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A Comparative Analysis of Fractional-Order Fokker–Planck Equation

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Abstract: The importance of partial differential equations in physics, mathematics and engineering cannot be emphasized enough. Partial differential equations are used to represent physical processes, which are then solved analytically or numerically to examine the dynamical behaviour of the system. The new iterative approach and the Homotopy perturbation method are used in this article to solve the fractional order Fokker–Planck equation numerically. The Caputo sense is used to characterize the fractional derivatives. The suggested approach's accuracy and applicability are demonstrated using illustrations. The proposed method's accuracy is expressed in terms of absolute error. The proposed methods are found to be in good agreement with the exact solution of the problems using graphs and tables. The results acquired using the given approaches are also obtained at various fractional orders of the derivative. It is observed from the graphs and tables that fractional order solutions converge to an integer solution when the fractional orders approach the integer-order of the problems. The tabular and graphical view for the given problems is obtained through Maple. The presented approaches can be applied to existing non-linear fractional partial differential equations due to their accurate, simple and straightforward implementation.

Keywords: Elzaki transform; new iterative method; Caputo derivatives; homotopy perturbation method; Fokker–Planck equation

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1. Introduction

The theory of fractional calculus (FC) has received a lot of interest in recent years because of its applications to complex systems. The simulation of significant-world issues employing fractional order derivatives gives higher accuracy than modeling involving integer-order derivatives, according to fractional derivative principles. FC refers to the background and non-local dispersed effects of any physical system in phenomena such as wave motion analysis, solitary waves, phase turbulence in reaction–diffusion schemes [1–4], chaotic drifting waves induced by photon collision [5], wrinkled flame front propagation [6], time fractional-coupled mKdV equation [7–9], fractional order wave equations [10] and fractional space-time diffusion equation [11–13]. Fractional Differential Equations (FDEs) have received a lot of interest from mathematicians because they enable fractional modeling of various natural processes [14–16]. As a result, the use of FDEs to represent many physical systems and processes has increased, such as coloured noise [17], economics [18], earthquake oscillation [19] and bioengineering [20]. Rheology [21], control theory [22], visco-elastic materials [23], signal processing [24], polymers [25], damping method [26] and so on are some of the additional applications [27–30]. The fractional differential equation is an effective tool for modeling nonlinear phenomena in scientific and engineering models. In applied mathematics and engineering, partial differential equations, especially nonlinear ones, have been utilized to model a vast array of scientific phenomena [31–33]. Parallel to their work in the physical sciences, researchers were able to identify and model a vast array of relevant and real-world physical problems using fractional order partial differential equations (FPDEs). It has long been asserted how crucial it is to establish estimates for them using numerical or analytical techniques. Consequently, symmetry analysis is a useful method for comprehending partial differential equations, particularly when examining equations generated from accounting-related mathematical concepts [34–36]. Even though symmetry is the cornerstone of nature, the majority of natural observations lack symmetry. A sophisticated strategy for concealing symmetry is the appearance of unexpected symmetry-breaking events. Two types of symmetry exist: finite and infinitesimal. Both discrete and continuous finite symmetries exist. Space is a continuous transformation, but parity and temporal inversion are discrete natural symmetries. Patterns have always intrigued mathematicians. In the seventeenth century, classification of spatial and planar patterns gained significant traction. Unfortunately, exact solution of fractional nonlinear differential equations has proven to be quite difficult.

The Fokker–Planck equation (FPE) was developed by Adriaan Fokker and Max Planck to describe the time evolution of the probability density function of a particle's position and velocity and it is one of the most extensively used statistical physics equations [37]. FPE appears in a variety of natural science domains; Brownian motion [38] and the diffusion model of chemical reactions [39] are now widely used in physics, chemistry, engineering and biology in various modified forms. The FPE first appears in kinetic theory [40], where it represents the behavior of one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. Some applications of this type of equation can be found in research by He and Wu [41], Jumarie [42], Kamitani and Matsuba [43], Xu et al. [44] and Zak [45].

Among these applications, we considered Fokker–Planck equations of fractional order with the general form

$$\xi_{\vartheta}^{\beta}(\phi,\vartheta) = L\big(\xi_{\phi}(\phi,\vartheta) + \xi_{\phi\phi}(\phi,\vartheta)\big) + N\xi_{\phi\phi}(\phi,\vartheta), \;\; \phi,\vartheta > 0, \; \beta \in (0,1], \tag{1}$$

with initial source

$$\xi(\phi,0) = \zeta(\phi).$$

The function (ϕ,ϑ) is assumed to be a causal function of time and space, i.e., vanishing for $\phi < 0$ and $\vartheta < 0$, β is the parameter describing the order of the fractional time and space derivative. Fokker and Planck proposed the Fokker–Planck equation (Equation (1)) to describe brownian motion of particles [46]. The Fokker–Plank equation, which explains solute transport, depicts the change in probability of a random function in time and space. PDEs of both time and space fractional order are used to describe a variety of phenomena, including wave propagation, continuous random walks, charge carrier transport in amorphous semiconductors, anomalous diffusion, ribosome mobility along mRNA and pattern generation in polymeric networks [47]. In the current study [48], we use both the innovative iterative approach offered by Gejji and Jafari [49] and the homotopy perturbation transform method proposed by Madani et al. [50], Khan and Wu. Daftardar-Gejji and Jafari introduced a new iterative method for finding numerical solutions to nonlinear functional equations in 2006 [51]. Many nonlinear differential equations of integer and fractional order [52] and fractional boundary value problems have been solved using the iterative method. In a simple manner, the second strategy combines the Elzaki transformation, the homotopy perturbation method and He's polynomials. He [53,54] invented the homotopy perturbation technique (HPM), which is a series expansion approach for solving nonlinear partial differential equations. To ensure convergence of approximation series over a certain interval of physical parameters, the HPM employs a so-called convergence-control parameter.

The rest of this work is arranged in the following manner. The Abel–Riemann fractional derivative, Caputo fractional derivative and Elzaki transform are all defined in Section 1. The new iterative transform method for solving fractional partial differential equations is described in Section 2. The Homotopy perturbation transform is described in Section 3 for solving fractional partial differential equations. In Section 4, we show five examples of how the approaches can be used to solve Fokker–Planck equations. Section 5 presents the conclusion.

2. Basic Definitions

2.1. Definition

The fractional derivative D^{β} in the Abel–Riemann sense having order β is given as [55]

$$D^{\beta}\mu(\varphi) = \begin{cases} \frac{d^{\kappa}}{d\varphi^{\kappa}}\mu(\varphi), & \beta = \kappa, \\ \frac{1}{\Gamma(\kappa-\beta)}\frac{d}{d\varphi^{\kappa}}\int_0^{\varphi}\frac{\mu(\varphi)}{(\varphi-\phi)^{\beta-\kappa+1}}d\phi, & \kappa-1<\beta<\kappa, \end{cases}$$

where $\kappa \in Z^{+}$, $\beta \in R^{+}$ and

$$D^{-\beta}\mu(\varphi) = \frac{1}{\Gamma(\beta)}\int_0^{\varphi}(\varphi-\phi)^{\beta-1}\mu(\phi)d\phi, \ \ 0<\beta\leq 1.$$

2.2. Definition

The fractional integration operator κ^{ϕ} in Abel–Riemann sense is defined as [55]

$$\kappa^{\beta}\mu(\varphi) = \frac{1}{\Gamma(\beta)}\int_0^{\varphi}(\varphi-\phi)^{\beta-1}\mu(\varphi)d\varphi, \ \ \varphi>0, \ \ \beta>0.$$

having properties:

$$\begin{aligned} \kappa^{\beta}\varphi^{\kappa} &= \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\beta+1)}\varphi^{\kappa+\phi}, \\ D^{\beta}\varphi^{\kappa} &= \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\beta+1)}\varphi^{\kappa-\phi}. \end{aligned}$$

2.3. Definition

The Caputo derivative D^{β} of fractional order β is given as [55]

$$^{C}D^{\beta}\mu(\varphi) = \begin{cases} \frac{1}{\Gamma(\kappa-\beta)}\int_0^{\varphi}\frac{\mu^{\kappa}(\phi)}{(\varphi-\phi)^{\beta-\kappa+1}}d\phi, & \kappa-1<\beta<\kappa, \\ \frac{d^{\kappa}}{d\varphi^{\kappa}}\mu(\varphi), & \kappa=\beta. \end{cases} \tag{2}$$

with the properties

$$\begin{aligned} &\kappa_{\varphi}^{\beta}D_{\varphi}^{\beta}g(\varphi) = g(\varphi) - \sum_{k=0}^{m} g^{k}(0^{+})\frac{\varphi^{k}}{k!}, \ \ for \ \ \varphi>0, \ \ and \ \kappa-1<\beta\leq\kappa, \ \ \kappa\in N. \\ &D_{\varphi}^{\beta}\kappa_{\varphi}^{\beta}g(\varphi) = g(\varphi). \end{aligned} \tag{3}$$

2.4. Definition

The Caputo operator in terms of Elzaki transform is [55]:

$$E[D_{\varphi}^{\beta}g(\varphi)] = s^{-\beta}E[g(\varphi)] - \sum_{k=0}^{\kappa-1} s^{2-\beta+k}g^{(k)}(0), \ \ where \ \ \kappa-1<\beta<\kappa.$$

3. Methodology of NITM

Consider fractional order PDE of the form

$$D_{\vartheta}^{\beta}\xi(\phi,\vartheta) + N\xi(\phi,\vartheta) + M\xi(\phi,\vartheta) = h(\phi,\vartheta), \quad n\in N, \ \ n-1<\beta\leq n, \tag{4}$$

subjected to initial condition

$$\xi^k(\phi, 0) = g_k(\phi), \qquad k = 0, 1, 2, ..., n-1, \tag{5}$$

where N and M represents linear and non-linear terms.

