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Potentials from the Polynomial Solutions of the Confluent Heun Equation

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Abstract: Polynomial solutions of the confluent Heun differential equation (CHE) are derived by identifying conditions under which the infinite power series expansions around the $z = 0$ singular point can be terminated. Assuming a specific structure of the expansion coefficients, these conditions lead to four non-trivial polynomials that can be expressed as special cases of the confluent Heun function $Hc(p, \beta, \gamma, \delta, \sigma; z)$. One of these recovers the generalized Laguerre polynomials $L_N^{(\alpha)}$, and another one the rationally extended X_1 type Laguerre polynomials $\hat{L}_N^{(\alpha)}$. The two remaining solutions represent previously unknown polynomials that do not form an orthogonal set and exhibit features characteristic of semi-classical orthogonal polynomials. A standard method of generating exactly solvable potentials in the one-dimensional Schrödinger equation is applied to the CHE, and all known potentials with solutions expressed in terms of the generalized Laguerre polynomials within, or outside the Natanzon confluent potential class, are recovered. It is also found that the potentials generated from the two new polynomial systems necessarily depend on the N quantum number. General considerations on the application of the Heun type differential differential equations within the present framework are also discussed.

Keywords: confluent Heun differential equation; polynomial solutions; supersymmetric quantum mechanics; exceptional orthogonal polynomials; solvable potentials

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1. Introduction

Differential equations play a central role in practically any branch of physics. They describe the spatial and temporal variations of physical quantities, and as such, they are essential to formulate physical laws and models. Typically the physical equations depend on several variables, so one has to deal with partial differential equations, or systems of differential equations. However, it is often possible to reduce the problem to ordinary differential equations by the separation of the variables. Among these, linear second-order differential equations play a special role because some of the most important physical equations (e.g., the one-dimensional Schrödinger equation) are of this type. In the analysis of these equations, one can rely on the general knowledge accumulated on the special functions of mathematical physics. Perhaps the most important of these is the hypergeometric function. The general theory of this function was developed already in the second half of the 19th century, so this knowledge was available by the time that the formalism of quantum mechanics was developed.

The one-dimensional stationary Schrödinger equation has been solved for a number of potentials by transforming it into the hypergeometric (or confluent hypergeometric) differential equation. The general solutions contain the linear combination of two functions of this type; however, bound-state solutions, which have to satisfy well-defined boundary conditions, are usually written in terms of a single classical orthogonal polynomial (Jacobi, generalized Laguerre and Hermite). The most general form of potentials solved by the hypergeometric function was identified as the Natanzon class [1]. The concept of shape-invariance has also been used to classify the most well-known exactly solvable potentials [2]. This concept is based on the formalism of supersymmetric quantum mechanics [3,4], which evolved from the factorization method [5,6]. Its roots also date back to the 19th century, when the Darboux transformation was introduced [7]. This method turned out to be an invaluable tool to generate new solvable potentials from known ones by rewriting the one-dimensional Schrödinger equation containing a second-order differential operator into the product of two first-order differential operators. These first-order differential operators act naturally as ladder operators connecting the solutions of different potentials. Their application is essentially based on the differential forms and recursion relations of the orthogonal polynomials. This formalism also allows associating group theoretical and algebraic structures to special functions [8]. These mathematical developments represent a natural framework to implementing symmetry considerations in physical theories (see, for example, [9]).

The theory of Natanzon potentials (including also Natanzon confluent potentials) is well established (see, for example, Refs. [10,11] and Chapter 7 of Ref. [12]), so it is a natural endeavor to attempt to extend this range to more general classes of solvable potentials. This requires considering more general special functions satisfying second-order differential equations.

One approach focuses on generalizing Bochner-type differential equations:

$$p(z)\frac{\mathrm{d}^2 P_N}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}P_N}{\mathrm{d}z} + r(z)P_N(z) = \lambda_N P_N(z)\,, \tag{1}$$

which satisfy the condition that $p(z)$, $q(z)$ and $r(z)$ are polynomials of degree 2, 1 and 0, respectively [13]. It is known that under these conditions, $P_N(z)$ is one of the classical orthogonal polynomials, i.e., Jacobi, Laguerre or Hermite. Allowing rational, rather than polynomial coefficients in Equation (1), a new type of orthogonal polynomials was introduced [14]. These exceptional orthogonal polynomials share most features of their classical counterparts, except that at least one of their zeroes fall outside their interval of orthogonality. This implies that their sequence does not start with a degree 0 polynomial. The first examples were the rationally extended X_1-Laguerre and X_1-Jacobi polynomials, the sequence of which start with a degree 1 polynomial. More general forms of these polynomials, the X_m-Laguerre and X_m-Jacobi polynomials have also been introduced [15,16]. These mathematical results were soon employed to generate new types of exactly solvable potentials, and the rational extension of certain shape-invariant potentials, i.e., the harmonic oscillator and the Scarf I potential, was introduced [17]. It was also proven that these potentials can be obtained from their ordinary counterparts by SUSYQM transformations; furthermore, they also exhibit the property of shape invariance. These potentials are clearly outside the Natanzon class, as their solutions contain X_1-type Jacobi or Laguerre polynomials, which can be expressed in terms of two ordinary orthogonal polynomials of the same type. These findings also gave further inspiration to the investigation of the mathematical aspects of exceptional orthogonal polynomials. Multistep transformations were formulated to generate further types of orthogonal polynomials [18] and solvable potentials related to them [19]. It has been proven that the exceptional orthogonal polynomials can be obtained by applying a finite sequence of Darboux transformations to classical orthogonal polynomials [20].

Another generalization of the hypergeometric function (and also of further special functions of mathematical physics) is the Heun function and its four confluent (confluent, biconfluent, double confluent and triconfluent) versions [21]. In this approach, the singular points of the corresponding differential equations play a central role. In contrast with the approach based on the rational extension of Bochner type differential equations, here, the solutions are not polynomials in general; rather, they are expanded in terms of power series or of some known special functions. Although the Heun equation was introduced toward the end of the 19th century [22], due to the technical complications, its theory is far less elaborated than that of the hypergeometric differential equation. Despite these circumstances, the Heun-type equations have been applied to derive bound-state solutions of exactly solvable potentials by transforming them into the one-dimensional stationary

Schrödinger equation. A classification of possible potentials is given in Ref. [23], without a detailed analysis of the bound-state solutions and the bound-state energy eigenvalues.

Recently, a systematic survey of solvable potentials related to the Heun-type differential equations was carried out: see Refs. [24] for the general, [25] for the confluent and [26] for the biconfluent Heun equation. See also Ref. [27]. The solutions of these potentials are usually expanded in terms of simpler special functions, but some reduce to polynomial forms. This is the case, for example, for the biconfluent Heun equation. Certain polynomial solutions recover [28,29] the sextic oscillator, which is known to belong to the quasi-exactly solvable (QES) potential family [30]. This potential is a special subset of the general sextic oscillator, as its parameters satisfy certain restrictions. Furthermore, although it possesses an infinite number of bound states, only a finite number of the lowest solutions can be obtained in closed polynomial form. This is because the coefficients appearing in the power series solutions satisfy a three-term recurrence relation, which can be terminated by a specific choice of the model parameters, leading to a polynomial solution. It is also known that the X_1-Jacobi polynomials satisfy the Heun differential equation, which has four singularities, and that they can also be expressed in terms of generalized hypergeometric functions [31].

These findings indicate that the polynomial solutions of the Heun-type differential equations can lead to already known solvable potentials, and perhaps to further unknown ones. Here, we investigate the case of the confluent Heun differential equation, because it can naturally be reduced to the hypergeometric and the confluent hypergeometric differential equations, so its polynomial solutions can be expected to recover those of the generalizations of Natanzon-class potentials. The procedure presented here can serve as a framework to discuss a wide range of exactly solvable potentials in a unified way.

The arrangement of the present work is as follows. The confluent Heun equation is presented in Section 2, and four non-trivial polynomial solutions are derived, most of them with two possible weight functions each. In Section 3, the possibility of generating exactly solvable potentials from the polynomial solutions is studied. Finally, in Section 4, the results are summarized, and further possible considerations are outlined.

