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Blow-Up Criterion and Persistence Property to a Generalized Camassa–Holm Equation

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Abstract: In this paper, a generalized Camassa–Holm equation, which may be used to describe wave motion in the shallow water, is considered. Some dynamic properties are studied for the model. Firstly, a new blow-up criterion for the equation is established. Then, analytical solutions are presented for the first time by using a new method. Finally, we investigate the persistence property for strong solutions. The results we obtain complement earlier results in this direction.

Keywords: wave-breaking criterion; analytical solution; persistence property; a generalized Camassa–Holm equation

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1. Introduction

A series of generalized Camassa–Holm equations, including both quadratic nonlinearity and cubic nonlinearity, which admit integrability and an infinite hierarchy of quasi-local higher symmetries, are derived by Novikov [1] using the perturbative symmetry approach. They are of the following structure

$$(1-\partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}, \ldots), \quad u = u(t,x), \tag{1}$$

where F is some function of u and its derivatives with respect to x, and the subscript denotes a partial derivative. Among them, one of the most famous examples is the Camassa–Holm equation

$$(1-\partial_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \tag{2}$$

which was deduced by Fokas and Fuchssteiner [2] and Camassa and Holm [3], respectively, to describe the wave motion of shallow water. It displays many remarkable properties, which include a Lax pair, a bi-Hamiltonian structure, and infinitely many conserved integrals [3]. Additionally, it can be solved by the inverse scattering method. The unusual features of the Camassa–Holm equation are that it has the peakon solutions [3] and the so-called wave breaking phenomena, that is, the wave profile remains bounded while its lope becomes unbounded in finite time [4]. More information about the Camassa–Holm equation can be found in [5–11]

The other celebrated example is the Degasperis–Procesi equation

$$(1-\partial_x^2)u_t = -4uu_x + 3u_x u_{xx} + uu_{xxx}. \tag{3}$$

Degasperis, Holm, and Hone [12] derived the formal integrability of Equation (3) by constructing a Lax pair. Equation (3) also admits many unusual properties, such as a bi-Hamiltonian structure and an infinite sequence of conserved quantities. Since the Degasperis–Procesi equation was derived, many works have been carried out to study its dynamics; for example, the local well-posedness of Equation (3) on the line [13] and on the circle [14] were established. In addition, Yin [13,14] derived the precise blow-up scenario and blow-up structure for the equation. All weak traveling wave solutions were classified [15]. Like for the Camassa–Holm equation, multsolton solutions [16] and the blow-up phenomena [17] for the Degasperis–Procesi equation were found. In a different direction, a study of discontinuous solutions (a shock wave) to the Degasperis–Procesi Equation (3) was initiated by Coclite and Karlsen [18–20], and Lundmark [21]. It is worth noting that a new blow-up quantity [22] among the characteristics is established for the Degasperis–Procesi Equation (3). The other equations related to shallow water waves, such as the Novikov equation and the Modified Camassa–Holm equation with cubic nonlinearity, can be found in [1,3,10,22–27] and the references therein.

In this paper, we consider the Cauchy problem of the integrable dispersive wave equation

$$\begin{cases} (1-\partial_x^2)u_t = 4uu_x - 6u_xu_{xx} - 2uu_{xxx} + 2u_x^2 + 2uu_{xx}, \\ u(0,x) = u_0(x), \end{cases} \tag{4}$$

which is presented in Novikov [1]. $u = u(t,x)$ denotes the fluid velocity at time $t > 0$ in the spatial direction; the problem (4) may be used to describe wave motion in the shallow water. It is shown in [1] that problem (4) possesses a hierarchy of local higher symmetries. Problem (4) is also regarded as a generalized Degasperis–Procesi equation because it has a similar structure to the Degasperis–Procesi equation [28]. Using the Littlewood–Paley theory, the local existence and uniqueness of strong solutions for the problem (4) were established in nonhomogeneous Besov spaces [28]. In [29], Mi etc. studied the well-posedness of (4) for the periodic and nonperiodic cases in the sense of Hadamard. In addition, the authors also proved nonuniform dependence by applying the method of approximate solutions, and well-posedness estimates.

