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Existence and Nonexistence of Positive Solutions for Perturbations of the Anisotropic Eigenvalue Problem

Olena Andrusenko [1], Leszek Gasiński [1,*] and Nikolaos S. Papageorgiou [2]

[1] Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland
[2] Department of Mathematics, Zografou Campus, National Technical University, 15780 Athens, Greece
* Correspondence: leszek.gasinski@up.krakow.pl

Abstract: We consider a Dirichlet problem, which is a perturbation of the eigenvalue problem for the anisotropic p-Laplacian. We assume that the perturbation is $(p(z)-1)$-sublinear, and we prove an existence and nonexistence theorem for positive solutions as the parameter λ moves on the positive semiaxis. We also show the existence of a smallest positive solution and determine the monotonicity and continuity properties of the minimal solution map.

Keywords: anisotropic eigenvalues and eigenvectors; regularity theory; maximum principle; truncations; minimal positive solutions

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric anisotropic Dirichlet problem (P_λ)

$$\begin{cases} -\Delta_{p(z)}u(z) = \lambda|u(z)|^{p(z)-2}u(z) + f(z,u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases}$$

For $p \in L^\infty(\Omega)$ with $1 < \operatorname*{ess\,inf}_{\Omega} p$, by $\Delta_{p(z)}$ we denote the anisotropic p-Laplacian differential operator defined by

$$\Delta_{p(z)}u = \operatorname{div}\left(|Du|^{p(z)-2}|Du\right) \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Problem (P_λ) is a perturbation of the eigenvalue problem for the anisotropic p-Laplacian, with the perturbation $f(z,x)$ being a Carathéodory function, which exhibits $(p(z)-1)$-sublinear growth as $x \to +\infty$.

Our goal in this paper is to give a complete description of the set of positive solutions of problem (P_λ) as the parameter λ varies on the positive semiaxis $(0,+\infty)$.

In the past, such perturbed versions of eigenvalue problems, were studied only in the context of isotropic equations. We mention the works of Papageorgiou–Rădulescu–Repovš [1] (semilinear Robin problems), Papageorgiou–Rădulescu–Zhang [2] (nonlinear Robin problems), Papageorgiou–Scapellato [3] (Dirichlet $(p,2)$-equations), Gasiński–Papageorgiou [4] (weighted (p,q)-equations). To the best of our knowledge there are no such works for anisotropic equations.

The reason for this gap in the literature is that the spectrum properties of the $p(z)$-Laplacian can be very "bad" depending on the exponent $p(\cdot)$. This is illustrated in the work of Fan–Zhang–Zhao [5]. These difficulties are a consequence of the fact that the anisotropic p-Laplacian is not homogeneous and so many of the tools and techniques available in the isotropic case fail in the anisotropic one. Nevertheless, under a monotonicity type condition on the exponent $p(\cdot)$ and using the results of Fan–Zhang–Zhao [5], we are able to give a precise description with respect to the parameter $\lambda > 0$, of the set of positive solutions. Moreover, we show that there exists a smallest positive solution and we determine the kind of dependence on the parameter $\lambda > 0$ of this minimal solution.

The $p(z)$-Laplacian has many physical applications. This includes usage in electrorheological fluids, which are fluids that can solidify into a jelly-like state almost instantaneously when subjected to an externally applied electric field of moderate strength with stiffness varying proportionally to the field strength. This transformation is reversible and once the applied field is removed, the original flow state is recovered (Winslow effect). Such processes have been modelled by using anisotropic operators (see Růžička [6]).

Because we look for the positive solutions, the problem is by its nature asymmetric. All the conditions on the reaction concern the positive semiaxis $\mathbb{R}_+$. However, even in its framework we should mention another asymmetry of the problem. Namely, the reaction $f(z,\cdot)$ is $(p(z)-1)$-sublinear near $+\infty$, but $(p(z)-1)$-superlinear near 0^+. This different behaviour of $f(z,\cdot)$ at the two ends of $\mathbb{R}_+$, is the reason that leads to a complete description of the set of positive solutions as the parameter $\lambda > 0$ varies.