By employing Elzaki transform for Equation (4), we obtain

$$E[D_\vartheta^\beta \xi(\phi, \vartheta)] + E[N\xi(\phi, \vartheta) + M\xi(\phi, \vartheta)] = E[h(\phi, \vartheta)]. \tag{6}$$

By employing the Elzaki differentiation property

$$E[\xi(\phi, \vartheta)] = \sum_{k=0}^{m} s^{2-\beta+k} u^{(k)}(\phi, 0) + s^\beta E[h(\phi, \vartheta)] - s^\beta E[N\xi(\phi, \vartheta) + M\xi(\phi, \vartheta)], \tag{7}$$

By applying the inverse Elzaki transform to Equation (7),

$$\xi(\phi, \vartheta) = E^{-1}\left[\left\{\sum_{k=0}^{m} s^{2-\beta+k} u^k(\phi, 0) + s^\beta E[h(\phi, \vartheta)]\right\}\right] - E^{-1}\left[s^\beta E[N\xi(\phi, \vartheta) + M\xi(\phi, \vartheta)]\right]. \tag{8}$$

By means of the iterative technique, we have

$$\xi(\phi, \vartheta) = \sum_{m=0}^{\infty} \xi_m(\phi, \vartheta). \tag{9}$$

$$N\left(\sum_{m=0}^{\infty} \xi_m(\phi, \vartheta)\right) = \sum_{m=0}^{\infty} N[\xi_m(\phi, \vartheta)], \tag{10}$$

the non-linear term N is determined as

$$N\left(\sum_{m=0}^{\infty} \xi_m(\phi, \vartheta)\right) = \xi_0(\phi, \vartheta) + N\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right) - M\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right). \tag{11}$$

On putting (9)–(11) into Equation (8), we have

$$\begin{aligned} \sum_{m=0}^{\infty} \xi_m(\phi, \vartheta) &= E^{-1}\left[s^\beta\left(\sum_{k=0}^{m} s^{2-\phi+k} u^k(\phi, 0) + E[h(\phi, \vartheta)]\right)\right] \\ &- E^{-1}\left[s^\beta E\left[N\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right) - M\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right)\right]\right]. \end{aligned} \tag{12}$$

Hence, the iteration formula is defined as

$$\xi_0(\phi, \vartheta) = E^{-1}\left[s^\beta\left(\sum_{k=0}^{m} s^{2-\phi+k} u^k(\phi, 0) + s^\beta E(g(\phi, \vartheta))\right)\right], \tag{13}$$

$$\xi_1(\phi, \vartheta) = -E^{-1}\left[s^\beta E[N[\xi_0(\phi, \vartheta)] + M[\xi_0(\phi, \vartheta)]]\right], \tag{14}$$

$$\xi_{m+1}(\phi, \vartheta) = -E^{-1}\left[s^\beta E\left[-N\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right) - M\left(\sum_{k=0}^{m} \xi_k(\phi, \vartheta)\right)\right]\right], \quad m \geq 1. \tag{15}$$

Thus, the solution for the m-term in the series is obtained by means of Equations (4) and (5)

$$\xi(\phi, \vartheta) \cong \xi_0(\phi, \vartheta) + \xi_1(\phi, \vartheta) + \xi_2(\phi, \vartheta) + ..., + \xi_m(\phi, \vartheta), \quad m = 1, 2, \ldots. \tag{16}$$

4. Methodology of HPTM

Consider the fractional order PDE of the form

$$\begin{aligned} &D_{\vartheta}^{\beta}\xi(\phi,\vartheta)+M\xi(\phi,\vartheta)+N\xi(\phi,\vartheta)=h(\phi,\vartheta), \quad \vartheta>0, \quad 0<\beta\leq 1, \\ &\xi(\phi,0)=g(\phi), \quad \phi\in\Re. \end{aligned} \tag{17}$$

By employing the Elzaki transform in Equation (17), we obtain

$$\begin{aligned} &E[D_{\vartheta}^{\beta}\xi(\phi,\vartheta)+M\xi(\phi,\vartheta)+N\xi(\phi,\vartheta)]=E[h(\phi,\vartheta)], \quad \vartheta>0, 0<\beta\leq 1, \\ &\xi(\phi,\vartheta)=s^2g(\phi)+s^{\beta}E[h(\phi,\vartheta)]-s^{\beta}E[M\xi(\phi,\vartheta)+N\xi(\phi,\vartheta)]. \end{aligned} \tag{18}$$

By applying inverse Elzaki transform, we obtain

$$\xi(\phi,\vartheta)=F(\phi,\vartheta)-E^{-1}\left[s^{\beta}E\{M\xi(\phi,\vartheta)+N\xi(\phi,\vartheta)\}\right], \tag{19}$$

where

$$F(\phi,\vartheta)=E^{-1}\left[s^2g(\phi)+s^{\beta}E[h(\phi,\vartheta)]\right]=g(\phi)+E^{-1}\left[s^{\beta}E[h(\phi,\vartheta)]\right]. \tag{20}$$

For parameter p, the perturbation technique is determined as

$$\xi(\phi,\vartheta)=\sum_{k=0}^{\infty}p^k\xi_k(\phi,\vartheta), \tag{21}$$

here p is the perturbation parameter and $p\in[0,1]$.

The nonlinear components are defined as

$$N\xi(\phi,\vartheta)=\sum_{k=0}^{\infty}p^kH_k(\xi_k), \tag{22}$$

where He's polynomials are represented by H_n with $\xi_0,\xi_1,\xi_2,...,\xi_n$, and are given as

$$H_n(\xi_0,\xi_1,\cdots,\xi_n)=\frac{1}{\beta(n+1)}D_p^k\left[N\left(\sum_{k=0}^{\infty}p^k\xi_k\right)\right]_{p=0}, \tag{23}$$

where $D_p^k=\frac{\partial^k}{\partial p^k}$.

On putting Equations (22) and (23) into Equation (19), we obtain

$$\sum_{k=0}^{\infty}p^k\xi_k(\phi,\vartheta)=F(\phi,\vartheta)-p\times\left[E^{-1}\left\{s^{\beta}E\{M\sum_{k=0}^{\infty}p^k\xi_k(\phi,\vartheta)+\sum_{k=0}^{\infty}p^kH_k(\xi_k)\}\right\}\right]. \tag{24}$$

By comparing both sides of the coefficient of p, we have

$$\begin{aligned} &p^0:\xi_0(\phi,\vartheta)=F(\phi,\vartheta), \\ &p^1:\xi_1(\phi,\vartheta)=E^{-1}\left[s^{\beta}E(M\xi_0(\phi,\vartheta)+H_0(\xi))\right], \\ &p^2:\xi_2(\phi,\vartheta)=E^{-1}\left[s^{\beta}E(M\xi_1(\phi,\vartheta)+H_1(\xi))\right], \\ &\vdots \\ &p^k:\xi_k(\phi,\vartheta)=E^{-1}\left[s^{\beta}E(M\xi_{k-1}(\phi,\vartheta)+H_{k-1}(\xi))\right], \quad k>0, k\in N. \end{aligned} \tag{25}$$

The terms $\xi_k(\phi,\vartheta)$ are easily computable giving convergent series. On taking $p \to 1$,

$$\xi(\phi,\vartheta) = \lim_{M\to\infty} \sum_{k=1}^{M} \xi_k(\phi,\vartheta). \tag{26}$$

4.1. Example

Let us consider the time-fractional Fokker–Planck equation as

$$\frac{\partial^\beta}{\partial\vartheta^\beta}(\xi(\phi,\vartheta)) + \frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi(\phi,\vartheta)\right) - \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi(\phi,\vartheta)\right) = 0, \ \ \phi,\vartheta > 0, \ \beta \in (0,1], \tag{27}$$

subjected to initial condition

$$\xi(\phi,0) = \phi^2 \tag{28}$$

for special value $\beta = 1$; the exact solution is

$$\xi(\phi,\vartheta) = \phi^2 \exp^{\frac{\vartheta}{2}} \tag{29}$$

By employing the Elzaki transform in Equation (27), we have

$$E[\xi(\phi,\vartheta)] = s^2(\phi^2) + s^\beta E\left[-\frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi(\phi,\vartheta)\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi(\phi,\vartheta)\right)\right], \tag{30}$$

By applying the inverse Elzaki transform, we obtain

$$\xi(\phi,\vartheta) = e^{-\phi} + E^{-1}\left(s^\beta E\left[-\frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi(\phi,\vartheta)\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi(\phi,\vartheta)\right)\right]\right). \tag{31}$$