Our aim in this paper is to investigate whether or not problem (4) with nonlocal nonlinearities has similar remarkable properties to Equation (2). More precisely, we firstly establish a new blow-up criterion for the problem (4), which is different from the one in [28]. Then, we use a new method, which differs from other methods [30–35], to obtain some analytical solutions for the first time. To seek analytical solutions, one of the difficult issues is that we have to deal with complicated computation. Luckily, we overcome the difficulties. Finally, we study the persistence property of strong solutions for the problem (4). The results we obtained in this paper complement earlier results in this direction.

Notation

We firstly give some notations.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times\mathbb{R}$ is denoted by C_0^∞. Let $L^p = L^p(\mathbb{R})(1 \le p < +\infty)$ be the space of all measurable functions h such that $\parallel h \parallel_{L^p}^p = \int_{\mathbb{R}} |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(\mathbb{R})$ with the standard norm $\parallel h \parallel_{L^\infty} = \inf_{m(e)=0} \sup_{x\in\mathbb{R}\setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space with the norm defined by

$$\parallel h \parallel_{H^s} = \left(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{h}(t,\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{h}(t,\xi) = \int_{\mathbb{R}} e^{-ix\xi} h(t,x)dx$.

We denote by $*$ the convolution, and the convolution product on $\mathbb{R}$ is defined by

$$(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy. \tag{5}$$

Using the Green function $G(x) = \frac{1}{2}e^{-|x|}$, we have $(1-\partial_x^2)^{-1}f = G(x)*f$ for all $f \in L^2$, and $G*(u-u_{xx}) = u$. For $T > 0$ and nonnegative number s, $C([0,T);H^s(\mathbb{R}))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. For simplicity, throughout this article, we let c denote any positive constant.

We rewrite the equivalent form of the problem (4) as follows:

$$\begin{cases} u_t - 2uu_x = \partial_x(1-\partial_x^2)^{-1}(u^2+(u^2)_x), \\ u(0,x) = u_0(x). \end{cases} \quad (6)$$

2. Blow-Up Criterion

Proposition 1 (see [36])**.** *Given $u(x,0) = u_0 \in H^s(\mathbb{R}), s > 3/2$, then there exist a maximal $T = T(u_0)$ and a unique solution u to the problem (4) such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s(\mathbb{R})) \bigcap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping $u_0 \to u(\cdot, u_0)$: $H^s \to C([0,T); H^s(\mathbb{R})) \bigcap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

Proposition 2. *Let $u_0 \in H^r(\mathbb{R})$ with $r > \frac{3}{2}$. Then, the corresponding solution u to problem (4) blows up in finite time if and only if*

$$\| u \|_{L^\infty} + \| u_x \|_{L^\infty} = +\infty. \quad (7)$$

Proof. Applying Proposition 1 and a simple density argument, it suffices to consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of solution u to the problem (4) with initial data $u_0 \in H^3(\mathbb{R})$. From Proposition 1, we know that $u \in C([0,T); H^3(\mathbb{R})) \bigcap C^1([0,T); H^2(\mathbb{R}))$. Due to $y = u - u_{xx}$, by direct computation, one has

$$\| y \|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 dx = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) dx. \quad (8)$$

So,

$$\| u \|_{H^2}^2 \leq \| y \|_{L^2}^2 \leq 2 \| u \|_{H^2}^2 . \quad (9)$$

The first equation of problem (6) is rewritten as

$$u_t - u_{txx} - 4uu_x + 6u_x u_{xx} + 2uu_{xxx} - 2u_x^2 - 2uu_{xx} = 0. \quad (10)$$

Multiplying both sides of (10) by u and integrating with respect to x on $\mathbb{R}$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) dx = \int_{\mathbb{R}} 2u_x^3 dx - \int_{\mathbb{R}} 4uu_x^2 dx. \quad (11)$$

Differentiating (10) with respect to x, we have

$$u_{tx} - u_{txxx} - 4u_x^2 - 4uu_{xx} + 6u_{xx}^2 + 8u_x u_{xxx} + 2uu_{xxxx} - 6u_x u_{xx} - 2uu_{xxx} = 0. \quad (12)$$