2. Mathematical Background Hypotheses

The analysis of problem (P_λ) uses variable Lebesgue and Sobolev spaces. A comprehensive presentation of the theory of the spaces can be found in the books of Cruz Uribe–Fiorenza [7] and of Diening–Harjulehto-Hästö–Růžička [8].

Let
$$E_1 = \{r \in C(\overline{\Omega}) : \ 1 < r(z) \text{ for all } z \in \overline{\Omega}\}.$$

For every $r \in E_1$, we set
$$r_+ = \max_{\overline{\Omega}} r, \quad r_- = \min_{\overline{\Omega}} r.$$

By $L^0(\Omega)$, we denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null set of Ω.

For every $r \in E_1$, the variable Lebesgue space $L^{r(z)}(\Omega)$ is defined by
$$L^{r(z)}(\Omega) = \left\{ u \in L^0(\Omega) : \ \varrho_r(u) = \int_\Omega |u|^{r(z)}\, dz < +\infty \right\}.$$

We endow this space with the so-called "Luxemburg norm" defined by
$$\|u\|_{r(z)} = \inf\left\{ \vartheta > 0 : \ \varrho_r\left(\frac{u}{\vartheta}\right) = \int_\Omega \left(\frac{|u|}{\vartheta}\right)^{r(z)} dz \leqslant 1 \right\}.$$

With this norm, $L^{r(z)}(\Omega)$ becomes a separable Banach space which is reflexive (in fact uniformly convex). Let $p' \in E_1$ be defined by
$$p'(z) = \frac{p(z)}{p(z)-1} \quad \forall z \in \overline{\Omega}$$
(that is $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for all $z \in \overline{\Omega}$). Then we have
$$L^{p'(z)}(\Omega) = L^{p(z)}(\Omega)^*.$$

Moreover, the following Hölder-type inequality holds:
$$\int_\Omega |uh|\, dz \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(z)} \|h\|_{p'(z)} \quad \forall u \in L^{p(z)}(\Omega),\ h \in L^{p'(z)}(\Omega).$$

In addition, if $q \in E_1$ and $q(z) \leqslant p(z)$ for all $z \in \overline{\Omega}$, then the embedding $L^{p(z)}(\Omega) \subseteq L^{q(z)}(\Omega)$ is continuous.

By using the variable Lebesgue spaces, we can define the corresponding variable Sobolev spaces. Consequently, if $p \in E$, then the variable Sobolev space $W^{1,p(z)}(\Omega)$ is defined by

$$W^{1,p(z)}(\Omega) = \{u \in L^{p(z)}(\Omega) : \ |Du| \in L^{p(z)}(\Omega)\}$$

with Du being the weak gradient of u. We endow this space with the following norm

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|Du\|_{p(z)},$$

where

$$\|Du\|_{p(z)} = \||Du|\|_{p(z)}.$$

Let $C^{0,1}(\overline{\Omega})$ denote the space of all Lipschitz functions from $\overline{\Omega}$ into $\mathbb{R}$. Let $p \in E_1 \cap C^{0,1}(\overline{\Omega})$. We define

$$W_0^{1,p(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p(z)}}.$$

The spaces $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$ are both separable Banach spaces which are reflexive (in fact uniformly convex). The Poincaré inequality holds for the space $W_0^{1,p(z)}(\Omega)$; namely, we can find $c_0 = c_0(\Omega) > 0$ such that

$$\|u\|_{p(z)} \leqslant c_0 \|Du\|_{p(z)} \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Therefore, on $W_0^{1,p(z)}(\Omega)$ we can consider the following equivalent norm:

$$\|u\| = \|Du\|_{p(z)} \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Let $p \in E_1 \cap C^{0,1}(\overline{\Omega})$. We set

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)} & \text{if} \quad p(z) < N, \\ +\infty & \text{if} \quad N \leqslant p(z) \end{cases} \qquad \forall z \in \overline{\Omega}.$$

This is the critical variable Sobolev exponent corresponding to p. Let $r \in C(\overline{\Omega})$ satisfy $1 \leqslant r(z) \leqslant p^*(z)$ (respectively $1 \leqslant r(z) < p^*(z)$) for all $z \in \overline{\Omega}$. We have that the embedding $W_0^{1,p(z)}(\Omega) \subseteq L^{r(z)}(\Omega)$ is continuous (respectively compact). This is the so-called "anisotropic Sobolev embedding theorem". If $p \in E_1 \cap C^{0,1}(\overline{\Omega})$, then

$$W_0^{1.p(z)}(\Omega)^* = W^{-1,p'(z)}(\Omega).$$

There is a close relation between the modular function $\varrho_p(Du) = \int_\Omega |Du|^{p(z)}\, dz$ and the norm $\|\cdot\|$, which we specify in the next theorem.