2. Polynomial Solutions of the Confluent Heun Equation

The non-symmetrical canonical form of the confluent Heun equation is written as [21]

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + Q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + R(z)F(z) = 0 \tag{2}$$

with

$$Q(z) = 4p + \frac{\gamma}{z} + \frac{\delta}{z-1} \tag{3}$$

and

$$R(z) = \frac{4p\beta z - \sigma}{z(z-1)} \,. \tag{4}$$

The solutions are formally written in terms of the $F(z) = Hc(p, \beta, \gamma, \delta, \sigma; z)$ functions that depend on five parameters [21]. Note that the differential equation of the hypergeometric and confluent hypergeometric functions ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$ can be obtained [32] for the choices $p = 0$, $\gamma = c$, $\delta = a + b + 1 - c$, $\sigma = -ab$, and $p = -1/4$, $\gamma = c$, $\delta = 0$, $\beta = a$, $\sigma = -a$, respectively.

Let us assume that the Hc function can be expressed in terms of a power series expansion around the singular point $z = 0$:

$$Hc(p, \beta, \gamma, \delta, \sigma; z) = \sum_{k=0}^{\infty} C_k z^k \,.$$

This construction leads to the following three-term recursion relation [21] for the C_k coefficients:

$$(k+1)(\gamma+k)C_{k+1} = 4p(k-1+\beta)C_{k-1} + [k(k-1)+k(\gamma+\delta-4p)-\sigma]C_k\,. \tag{5}$$

With the assumption that $\gamma \neq -N$ and $\gamma \neq -N-1$, the recursion terminates at $k=N$ under the following conditions:

$$0 = (N+1)(\gamma+N)C^{(N)}_{N+1} = 4p(N-1+\beta)C^{(N)}_{N-1} + (N(N-1)+N(\gamma+\delta-4p)-\sigma)C^{(N)}_{N} \tag{6}$$

and

$$0 = (N+2)(\gamma+N+1)C^{(N)}_{N+2} = 4p(N+\beta)C^{(N)}_{N} + ((N+1)N+(N+1)(\gamma+\delta-4p)-\sigma)C^{(N)}_{N+1} \tag{7}$$

Equation (6) prescribes a relation between $C^{(N)}_{N-1}$ and $C^{(N)}_{N}$ and secures that $C^{(N)}_{N+1}=0$. With this, and the $\beta=-N$ choice, $C^{(N)}_{N+2}=0$ follows from Equation (7), so the termination of the series can be reached. Assume that $p \neq 0$ the conditions can be summarized as follows:

$$\begin{aligned} \beta &= -N\,, \\ C^{(N)}_{N-1} &= \frac{1}{4p}[N(N-1)+N(\gamma+\delta-4p)-\sigma]C^{(N)}_{N}\,. \end{aligned} \tag{8}$$

Under these conditions, the confluent Heun function will reduce to a polynomial form:

$$Hc(p,\beta=-N,\gamma,\delta,\sigma;z) = \sum_{k=0}^{N} C^{(N)}_k z^k\,. \tag{9}$$

Note that a constant function, i.e., a polynomial of order $N=0$, can be the solution of Equation (2) only if $R(z)=0$ holds for $N=0$. This requirement is fulfilled for the classical orthogonal polynomials, the Jacobi, generalized Laguerre and Hermite polynomials [32], but it need not hold in the present case. In particular, the necessary condition is $\sigma=0$ for $N=0$.

Equations (5) to (8) may give a hint at the structure of $C^{(N)}_k$. First, let us introduce the notation $\alpha \equiv 4p$. It seems reasonable to assume that $C^{(N)}_k$ depends on α^k and on various factorials depending on k, N and α:

$$C^{(N)}_k = \frac{\alpha^{k-1}}{k!(N-k)!(\alpha+\mu+k)!}D^{(N)}_k\,, \tag{10}$$

where $D^{(N)}_k$ represents any further dependence of $C^{(N)}_k$ on the parameters.

We may also assume that the remaining parameters of $Hc(p=\alpha/4,\beta=-N,\gamma,\delta,\sigma;z)$ depend on α as first-order polynomials:

$$4p=\alpha\,, \quad \gamma=\alpha+b\,, \quad \delta=c\alpha+d\,, \quad \sigma=e\alpha+f\,. \tag{11}$$

Substituting (10) and (11) in (5), we arrive at a recursion relation on $D^{(N)}_k$, which simplifies to

$$\alpha(N-k)D^{(N)}_{k+1} = -k(\alpha+b-1+k)D^{(N)}_{k-1} + [k(k-1+b+d)-f-\alpha(ck-e)]D^{(N)}_k \tag{12}$$

provided that we make the choice

$$\mu = b-1\,. \tag{13}$$

Now a further assumption can be made on the structure of $D_k^{(N)}$. We may assume that it is also a first-order polynomial of α, where the coefficients depend on the remaining parameters only:

$$D_k^{(N)} = \alpha A_k^{(N)} + B_k^{(N)} \,. \tag{14}$$

Substituting this equation into (12) and collecting similar powers of on the two sides of the equation, one finds that

$$
\begin{aligned}
(N-k)A_{k+1}^{(N)} &= -kA_{k-1}^{(N)} + (kc-e)A_k^{(N)} \,, && (15)\\
(N-k)B_{k+1}^{(N)} &= -kB_{k-1}^{(N)} - k(b-1+k)A_{k-1}^{(N)} + (k(k-1+b+d)-f)A_k^{(N)} + (kc-e)B_k^{(N)} \,, && (16)\\
0 &= -k(b-1+k)B_{k-1}^{(N)} + (k(k-1+b+d)-f)B_k^{(N)} \,. && (17)
\end{aligned}
$$

In the next step, one can factor out the k-dependence from these equations by assuming that $A_k^{(N)}$ and $B_k^{(N)}$ depend on k in a polynomial form. Inspecting the left side of (16), it is reasonable to assume that the order of $A_k^{(N)}$ is one unit lower than that of $B_k^{(N)}$. A first guess might be assuming that they are first- and second-order polynomials of k, respectively:

$$A_k^{(N)} = a_0^{(N)} + a_1^{(N)}k \qquad B_k^{(N)} = b_0^{(N)} + b_1^{(N)}k + b_2^{(N)}k^2 \,. \tag{18}$$

The coefficients $a_i^{(N)}$ and $b_i^{(N)}$ depend only on the parameters b, c, d, e and f introduced in Equation (11). Substituting (18) into Equations (15) to (17) and collecting similar powers of k on the two sides, we obtain three equations from Equation (15):

$$
\begin{aligned}
ca_1^{(N)} &= 0 && (19)\\
ca_0^{(N)} + (2-e-N)a_1^{(N)} &= 0 && (20)\\
(N+e)a_0^{(N)} + Na_1^{(N)} &= 0 \,, && (21)
\end{aligned}
$$

four equations from Equation (16):

$$
\begin{aligned}
cb_2^{(N)} &= 0 && (22)\\
(d+1)a_1^{(N)} + cb_1^{(N)} + (4-e-N)b_2^{(N)} &= 0 && (23)\\
da_0^{(N)} + (b-1-f)a_1^{(N)} + cb_0^{(N)} + (2-N-e)b_1^{(N)} - 2Nb_2^{(N)} &= 0 && (24)\\
fa_0^{(N)} + (N+e)b_0^{(N)} + Nb_1^{(N)} + Nb_2^{(N)} &= 0 && (25)
\end{aligned}
$$

and also four equations from Equation (17):

$$
\begin{aligned}
(d+2)b_2^{(N)} &= 0 && (26)\\
(d+1)b_1^{(N)} + (2b-f-3)b_2^{(N)} &= 0 && (27)\\
db_0^{(N)} + (b-1-f)b_1^{(N)} + (1-b)b_2^{(N)} &= 0 && (28)\\
fb_0^{(N)} &= 0 && (29)
\end{aligned}
$$