Multiplying both sides of (12) by $2u_x$ and integrating with respect to x on $\mathbb{R}$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2) dx = -4 \int_{\mathbb{R}} u_x^3 dx - 10 \int_{\mathbb{R}} u_x u_{xx}^2 dx + 4 \int_{\mathbb{R}} uu_{xx}^2 dx. \quad (13)$$

Adding up (11) and (13), we arrive at

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) dx &= -2 \int_{\mathbb{R}} u_x^3 dx - 4 \int_{\mathbb{R}} uu_x^2 dx - 10 \int_{\mathbb{R}} u_x u_{xx}^2 dx + 4 \int_{\mathbb{R}} uu_{xx}^2 dx \\ &\leq c(\| u \|_{L^\infty} + \| u_x \|_{L^\infty}) \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) dx. \end{aligned} \quad (14)$$

If there exists a constant $M > 0$ such that $\| u \|_{L^\infty} + \| u_x \|_{L^\infty} < M$, from (14) we deduce that

$$\frac{d}{dt} \| u \|_{H_0^2}^2 \leq cM \| u \|_{H_0^2}^2, \quad (15)$$

where $\| u \|_{H_0^2}^2 = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2)dx$. By virtue of Gronwall's inequality, one has

$$\| u \|_{H_0^2}^2 \leq \| u_0 \|_{H_0^2}^2 e^{cMt}. \tag{16}$$

On the other hand, due to $u = G * y$ and $u_x = G_x * y$, then

$$\begin{aligned} \| u \|_{L^\infty} + \| u_x \|_{L^\infty} &\leq \| G * y \|_{L^\infty} + \| G_x * y \|_{L^\infty} \\ &\leq (\| G \|_{L^2} + \| G_x \|_{L^2}) \| y \|_{L^2} \\ &\leq \sqrt{2}(\| G \|_{L^2} + \| G_x \|_{L^2}) \| u \|_{H^2}. \end{aligned} \tag{17}$$

This completes the proof of Proposition 2. □

3. Analytical Solutions

In this section, we will use the definition of the weak solution to discuss analytical solutions for the problem (4).

Definition 1. *Given the initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function u is said to be a weak solution to the initial-value problem (6) if it satisfies the following identity:*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t - u^2\varphi_x - G * (u^2 + 2uu_x)\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0 \tag{18}$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Proposition 3. *The peakon function*

$$u(t,x) = p(t)e^{-|x-q(t)|}. \tag{19}$$

is not a global weak solution to problem (4) in the sense of Definition 1, where $p(t)$ and $q(t)$ are uncertain functions.

Proof. Assume that $u(t,x) = p(t)e^{-|x-q(t)|}$ is a global weak solution to problem (4) in the sense of Definition 1. We firstly claim that

$$u_t = p'(t)e^{-|x-q(t)|} + p(t)q'(t)sign(x-ct)e^{-|x-q(t)|}, \quad u_x = -p(t)sign(x-ct)e^{-|x-q(t)|}. \tag{20}$$

Hence, using (18), (20) and integration by parts, we derive that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} u\varphi_t - u^2\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx \\ &= -\int_0^T \int_{\mathbb{R}} \varphi(u_t - 2uu_x)dxdt \\ &= -\int_0^T \int_{\mathbb{R}} \varphi[p'(t)e^{-|x-q(t)|} + p(t)q'(t)sign(x-q(t))e^{-|x-q(t)|} \\ &\quad +2p^2(t)sign(x-q(t))e^{-|x-q(t)|}]dxdt. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} -G * (u^2 + 2uu_x)\varphi_x dxdt \\ &= \int_0^T \int_{\mathbb{R}} \varphi G_x * [p^2(t)(1 - 2sign(x-q(t)))e^{-2|x-q(t)|}]dxdt. \end{aligned} \tag{22}$$

Note that $G_x = -\frac{1}{2}sign(x)e^{-|x|}$. For $x \leq q(t)$,

$$
\begin{aligned}
G_x * (u^2 + 2uu_x) &= -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= -\frac{1}{2}(\int_{-\infty}^{x} + \int_{x}^{q(t)} + \int_{q(t)}^{\infty})sign(x-y)e^{-|x-y|} \\
&\qquad \times p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= I_1 + I_2 + I_3. \qquad (23)
\end{aligned}
$$