Theorem 1. *If $p \in E_1$, then*
(a) $\|u\| = \vartheta \Longleftrightarrow \varrho_p(\frac{Du}{\vartheta}) = 1$;
(b) $\|u\| < 1$ *(respectively* $= 1, > 1$*)* $\Longleftrightarrow \varrho_p(Du) < 1$ *(respectively* $= 1, > 1$*)*;
(c) $\|u\| < 1 \Longrightarrow \|u\|^{p_+} \leqslant \varrho_p(Du) \leqslant \|u\|^{p_-}$;
(d) $\|u\| > 1 \Longrightarrow \|u\|^{p_-} \leqslant \varrho_p(Du) \leqslant \|u\|^{p_+}$;
(e) $\|u\| \to 0$ *(respectively* $\to +\infty$*)* $\Longleftrightarrow \varrho_p(Du) \to 0$ *(respectively* $\to +\infty$*)*;
(f) $\|u_h - u\| \to 0 \Longleftrightarrow \varrho_p(Du_n - Du) \to 0$.

Let $A_p\colon W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ be defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p(z)}(\Omega).$$

This operator has the following properties (see Gasiński–Papageorgiou [9] (Proposition 2.5)).

Theorem 2. *The operator $A_p\colon W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_p(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W_0^{1,p(z)}(\Omega)$).*

The anisotropic regularity theorem of Fan [10] will lead us to the space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) :\ u|_{\partial\Omega} = 0\right\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) :\ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ :\ u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with n being the outward unit normal on $\partial\Omega$.

Next, we recall some basic facts about the spectrum of $(-\Delta_{p(z)}, W_0^{1,p(z)}(\Omega))$. Consequently, we consider the following nonlinear eigenvalue problem (E_λ)

$$\begin{cases} -\Delta_{p(z)}u(z) = \widehat{\lambda}|u(z)|^{p(z)-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of the anisotropic p-Laplacian, if problem (E_λ) has a nontrivial weak solution $\widehat{u}_\lambda \in W_0^{1,p(z)}(\Omega)$. Evidently,

$$\widehat{\lambda} = \frac{\varrho_p(D\widehat{u}_\lambda)}{\varphi_p(u_\lambda)} > 0.$$

By Λ, we denote the set of eigenvalues of (E_λ). In the isotropic case (that is, when p is constant), then $\inf\Lambda = \widehat{\lambda}_1(p) > 0$, with $\widehat{\lambda}_1(p)$ being the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. In contrast, in the anisotropic case, it can happen that $\inf\Lambda = 0$. In order for $\inf\Lambda > 0$ and therefore to have a first eigenvalue $\widehat{\lambda}_1(p) > 0$, we need to impose a monotonicity condition on the variable exponent $p(\cdot)$.

We introduce the following quantities:

$$\widehat{\lambda}_1 = \inf\Lambda,$$

$$\mu_1 = \inf\left\{\frac{\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\, dz}{\int_\Omega \frac{1}{p(z)}|u|^{p(z)}\, dz} :\ u \in W_0^{1,p(z)}(\Omega),\ u \neq 0\right\}, \tag{1}$$

$$\widetilde{\mu}_1 = \left\{\frac{\varrho_p(Du)}{\varrho_p(u)} :\ W_0^{1,p(z)}(\Omega),\ u \neq 0\right\}. \tag{2}$$

It is easy to see that

$$\frac{p_-}{p_+}\widetilde{\mu}_1 \leqslant \mu_1 \leqslant \frac{p_+}{p_-}\widetilde{\mu}_1 \quad \text{and} \quad \widetilde{\mu}_1 \leqslant \widehat{\lambda}_1.$$

From Fan–Zhang–Zhao [5] (Lemma 3.1), we know that

$$\widehat{\lambda}_1 > 0 \iff \mu_1 > 0 \iff \widetilde{\mu}_1 > 0.$$

Moreover, if $N \geqslant 2$ and there exists $d \in \mathbb{R}^N \setminus \{0\}$ such that for any $z \in \Omega$ the function $t \longmapsto p(z+td)$ is monotone on $T_z = \{t : z+td \in \Omega\}$, then $\widehat{\lambda}_1 > 0$ (Fan–Zhang–Zhao [5] (Theorem 3.3)).