From Equations (19) and (20), it follows that $c \neq 0$, and $a_1^{(N)} = 0$ implies $a_0^{(N)} = 0$, which, together with Equations (22) to (24) leads to $b_i^{(N)} = 0$, $i = 1,\ 2,\ 3$, i.e., to the trivial solution $C_k^{(N)} = 0$. Further solutions of Equations (19) to (21) arise from $c = 0$ and $a_1^{(N)} \neq 0$:

$$c = 0 \,, \qquad e = 2-N \,, \qquad a_0^{(N)} = -\frac{N}{2}a_1^{(N)} \,.$$

Substituting these equations into (22) to (25), one obtains

$$
\begin{aligned}
(d+1)a_1^{(N)} + 2b_2^{(N)} &= 0\,, \\
\left(-\frac{Nd}{2} + b - 1 - f\right)a_1^{(N)} - 2Nb_2^{(N)} &= 0\,, \\
-\frac{fN}{2}a_1^{(N)} + 2b_0^{(N)} + Nb_1^{(N)} + Nb_2^{(N)} &= 0\,,
\end{aligned} \tag{30}
$$

while Equations (26) to (29) remain unchanged. It is straightforward to express $b_2^{(N)}$ from Equation (30) as

$$
b_2^{(N)} = -\frac{d+1}{2}a_1^{(N)}\,. \tag{31}
$$

Substituting this into the remaining equations, one obtains

$$
\begin{aligned}
\left(\frac{Nd}{2} + n + b - 1 - f\right)a_1^{(N)} &= 0\,, \\
-\frac{N}{2}(f+d+1)a_1^{(N)} + 2b_0^{(N)} + Nb_1^{(N)} &= 0\,, \\
(d+1)(d+2)a_1^{(N)} &= 0\,, \\
b_1^{(N)} - \frac{1}{2}(2b-f-3)a_1^{(N)} &= 0\,, \\
db_0^{(N)} + (b-1-f)b_1^{(N)} + \frac{1}{2}(b-1)(d+1)a_1^{(N)} &= 0\,, \\
fb_0^{(N)} &= 0\,.
\end{aligned} \tag{32}
$$

Remembering that we assumed previously that $a_1^{(N)} \neq 0$, Equation (32) allows two possible solutions corresponding to $d = -2$ and $d = -1$.

On returning to Equations (19) to (21), we may notice that two further solutions follow from the assumption that *both* $c = 0$ and $a_1^{(N)} = 0$ hold simultaneously. Without omitting the detailed derivations, we note that in these latter two cases, the $e = -N$ choice has to be made. In what follows, we refer to the four solutions as $d = -2$, $d = -1$, $d = 0$ and $f \neq 0$.

2.1. The First Solution: $D = -2$

This solution obtained after some straightforward algebra is

$$
b = 1\,, \quad c = 0\,, \quad d = -2\,, \quad e = 2 - N\,, \quad f = 0 \tag{33}
$$

and

$$
a_0^{(N)} = -\frac{N}{2}a_1^{(N)}\,, \quad b_0^{(N)} = 0\,, \quad b_1^{(N)} = -\frac{1}{2}a_1^{(N)}\,, \quad b_2^{(N)} = \frac{1}{2}a_1^{(N)}\,.
$$

Here, $a_1^{(N)}$ is a freely choosable coefficient that determines the remaining coefficients. Equation (33) leads to the following set of parameters appearing in the confluent Heun equation:

$$
\gamma = \alpha + 1\,, \quad \delta = -2\,, \quad \sigma = (2-N)\alpha\,, \tag{34}
$$

where $\alpha = 4p$, appearing in Equation (11). Substituting all these results into (18), (14), and eventually in (10), one obtains

$$
C_k^{(N)} = \frac{\alpha^{k-1}(N-1)!\alpha!}{k!(N-k)!(\alpha+k)!}[\alpha(N-2k) - k(k-1)]\,, \tag{35}
$$

where $C_k^{(N)}$ is normalized such that $C_0^{(N)} = 1$ holds. Note that for $\alpha > 0$ the sign of $C_k^{(N)}$ is determined by $\alpha(N-2k) - k(k-1)$, which is a quadratic function of k. For $k = 0$ it is αN, due to the chosen normalization, while for $k = N$ it is negative: $-\alpha N - N(N-1)$. This means that $C_k^{(N)}$ changes sign exactly once as k proceeds from 0 to N, implying also that the polynomial has one root on the positive real z axis. Defining it on $z \leq 0$, the sign of the individual terms keeps alternating, except once, where $\alpha(N-2k) - k(k-1)$ changes sign, so there can be up to $N-1$ real roots there.

Applying the specific parameters obtained in this case in Equations (3) and (4), the confluent Heun differential equation reduces to

$$z(z-1)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + [\alpha z(z-1) + (\alpha+1)(z-1) - 2z]\frac{\mathrm{d}F}{\mathrm{d}z} + \alpha(-Nz + N - 2)F(z) = 0\,. \tag{36}$$

This equation depends on one parameter, α, and the non-negative integer N that sets the order of the polynomial solution.

It is instructive to introduce a scaling of the z variable as $z = -y/\alpha$. With this choice, the only root for $z > 0$ moves to the $y < 0$ domain. Then, Equation (36) is converted into

$$-y(y+\alpha)\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + (y-\alpha)(y+\alpha+1)\frac{\mathrm{d}F}{\mathrm{d}y} + [-Ny + \alpha(2-N)]F(z) = 0\,. \tag{37}$$

This differential equation can be recognized as that of the X_1 type exceptional Laguerre polynomials $F(y) = \hat{L}_N^{(\alpha)}(y)$ [17,33]. We may thus conclude that these exceptional polynomials represent a special case of the confluent Heun function

$$\hat{L}_N^{(\alpha)}(y) = Hc(p = \alpha/4, \beta = -N, \gamma = \alpha+1, \delta = -2, \sigma = \alpha(2-N); z = -y/\alpha)$$

up to a normalization factor. Note that the sequence of the X_1-type exceptional Laguerre polynomials starts with $N = 1$, so in contrast with classical orthogonal polynomials (generalized Laguerre, Hermite, Jacobi), it does not contain the constant function. This result can be interpreted in a simple way in the present setting. As discussed previously, $N = 0$ can occur in (9) only if $\sigma = 0$ also holds; otherwise it cannot satisfy the confluent Heun differential equation. However, we obtained that $\sigma = \alpha(2-N) \neq 0$, so the 0'th-order polynomial cannot occur.

One may compare the (35) coefficients with those appearing in the X_1-type exceptional Laguerre polynomials:

$$\hat{L}_N^{(\alpha)}(y) = \sum_{k=0}^{N} \hat{C}_k^{(N)} y^k\,.$$

These coefficients can be obtained from the relation that expresses $\hat{L}_N^{(\alpha)}(y)$ in terms of two classical generalized Laguerre polynomials [14]:

$$\hat{L}_N^{(\alpha)}(y) = -(y+\alpha+1)L_{N-1}^{(\alpha)}(y) + L_{N-2}^{(\alpha)}(y)\,. \tag{38}$$

These latter polynomials are expanded as

$$L_n^{(\alpha)}(y) = \sum_{k=0}^{n}(-1)^k \begin{pmatrix} \alpha+n \\ n-k \end{pmatrix} \frac{y^k}{k!}\,. \tag{39}$$

After some algebra, one finds that

$$\begin{aligned} \hat{C}_k^{(N)} &= \frac{(-1)^k(\alpha+N-2)!(\alpha+N)}{k!(N-k)!(\alpha+k)!}[\alpha(2k-N) + k(k-1)] \\ &= \frac{(-1)^{k+1}(\alpha+N)!}{\alpha^{k-1}\alpha!(N-1)!(\alpha+N-1)}C_k^{(N)}\,. \end{aligned} \tag{40}$$

The factors in (40) originate from two sources: $(-1)^k \alpha^{-k}$ appears due to the relation between z and y, i.e., $z = -y/\alpha$. The remaining factors are due to the different normalization used in the two cases. Note also that the sign of $\hat{C}_k^{(N)}$ alternates with k, except where $\alpha(N-2k) - k(k-1)$ changes sign, so there are $N-1$ roots for $y \geq 0$, in accordance with the basic properties of the X_1-type exceptional Laguerre polynomials.

It is instructive to examine the relation of these results with the formalism of the Sturm–Liouville approach outlined in the Appendix A. Substituting the actual parameters into (A5), one obtains

$$p(z) = z(z-1)\,, \quad q(z) = \alpha z(z-1) + (\alpha+1)(z-1) - 2z\,, \quad r(z) = -\alpha N z\,, \quad \lambda = \alpha(2-N)\,.$$

It is seen that besides λ, $r(z)$ also depends on N. However, in the derivation of the orthogonality relation (A3) and (A4), it was assumed that the $p(z)$, $q(z)$ and $r(z)$ functions are the same for all the solutions, so we conclude that the choice (A5) does not lead to a set of orthogonal polynomials. See also Equation (A9). The situation is similar to semi-classical orthogonal polynomials, for which the $p(z)$, $q(z)$ and $r(z)$ depend on N: see [34,35] and references.