We directly compute I_1 as follows:

$$
\begin{aligned}
I_1 &= -\frac{1}{2}\int_{-\infty}^{x} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= -\frac{3}{2}p^2(t)\int_{-\infty}^{x} e^{-x-2q(t)+3y}dy \\
&= -\frac{3}{2}p^2(t)e^{-x-2q(t)}\int_{-\infty}^{x} e^{3y}dy \\
&= -\frac{1}{2}p^2(t)e^{2x-2q(t)}. \qquad (24)
\end{aligned}
$$

Applying a similar procedure, we obtain

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int_{x}^{q(t)} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= \frac{3}{2}p^2(t)\int_{x}^{q(t)} e^{x-2q(t)+y}dy \\
&= \frac{3}{2}p^2(t)e^{x+\frac{8}{3}t}\int_{x}^{q(t)} e^{y}dy \\
&= \frac{3}{2}p^2(t)e^{x-q(t)} - \frac{3}{2}p^2(t)e^{2x-2q(t)}. \qquad (25)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= -\frac{1}{2}\int_{q(t)}^{\infty} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= -\frac{1}{2}p^2(t)\int_{q(t)}^{\infty} e^{x+2q(t)-3y}dy \\
&= -\frac{1}{2}p^2(t)e^{x+2q(t)}\int_{q(t)}^{\infty} e^{-3y}dy \\
&= -\frac{1}{6}p^2(t)e^{x-q(t)}. \qquad (26)
\end{aligned}
$$

Therefore, from (24)–(26), we deduce that for $x \le q(t)$

$$
G_x * (u^2 + 2uu_x)(t,x) = -2p^2(t)e^{2x-2q(t)} + \frac{4}{3}p^2(t)e^{x-q(t)}. \qquad (27)
$$

For $x > q(t)$,

$$
\begin{aligned}
G_x * (u^2 + 2uu_x) &= -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= -\frac{1}{2}(\int_{-\infty}^{q(t)} + \int_{q(t)}^{x} + \int_{x}^{\infty})sign(x-y)e^{-|x-y|} \\
&\qquad \times p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\
&= II_1 + II_2 + II_3. \qquad (28)
\end{aligned}
$$

We directly compute II_1 as follows:

$$\begin{aligned} II_1 &= -\frac{1}{2}\int_{-\infty}^{q(t)} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\ &= -\frac{3}{2}p^2(t)\int_{-\infty}^{q(t)} e^{-x-2q(t)+3y}dy \\ &= -\frac{3}{2}p^2(t)e^{-x-2q(t)}\int_{-\infty}^{x} e^{3y}dy \\ &= -\frac{1}{2}p^2(t)e^{-x+q(t)}. \end{aligned} \tag{29}$$

Using a similar procedure, we have

$$\begin{aligned} II_2 &= -\frac{1}{2}\int_{q(t)}^{x} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\ &= \frac{1}{2}p^2(t)\int_{q(t)}^{x} e^{-x+2q(t)-y}dy \\ &= \frac{1}{2}p^2(t)e^{-x+2q(t)}\int_{q(t)}^{x} e^{-y}dy \\ &= -\frac{1}{2}p^2(t)e^{-2x+2q(t)} + \frac{1}{2}p^2(t)e^{-x+q(t)}. \end{aligned} \tag{30}$$

and

$$\begin{aligned} II_3 &= -\frac{1}{2}\int_{x}^{\infty} sign(x-y)e^{-|x-y|}p^2(t)(1-2sign(y-q(t)))e^{-2|y-q(t)|}dy \\ &= -\frac{1}{2}p^2(t)\int_{x}^{\infty} e^{x+2q(t)-3y}dy \\ &= -\frac{1}{2}p^2(t)e^{x+2q(t)}\int_{x}^{\infty} e^{-3y}dy \\ &= -\frac{1}{6}p^2(t)e^{-2x+2q(t)}. \end{aligned} \tag{31}$$

Therefore, from (29)–(31) we deduce that for $x > q(t)$

$$G_x * (u^2 + 2uu_x)(t,x) = -\frac{2}{3}p^2(t)e^{-2x+2q(t)}. \tag{32}$$

Hence, we obtain from two cases mentioned above that

$$G_x * (u^2 + 2uu_x)(t,x) = \begin{cases} -\frac{2}{3}p^2(t)e^{-2x+2q(t)}, & if \quad x > q(t), \\ -2p^2(t)e^{2x-2q(t)} + \frac{4}{3}p^2(t)e^{x-q(t)}, & if \quad x \le q(t). \end{cases} \tag{33}$$