Let $u, v \colon \Omega \longrightarrow \mathbb{R}$ be measurable functions, such that $u(z) \leqslant v(z)$ for almost all $z \in \Omega$. We introduce the following sets:

$$\begin{aligned} [u,v] &= \{h \in W_0^{1,p(z)}(\Omega) : u(z) \leqslant h(z) \leqslant v(z) \text{ for a.a. } z \in \Omega\}, \\ \mathrm{int}_{C_0^1(\overline{\Omega})}[u,v] &- \text{the interior of } [u,v] \cap C_0^1(\overline{\Omega}) \text{ in } C_0^1(\overline{\Omega}). \end{aligned}$$

Moreover, we define $u^+(z) = \max\{u(z), 0\}$, $u^- = \max\{-u(z), 0\}$ for all $z \in \Omega$. We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p(z)}(\Omega)$, then $u^{\pm} \in W_0^{1,p(z)}(\Omega)$. If $h_1, h_2 \colon \Omega \to \mathbb{R}$ are measurable, then $h_1 \prec h_2$ if for all compact sets $K \subseteq \Omega$, $0 < c_K \leqslant h_2(z) - h_1(z)$ for a.a. $z \in K$. Finally, if X is a Banach space and $\varphi \in C^1(X;\mathbb{R})$, then $K_\varphi = \{u \in X : \varphi'(u) = 0\}$.

Our hypotheses on the data of (P_λ) are the following:

$\underline{H_0:}$ $p \in C^{0,1}(\overline{\Omega})$, $1 < p_- \leqslant p_+ < N$ and there exists $d \in \mathbb{R}^N \setminus \{0\}$ such that for all $z \in \Omega$, the function $t \longmapsto p(z+td)$ is monotone on $T_z = \{t : z+td \in \Omega\}$.

Remark 1. *As we already mentioned, these hypotheses imply that* $\widehat{\lambda}_1 > 0$.

$\underline{H_1:}$ $f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nontrivial Carathéodory function, such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)$, $q \in C(\overline{\Omega})$, $q_+ < p_-$, such that

$$0 \leqslant f(z,x) \leqslant a(z)(1 + x^{q(z)-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0;$$

(ii) we have that

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p(z)-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\tau \in C(\overline{\Omega})$ with $\tau_+ < p_-$, $\delta > 0$ and $c_0 > 0$ such that

$$c_0 x^{\tau(z)-1} \leqslant f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta;$$

(iv) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$, the map $x \longmapsto f(z,x) + \widehat{\xi}_\varrho x^{p(z)-1}$ is nondecreasing on $[0, \varrho]$.

Remark 2. *Because we look for positive solutions and the above hypotheses concern the positive semiaxes (lack of symmetry)* $\mathbb{R}_+ = [0, +\infty)$, *we may assume without any loss of generality that* $f(z,x) = 0$ *for a.a.* $z \in \Omega$, *all* $x \leqslant 0$.

We introduce the following two sets:

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\}, \\ S_\lambda &= \text{set of positive solutions of } (P_\lambda). \end{aligned}$$

3. Positive Solutions

In this section, we prove an existence theorem for problem (P_λ), which describes the set of positive solutions as λ varies in $(0, +\infty)$.