With the (A7) choice, these functions are

$$p(z) = z\,, \quad q(z) = z\left(\alpha + \frac{\alpha+1}{z} - \frac{2}{z-1}\right), \quad r(z) = -\frac{2\alpha}{z-1}\,, \quad \lambda = \alpha N\,.$$

Now N appears only in the constant term λ, so the conditions for orthogonality are satisfied. In fact, this choice recovers Equation (36). The weight function is

$$w(z) = z^{\alpha}(z-1)^{-2}\exp(\alpha z)\,. \tag{41}$$

The substitution $z = -y/\alpha$ leads to the differential equation of the X_1-type exceptional Laguerre polynomials (37), while Equation (41) recovers the corresponding weight function (up to an unimportant scaling factor) [17,33]:

$$w(y) = \frac{y^{\alpha}}{(y+\alpha)^2}\exp(-y)\,.$$

The results are summarized in Table 1.

Table 1. The polynomial systems obtained for the four parameter sets and the different realizations of the differential operator T in Equation (A1).

Solution	$p(z)$	$q(z)$	$r(z)$	λ	z	**Polynomial**	$N_{\min}$	**Orthogonality**
$d=-2$	$z(z-1)$	$\alpha z(z-1) - 2z$ $+(\alpha+1)(z-1)$	$-\alpha N z$	$(2-N)\alpha$	$-\frac{y}{\alpha}$	$\hat{L}_N^{(\alpha)}(y)$	1	–
$d=-2$	z	$\alpha z + \alpha + 1 - \frac{2z}{z-1}$	$-\frac{2\alpha}{z-1}$	$N\alpha$	$-\frac{y}{\alpha}$	$\hat{L}_N^{(\alpha)}(y)$	1	+
$d=-1$	$z(z-1)$	$\alpha z(z-1) - z$ $+\left(\alpha+1-\frac{N}{2}\right)(z-1)$	$-\alpha N z$	$(2-N)\alpha$	$-\frac{y}{\alpha}$	$K_N(y)$	1	–
$d=-1$	z	$\alpha z + \alpha + 1 - \frac{N}{2} - \frac{z}{z-1}$	$-\frac{2\alpha}{z-1}$	$N\alpha$	$-\frac{y}{\alpha}$	$K_N(y)$	1	–
$d=0$	z	$-z+\gamma$	0	N	y	$L_N^{(\gamma-1)}(y)$	0	+
$f\neq 0$	$z(z-1)$	$\alpha z(z-1) - z$ $+\left(\alpha+1-\frac{N}{2}\right)(z-1)$	$-\alpha N z$	$-N\left(\alpha+\frac{1}{2}\right)$	$-\frac{y}{\alpha}$	$M_N(y)$	0	–
$f\neq 0$	z	$\alpha z + \alpha + 1 - \frac{N}{2} - \frac{z}{z-1}$	$\frac{N}{2(z-1)}$	$N\alpha$	$-\frac{y}{\alpha}$	$M_N(y)$	0	–

2.2. *The Second Solution: $D=-1$*

This solution is

$$b = 1 - \frac{N}{2}\,, \quad c = 0\,, \quad d = -1\,, \quad e = 2 - N \quad f = 0$$

and

$$a_0^{(N)} = -\frac{N}{2}a_1^{(N)}\,, \quad b_0^{(N)} = -\frac{N}{2}b_1^{(N)}\,, \quad b_2^{(N)} = 0\,.$$

Note that here, there are two freely choosable independent coefficients, $a_1^{(N)}$ and $b_1^{(N)}$. The parameters of the confluent Heun equation are now

$$\gamma = \alpha + 1 - \frac{N}{2}\,, \quad \delta = -1\,, \quad \sigma = (2-N)\alpha\,. \tag{42}$$

Note also that although now there are two independent coefficients, the $A_k^{(N)}$ and $B_k^{(N)}$ they generate in (18) are linearly dependent, so they can be merged in (14) into a single expression depending on $(2k-N)$, while the remaining constant factors contribute only to the normalization of $C_k^{(N)}$. Prescribing $C_0^{(N)} = 1$, one obtains

$$C_k^{(N)} = \frac{\alpha^k (N-1)!(\alpha - \frac{N}{2})!}{k!(N-k)!(\alpha + k - \frac{N}{2})!}(N-2k)\,. \tag{43}$$

Similar to the case discussed in Section 2.1, this polynomial is not defined for $N=0$ because $N=0$ and $\sigma=0$ cannot hold simultaneously. If we assume that $\alpha > 0$ holds, then the sign of $C_k^{(N)}$ is determined by that of $N-2k$, $(\alpha - N/2)!$ and $(\alpha + k - N/2)!$. If $\alpha \geq N/2$, then the latter two expressions are positive, so the sign is determined by $N-2k$. This quantity will change sign once as k proceeds from 0 to N (taking up also 0 at $k = N/2$ for even values of N). This means that for $z > 0$ the polynomial will start with the $C_0^{(N)} = 1$ value at $z = 0$, then tend to $-\infty$ asymptotically, so it has one node for $z > 0$. For $z < 0$, the sign of the individual terms will alternate, except at $k = N/2$, so there can be up to $N-1$ nodes. The situation is more complicated if $\alpha < N/2$, because in that case, $(\alpha - N/2)!/(\alpha + k - N/2)! = [(\alpha + 1 - N/2)(\alpha + 2 - N/2)\ldots(\alpha + k - N/2)]^{-1}$ may change sign several times as k proceeds from 0 to N, so the sign of $C_k^{(N)}$ may change in an irregular way, leading to unusual polynomial patterns.

Let us denote this polynomial as K_N, and express it as the function $y = -\alpha z$, similarly to the $d = -2$ case. This changes the coefficients (43) by a factor of $(-1)^k \alpha^{-k}$:

$$\tilde{C}_k^{(N)} = (-1)^k \frac{(N-1)!(\alpha - \frac{N}{2})!}{k!(N-k)!(\alpha + k - \frac{N}{2})!}(N-2k)\,, \tag{44}$$

so $K_N(y)$ is expressed as

$$K_N(y) = Hc(p = \alpha/4, \beta = -N, \gamma = \alpha + 1 - N/2, \delta = -1, \sigma = \alpha(2-N); z = -y/\alpha) \tag{45}$$

$$= \sum_{k=0}^{N} \tilde{C}_k^{(N)} y^k\,. \tag{46}$$

Note that $K_N(y)$ is at least a first-order polynomial, so $N > 0$ is prescribed. We can also expand $K_N(y)$ in terms of generalized Laguerre polynomials:

$$\begin{aligned} K_N(y) &= \sum_{n=0}^{N} s_n^{(N)} L_n^{(\alpha)}(y) \\ &= \sum_{n=0}^{N} s_n^{(N)} \sum_{k=0}^{N} (-1)^k \binom{\alpha+n}{n-k} \frac{y^k}{k!} \\ &= \sum_{k=0}^{N} (-1)^k y^k \frac{\alpha!}{(\alpha+k)!k!} \sum_{n=0}^{N} s_n^{(N)} \frac{(\alpha+n)!}{\alpha!(n-k)!}\,. \end{aligned} \tag{47}$$

Substituting (44) in Equation (46) and comparing the coefficients of y^k with those appearing in Equation (47), one arrives at the relation

$$\sum_{n=0}^{N} s_n^{(N)} \frac{(\alpha+n)!}{\alpha!} \frac{(N-k)!}{(n-k)!} = \frac{(\alpha+k)!}{\alpha!} \frac{(\alpha-N/2)!}{(\alpha-N/2+k)!} (N-1)!(N-2k) \,. \tag{48}$$

This equation holds for any allowed value of k, i.e., $k = 0, 1, \ldots N$, so it represents a set of a set of $N+1$ algebraic equations for the coefficients $s_n^{(N)}$. Setting $k = N$ implies that all the terms on the left handside of Equation (48) vanish due to the $(n-N)!$ factor in the denominator, except for that with $n = N$. So we immediately find that

$$s_N^{(N)} = -N! \frac{(\alpha-N/2)!}{(\alpha+N/2)!} \,. \tag{49}$$

Similarly, taking $k = N-1$ only the terms with $s_N^{(N)}$ and $s_{N-1}^{(N)}$ remain. Substituting (49) into (48), the relation

$$s_{N-1}^{(N)} = (N-1)! \frac{(\alpha-N/2)!}{(\alpha+N/2)!} \left(\frac{N^2}{2} + N + 2\alpha \right) \tag{50}$$

is obtained. The same algorithm can be used to determine any further coefficient $s_n^{(N)}$.