Due to $u = p(t)e^{-|x-q(t)|}$,

$$\begin{aligned} &p'(t)e^{-|x-q(t)|} + p(t)q'(t)sign(x-q(t))e^{-|x-q(t)|} + 2p^2(t)sign(x-q(t))e^{-|x-q(t)|} \\ &= \begin{cases} p'(t)e^{-x+q(t)} + p(t)q'(t)e^{-x+q(t)} + 2p^2(t)e^{-2x+2q(t)}, & if x > q(t), \\ p'(t)e^{x-q(t)} - p(t)q'(t)e^{x-q(t)} - 2p^2(t)e^{2x-2q(t)}, & if x \le q(t). \end{cases} \end{aligned} \tag{34}$$

To ensure that $u = p(t)e^{-|x-q(t)|}$ is a global weak solution 0f the problem (4), then the functions $p(t)$ and $q(t)$ satisfy the following conditions:

$$\begin{cases} p' + pq' = 0, p^2 = 0 & if x > q(t), \\ p' - pq' = \frac{4}{3}p^2, & if x \le q(t). \end{cases} \tag{35}$$

Obviously, $u = 0$ is a general solution for the problem (4). So, there is not global weak solution in the sense of Definition 1.

This completes the proof of Proposition 3. □

Remark 1. *If $x \leq q(t)$, the analytical solution of the problem (4) is of following form:*

$$u = p(t)e^{x-q(t)}, \tag{36}$$

where $p(t)$ and $q(t)$ satisfy

$$p' - pq' = \frac{4}{3}p^2. \tag{37}$$

Example 1. *For $x \leq q(t)$, letting $q(t) = \sqrt{t} + c, c > 0$, from (37) we derive that*

$$p' - \frac{1}{2\sqrt{t}}p - \frac{4}{3}p^2 = 0. \tag{38}$$

(38) implies that

$$p = -\frac{3}{8(\sqrt{t} - 1)}. \tag{39}$$

Hence, we obtain from (36) the solution of (4) for $x \leq q(t)$.

$$u = -\frac{3}{8(\sqrt{t} - 1)}e^{x-\sqrt{t}+c}. \tag{40}$$

Example 2. *For $x \leq q(t)$, letting $q(t) = ct + x_0, c > 0$, from (37) we derive that*

$$p' - cp - \frac{4}{3}p^2 = 0. \tag{41}$$

(41) implies that

$$p = -\frac{3c}{4}. \tag{42}$$

Therefore, we obtain from (36) the solution of (4) for $x \leq q(t)$.

$$u = -\frac{3c}{4}e^{x-ct-x_0}. \tag{43}$$

4. Persistence Property

The task of this section is to study persistence property of the strong solutions to problem (4) in L^∞-space.

Proposition 4. *Provided that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ satisfies*

$$\mid u_0(x) \mid, \mid u_{0x}(x) \mid \sim O(e^{-\theta x}) \quad as \quad x \to \infty, \quad for \quad \theta \in (0,1),$$

then the corresponding strong solution $u(t,x) \in C([0,T]; H^s(\mathbb{R}))$ to the problem (4) satisfies

$$\mid u(t,x) \mid, \mid u_x(t,x) \mid \sim O(e^{-\theta x}) \quad as \quad x \to \infty$$

uniformly in the time interval $[0,T]$.

Notation 1.

$$| u(t,x) | \sim O(e^{-\theta x}) \quad as \quad x \to \infty \quad if \quad \lim_{x\to\infty} \frac{| u(x) |}{e^{-\theta x}} = L.$$

Proof. We introduce the notations

$$F(u) = u^2 + 2uu_x, \qquad M = \sup_{t\in[0,T]} \{\| u(t) \|_{L^\infty} + \| u_x(t) \|_{L^\infty}\} < 0$$

and the weight function $\varphi_N(x)$ which is independent of t

$$\varphi_N(x) = \begin{cases} 1, & x \le 0, \\ e^{\theta x}, & x \in (0, N], \\ e^{\theta N}, & x \ge N, \end{cases}$$

where $N \in \mathbb{Z}^+$.