Hence, by implementing NITM, we obtain

$$\xi_0(\phi,\vartheta) = \phi^2,$$
$$\xi_1(\phi,\vartheta) = E^{-1}\left[s^\beta E\left\{-\frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi_0(\phi,\vartheta)\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi_0(\phi,\vartheta)\right)\right\}\right] = \phi^2\frac{\vartheta^\beta}{2\Gamma(\beta+1)},$$
$$\xi_2(\phi,\vartheta) = E^{-1}\left[s^\beta E\left\{-\frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi_1(\phi,\vartheta)\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi_1(\phi,\vartheta)\right)\right\}\right] = \phi^2\frac{(\vartheta^\beta)^2}{4\Gamma(2\beta+1)},$$
$$\xi_3(\phi,\vartheta) = E^{-1}\left[s^\beta E\left\{-\frac{\partial}{\partial\phi}\left(\frac{\phi}{6}\xi_2(\phi,\vartheta)\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{12}\xi_2(\phi,\vartheta)\right)\right\}\right] = \phi^2\frac{(\vartheta^\beta)^3}{8\Gamma(3\beta+1)},$$
$$\vdots$$

Thus, we obtain solution in series form as

$$\xi(\phi,\vartheta) = \xi_0(\phi,\vartheta) + \xi_1(\phi,\vartheta) + \xi_2(\phi,\vartheta) + \xi_3(\phi,\vartheta) + \cdots. \tag{32}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi^2\left\{1 + \frac{\vartheta^\beta}{2\Gamma(\beta+1)} + \frac{\vartheta^{2\beta}}{4\Gamma(2\beta+1)} + \frac{\vartheta^{3\beta}}{8\Gamma(3\beta+1)} + \cdots\right\}. \tag{33}$$

Now, by implementing HPETM, we obtain

$$\sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta) = (\phi^2) + p\{E^{-1}(s^\beta E[\sum_{n=0}^{\infty} p^n H_n(w)])\}. \tag{34}$$

By comparing both sides of the coefficient of p, we obtain:

$$
\begin{aligned}
p^0 &: w_0(\phi,\vartheta) = \phi^2,\\
p^1 &: w_1(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_0(w))\}] = \phi^2 \frac{\vartheta^\beta}{2\Gamma(\beta+1)},\\
p^2 &: w_2(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_1(w))\}] = \phi^2 \frac{(\vartheta^\beta)^2}{4\Gamma(2\beta+1)},\\
p^3 &: w_3(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_2(w))\}] = \phi^2 \frac{(\vartheta^\beta)^3}{8\Gamma(3\beta+1)},\\
&\vdots
\end{aligned} \tag{35}
$$

Thus, we obtain the solution in series form in terms of HPM as

$$\xi(\phi,\vartheta) = \sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta). \tag{36}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi^2\left\{1 + \frac{\vartheta^\beta}{2\Gamma(\beta+1)} + \frac{\vartheta^{2\beta}}{4\Gamma(2\beta+1)} + \frac{\vartheta^{3\beta}}{8\Gamma(3\beta+1)} + \cdots\right\}. \tag{37}$$

The graphs in Figure 1 depict how the exact and suggested techniques solved the problem when $\beta = 1$. Figure 1 depicts our method's solution at various fractional orders of $\beta = 1, 0.75, 0.50, 0.25$ inside the domain of $0 \le \phi, \vartheta \ge 1$, while Figure 1 depicts the solution for problem 1 at $\vartheta = 0.5$ and $0 \le \phi \ge 1$, respectively. Additionally, Table 1 compares the proposed method results in terms of absolute error at various fractional orders.

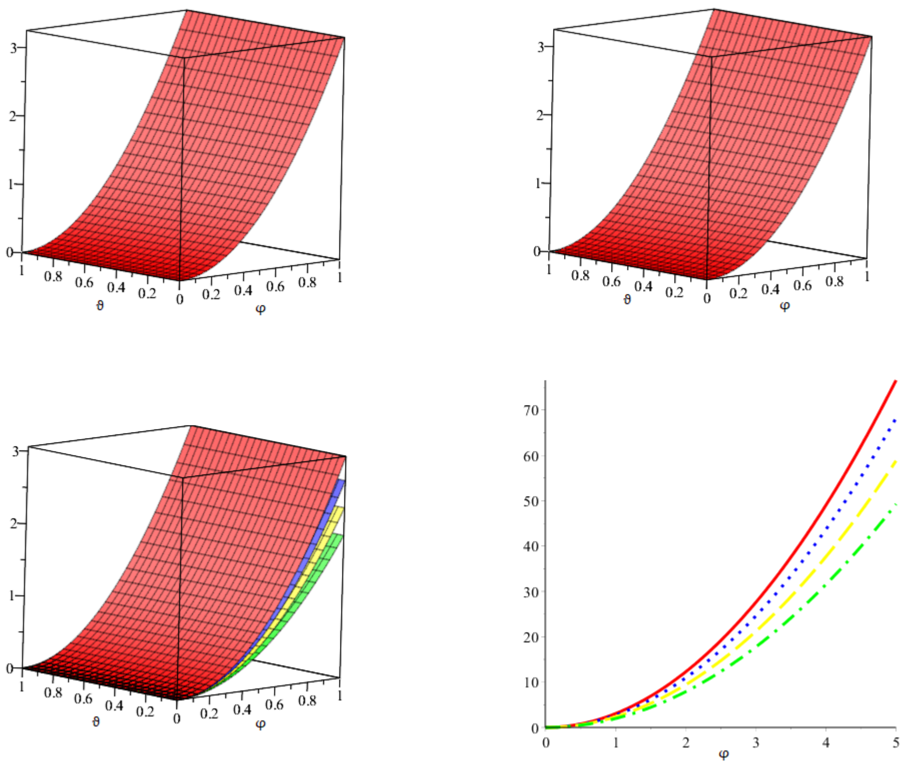

Figure 1. Graphical layout of exact solution, proposed method solution, and 3D and 2D behavior at various fractional orders of example 1.

Table 1. Comparative analysis of NITM and HPTM solution of example 1.

η	ξ	$\lvert Exact - NITM \rvert$	$\lvert Exact - NITM \rvert$	$\lvert Exact - HPTM \rvert$	$\lvert Exact - HPTM \rvert$
		$\beta = 0.6$	$\beta = 1$	$\beta = 0.8$	$\beta = 1$
	0.5	$8.48379150 \times 10^{-03}$	$1.5700000 \times 10^{-08}$	$4.09610800 \times 10^{-04}$	$1.5700000 \times 10^{-08}$
	1	$1.69675830 \times 10^{-02}$	$6.3000000 \times 10^{-08}$	$1.63844300 \times 10^{-03}$	$6.3000000 \times 10^{-08}$
	1.5	$2.54513740 \times 10^{-02}$	$1.4100000 \times 10^{-07}$	$3.68649700 \times 10^{-03}$	$1.4100000 \times 10^{-07}$
	2	$3.39351660 \times 10^{-02}$	$2.5200000 \times 10^{-07}$	$6.55377200 \times 10^{-03}$	$2.5200000 \times 10^{-07}$
	2.5	$4.24189580 \times 10^{-02}$	$3.9300000 \times 10^{-07}$	$1.02402690 \times 10^{-02}$	$3.9300000 \times 10^{-07}$
0.001	3	$5.09027490 \times 10^{-02}$	$5.6700000 \times 10^{-07}$	$1.47459870 \times 10^{-02}$	$5.6700000 \times 10^{-07}$
	3.5	$5.93865400 \times 10^{-02}$	$7.7000000 \times 10^{-07}$	$2.00709300 \times 10^{-02}$	$7.7000000 \times 10^{-07}$
	4	$6.78703320 \times 10^{-02}$	$1.0100000 \times 10^{-06}$	$2.62150900 \times 10^{-02}$	$1.0100000 \times 10^{-06}$
	4.5	$7.63541240 \times 10^{-02}$	$1.2700000 \times 10^{-06}$	$3.31784700 \times 10^{-02}$	$1.2700000 \times 10^{-06}$
	5	$8.48379150 \times 10^{-02}$	$1.5700000 \times 10^{-06}$	$4.09610800 \times 10^{-02}$	$1.5700000 \times 10^{-06}$
	0.5	$1.27076980 \times 10^{-02}$	$6.2500000 \times 10^{-08}$	$6.81188800 \times 10^{-04}$	$6.2500000 \times 10^{-08}$
	1	$2.54153960 \times 10^{-02}$	$2.5000000 \times 10^{-07}$	$2.72475500 \times 10^{-03}$	$2.5000000 \times 10^{-07}$
	1.5	$3.81230940 \times 10^{-02}$	$5.6300000 \times 10^{-07}$	$6.13069900 \times 10^{-03}$	$5.6300000 \times 10^{-07}$
	2	$5.08307920 \times 10^{-02}$	$1.0000000 \times 10^{-06}$	$1.08990200 \times 10^{-02}$	$1.0000000 \times 10^{-06}$
	2.5	$6.35384900 \times 10^{-02}$	$1.5630000 \times 10^{-06}$	$1.70297190 \times 10^{-02}$	$1.5630000 \times 10^{-06}$
0.002	3	$7.62461880 \times 10^{-02}$	$2.2500000 \times 10^{-06}$	$2.45227950 \times 10^{-02}$	$2.2500000 \times 10^{-06}$
	3.5	$8.89538860 \times 10^{-02}$	$3.0600000 \times 10^{-06}$	$3.33782500 \times 10^{-02}$	$3.0600000 \times 10^{-06}$
	4	$1.01661584 \times 10^{-01}$	$4.0000000 \times 10^{-06}$	$4.35960800 \times 10^{-02}$	$4.0000000 \times 10^{-06}$
	4.5	$1.14369282 \times 10^{-01}$	$5.0600000 \times 10^{-06}$	$5.51762900 \times 10^{-02}$	$5.0600000 \times 10^{-06}$
	5	$1.2707698 \times 10^{-01}$	$6.2500000 \times 10^{-06}$	$6.81188800 \times 10^{-02}$	$6.2500000 \times 10^{-06}$