The comparison with the Sturm–Liouville approach outlined in the Appendix A reveals that this set of polynomials does not correspond to an orthogonal set. This is because γ in Equation (42) depends on N, so the $q(z)$ function will not be the same for all the solutions. In fact, the weight function (A2) would also be solution-dependent, as can be seen in Table 1. For this reason, the orthogonality relation in Equations (A3) and (A4) cannot be derived in this case. See also Equation (A9). This is another example reminiscent of semi-classical orthogonal polynomials [34,35].

2.3. The Third Solution: $D = 0$

This solution follows from the choice $c = 0$, $a_1^{(N)} = 0$ and $a_0^{(N)} \neq 0$ in Equations (19) to (21). One of the two possible solutions is

$$c = 0\,, \quad d = 0\,, \quad e = -N\,, \quad f = 0\,.$$

with

$$a_1^{(N)} = 0\,, \quad b_1^{(N)} = 0\,, \quad b_2^{(N)} = 0\,.$$

Now $a_0^{(N)}$ and $b_0^{(N)}$ can be chosen freely, so $D_k^{(N)}$ in Equation (14) is a constant, independent of k, and thus can be adjusted to the required normalization of the $C_k^{(N)}$ expansion coefficients. There is no restriction on b either, which means that the γ parameter is also unrestricted (apart from the requirement that it has to be different from a negative integer). In addition to the general requirement $\beta = -N$, the remaining parameters obey the relations

$$\delta = 0\,, \quad \sigma = -N\alpha \tag{51}$$

where $\alpha = 4p$. With these, and choosing $D_k^{(N)} = 1$, the expansion coefficients (10) take the form

$$C_k^{(N)} = \frac{\alpha^{k-1}}{k!(N-k)!(\gamma-1+k)!} \,.$$

It may be noted that for $\alpha = -1$, these coefficients recover those of the generalized Laguerre polynomials [32]: see Equation (39) with the α and n used there replaced with $\gamma - 1$ and N. The usual variable transformation $z = -y/\alpha$ reduces now to the identity $z = y$, so Equations (3) and (4) recover the differential equation of the generalized Laguerre polynomials with $Q(y) = -1 + \gamma/y$ and $R(y) = N/y$. Note that the expressions with $(y-1)^{-1}$ are canceled both from $Q(y)$ (due to $\delta = 0$) and from $R(z)$ (due to $4p\beta = \sigma$). The fact that $R(y)$ vanishes for $N = 0$ means that in contrast with the previous two cases, the polynomial series starts with a first-order (constant) member, as it should, in the case of a classical orthogonal polynomial. In summary, one finds that for this set of parameters, the confluent Heun equation reduces to the generalized Laguerre polynomials

$$L_N^{(\gamma-1)}(y) = Hc(p = -1/4, \beta = -N, \gamma, \delta = 0, \sigma = N; y)$$

up to a normalization factor. Note that this case appears in Table 1 only once because the two choices of the $p(z)$, $q(z)$ and $r(z)$ functions (see Equations (A5) and (A7)) degenerate.

2.4. The Fourth Solution: $F \neq 0$

This solution is the other one following from the $c = 0$, $a_1^{(N)} = 0$ and $a_0^{(N)} \neq 0$ choice:

$$b = 1 - \frac{N}{2}\,, \quad c = 0\,, \quad d = -1\,, \quad e = -N \quad f = -\frac{N}{2}$$

and

$$a_1^{(N)} = 0\,, \quad b_0^{(N)} = 0\,, \quad b_1^{(N)} = \frac{1}{2}a_0^{(N)}\,, \quad b_2^{(N)} = 0\,.$$

The parameters of the confluent Heun equation are

$$\gamma = \alpha + 1 - \frac{N}{2}\,, \quad \delta = -1\,, \quad \sigma = -N\left(\alpha + \frac{1}{2}\right),$$

in addition to the generally valid $\beta = -N$ and $4p = \alpha$ relations. In contrast with the first and the second solutions, i.e., the case of the X_1-type exceptional Laguerre and the $K_N(y)$ polynomials, this series starts with $N = 0$ because $N = 0$ and $\sigma = 0$ can occur simultaneously, i.e., the $R(y)$ function in Equation (4) vanishes for $N = 0$.

Taking the normalization $C_0^{(N)} = 1$, the expansion coefficients turn out to be

$$C_k^{(N)} = \frac{\alpha^k N!(\alpha - \frac{N}{2})!}{k!(N-k)!(\alpha + k - \frac{N}{2})!}\left(\alpha + \frac{1}{2}\right). \tag{52}$$

Note that if $\alpha > N/2$ holds, then each term in (52) is positive, so the polynomial (9) will be strictly positive in the $z > 0$ domain and will have no nodes there. (For $\alpha < N/2$, the sign of the coefficients will be determined by the product $(\alpha - N/2)!/(\alpha + k - N/2)! = [(\alpha + 1 - N/2)(\alpha + 2 - N/2)\ldots(\alpha + k - N/2)]^{-1}$, which may change sign as k proceeds from 0 to N).

With the usual variable transformation $z = -y/\alpha$, we can define the polynomial containing expansion coefficients with an alternating sign and in the $y > 0$ domain, where its roots will be located. With this transformation, the coefficients are transformed into

$$\tilde{C}_k^{(N)} = \frac{(-1)^k N!(\alpha - \frac{N}{2})!}{k!(N-k)!(\alpha + k - \frac{N}{2})!}\left(\alpha + \frac{1}{2}\right).$$

Denoting this polynomial as $M_N(y)$, it can be expressed in terms of the confluent Heun function as

$$M_N(y) = Hc(p = \alpha/4, \beta = -N, \gamma = \alpha + 1 - N/2, \delta = -1, \sigma = -N(\alpha + 1/2); z = -y/\alpha)\,.$$

Similar to $K_N(y)$, this polynomial can also be expanded in terms of generalized Laguerre polynomials. The equations corresponding to (48), (49) and (50) are

$$\sum_{n=0}^{N} s_n^{(N)} \frac{(\alpha+n)!}{\alpha!} \frac{(N-k)!}{(n-k)!} = \frac{(\alpha+k)!}{\alpha!} \frac{(\alpha-N/2)!}{(\alpha-N/2+k)!} \frac{N!}{\alpha} \left(\alpha + \frac{1}{2}\right) ,$$

$$s_N^{(N)} = \frac{N!(\alpha-N/2)!}{\alpha(\alpha+N/2-1)!}$$

and

$$s_{N-1}^{(N)} = -\frac{(N+1)!(\alpha-N/2)!}{2\alpha(\alpha+N/2-1)!} .$$

Again, any further coefficient $s_n^{(N)}$ can be determined by the same algorithm.

The M_N polynomials show some similarity to the K_N set. Since p, γ and δ are the same as in the two cases, so is the $q(z)$ function appearing in the actual form of Equation (1). Furthermore, since $p(z)$ is also the same (see Equations (A5) and (A7)), the weight functions (A6) and (A8) will also be the same. However, the $r(z)$ functions and the λ eigenvalue will be different. What is also common is that both $q(z)$ and $r(z)$ depend on N, so M_N represents another example for properties similar to semi-classical orthogonal polynomials [35].

3. Potentials with Polynomial Solutions of the Confluent Heun Equation

The general method of transforming the second-order differential equation of the form (2) into the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 ,$$

makes use of a variable transformation $z(x)$ and the substitution

$$\psi(x) = f(x)F(z(x)) .$$

It is then straightforward to show that E, $V(x)$ and $\psi(x)$ are obtained in terms of $Q(z)$, $R(z)$, $F(z)$ and $z(x)$ as

$$\begin{aligned} E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ &\quad + (z'(x))^2\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right) \end{aligned} \tag{53}$$

and

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)\mathrm{d}z\right) F(z(x)) . \tag{54}$$

In the next step, constant terms are defined on the right-hand side of (53) to account for E on its left-hand side. This requirement defines a first-order differential equation for $z(x)$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \Phi(z) = C \tag{55}$$

such that the inverse $x(z)$ function can be obtained by direct integration:

$$\int \Phi^{1/2}(z)\mathrm{d}z = C^{1/2}x + x_0 .$$

where $\Phi(z)$ is chosen in such a way that it coincides with one (ore more) terms of Equation (53) originating from $R(z)$ and $Q(z)$. Combining these equations, one arrives at

$$\begin{aligned} E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ &+ \frac{C}{\Phi(z(x))}\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right), \end{aligned} \tag{56}$$

where the construction guarantees that there will be a constant term on the right-hand side of Equation (56). The general formalism introduced in Ref. [36] has been applied to the classical orthogonal polynomials [37], exceptional orthogonal polynomials [17], the hypergeometric function [10] and the symmetrical canonical form of the confluent Heun equation [38]. Other transformation methods with a somewhat different approach have also been introduced for the (confluent) hypergeometric function, leading to the Natanzon potential class [1] (the relation of these methods is discussed in Ref. [10] and Chapter 7 of Ref. [12]) and the non-symmetrical canonical form of the confluent Heun equation [25] (see Ref. [38] for the connection with the present approach).