From the first equation of the problem (4), we obtain

$$\partial_t(\varphi_N u) - 2\varphi_N u u_x - \varphi_N \partial_x G * F(u) = 0. \tag{44}$$

Multiplying Equation (44) by $(\varphi_N u)^{2n-1}$ with $n \in \mathbb{Z}^+$ and integrating the both sides with respect to x, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\varphi_N u)^{2n-1} \partial_t(\varphi_N u) dx - 2\int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N u u_x dx \\ &- \int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N \partial_x G * F(u) dx = 0. \end{aligned} \tag{45}$$

The first term of the above identity is

$$\begin{aligned} \int_{\mathbb{R}} (\varphi_N u)^{2n-1} \partial_t(\varphi_N u) dx &= \frac{1}{2n}\frac{d}{dt} \| \varphi_N u \|_{L^{2n}}^{2n} \\ &= \| \varphi_N u \|_{L^{2n}}^{2n-1} \frac{d}{dt} \| \varphi_N u \|_{L^{2n}} . \end{aligned} \tag{46}$$

Using Hölder's inequality, for the second and third term, we have

$$2 \left| \int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N u u_x dx \right| \le 2 \| u_x \|_{L^\infty} \| \varphi_N u \|_{L^{2n}}^{2n} \tag{47}$$

and

$$\left| \int_{\mathbb{R}} (\varphi_N u)^{2n-1} \varphi_N \partial_x G * F(u) dx \right| \le \| \varphi_N u \|_{L^{2n}}^{2n-1} \| \varphi_N \partial_x G * F(u) \|_{L^{2n}} . \tag{48}$$

It follows from (45)–(48) that

$$\frac{d}{dt} \| \varphi_N u \|_{L^{2n}} \le 2 \| u_x \|_{L^\infty} \| \varphi_N u \|_{L^{2n}} + \| \varphi_N \partial_x G * F(u) \|_{L^{2n}} . \tag{49}$$

From the Gronwall's inequality, we obtain

$$\| \varphi_N u \|_{L^{2n}} \le e^{2Mt} (\| \varphi_N u_0 \|_{L^{2n}} + \int_0^t \| \varphi_N \partial_x G * F(u) \|_{L^{2n}} ds). \tag{50}$$

Due to $\lim_{r\to\infty} \| f \|_{L^r} = \| f \|_{L^\infty}$, when $f \in L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$.

Taking limits to both sides of (50) results in

$$\| \varphi_N u \|_{L^\infty} \le e^{2Mt} (\| \varphi_N u_0 \|_{L^\infty} + \int_0^t \| \varphi_N \partial_x G * F(u) \|_{L^\infty} ds). \tag{51}$$

Next, we will establish an estimate on $\| \varphi_N u_x \|_{L^\infty}$. Differentiating the first equation of the problem (6) with respect to x and multiplying the resultant equation by φ_N, one has

$$\partial_t(\varphi_N u_x) - 2\varphi_N u_x u_x - 2\varphi_N u u_{xx} - \varphi_N \partial_x^2 G * F(u) = 0. \tag{52}$$

Multiplying Equation (52) by $(\varphi_N u_x)^{2n-1}$ with $n \in \mathbb{Z}^+$ and integrating the both sides with respect to x, we have

$$\begin{aligned}&\int_{\mathbb{R}} (\varphi_N u_x)^{2n-1} \partial_t(\varphi_N u_x) dx - 2\int_{\mathbb{R}} (\varphi_N u_x)^{2n-1} \varphi_N u_x u_x dx \\ &-2\int_{\mathbb{R}} (\varphi_N u_x)^{2n-1} \varphi_N u u_{xx} dx - \int_R (\varphi_N u_x)^{2n-1} \varphi_N \partial_x^2 G * F(u) dx = 0.\end{aligned} \tag{53}$$