4.2. Example

Let us consider the time-fractional Fokker–Planck equation as

$$\frac{\partial}{\partial\vartheta^{\beta}}(\xi(\phi,\vartheta)) + \frac{\partial}{\partial\phi}(\phi\xi(\phi,\vartheta)) - \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi(\phi,\vartheta)\right) = 0, \quad \phi,\vartheta > 0, \ \beta \in (0,1], \tag{38}$$

subjected to initial condition

$$\xi(\phi,0) = \phi, \tag{39}$$

for special value $\beta = 1$, the exact solution is

$$\xi(\phi,\vartheta) = \phi \exp^{\vartheta}. \tag{40}$$

By employing Elzaki transform in Equation (38), we have

$$E[\xi(\phi,\phi,\vartheta)] = s^2(\phi) + s^{\beta}E\left[-\frac{\partial}{\partial\phi}(\phi\xi(\phi,\vartheta)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi(\phi,\vartheta)\right)\right], \tag{41}$$

By applying inverse Elzaki transform, we obtain

$$\xi(\phi,\phi,\vartheta) = (\phi) + E^{-1}\left(s^{\beta}E\left[-\frac{\partial}{\partial\phi}(\phi\xi(\phi,\vartheta)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi(\phi,\vartheta)\right)\right]\right). \tag{42}$$

Hence, by implementing NITM, we obtain

$$\xi_0(\phi,\vartheta) = \phi,$$

$$\xi_1(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{-\frac{\partial}{\partial\phi}(\phi\xi_0(\phi,\vartheta)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi_0(\phi,\vartheta)\right)\right\}\right] = \phi\frac{\vartheta^{\beta}}{\Gamma(\beta+1)},$$

$$\xi_2(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{-\frac{\partial}{\partial\phi}(\phi\xi_1(\phi,\vartheta)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi_1(\phi,\vartheta)\right)\right\}\right] = \phi\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)},$$
$$\xi_3(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{-\frac{\partial}{\partial\phi}(\phi\xi_2(\phi,\vartheta)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{\phi^2}{2}\xi_2(\phi,\vartheta)\right)\right\}\right] = \phi\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)},$$
$$\vdots$$

Thus, we obtain the solution in series form as

$$\xi(\phi,\vartheta) = \xi_0(\phi,\vartheta) + \xi_1(\phi,\vartheta) + \xi_2(\phi,\vartheta) + \xi_3(\phi,\vartheta) + \cdots \xi_n(\phi,\vartheta). \tag{43}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi + \phi\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + \phi\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)} + \phi\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)} + \cdots. \tag{44}$$

Now, by implementing HPETM, we obtain

$$\sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta) = (e^{-\phi}) + p\{E^{-1}(s^{\beta}E[\sum_{n=0}^{\infty} p^n H_n(w)])\}. \tag{45}$$

By comparing both sides of the coefficient of p, we obtain:

$$\begin{aligned}
&p^0 : w_0(\phi,\vartheta) = \phi,\\
&p^1 : w_1(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_0(w))\}] = \phi\frac{\vartheta^{\beta}}{\Gamma(\beta+1)},\\
&p^2 : w_2(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_1(w))\}] = \phi\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)},\\
&p^3 : w_3(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_2(w))\}] = \phi\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)},\\
&\qquad\vdots
\end{aligned} \tag{46}$$

Thus, we obtain solution in series form in terms of HPM as

$$\xi(\phi,\vartheta) = \sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta). \tag{47}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi + \phi\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + \phi\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)} + \phi\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)} + \cdots. \tag{48}$$

The graphs in Figure 2 depict how the exact and suggested techniques solved the problem when $\beta = 1$. Figure 2 depicts our method's solution at various fractional orders of $\beta = 1, 0.75, 0.50, 0.25$ inside the domain of $0 \leq \phi, \vartheta \geq 5$, while Figure 2 depicts the solution for problem 2 at $\vartheta = 0.5$ and $0 \leq \phi \geq 5$, respectively. Additionally, Table 2 compares the proposed method results in terms of absolute error at various fractional orders.

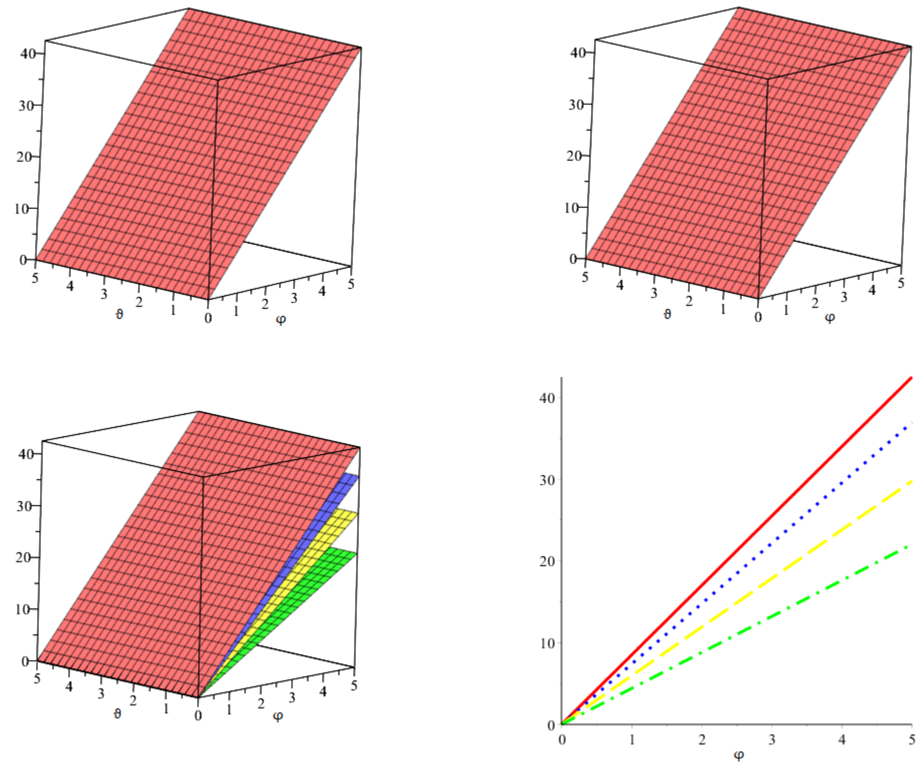

Figure 2. Graphical layout of exact solution, proposed method solution, and 3D and 2D behavior at various fractional orders of example 2.

Table 2. Comparative analysis of NITM and HPTM solution of example 2.

