent polynomials. Further, we established their quasi-monomial properties and operational rule. Additionally, symmetric identities are given.

Further, future observations may be to derive the determinant forms and summation formulae for these polynomials.

Funding: The author was supported by Researchers Supporting Project number (RSPD2023R640), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the referees for their comments and suggestions, which helped to improve the original manuscript greatly.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Appell, P. Sur Une Classe de Polynômes. *Ann. Sci. École. Norm. Sup.* **1880**, *9*, 119–144. [CrossRef]
2. Hermite, C. Sur un nouveau dvelopment en sries de functions. *Compt. Rend. Acad. Sci. Paris* **1864**, *58*, 93–100.
3. Ryoo, C.S. Notes on degenerate tangent polynomials. *Glob. J. Pure Appl. Math.* **2015**, *11*, 3631–3637.
4. Hwang, K.W.; Ryoo, C.S. Differential equations associated with two variable degenerate Hermite polynomials. *Mathematics* **2020**, *8*, 228. [CrossRef]
5. Kim, T. A Note on the Degenerate Type of Complex Appell Polynomials. *Symmetry* **2019**, *11*, 1339. [CrossRef]
6. Kim, T.; Yao, Y.; Kim, D.S.; Jang, G.-W. Degenerate r-Stirling numbers and r-Bell polynomials. *Russ. J. Math. Phys.* **2018**, *25*, 44–58. [CrossRef]
7. Kim, D.S.; Kim, T.; Lee, H. A note on degenerate Euler and Bernoulli polynomials of complex variable. *Symmetry* **2019**, *11*, 1168. [CrossRef]
8. Wani, S.A.; Khan, S. Properties and applications of the Gould-Hopper-Frobenius-Euler polynomials. *Tbil. Math. J.* **2019**, *12*, 93–104. [CrossRef]
9. Wani, S.A.; Khan, S.; Naikoo, S. Differential and integral equations for the Laguerre-Gould-Hopper based Appell and related polynomials. *Boletãn Soc. Matemãtica Mex.* **2019**, *26*, 617–646. [CrossRef]
10. Khan, S.; Riyasat, M.; Wani, S.A. On some classes of differential equations and associated integral equations for the Laguerre-Appell polynomials. *Adv. Pure Appl. Math.* **2017**, *5*, 185–194. [CrossRef]
11. Khan, S.; Wani, S.A. Differential and integral equations associated with some hybrid families of Legendre polynomials. *Tbil. Math. J.* **2018**, *11*, 127–139.
12. Khan, S.; Wani, S.A. A note on Differential and integral equations for the Laguerre-Hermite polynomials. In Proceedings of the Second International Conference on Computer and Communication Technologies, IC3T 2017. *Adv. Intell. Syst. Comput.* **2016**, *381*, 547–555.
13. Khan, S.; Wani, S.A. A note on differential and integral equations for the Legendre-Hermite polynomials. *Int. J. Adv. Res. Sci. Eng.* **2018**, *7*, 514–520.
14. Araci, S.; Riyasat, M.; Wani, S.A.; Khan, S. Differential and integral equations for the 3-variable Hermite-Frobenius-Euler and Frobenius-Genocchi polynomials. *App. Math. Inf. Sci.* **2017**, *11*, 1335–1346. [CrossRef]
15. Hwang, K.W.; Seol, Y.; Ryoo, C.S. Explicit Identities for 3-Variable Degenerate Hermite Kampe deFeriet Polynomials and Differential Equation Derived from Generating Function. *Symmetry* **2021**, *13*, 7. [CrossRef]
16. Dattoli, G. Generalized polynomials operational identities and their applications. *J. Comput. Appl. Math.* **2000**, *118*, 111–123. [CrossRef]
17. Dattoli, G. Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, Advanced Special functions and applications, (Melfi, 1999). *Melfi Sch. Adv. Top. Math. Phys.* **2000**, *1*, 147–164.
18. Steffensen, J.F. The poweriod, an extension of the mathematical notion of power *Acta Math.* **1941**, *73*, 333–366. [CrossRef]
19. Appell, P.; Kampé de Fériet, J. *Fonctions Hypergéométriques et Hypersphériques: Polynômes d'Hermite*; Gauthier-Villars: Paris, France, 1926.
20. Bretti, G.; Cesarano, C.; Ricci, P.E. Laguerre-Type Exponentials and Generalized Appell Polynomials *Comput. Math. Appl.* **2004**, *48*, 833–839.
21. Erdélyi, A., Magnus, W.; Oberhettinger, F. Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA, 1955; Volume III.
22. Sandor, J.; Crstici, B. *Handbook of Number Theory*; Kluwer Academic: London, UK, 2004; Volume II.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.