Applying the general method to the non-symmetrical canonical form of the confluent Heun equation with $Q(z)$ and $R(z)$ appearing in Equations (3) and (4), one arrives at

$$\begin{aligned} E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\ &+ (z'(x))^2\left(-\frac{\alpha^2}{4} + \frac{2\sigma - \alpha\gamma + \delta\gamma}{2z(x)} + \frac{2\alpha\beta - 2\sigma - \alpha\delta - \gamma\delta}{2(z(x)-1)} \right. \\ &\left. + \frac{\gamma(2-\gamma)}{4z^2(x)} + \frac{\delta(2-\delta)}{4(z(x)-1)^2}\right) \end{aligned} \tag{57}$$

and

$$f(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{\alpha}{2}z(x)\right)(z(x))^{\gamma/2}(z(x)-1)^{\delta/2}. \tag{58}$$

In the general case, the constant (E) term will be the linear combination of the five terms appearing in the parentheses on the right-hand side of Equation (57), so the equivalent of Equation (55) will be now

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \Phi(z(x)) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \frac{\phi(z(x))}{[z(x)(z(x)-1)]^2} = C\,, \tag{59}$$

i.e.,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \pm C^{1/2} z(x)(z(x)-1)[\phi(z(x))]^{-1/2}\,, \tag{60}$$

where

$$\phi(z(x)) = p_1 z^2(x)(z(x)-1)^2 + p_2 z(x)(z(x)-1)^2 + p_3 z^2(x)(z(x)-1) + p_4(z(x)-1)^2 + p_5 z^2(x)\,. \tag{61}$$

Here, the p_i coefficients determine the $x(z)$ function after the integration of Equation (60). Note that $\phi(z(x))$ also appears in $f(x)$ after combining Equations (58) and (59):

$$f(x) \sim \phi^{1/4}(z(x)) \exp\left(\frac{\alpha}{2}z(x)\right)(z(x))^{(\gamma-1)/2}(z(x)-1)^{(\delta-1)/2}. \tag{62}$$

Note that for $\alpha > 0$, the exponential factor becomes unbound in the $z \to \infty$ limit. Therefore, normalizable functions are expected either if $z(x)$ remains bounded, or if $z(x) < 0$.

The general form of the potential will contain the same terms, supplemented with those originating from the two terms of the Schwartzian derivative containing higher derivatives of $z(x)$:

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{C}{\phi(z(x))}\Big[s_1 z^2(x)(z(x)-1)^2 + s_2 z(x)(z(x)-1)^2 + s_3 z^2(x)(z(x)-1) + s_4(z(x)-1)^2 + s_5 z^2(x)\Big]. \tag{63}$$

Substituting $V(x)$ from Equation (63) and $(z'(x))^2$ from Equation (59) into Equation (57) and comparing the corresponding terms, one finds that the following five equations have to be satisfied simultaneously:

$$s_1 - p_1\frac{E}{C} - \frac{\alpha^2}{4} = 0\,, \tag{64}$$

$$s_2 - p_2\frac{E}{C} + \sigma - \frac{\alpha\gamma}{2} + \frac{\delta\gamma}{2} = 0\,, \tag{65}$$

$$s_3 - p_3\frac{E}{C} + \alpha\beta - \sigma - \frac{\alpha\delta}{2} - \frac{\gamma\delta}{2} = 0\,, \tag{66}$$

$$s_4 - p_4\frac{E}{C} - \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right) = 0\,, \tag{67}$$

$$s_5 - p_5\frac{E}{C} - \frac{\delta}{2}\left(\frac{\delta}{2} - 1\right) = 0\,. \tag{68}$$

This set of equations connects the parameters p_i appearing in the $z(x)$ function, the coupling coefficients (s_i) of the potential (63), the energy eigenvalue E and the parameters of the confluent Heun equation. It has to be solved under the condition that in the case of polynomial solutions, N, the degree of the polynomial does not appear in the s_i coupling coefficients and the p_i parameters; otherwise, the potential $V(x)$ will be state-dependent.

3.1. The Potential Obtained from the First Solution

Specifying this result to the polynomial solution with $\beta = -N$ and the parameter set obtained for $d = -2$, i.e., Equation (34), one finds that

$$s_1 - p_1\frac{E}{C} - \frac{\alpha^2}{4} = 0\,, \tag{69}$$

$$s_2 - p_2\frac{E}{C} - \frac{\alpha^2}{2} + \frac{\alpha}{2} - 1 - N\alpha = 0\,, \tag{70}$$

$$s_3 - p_3\frac{E}{C} + 1 = 0\,, \tag{71}$$

$$s_4 - p_4\frac{E}{C} - \frac{\alpha+1}{2}\frac{\alpha-1}{2} = 0\,, \tag{72}$$

$$s_5 - p_5\frac{E}{C} - 2 = 0\,. \tag{73}$$

Note that N appears only in Equation (70), so it is reasonable to determine E from this expression. This means that $p_i = 0$ has to hold for $i \neq 2$, while the $p_2 = 1$ choice can be made without the loss of generality. (C remains in Equation (70) and can be used to rescale E arbitrarily). Note that this selection immediately sets the coupling coefficients s_1, s_3, s_4 and s_5 in terms of α and some numerical factors.

Furthermore, from Equation (61) one has $\phi(z) = z(z-1)^2$, i.e., Equation (60) leads to

$$\frac{\mathrm{d}z}{\mathrm{d}x} = C^{1/2} z^{1/2}(x)\,.$$

The resulting potential corresponds to the choice $m_1 = 1/2$, $m_2 = 0$ in Table 1 of Ref. [25], where this potential is displayed without specifying the coupling coefficients of the individual potential terms.

However, it is reasonable to consider the rescaling $z = -y/\alpha$ introduced in Section 2.1 in order to obtain the X_1 type exceptional Laguerre polynomial as solutions. With this, and the following choice of the parameters

$$C = -2\omega/\alpha\,, \qquad \alpha = l + 1/2\,, \qquad s_2 = 1$$

one obtains $y(x) = \frac{\omega}{2}x^2$ and the formulae relevant to the rationally extended harmonic oscillator [17,33]:

$$V(x) = \frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2} + \frac{4\omega}{\omega x^2 + 2l + 1} - \frac{8\omega(2l+1)}{(\omega x^2 + 2l + 1)^2}\,, \tag{74}$$

$$E_N = \omega(2N + l - 1/2)$$

$$\psi_N(x) \sim \frac{x^{l+1}}{\omega x^2 + 2l + 1}\exp(-\frac{\omega}{4}x^2)\hat{L}_N^{(l+1/2)}(\frac{\omega}{2}x^2)\,.$$

Note that N corresponds to $N = \nu + 1$ in the notation of Ref. [17], where $\nu = 0,\ 1, \ldots$ labels the actual degree of the exceptional Laguerre polynomial. Similar to other potentials with solutions containing exceptional orthogonal plynomials (see Ref. [39]), this potential can also be obtained from the conventional harmonic oscillator by a supersymmetric transformation with broken supersymmetry. For a pedagogical review, see Ref. [40]. Actually, potential (74) was derived [41] by SUSY transformations from the radial harmonic oscillator a decade before the concept of rationally extended potentials was introduced. However, its importance as a new shape-invariant potential class was not realized at that time.