η	ξ	$\lvert Exact - NITM \rvert$	$\lvert Exact - NITM \rvert$	$\lvert Exact - HPTM \rvert$	$\lvert Exact - HPTM \rvert$
		$\beta = 0.6$	$\beta = 1$	$\beta = 0.8$	$\beta = 1$
0.001	0.5	$4.36677680 \times 10^{-03}$	$3.5000000 \times 10^{-09}$	$1.64248450 \times 10^{-03}$	$3.5000000 \times 10^{-09}$
	1	$1.74671070 \times 10^{-02}$	$7.0000000 \times 10^{-09}$	$3.28496900 \times 10^{-03}$	$7.0000000 \times 10^{-09}$
	1.5	$3.93009910 \times 10^{-02}$	$1.0000000 \times 10^{-08}$	$4.92745400 \times 10^{-03}$	$1.0000000 \times 10^{-08}$
	2	$6.98684280 \times 10^{-02}$	$1.4000000 \times 10^{-08}$	$6.56993800 \times 10^{-03}$	$1.4000000 \times 10^{-08}$
	2.5	$1.09169419 \times 10^{-02}$	$1.8000000 \times 10^{-08}$	$8.21242200 \times 10^{-03}$	$1.8000000 \times 10^{-08}$
	3	$1.57203963 \times 10^{-02}$	$2.1000000 \times 10^{-08}$	$9.85490700 \times 10^{-03}$	$2.1000000 \times 10^{-08}$
	3.5	$2.13972060 \times 10^{-01}$	$2.4000000 \times 10^{-08}$	$1.14973920 \times 10^{-02}$	$2.4000000 \times 10^{-08}$
	4	$2.79473710 \times 10^{-01}$	$2.8000000 \times 10^{-08}$	$1.31398760 \times 10^{-02}$	$2.8000000 \times 10^{-08}$
	4.5	$3.53708920 \times 10^{-01}$	$3.2000000 \times 10^{-08}$	$1.47823600 \times 10^{-02}$	$3.2000000 \times 10^{-08}$
	5	$4.36677680 \times 10^{-01}$	$3.5000000 \times 10^{-08}$	$1.64248450 \times 10^{-02}$	$3.5000000 \times 10^{-08}$
0.002	0.5	$6.12090700 \times 10^{-03}$	$1.7000000 \times 10^{-08}$	$2.73690750 \times 10^{-03}$	$1.7000000 \times 10^{-08}$
	1	$2.44836280 \times 10^{-02}$	$3.4000000 \times 10^{-08}$	$5.47381500 \times 10^{-03}$	$3.4000000 \times 10^{-08}$
	1.5	$5.50881630 \times 10^{-02}$	$5.1000000 \times 10^{-08}$	$8.21072200 \times 10^{-03}$	$5.1000000 \times 10^{-08}$
	2	$9.79345120 \times 10^{-02}$	$6.8000000 \times 10^{-08}$	$1.09476300 \times 10^{-02}$	$6.8000000 \times 10^{-08}$
	2.5	$1.53022675 \times 10^{-02}$	$8.5000000 \times 10^{-08}$	$1.36845380 \times 10^{-02}$	$8.5000000 \times 10^{-08}$
	3	$2.20352652 \times 10^{-01}$	$1.0200000 \times 10^{-07}$	$1.64214450 \times 10^{-02}$	$1.0200000 \times 10^{-07}$
	3.5	$2.99924450 \times 10^{-01}$	$1.1900000 \times 10^{-07}$	$1.91583520 \times 10^{-02}$	$1.1900000 \times 10^{-07}$
	4	$3.91738050 \times 10^{-01}$	$1.3600000 \times 10^{-07}$	$2.18952600 \times 10^{-02}$	$1.3600000 \times 10^{-07}$
	4.5	$4.95793470 \times 10^{-01}$	$1.5300000 \times 10^{-07}$	$2.46321680 \times 10^{-02}$	$1.5300000 \times 10^{-07}$
	5	$6.12090700 \times 10^{-01}$	$1.7000000 \times 10^{-07}$	$2.73690750 \times 10^{-02}$	$1.7000000 \times 10^{-07}$

4.3. Example

Let us consider the time-fractional Fokker–Planck equation as

$$\frac{\partial}{\partial \vartheta^{\beta}}(\xi(\phi,\vartheta)) + \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi(\phi,\vartheta)\right) - \frac{\partial^2}{\partial \phi^2}\left(\xi^2(\phi,\vartheta)\right) = 0, \ \ \phi,\vartheta > 0, \ \beta \in (0,1], \tag{49}$$

subjected to initial condition

$$\xi(\phi,0) = \phi^2. \tag{50}$$

for special value $\beta = 1$; the exact solution is

$$\xi(\phi,\vartheta) = \phi^2 \exp^{\vartheta}. \tag{51}$$

By employing the Elzaki transform in Equation (49), we have

$$E[\xi(\phi,\phi,\vartheta)] = s^2(\phi^2) + s^{\beta} E\left[\frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi(\phi,\vartheta)\right) + \frac{\partial^2}{\partial \phi^2}\left(\xi^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi^2(\phi,\vartheta)\right)\right], \tag{52}$$

By applying the inverse Elzaki transform, we obtain

$$\xi(\phi,\phi,\vartheta) = \phi^2 + E^{-1}\left(s^{\beta} E\left[\frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi(\phi,\vartheta)\right) + \frac{\partial^2}{\partial \phi^2}\left(\xi^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi^2(\phi,\vartheta)\right)\right]\right). \tag{53}$$

Hence, by implementing NITM, we obtain

$$\xi_0(\phi,\vartheta) = \phi^2,$$
$$\xi_1(\phi,\vartheta) = E^{-1}[s^{\beta}E\{\frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi_0(\phi,\vartheta)\right) + \frac{\partial^2}{\partial \phi^2}\left(\xi_0^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi_0^2(\phi,\vartheta)\right)\}] = \phi^2\frac{\vartheta^{\beta}}{\Gamma(\beta+1)},$$

$$\xi_2(\phi,\vartheta) = E^{-1}[s^{\beta}E\{\frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi_1(\phi,\vartheta)\right) + \frac{\partial^2}{\partial \phi^2}\left(\xi_1^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi_1^2(\phi,\vartheta)\right)\}] = \phi^2\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)},$$

$$\xi_3(\phi,\vartheta) = E^{-1}[s^{\beta}E\{\frac{\partial}{\partial \phi}\left(\frac{\phi}{3}\xi_2(\phi,\vartheta)\right) + \frac{\partial^2}{\partial \phi^2}\left(\xi_2^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi_2^2(\phi,\vartheta)\right)\}] = \phi^2\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)},$$
$$\vdots$$

Thus, we obtain the solution in series form as

$$\xi(\phi,\vartheta) = \xi_0(\phi,\vartheta) + \xi_1(\phi,\vartheta) + \xi_2(\phi,\vartheta) + \xi_3(\phi,\vartheta) + \cdots \xi_n(\phi,\vartheta). \tag{54}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi^2 + \phi^2\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + \phi^2\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)} + \phi^2\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)} + \cdots. \tag{55}$$

Now, by implementing HPETM, we obtain

$$\sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta) = (e^{-\phi}) + p\{E^{-1}(s^{\beta}E[\sum_{n=0}^{\infty} p^n H_n(w)])\}. \tag{56}$$

The non-linear terms are represented by the polynomials $H_n(w)$. The elements of He's polynomials, for example, are obtained using the recursive relationship $H_n(w) = \frac{\partial^2}{\partial \phi^2}\left(\xi^2(\phi,\vartheta)\right) - \frac{\partial}{\partial \phi}\left(\frac{4}{\phi}\xi^2(\phi,\vartheta)\right)$, $\forall n \in N$. The following approximation is achieved by equating the equivalent power coefficient of p on both sides:

$$
\begin{aligned}
&p^0 : w_0(\phi, \vartheta) = \cos(\phi), \\
&p^1 : w_1(\phi, \vartheta) = [E^{-1}\{s^\beta E(H_0(w))\}] = \phi^2 \frac{\vartheta^\beta}{\Gamma(\beta+1)}, \\
&p^2 : w_2(\phi, \vartheta) = [E^{-1}\{s^\beta E(H_1(w))\}] = \phi^2 \frac{(\vartheta^\beta)^2}{\Gamma(2\beta+1)}, \\
&p^3 : w_3(\phi, \vartheta) = [E^{-1}\{s^\beta E(H_2(w))\}] = \phi^2 \frac{(\vartheta^\beta)^3}{\Gamma(3\beta+1)}, \\
&\vdots
\end{aligned} \tag{57}
$$

The solution in series form by means of HPM is given as

$$\xi(\phi, \vartheta) = \sum_{n=0}^{\infty} p^n w_n(\phi, \vartheta). \tag{58}$$

So we obtain

$$\xi(\phi, \vartheta) = \phi^2 + \phi^2 \frac{\vartheta^\beta}{\Gamma(\beta+1)} + \phi^2 \frac{(\vartheta^\beta)^2}{\Gamma(2\beta+1)} + \phi^2 \frac{(\vartheta^\beta)^3}{\Gamma(3\beta+1)}. \tag{59}$$

The graphs in Figure 3 depict how the exact and suggested techniques solved the problem when $\beta = 1$. Figure 3 depicts our method's solution at various fractional orders of $\beta = 1, 0.75, 0.50, 0.25$ inside the domain of $0 \leq \phi, \vartheta \geq 5$, while Figure 3 depicts the solution for problem 3 at $\vartheta = 0.5$ and $0 \leq \phi \geq 10$, respectively.

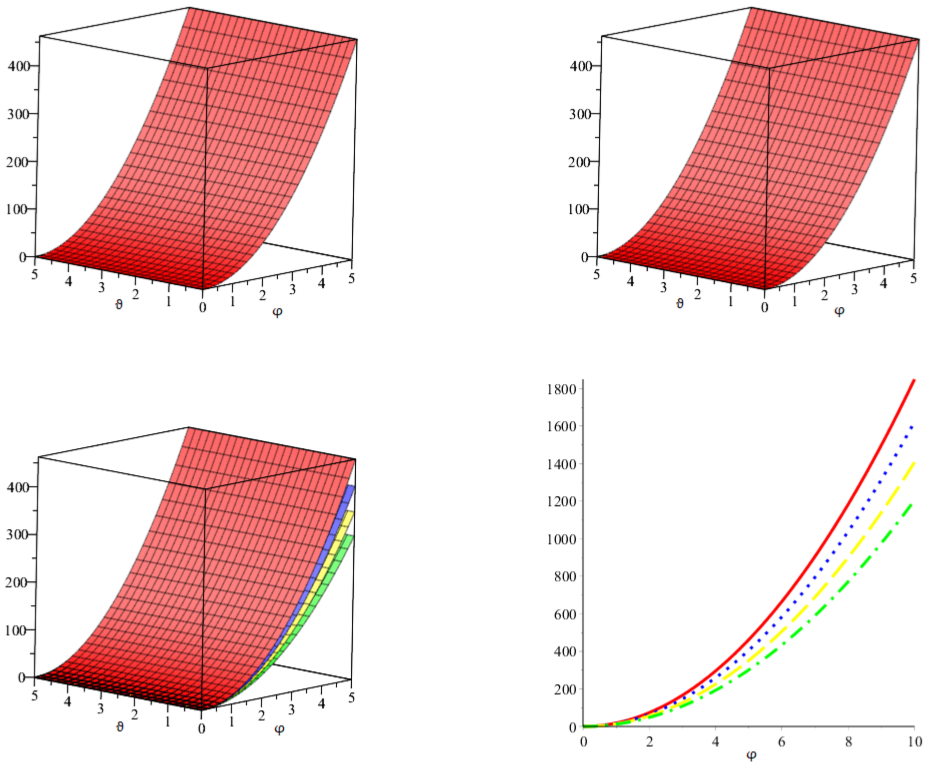

Figure 3. Graphical layout of exact solution, proposed method solution, and 3D and 2D behavior at various fractional orders of example 3.