Applying integration by parts and the Hölder's inequality, from the third term of Equation (53) one has

$$\begin{aligned}&\mid 2\int_{\mathbb{R}} (\varphi_N u_x)^{2n-1} \varphi_N u u_{xx} dx \mid \\ =&\mid 2\int_{\mathbb{R}} u(\varphi_N u_x)^{2n-1}((\varphi_N u_x)_x - \varphi_N' u_x) dx \mid \\ =&\mid 2\int_{\mathbb{R}} u\left(\frac{(\varphi_N u_x)^{2n}}{2n}\right)_x dx - 2\int_{\mathbb{R}} u(\varphi_N u_x)^{2n-1} \varphi_N' u_x dx \mid \\ \leq& \frac{1}{n} \| u_x \|_{L^\infty} \| \varphi_N u_x \|_{L^{2n}}^{2n} + 2 \| u \|_{L^\infty} \| \varphi_N u_x \|_{L^{2n}}^{2n} \\ \leq& (\frac{1}{n} \| u_x \|_{L^\infty} + 2 \| u \|_{L^\infty}) \| \varphi_N u_x \|_{L^{2n}}^{2n},\end{aligned} \tag{54}$$

where we have used the relation $0 \leq \varphi_N' \leq \varphi_N$ for a.e. $x \in \mathbb{R}$. For other terms of Equation (53), we use the same procedure as the above. Hence, we obtain

$$\begin{aligned}\frac{d}{dt} \| \varphi_N u_x \|_{L^{2n}} \leq& (\frac{2n+1}{n} \| u_x \|_{L^\infty} + \| u \|_{L^\infty}) \| \varphi_N u_x \|_{L^{2n}} \\ &+ \| \varphi_N \partial_x^2 G * F(u) \|_{L^{2n}} \\ \leq& \frac{2n+1}{n} M \| \varphi_N u_x \|_{L^{2n}} + \| \varphi_N \partial_x^2 G * F(u) \|_{L^{2n}}.\end{aligned} \tag{55}$$

From the Gronwall's inequality, we obtain

$$\| \varphi_N u_x \|_{L^{2n}} \leq e^{\frac{2n+1}{n} Mt} (\| \varphi_N u_{0x} \|_{L^{2n}} + \int_0^t \| \varphi_N \partial_x^2 G * F(u) \|_{L^{2n}} ds). \tag{56}$$

Taking limits on both sides of inequality (56) yields

$$\| \varphi_N u_x \|_{L^\infty} \leq e^{2Mt} (\| \varphi_N u_{0x} \|_{L^\infty} + \int_0^t \| \varphi_N \partial_x^2 G * F(u) \|_{L^\infty} ds). \tag{57}$$

It follows from (51) and (57) that

$$\begin{aligned}&\| \varphi_N u \|_{L^\infty} + \| \varphi_N u_x \|_{L^\infty} \leq e^{2Mt} (\| \varphi_N u_0 \|_{L^\infty} + \| \varphi_N u_{0x} \|_{L^\infty}) \\ &+ e^{2Mt} (\int_0^t \| \varphi_N \partial_x G * F(u) \|_{L^\infty} + \| \varphi_N \partial_x^2 G * F(u) \|_{L^\infty} ds).\end{aligned} \tag{58}$$

On the other hand, a calculation shows that there is a constant c_0 depending only on $\theta \in (0,1)$ such that for any $N \in \mathbb{Z}^+$

$$\varphi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} dy \leq c_0 = \frac{1}{1-\theta} + \frac{1}{1+\theta} + 2. \tag{59}$$

Therefore, for any function f, we have

$$\begin{aligned}
&| \varphi_N \partial_x G * f^2(x) | \\
&\quad =| \frac{1}{2}\varphi_N \int_{\mathbb{R}} sgn(x-y)e^{-|x-y|} f^2(y)dy | \\
&\quad \leq \frac{1}{2}\varphi_N \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} \varphi_N(y) f(y) f(y) dy \\
&\quad \leq \frac{1}{2}(\varphi_N \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} dy) \parallel \varphi_N(y) f(y) \parallel_{L^\infty} \parallel f(y) \parallel_{L^\infty} \\
&\quad \leq c_0 \parallel \varphi_N(y) f(y) \parallel_{L^\infty} \parallel f(y) \parallel_{L^\infty} .
\end{aligned} \tag{60}$$