3.2. The Cases of the Second and Fourth Solutions

It was discussed previously in Sections 2.2 and 2.4 that the polynomial solutions obtained there do not form an orthogonal set. This is because the $q(z)$ (and $Q(z)$) functions that appear in the linear derivative term of the Sturm–Liouville (and the confluent Heun) equation depends on N, the degree of the polynomial. This formally also makes the weight function state-dependent. This complication also appears in the actual form of Equations (64) to (68). Substituting the parameters (42) in them, one obtains

$$s_1 - p_1\frac{E}{C} - \frac{\alpha^2}{4} = 0\,, \tag{75}$$

$$s_2 - p_2\frac{E}{C} - \frac{1}{2}(\alpha-1)^2 - \frac{N}{4}(3\alpha - 1) = 0\,, \tag{76}$$

$$s_3 - p_3\frac{E}{C} - \alpha + \frac{1}{2} - \frac{N}{4} = 0\,, \tag{77}$$

$$s_4 - p_4\frac{E}{C} - \frac{1}{4}\left(\alpha - \frac{N}{2}\right)^2 + \frac{1}{4}\,, \tag{78}$$

$$s_5 - p_5\frac{E}{C} - \frac{3}{4} = 0\,. \tag{79}$$

for the second solution (the K_N polynomials) and

$$
\begin{aligned}
&s_1 - p_1\frac{E}{C} - \frac{\alpha^2}{4} = 0\,, \\
&s_2 - p_2\frac{E}{C} - \frac{1}{2}(\alpha+1)^2 - \frac{N}{4}(5\alpha+3) = 0\,, \\
&s_3 - p_3\frac{E}{C} + \alpha + \frac{1}{2} + \frac{N}{4} = 0\,, \\
&s_4 - p_4\frac{E}{C} - \frac{1}{4}\left(\alpha - \frac{N}{2}\right)^2 + \frac{1}{4}\,, \\
&s_5 - p_5\frac{E}{C} - \frac{3}{4} = 0\,.
\end{aligned}
$$

for the fourth solution (the M_N polynomials).

Three of the equations are identical, while the remaining two (the second and the third) differ only in some simple factors.

These equations have to be satisfied simultaneously under the condition that the s_i coupling coefficients cannot depend on N. The p_i parameters defining the variable $z(x)$ transformation function (see Equations (60) and (61)) are also not expected to depend on N, so N is allowed to appear only in the energy eigenvalue E and the α parameter.

It is seen that now three of the five equations depend on N in both cases. This situation is clearly different from that found for Natanzon-class potentials (including the shape-invariant ones too), where N entered the formulas exclusively through $R(z)$, so it appeared in a single term of Equation (53). This was the case also with the first polynomial solution discussed previously, as N occurred there only in the expression containing σ and β (see (34), i.e., in $R(z)$ in Equation (4)). Now besides σ and β, γ is also dependent on N, which means that N appears also in terms originating from $Q(z)$ in (53), as can be seen from Equation (57).

A non-zero p_5 in Equation (79) would lead to a constant E, so $p_5 = 0$ has to be taken, leading to $s_5 = 3/4$. The $p_1 = 0$ choice in Equation (75) leads to $\alpha = 2s_1^{1/2} = const.$, while $p_1 \neq 0$ implies that E depends on N though $\alpha = \alpha(N)$. The former choice leads to contradiction, as the remaining three equations result in different E_N expressions: in Equations (76) and (77), E is a linear function of N, while in Equation (78), it is quadratic. However, it turns out that by allowing α to be dependent on N (by the $p_1 \neq 0$ choice), there is no $\alpha(N)$ function that would satisfy Equations (76)–(78) simultaneously. The same holds for the corresponding equations for the fourth solution M_N. All these circumstances indicate that the second and fourth polynomial solutions of the confluent Heun equation discussed in Sections 2.2 and 2.4 do not form an appropriate basis to generate solvable potentials of the Schrödinger equation. Only state- and energy-dependent potentials could be constructed by employing them.

3.3. Potentials Obtained from the Third Solution

The parameters obtained for the third solution, i.e., $\beta = -N$, $p = -1/4$ and those in Equation (51), lead to

$$s_1 - p_1\frac{E}{C} - \frac{1}{4} = 0\,, \tag{80}$$

$$s_2 - p_2\frac{E}{C} + N + \frac{\gamma}{2} = 0\,, \tag{81}$$

$$s_3 - p_3\frac{E}{C} = 0\,, \tag{82}$$

$$s_4 - p_4\frac{E}{C} - \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right) = 0\,, \tag{83}$$

$$s_5 - p_5\frac{E}{C} = 0\,. \tag{84}$$

Here, Equations (82) and (84) can be fulfilled for arbitrary E only with the $s_3 = 0$, $p_3 = 0$, $s_5 = 0$ and $p_5 = 0$ choices, so there are only three equations to be satisfied simultaneously in this case. In fact, one finds that in Equation (57) the two potential terms containing $(z(x) - 1)^{-1}$ and $(z(x) - 1)^{-2}$ are canceled. This also implies that the corresponding two terms are also canceled in Equation (61). The remaining three equations define the three shape-invariant potentials related to the generalized Laguerre polynomial [37]. In particular, taking $p_1 \neq 0$, $p_2 \neq 0$ and $p_4 \neq 0$ leads to the Coulomb, radial harmonic oscillator and the Morse potentials. These choices denoted as the LII, LI and LIII cases in Ref. [37] correspond to $\phi(z) = z^2(z-1)^2$, $z^2(z-1)$ and $(z-1)^2$. Substituting these into Equation (60), the well-known transformation functions $z(x) \sim x$, $z(x) \sim x^2$ and $z(x) \sim \exp(-cx)$ are recovered.

It may be noted that taking two of the three parameters non-zero, i.e., with $p_1 \neq 0$, $p_2 \neq 0$ and $p_4 = 0$, one can recover the generalized Coulomb potential [42]. In this case $\phi(z) \sim (z-1)^2(p_1 z^2 + p_2 z)$, and the integration of (60) results in an inverse $x(z)$ function, i.e., $z(x)$ is an implicit function. In spite of this, all the formulas can be expressed in closed analytical form. This potential carries the features of both "parent" potentials: it is oscillator-like near the origin $x = 0$ and for the low-lying states with moderate N, while it is Coulomb-like asymptotically and for $N \to \infty$.

4. Summary and Outlook

The polynomial solutions of the confluent Heun differential equation (CHE) were investigated with the intention of generating exactly solvable potentials from them. The solutions were written in terms of a power series expansion around the $z = 0$ singular point. The expansion coefficients were found to satisfy a three-term recurrence relation. The conditions of terminating the series were established, and the coefficients $C_k^{(N)}$ were expressed in terms of a parameter set that was related to the CHE parameters p, γ, δ and σ. The recursion relation and the termination conditions resulted in a system of algebraic equations on the parameters. It was found that this system has four non-trivial solutions, corresponding to four different polynomial solutions of the confluent Heun differential equation.

The first solution resulted in a polynomial system that started with $N = 1$, i.e., its first element was a first-order polynomial. This finding could be given a natural explanation in the present framework. It was shown that two different weight functions could be defined for this polynomial system. Taking one of them, the X_1-type exceptional Laguerre polynomials $\hat{L}_N^{(\alpha)}(y)$ were recovered. With the other option, a non-orthogonal polynomial system was identified.

Another solution resulted in parameters that reduced the confluent Heun differential equation to that of the generalized Laguerre polynomials, and the polynomials also recovered the latter polynomials $L_N^{(\gamma-1)}(y)$.

The remaining two solutions resulted in two distinct polynomials that could be defined with two different weight functions each. One of these polynomials started with $N = 0$, while the other one with $N = 1$. However, neither of the four combinations represented an orthogonal set. This is because the γ parameter that appears in the linear differential term of the CHE showed dependence on N. This is a feature characteristic of semi-classical orthogonal polynomials.

The polynomials identified in this way were applied within a transformation method that generates exactly solvable potentials in the one-dimensional Schrödinger equation. The formalism was general enough to recover all known exactly solvable potentials related to the generalized Laguerre polynomials: the three shape-invariant potentials (Coulomb, harmonic oscillator, and Morse), the generalized Coulomb potential, which is a non-trivial member of the Natanzon confluent potential class, and the rationally extended harmonic oscillator, which is outside this class. However, it was found that the two remaining solutions with non-orthogonal polynomial systems were not suitable to generate further exactly solvable potentials, unless the potentials are allowed to be state-dependent. The present study revealed the importance of the confluent Heun equation: it offers a unified mathematical framework from which all known potentials solvable in terms of generalized Laguerre polynomials can be obtained in a systematic way.