4.4. Example

Let us consider the time-fractional Fokker–Planck equation as

$$\frac{\partial}{\partial \vartheta^{\beta}}(\xi(\phi,\vartheta)) - \frac{\partial}{\partial \phi}\xi(\phi,\vartheta) - \frac{\partial^2}{\partial \phi^2}\xi(\phi,\vartheta) = 0, \vartheta > 0,\ \beta \in (0,1], \tag{60}$$

subjected to initial condition

$$\xi(\phi,0) = \phi, \tag{61}$$

for special value $\beta = 1$; the exact solution is

$$\xi(\phi,\vartheta) = \phi + \vartheta. \tag{62}$$

By employing the Elzaki transform in Equation (60), we have

$$E[\xi(\phi,\phi,\vartheta)] = s^2(\phi) + s^{\beta}E\left[\frac{\partial}{\partial \phi}\xi(\phi,\vartheta) + \frac{\partial^2}{\partial \phi^2}\xi(\phi,\vartheta)\right], \tag{63}$$

By applying the inverse Elzaki transform, we obtain

$$\xi(\phi,\phi,\vartheta) = (\phi) + E^{-1}\left(s^{\beta}E\left[\frac{\partial}{\partial \phi}\xi(\phi,\vartheta) + \frac{\partial^2}{\partial \phi^2}\xi(\phi,\vartheta)\right]\right). \tag{64}$$

Hence, by implementing NITM, we obtain

$$\begin{aligned}
&\xi_0(\phi,\vartheta) = \phi,\\
&\xi_1(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{\frac{\partial}{\partial \phi}\xi_0(\phi,\vartheta) + \frac{\partial^2}{\partial \phi^2}\xi_0(\phi,\vartheta)\right\}\right] = \frac{\vartheta^{\beta}}{\Gamma(\beta+1)},\\
&\xi_2(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{\frac{\partial}{\partial \phi}\xi_1(\phi,\vartheta) + \frac{\partial^2}{\partial \phi^2}\xi_1(\phi,\vartheta)\right\}\right] = 0,\\
&\xi_3(\phi,\vartheta) = E^{-1}\left[s^{\beta}E\left\{\frac{\partial}{\partial \phi}\xi_2(\phi,\vartheta) + \frac{\partial^2}{\partial \phi^2}\xi_2(\phi,\vartheta)\right\}\right] = 0,\\
&\vdots
\end{aligned}$$

Thus, we obtain the solution in series form as

$$\xi(\phi,\vartheta) = \xi_0(\phi,\vartheta) + \xi_1(\phi,\vartheta) + \xi_2(\phi,\vartheta) + \xi_3(\phi,\vartheta) + \cdots \xi_n(\phi,\vartheta). \tag{65}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi + \frac{\vartheta^{\beta}}{\Gamma(\beta+1)}. \tag{66}$$

Now, by implementing HPETM, we obtain

$$\sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta) = (e^{-\phi}) + p\{E^{-1}(s^{\beta}E[\sum_{n=0}^{\infty} p^n H_n(w)])\}. \tag{67}$$

By comparing both sides of the coefficient of p, we obtain:

$$
\begin{aligned}
p^0 &: w_0(\phi,\vartheta) = \phi,\\
p^1 &: w_1(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_0(w))\}] = \frac{\vartheta^\beta}{\Gamma(\beta+1)},\\
p^2 &: w_2(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_1(w))\}] = 0,\\
p^3 &: w_3(\phi,\vartheta) = [E^{-1}\{s^\beta E(H_2(w))\}] = 0,\\
&\vdots
\end{aligned}
\tag{68}
$$

Thus, we obtain the solution in series form in terms of HPM as

$$\xi(\phi,\vartheta) = \sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta). \tag{69}$$

So we obtain

$$\xi(\phi,\vartheta) = \phi + \frac{\vartheta^\beta}{\Gamma(\beta+1)}. \tag{70}$$

The graphs in Figure 4 depict how the exact and suggested techniques solved the problem when $\beta = 1$. Figure 4 depicts our method's solution at various fractional orders of $\beta = 1, 0.75, 0.50, 0.25$ inside the domain of $0 \le \phi, \vartheta \ge 5$, while Figure 4 depicts the solution for problem 4 at $\vartheta = 0.5$ and $0 \le \phi \ge 5$, respectively.

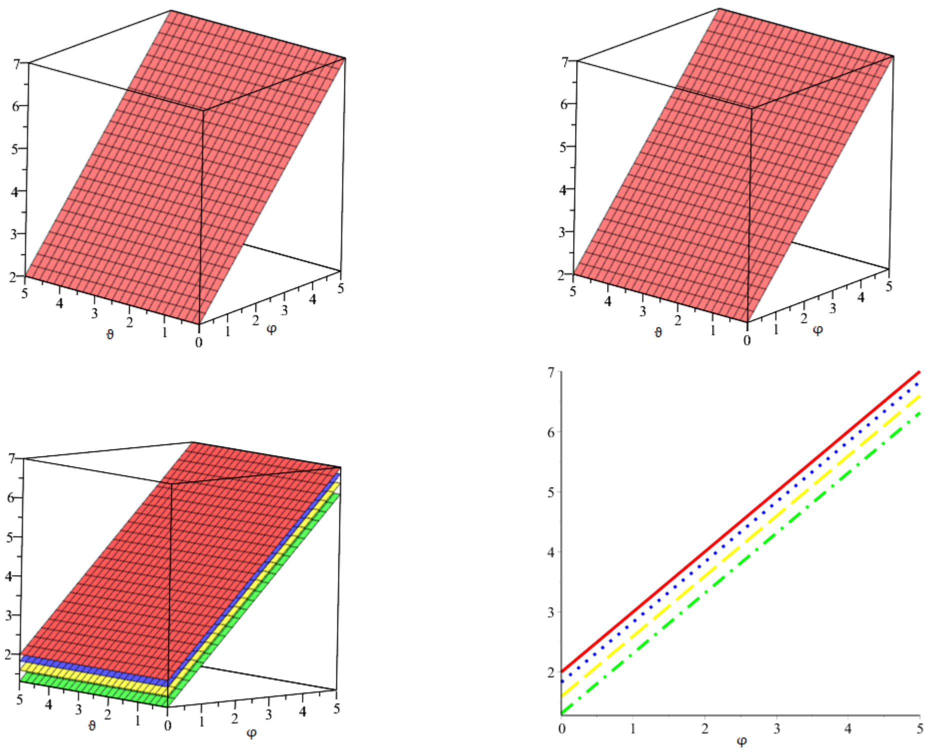

Figure 4. Graphical layout of exact solution, proposed method solution, and 3D and 2D behavior at various fractional orders of example 4.

4.5. Example

Let us consider the time-fractional Fokker–Planck equation as

$$\frac{\partial^\beta}{\partial\vartheta^\beta}(\xi(\phi,\vartheta)) - (1-\phi)\frac{\partial}{\partial\phi}\xi(\phi,\vartheta) - (e^\vartheta\phi^2)\frac{\partial^2}{\partial\phi^2}\xi(\phi,\vartheta) = 0, \quad \vartheta > 0,\ \beta \in (0,1], \tag{71}$$

subjected to initial condition

$$\xi(\phi, 0) = 1 + \phi, \tag{72}$$

for special value $\beta = 1$; the exact solution is

$$\xi(\phi, \vartheta) = \exp^{\vartheta}(1 + \phi). \tag{73}$$

By employing the Elzaki transform in Equation (71), we have

$$E[\xi(\phi, \phi, \vartheta)] = s^2(1 + \phi) + s^{\beta} E\left[(1 - \phi)\frac{\partial}{\partial \phi}\xi(\phi, \vartheta) + (e^{\vartheta}\phi^2)\frac{\partial^2}{\partial \phi^2}\xi(\phi, \vartheta)\right], \tag{74}$$

By applying the inverse Elzaki transform, we obtain

$$\xi(\phi, \phi, \vartheta) = (1 + \phi) + E^{-1}\left(s^{\beta} E\left[(1 - \phi)\frac{\partial}{\partial \phi}\xi(\phi, \vartheta) + (e^{\vartheta}\phi^2)\frac{\partial^2}{\partial \phi^2}\xi(\phi, \vartheta)\right]\right). \tag{75}$$