Noting $\partial_x^2 G * f^2 = G * f^2 - f^2$, it has

$$| \varphi_N \partial_x^2 G * f^2 |=| \varphi_N G * f^2 - \varphi_N f^2 | \leq c_0 \parallel \varphi_N f \parallel_{L^\infty} \parallel f \parallel_{L^\infty} . \tag{61}$$

Therefore, from (60) and (61), we have

$$\begin{aligned}
&| \varphi_N \partial_x G * [u^2 + 2uu_x] | \\
&\quad \leq | \varphi_N \partial_x G * u^2 | + | \varphi_N \partial_x^2 G * u^2 | \\
&\quad \leq 2c_0 \parallel \varphi_N u \parallel_{L^\infty} \parallel u \parallel_{L^\infty}
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
&| \varphi_N \partial_x^2 G * [u^2 + 2uu_x] | \\
&\quad \leq | \varphi_N \partial_x^2 G * u^2 | + | \varphi_N \partial_x^2 G * 2uu_x | \\
&\quad \leq c_0 \parallel \varphi_N u \parallel_{L^\infty} \parallel u \parallel_{L^\infty} + c_0 \parallel \varphi_N u_x \parallel_{L^\infty} \parallel u \parallel_{L^\infty} .
\end{aligned} \tag{63}$$

Substituting inequalities (62) and (63) into (58), we obtain

$$\begin{aligned}
&\parallel \varphi_N u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty} \\
&\quad \leq e^{2Mt}(\parallel \varphi_N u_0 \parallel_{L^\infty} + \parallel \varphi_N u_{0x} \parallel_{L^\infty}) + 2c_0 e^{2Mt} \\
&\qquad \times \int_0^t (\parallel \varphi_N u \parallel_{L^\infty} \parallel u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty} \parallel u \parallel_{L^\infty}) ds \\
&\quad \leq e^{2Mt}(\parallel \varphi_N u_0 \parallel_{L^\infty} + \parallel \varphi_N u_{0x} \parallel_{L^\infty}) + 2c_0 e^{2Mt} \\
&\qquad \times \int_0^t \parallel u \parallel_{L^\infty} (\parallel \varphi_N u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty}) ds \\
&\quad \leq e^{2Mt}(\parallel \varphi_N u_0 \parallel_{L^\infty} + \parallel \varphi_N u_{0x} \parallel_{L^\infty}) + 2c_0 M e^{2Mt} \\
&\qquad \times \int_0^t (\parallel \varphi_N u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty}) ds.
\end{aligned} \tag{64}$$

It follows from the Gronwall's inequality that

$$\begin{aligned}
&\parallel \varphi_N u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty} \\
&\quad \leq e^{[2M + c_0 M e^{2Mt}]t}(\parallel \varphi_N u_0 \parallel_{L^\infty} + \parallel \varphi_N u_{0x} \parallel_{L^\infty}).
\end{aligned} \tag{65}$$

Therefore, for any $N \in \mathbb{Z}^+$ and any $t \in [0, T]$, we obtain

$$\begin{aligned}
&\parallel \varphi_N u \parallel_{L^\infty} + \parallel \varphi_N u_x \parallel_{L^\infty} \\
&\quad \leq e^{[2M + c_0 M e^{2Mt}]t}(\parallel e^{\theta x} u_0 \parallel_{L^\infty} + \parallel e^{\theta x} u_{0x} \parallel_{L^\infty}).
\end{aligned} \tag{66}$$

Taking the limit as N goes to infinity in (66) leads to

$$\begin{aligned} &| e^{\theta x} u | + | e^{\theta x} u_x | \\ &\quad \leq e^{[2M + c_0 M e^{2Mt}]t} (\| e^{\theta x} u_0 \|_{L^\infty} + \| e^{\theta x} u_{0x} \|_{L^\infty}). \end{aligned} \tag{67}$$

This completes the proof of Proposition 4. □

5. Conclusions

In this paper, we focus on several dynamic properties of the problem (4). We first establish a new blow-up criterion for the equation; then, we study analytical solutions for the equation by using a new method. Here, we present two analytical solutions for the problem (4) for the first time. Finally, we study persistence property for strong solutions. The properties of the problem (4) not only present fundamental importance from a mathematical point of view but also are of great physical interest. In future paper, we will study blow-up structures and the stability of solitary waves for problem (4).

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