Proposition 1. *If hypotheses H_0 and H_1 hold and $\lambda \geqslant \widehat{\lambda}_1$, then $\lambda \notin \mathcal{L}$.*

Proof. Arguing by contradiction, suppose that $\lambda \in \mathcal{L}$. Then we can find $u \in S_\lambda$. The anisotropic regularity theory of Fan [10] implies that $u \in C_+ \setminus \{0\}$. We have $\Delta_{p(z)} u \leqslant 0$ in Ω and so Proposition A2 of Papageorgiou–Rădulescu–Zhang [11], implies that $u \in \operatorname{int} C_+$. Moreover, let $\widehat{u}_1$ be a positive eigenfunction for $\widehat{\lambda}_1$ (recall that $\widehat{\lambda}_1 > 0 \iff \mu_1 > 0 \iff \widetilde{\mu}_1 > 0$). Similarly, we have $\widehat{u}_1 \in \operatorname{int} C_+$.

Consider the Picone function $R(\widehat{u}_1, u)(\cdot)$ defined by

$$R(\widehat{u}_1, u)(z) = |D\widehat{u}_1(z)^{p(z)}| - |Du(z)|^{p(z)-1}\left(Du(z), D\left(\frac{\widehat{u}_1(z)^{p(z)}}{u(z)^{p(z)-1}}\right)\right)_{\mathbb{R}^N}.$$

From Jaroš [12], we know that

$$0 \leqslant R(\widehat{u}_1, u)(z) \quad \forall z \in \Omega,$$

so

$$0 \leqslant \varrho_p(D\widehat{u}_1) - \int_\Omega (-\Delta_{p(z)} u)\frac{\widehat{u}_1^{p(z)}}{u^{p(z)-1}}\, dz$$

(using the nonlinear Green's identity), thus

$$0 \leqslant \widehat{\lambda}_1 \varrho_p(\widehat{u}_1) - \int_\Omega \lambda \widehat{u}_1^{p(z)}\, dz$$

(see (2) and recall that $f \geqslant 0$), hence

$$0 \leqslant (\widehat{\lambda}_1 - \lambda)\varrho_p(\widehat{u}_1).$$

If $\lambda > \widehat{\lambda}_1$, then we have a contradiction.

If $\lambda = \widehat{\lambda}_1$, then $R(\widehat{u}_1, u)(z) = 0$ for a.a. $z \in \Omega$ and so by Lemma 2.2 of Jaroš [12], we have

$$\widehat{u}_1 Du = uD\widehat{u}_1,$$

so

$$D\left(\frac{\widehat{u}_1}{u}\right) = 0.$$

Hence, $\widehat{u}_1 = \vartheta u$ for some $\vartheta > 0$, a contradiction since $f \neq 0$. □

Let

$$\eta^* = \frac{p_-}{p_+}\widetilde{\mu}_1.$$

Proposition 2. *If hypotheses H_0 and H_1 hold and $\lambda < \eta^*$, then $\lambda \in \mathcal{L}$.*

Proof. Let $F(z, x) = \int_0^x f(z, s)\, ds$ and consider the C^1-functional $\varphi_\lambda \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\, dz - \int_\Omega \frac{\lambda}{p(z)}(u^+)^{p(z)}\, dz - \int_\Omega F(z, u^+(z))\, dz.$$

On account of hypotheses $H_1(i), (ii)$, given $\varepsilon > 0$, we can find $c_1 = c_1(\varepsilon) > 0$, such that

$$F(z, x) \leqslant \frac{\varepsilon}{p(z)}|x|^{p(z)} + c_1 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3}$$

For all $u \in W_0^{1,p(z)}(\Omega)$, we have

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz - \int_\Omega \frac{\lambda+\varepsilon}{p(z)}|u|^{p(z)}\,dz - c_2 \\ &\geqslant \left(\frac{1}{p_+} + \frac{\lambda+\varepsilon}{\widetilde{\mu}_1 p_-}\right)\varrho_p(Du) - c_2 \end{aligned} \tag{4}$$

for some $c_2 > 0$ (see (2) and (3)).

Because $\lambda < \eta^*$, we can choose $\varepsilon > 0$ small so that

$$\lambda + \varepsilon < \eta^* = \frac{p_-}{p_+}\widetilde{\mu}_1.$$

Consequently, from (4), we have

$$\varphi_\lambda(u) \geqslant c_3\varrho_p(Du) - c_2,$$

for some $c_3 > 0$; thus, φ_λ is coercive (see Theorem 1).