Hence, by implementing NITM, we obtain

$$\begin{aligned}
\xi_0(\phi, \vartheta) &= 1 + \phi, \\
\xi_1(\phi, \vartheta) &= E^{-1}\left[s^{\beta} E\left\{(1 - \phi)\frac{\partial}{\partial \phi}\xi_0(\phi, \vartheta) + (e^{\vartheta}\phi^2)\frac{\partial^2}{\partial \phi^2}\xi_0(\phi, \vartheta)\right\}\right] = (1 + \phi)\frac{\vartheta^{\beta}}{\Gamma(\beta + 1)}, \\
\xi_2(\phi, \vartheta) &= E^{-1}\left[s^{\beta} E\left\{(1 - \phi)\frac{\partial}{\partial \phi}\xi_1(\phi, \vartheta) + (e^{\vartheta}\phi^2)\frac{\partial^2}{\partial \phi^2}\xi_1(\phi, \vartheta)\right\}\right] = (1 + \phi)\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta + 1)}, \\
\xi_3(\phi, \vartheta) &= E^{-1}\left[s^{\beta} E\left\{(1 - \phi)\frac{\partial}{\partial \phi}\xi_2(\phi, \vartheta) + (e^{\vartheta}\phi^2)\frac{\partial^2}{\partial \phi^2}\xi_2(\phi, \vartheta)\right\}\right] = (1 + \phi)\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta + 1)}, \\
&\vdots
\end{aligned}$$

Thus, we obtain the solution in series form as

$$\xi(\phi, \vartheta) = \xi_0(\phi, \vartheta) + \xi_1(\phi, \vartheta) + \xi_2(\phi, \vartheta) + \xi_3(\phi, \vartheta) + \cdots \xi_n(\phi, \vartheta). \tag{76}$$

So we obtain

$$\xi(\phi, \vartheta) = (1 + \phi) + (1 + \phi)\frac{\vartheta^{\beta}}{\Gamma(\beta + 1)} + (1 + \phi)\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta + 1)} + (1 + \phi)\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta + 1)} + \cdots. \tag{77}$$

Now, by implementing HPETM, we obtain

$$\sum_{n=0}^{\infty} p^n w_n(\phi, \vartheta) = (e^{-\phi}) + p\{E^{-1}(s^{\beta} E[\sum_{n=0}^{\infty} p^n H_n(w)])\}. \tag{78}$$

By comparing both sides of the coefficient of p, we obtain:

$$\begin{aligned}
p^0 &: w_0(\phi,\vartheta) = 1+\phi,\\
p^1 &: w_1(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_0(w))\}] = (1+\phi)\frac{\vartheta^{\beta}}{\Gamma(\beta+1)},\\
p^2 &: w_2(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_1(w))\}] = (1+\phi)\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)},\\
p^3 &: w_3(\phi,\vartheta) = [E^{-1}\{s^{\beta}E(H_2(w))\}] = (1+\phi)\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)},\\
&\vdots
\end{aligned} \tag{79}$$

Thus, we obtain the solution in series form in terms of HPM as

$$\xi(\phi,\vartheta) = \sum_{n=0}^{\infty} p^n w_n(\phi,\vartheta). \tag{80}$$

So we obtain

$$\xi(\phi,\vartheta) = (1+\phi) + (1+\phi)\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + (1+\phi)\frac{(\vartheta^{\beta})^2}{\Gamma(2\beta+1)} + (1+\phi)\frac{(\vartheta^{\beta})^3}{\Gamma(3\beta+1)} + \cdots. \tag{81}$$

The graphs in Figure 5 depict how the exact and suggested techniques solved the problem when $\beta = 1$. Figure 5 depicts our method's solution at various fractional orders of $\beta = 1, 0.75, 0.50, 0.25$ inside the domain of $0 \le \phi, \vartheta \ge 5$, while Figure 5 depicts the solution for problem 5 at $\vartheta = 0.5$ and $0 \le \phi \ge 10$, respectively.

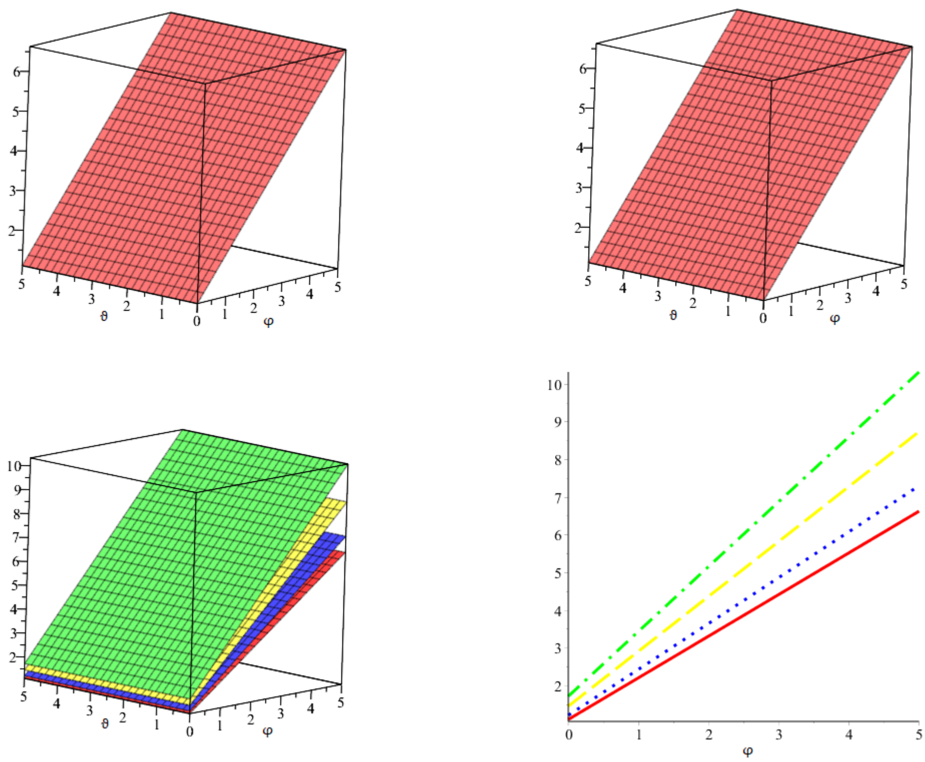

Figure 5. Graphical layout of exact solution, proposed method solution, and 3D and 2D behavior at various fractional orders of example 5.

5. Conclusions

To solve the space and time-fractional Fokker–Planck equation, the new iterative approach and the homotopy perturbation method are used in this article. The two methods are particularly powerful and efficient in finding analytical and numerical solutions for a wide range of space-time fractional partial differential equations. Without employing linearization, perturbation or limiting assumptions, they give results in terms of convergent series with easily computed components. The study demonstrates that the two methodologies need less computational effort than previous methods while providing quantitatively accurate results. In all examples, the excellent agreement of numerical findings between the two approaches is also evident and notable. Finally, the proposed approaches are more efficient and solve the complexity of calculating fractional order PDE solutions, which occurs frequently in science and engineering.

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References

1. Yépez-Martínez, H.; Gómez-Aguilar, F.; Sosa, I.O.; Reyes, J.M.; Torres-Jiménez, J. The Fengs first integral method applied to the nonlinear mKdV space-time fractional partial differential equation. *Rev. Mex. Fis.* **2016**, *62*, 310–316.
2. Baleanu, D.; Machado, J.A.; Luo, A.C. (Eds.) *Fractional Dynamics and Control*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2011.
3. Magin Richard, L. *Fractional Calculus in Bioengineering*; Begell House Redding: Chicago, IL, USA, 2006.
4. Ellahi, R.; Alamri, S.Z.; Basit, A.; Majeed, A. Effects of MHD and slip on heat transfer boundary layer flow over a moving plate based on specific entropy generation. *J. Taibah Univ. Sci.* **2018**, *12*, 476–482. [CrossRef]
5. Tenreiro, Machado, J.A.; Silva, M.F.; Barbosa, R.S.; Jesus, I.S.; Reis, C.M.; Marcos, M.G.; Galhano, A.F. Some applications of fractional calculus in engineering. *Math. Probl. Eng.* **2010**, *2010*, 639801.
6. Li, C.; Qian, D.; Chen, Y. On Riemann-Liouville and Caputo derivatives. *Discret. Dyn. Nat. Soc.* **2011**, *2011*, 562494. [CrossRef]
7. Kirane, M.; Abdeljabbar, A. Nonexistence of Global Solutions of Systems of Time Fractional Differential equations posed on the Heisenberg group. *Math. Methods Appl. Sci.* **2022**, *45*, 7336–7345. [CrossRef]
8. Al-Sawalha, M.M.; Khan, A.; Ababneh, O.Y.; Botmart, T. Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives. *AIMS Math.* **2022**, *7*, 18334–18359. [CrossRef]
9. Rahman, Z.; Abdeljabbar, A.; Harun-Or-Roshid; Ali, M.Z. Novel Precise Solitary Wave Solutions of Two Time Fractional Nonlinear Evolution Models via the MSE Scheme. *Fractal Fract.* **2022**, *6*, 444. [CrossRef]
10. Cuahutenango-Barro, B.; Taneco-Hernández, M.A.; Gómez-Aguilar, J.F. On the solutions of fractional-time wave equation with memory effect involving operators with regular kernel. *Chaos Solitons Fractals* **2018**, *115*, 283–299. [CrossRef]
11. Gómez-Aguilar, J.F.; Mir, a-Hernández, M.; López-López, M.G.; Alvarado-Martínez, V.M.; Baleanu, D. Modeling and simulation of the fractional space-time diffusion equation. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *30*, 115–127. [CrossRef]
12. Eftekhari, T.; Rashidinia, J. A new operational vector approach for time-fractional subdiffusion equations of distributed order based on hybrid functions. *Math. Methods Appl. Sci.* **2022**, *46*, 388–407. [CrossRef]
13. Sunthrayuth, P.; Alyousef, H.A.; El-Tantawy, S.A.; Khan, A.; Wyal, N. Solving Fractional-Order Diffusion Equations in a Plasma and Fluids via a Novel Transform. *J. Funct. Spaces* **2022**, *2022*, 1899130. [CrossRef]
14. Wang, L.; Liu, G.; Xue, J.; Wong, K. Channel Prediction Using Ordinary Differential Equations for MIMO systems. *IEEE Trans. Veh. Technol.* **2022**, 1–9. [CrossRef]