These studies gave an opportunity to combine and compare the formalism of the confluent Heun differential equation and that of the rationally extended Laguerre polynomials. First, it can be established that the polynomial solutions of the CHE represent a natural framework to apply the rational extension of Bochner-type differential equations. This extension is clearly established for the X_1-type Laguerre polynomials. However, for the X_m-type Laguerre polynomials, the differential equation contains rational expressions that cannot be reproduced using the confluent Heun differential equation.

The fact that the X_1-type rationally extended Laguerre polynomials are expressed as the combination of two ordinary generalized Laguerre polynomials can also be interpreted in a natural way. The $\hat{L}_N^{(\alpha)}(y)$ X_1-type Laguerre polynomials should be expandable in terms of the generalized Laguerre polynomials $L_N^{(\alpha)}(y)$ up to the order N. However, since the latter polynomials satisfy a three-term recurrence relation, it is always possible to rewrite the expansion in terms of only two of the generalized Laguerre polynomials, for example, as in Equation (38), or in terms $L_N^{(\alpha)}(y)$ and $L_{N-1}^{(\alpha)}(y)$. Such a structure also arises naturally in a wave function obtained from a supersymmetry transformation applied to a harmonic oscillator wave function: there, the first-order differential operator produces two terms, one with $L_N^{(\alpha)}(y)$ and one with its derivative, which is expressed in terms of lower-order polynomials.

The structure of the $Q(z)$ function (3) also determines the structure of the possible wave functions derived from the CHE: see Equations (54) and (58). The exponential term in (58) originates from the constant term $4p = \alpha$ in Equation (3). This term naturally restricts any physical solution either to $z \to \infty$ or $z \to -\infty$. The remaining two terms in Equation (3) lead to power-like expressions in Equation (58).

The fact that the potentials generated from the CHE have more general structure than the Natanzon and Natanzon confluent potentials also has a straightforward interpretation in the present formalism. The $Q(z)$ function appearing in the linear derivative term of the CHE contains three independent terms, while that appearing in the (confluent) hypergeometric function has only two. According to Equation (53), this results in five significant terms in the potential in the CHE case (see Equation (63)) so the key set of Equations (64)–(68) contains five members. In the case of Natanzon (confluent) potentials, one has two terms in $Q(z)$ and thus three significant terms in the potential and three equations [10]. As seen in Section 3.3, the CHE can reduce to the differential equation of the generalized Laguerre polynomial, and in that case, the formulas simplify.

All these considerations present a suitable illustration to the concepts along which the present field has evolved: it has received impact both from the mathematical side (the theory of ordinary linear differential equations) and the physical one (generating acceptable solvable quantum mechanical potentials). The combination of these two approaches turned out to be mutually beneficial for both communities.

The present findings may give inspiration to further investigations. More general forms of the expansion coefficients can be assumed than those in Equations (11), (14) and (18). The polynomial solutions of the remaining Heun-type differential equations can also be studied in a similar fashion.

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Appendix A

Here, the formalism of the Sturm–Liouville theory is linked to that applied in the present work. The results are used to derive a weight function with respect to which the orthogonality of the polynomials can be inspected.

Following the formalism of Ref. [20], the general form of the differential equation can be written as

$$T(z) \equiv p(z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + r(z)F(z) = \lambda F(z)\,. \tag{A1}$$

This equation is related to Equation (2) via $Q(z) = q(z)/p(z)$ and $R(z) = (r(z) - \lambda)/p(z)$. Defining the $w(z)$ function as

$$w(z) = p^{-1}(z)\exp\left(\int^z \frac{q(t)}{p(t)}\mathrm{d}t\right), \tag{A2}$$

Equation (A1) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)w(z)\frac{\mathrm{d}F}{\mathrm{d}z}\right) + r(z)w(z)F(z) = \lambda w(z)F(z)\,.$$

Applying the T operator defined in Equation (A1) to two different solutions $F_N(z)$ and $F_M(z)$, one finds

$$(T(F_N)F_M(z) - T(F_M)F_N(z))w(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(p(z)w(z)\left(\frac{\mathrm{d}F_N}{\mathrm{d}z}F_M(z) - \frac{\mathrm{d}F_M}{\mathrm{d}z}F_N(z)\right)\right).$$

Integration on the domain $[z_1, z_2]$ leads to

$$\int_{z_1}^{z_2}(T(F_N)F_M(z) - T(F_M)F_N(z))w(z)\mathrm{d}z = \left[p(z)w(z)\left(\frac{\mathrm{d}F_N}{\mathrm{d}z}F_M(z) - \frac{\mathrm{d}F_M}{\mathrm{d}z}F_N(z)\right)\right]_{z_1}^{z_2} \tag{A3}$$

$$= (\lambda_N - \lambda_M)\int_{z_1}^{z_2} F_N(z)F_M(z)w(z)\mathrm{d}z\,. \tag{A4}$$

If the expression in Equation (A3) vanishes, then the orthogonality of the $F_N(z)$ and $F_M(z)$ functions with respect to the $w(z)$ weight function is secured. This occurs, for example, if the expression within the square brackets vanishes at the boundaries.

When applying the present formalism to the confluent Heun differential equation, there are several options to define the $p(z)$, $q(z)$ and $r(z)$ functions in Equation (A1).

The first choice is using polynomials:

$$p(z) = z(z-1)\,, \qquad q(z) = 4pz(z-1) + \gamma(z-1) + \delta z\,, \qquad r(z) = 4p\beta z\,, \qquad \lambda = \sigma\,. \tag{A5}$$

This choice leads to the weight function

$$w(z) = z^{\gamma-1}(z-1)^{\delta-1}\exp(4pz)\,. \tag{A6}$$

In the second choice, $q(z)$ and $r(z)$ are rational functions:

$$p(z) = z\,, \qquad q(z) = z\left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right), \qquad r(z) = \frac{4p\beta - \sigma}{z-1}\,, \qquad \lambda = 4p\beta\,. \tag{A7}$$

Now the weight function is

$$w(z) = z^{\gamma-1}(z-1)^{\delta}\exp(4pz)\,. \tag{A8}$$

z_1 and z_2 determine whether the expression in (A3) vanishes or not. In both cases $p(z)w(z) = z^{\gamma}(z-1)^{\delta}\exp(4pz)$ holds. We assume that the $F_N(z)$ solutions are polynomials in z. Under these conditions, the expression vanishes at $z = 0$ provided that $\gamma > 0$ and it also vanishes at $z = \infty$ or $z = -\infty$, depending on the sign of $4p$.

It is worthwhile to discuss the case when the $p(z)$, $q(z)$ and $r(z)$ functions depend on N. This corresponds to semi-classical orthogonal polynomials [34,35]. Here, we consider the case when $q(z)$ and $r(z)$ show this dependence:

$$q_N(z) = q(z) + q(z,N) \qquad\qquad r_N(z) = r(z) + r(z,N)\,,$$

where the dependence on N is separated into $q(z,N)$ and $r(z,N)$. In this case, Equation (A3) picks up a new term:

$$\begin{aligned}\int_{z_1}^{z_2}(T(F_N)F_M(z) \quad &- \quad T(F_M)F_N(z))w(z)\mathrm{d}z = \left[p(z)w(z)\left(\frac{\mathrm{d}F_N}{\mathrm{d}z}F_M(z) - \frac{\mathrm{d}F_M}{\mathrm{d}z}F_N(z)\right)\right]_{z_1}^{z_2} \\ &+ \int_{z_1}^{z_2} w(z)[q(z,N)\frac{\mathrm{d}F_N}{\mathrm{d}z}F_M(z) - q(z,M)\frac{\mathrm{d}F_M}{\mathrm{d}z}F_N(z) \\ &+ (r(z,N) - r(z,M))F_N(z)F_M(z)]\mathrm{d}z \\ &= \quad (\lambda_N - \lambda_M)\int_{z_1}^{z_2} F_N(z)F_M(z)w(z)\mathrm{d}z\,.\end{aligned} \tag{A9}$$

Due to the new term, orthogonality is canceled in general.

Note that the one-dimensional Schrödinger equation can also be inspected in the present formalism. This corresponds to $p(z) = -1$, $q(z) = 0$ and $r(z) = V(z)$, from which $w(z) = const.$ follows directly. The orthogonality of the states then follows automatically, provided that the potential $V(z)$ is independent of N.

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