15. Katsikis, V.N.; Mourtas, S.D.; Simos, T.E. Zeroing Neural Network for Pseudoinversionof an Arbitrary Time-Varying Matrix Based on Singular Value Decomposition. *Mathematics* **2022**, *10*, 1208. [CrossRef]
16. Chen, H.; Li, S. Multi-Sensor Fusion by CWT-PARAFAC-IPSO-SVM for Intelligent Mechanical Fault Diagnosis. *Sensors* **2022**, *22*, 3647. [CrossRef] [PubMed]
17. Xu, Y.; Li, Y.; Liu, D.; Jia, W.; Huang, H. Responses of Duffing oscillator with fractional damping and random phase. *Nonlinear Dyn.* **2013**, *74*, 745–753. [CrossRef]
18. Caputo, M. Linear models of dissipation whose Q is almost frequency independent II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
19. Ford, N.J.; Simpson, A.C. The numerical solution of fractional differential equations: Speed versus accuracy. *Numer. Algorithms* **2001**, *26*, 333–346. [CrossRef]
20. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
21. Liu, L.; Zhang, S.; Zhang, L.; Pan, G.; Yu, J. Multi-UUV Maneuvering Counter-Game for Dynamic Target Scenario Based on Fractional-Order Recurrent Neural Network. *IEEE Trans. Cybern.* **2022**, 1–14. [CrossRef]
22. Shah, N.A.; Dassios, I.; Chung, J.D. A decomposition method for a fractional order multi-dimensional telegraph equation via the Elzaki transform. *Symmetry* **2021**, *13*, 8. [CrossRef]
23. Saadeh, R.; Qazza, A.; Burqan, A. A new integral transform: ARA transform and its properties and applications. *Symmetry* **2020**, *12*, 925. [CrossRef]
24. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Fract. Differ. Appl.* **2015**, *2*, 731–785.
25. Yang, X.J.; Baleanu, D.; Srivastava, H.M. *Local Fractional Integral Transforms and Their Applications*; Academic Press: Cambridge, MA, USA, 2015.
26. Losada, J.; Nieto, J.J. Properties of the new fractional derivative without singular kernel. *Fract. Differ. Appl.* **2015**, *2*, 87–92.
27. Zidan, A.M.; Khan, A.; Shah, R.; Alaoui, M.K.; Weera, W. Evaluation of time-fractional Fisher's equations with the help of analytical methods. *AIMS Math.* **2022**, *7*, 18746–18766. [CrossRef]
28. Kbiri Alaoui, M.; Nonlaopon, K.; Zidan, A.M.; Khan, A.; Shah, R. Analytical investigation of fractional order cahn-hilliard and gardner equations using two novel techniques. *Mathematics* **2022**, *10*, 1643. [CrossRef]
29. Areshi, M.; Khan, A.; Shah, R.; Nonlaopon, K. Analytical investigation of fractional order Newell-Whitehead-Segel equations via a novel transform. *AIMS Math.* **2022**, *7*, 6936–6958. [CrossRef]
30. Alyobi, S.; Shah, R.; Khan, A.; Shah, N.A.; Nonlaopon, K. Fractional Analysis of Nonlinear Boussinesq Equation under Atangana-Baleanu-Caputo Operator. *Symmetry* **2022**, *14*, 2417. [CrossRef]
31. Fan, X.; Wei, G.; Lin, X.; Wang, X.; Si, Z.; Zhang, X.; Zhao, W. Reversible Switching of Interlayer Exchange Coupling through Atomically Thin VO2 via Electronic State Modulation. *Matter* **2020**, *2*, 1582–1593. [CrossRef]
32. Meng, F.; Pang, A.; Dong, X.; Han, C.; Sha, X.; Jing, N.; Na, J. H-infinity Optimal Performance Design of an Unstable Plant under Bode Integral Constraint. *Complexity* **2018**, *2018*, 1–10. [CrossRef]
33. Meng, F.; Wang, D.; Yang, P.; Xie, G.; Cutberto, R.; Romero-Melendez, C. Application of Sum of Squares Method in Nonlinear H Control for Satellite Attitude Maneuvers. *Complexity* **2019**, *2019*, 1–10. [CrossRef]
34. Jin, H.; Wang, Z. Boundedness, blowup and critical mass phenomenon in competing chemotaxis. *J. Differ. Equ.* **2016**, *260*, 162–196. [CrossRef]
35. Liu, P.; Shi, J.; Wang, Z.-A. Pattern formation of the attraction-repulsion Keller-Segel system. *Discret. Contin. Dyn. Syst. B* **2013**, *18*, 2597–2625. [CrossRef]
36. He, H.M.; Peng, J.G.; Li, H.Y. Iterative approximation of fixed point problems and variational inequality problems on Hadamard manifolds. *UPB Bull. Ser. A* **2022**, *84*, 25–36.
37. Risken, H. *The FokkerPlanck Equation: Method of Solution and Applications*; Springer: Berlin/Heidelberg, Germany, 1989.
38. Chandresekhar, S. Stochastic problems in physics and astronomy. *Rev. Mod. Phys.* **1943**, *15*, 1–89. [CrossRef]
39. Kramers, H.A. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica* **1940**, *7*, 284–304. [CrossRef]
40. Fokker, A. The median energy of rotating electrical dipoles in radiation fields. *Annalen Der Physik* **1914**, *43*, 810–820. [CrossRef]
41. He, J.H.; Wu, X.H. Construction of solitary solution and compaction-like solution by variational iteration method. *Chaos Solitons Fractals* **2006**, *29*, 108–113. [CrossRef]
42. Jumarie, G. Fractional Brownian motions via random walk in the complex plane and via fractional derivative, comparison and further results on their Fokker–Planck equations. *Chaos Solitons Fractals* **2004**, *22*, 907–925. [CrossRef]
43. Kamitani, Y.; Matsuba, I. Self-similar characteristics of neural networks based on Fokker–Planck equation. *Chaos Solitons Fractals* **2004**, *20*, 329–335. [CrossRef]
44. Xu, Y.; Ren, F.Y.; Liang, J.R.; Qiu, W.Y. Stretched Gaussian asymptotic behavior for fractional Fokker–Planck equation on fractal structure in external force fields. *Chaos Solitons Fractals* **2004**, *20*, 581–586. [CrossRef]
45. Zak, M. Expectation-based intelligent control. *Chaos Solitons Fractals* **2006**, *28*, 616–626. [CrossRef]
46. Risken, H. *Fokker–Planck Equation*; Springer: Berlin/Heidelberg, Germany, 1996; pp. 63–95.
47. Herau, F. Short and long time behavior of the Fokker–Planck equation in a confining potential and applications. *J. Funct. Anal.* **2007**, *244*, 95–118. [CrossRef]
48. Daftardar-Gejji, V.; Bhalekar, V.S. Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method. *Comput. Math. Appl.* **2010**, *59*, 1801–1809. [CrossRef]

49. Daftardar-Gejji, V.; Jafari, H. An iterative method for solving nonlinear functional equations. *J. Math. Anal. Appl.* **2006**, *316*, 753–763. [CrossRef]
50. Bhalekar, S.; Daftardar-Gejji, V. New iterative method: Application to partial differential equations. *Appl. Math. Comput.* **2008**, *203*, 778–783. [CrossRef]
51. Jafari, H. Iterative Methods for Solving System of Fractional Differential Equations. Ph.D. Thesis, Pune University, Pune, India, 2006.
52. Bhalekar, S.; Daftardar-Gejji, V. Solving evolution equations using a new iterative method. *Numer. Methods Partial. Differ. Equ.* **2010**, *26*, 906–916. [CrossRef]
53. He, J.-H. The homotopy perturbation method nonlinear oscillators with discontinuities. *Appl. Math. Comput.* **2004**, *151*, 287–292. [CrossRef]
54. He, H. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Eng.* **1999**, *178*, 257–262. [CrossRef]
55. Nonlaopon, K.; Alsharif, A.M.; Zidan, A.M.; Khan, A.; Hamed, Y.S.; Shah, R. Numerical investigation of fractional order Swift-Hohenberg equations via a novel transform. *Symmetry* **2021**, *13*, 1263. [CrossRef]