By using the anisotropic Sobolev embedding theorem, we see that φ_λ is sequentially weakly lower semicontinuous.

Consequently, we can find $u_\lambda \in W_0^{1,p(z)}(\Omega)$, such that

$$\varphi_\lambda(u_\lambda) = \inf\{\varphi_\lambda(u) : \ u \in W_0^{1,p(z)}(\Omega)\}. \tag{5}$$

On account of hypothesis $H_1(iii)$, we have

$$\frac{c_0}{\tau(z)}x^{\tau(z)} \leqslant F(z,x) \quad \text{for a.a.} z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{6}$$

Let $\widehat{u}_1 \in \operatorname{int} C_+$ be an eigenfunction corresponding to $\widehat{\lambda}_1 > 0$. We choose $t \in (0,1)$ small, so that

$$0 \leqslant t\widehat{u}_1(z) \leqslant \delta \quad \forall z \in \overline{\Omega}. \tag{7}$$

From (6) and (7), we have

$$\begin{aligned} \varphi_\lambda(t\widehat{u}_1) &\leqslant \frac{t^{p_-}}{p_-}\varrho_p(D\widehat{u}_1) - \frac{c_0 t^{\tau_+}}{\tau_+}\varrho_p(\widehat{u}_1) \\ &= \left(\frac{t^{p_-}}{p_-} - \frac{c_0 t^{\tau_+}}{\widehat{\lambda}_1\tau_+}\right)\varrho_p(D\widehat{u}_1) \\ &= \left(\frac{1}{p_-} - \frac{c_0}{\widehat{\lambda}_1\tau_+ t^{p_- - \tau_+}}\right)t^{p_-}\varrho_p(D\widehat{u}_1). \end{aligned}$$

Because $\tau_+ < p_-$, we see that if we choose $t \in (0,1)$ even smaller, we have

$$\varphi_\lambda(t\widehat{u}_1) < 0,$$

so

$$\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0)$$

(see (5)), thus $u_\lambda \neq 0$.

From (5), we have

$$\langle \varphi'_\lambda(u_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle A_p(u_\lambda), h\rangle = \int_\Omega \left(\lambda(u_\lambda^+)^{p(z)-1} + f(z,u_\lambda^+)\right)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{8}$$

In (8), we choose $h = -u_\lambda^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(Du_\lambda^-) = 0,$$

so $u_\lambda \geqslant 0$, $u_\lambda \neq 0$.

Then from (8), we have

$$-\Delta_{p(z)} u_\lambda = \lambda u_\lambda^{p(z)-1} + f(z, u_\lambda) \geqslant 0 \quad \text{in } \Omega.$$

The anisotropic regularity theory (see Fan [10]) and the anisotropic maximum principle (see Papageorgiou–Rădulescu–Zhang [11]), imply that $u_\lambda \in \operatorname{int} C_+$. □

Corollary 1. *If hypotheses H_0 and H_1 hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq \operatorname{int} C_+$.*

Next, we show that $\mathcal{L}$ is connected (an interval).

Theorem 3. *If hypotheses H_0 and H_1 hold, $0 < \mu < \lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu$, such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. We introduce the Carathéodory function $\widehat{k}_\mu(z, x)$ defined by

$$\widehat{k}_\mu(z, x) = \begin{cases} \mu(x^+) + f(z, x^+) & \text{if} \quad x \leqslant u_\lambda(z), \\ \mu u_\lambda(z) + f(z, u_\lambda(z)) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{9}$$

Let $\widehat{K}_\mu(z, x) = \int_0^x \widehat{k}_\mu(z, s)\, ds$ and consider the C^1-functional $\widehat{\varphi}_\mu \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz - \int_\Omega \widehat{K}_\mu(z, u)\, dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

We have

$$\widehat{\varphi}_\mu(u) \geqslant \frac{1}{p_+} \varrho_p(Du) - c_4, \tag{10}$$

for some $c_4 > 0$ (see (9)). By using Theorem 1, we infer that $\widehat{\varphi}_\mu$ is coercive.

Moreover, the anisotropic Sobolev embedding theorem implies that $\widehat{\varphi}_\mu$ is sequentially weakly lower semicontinuous.

Therefore, we can find $u_\mu \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\varphi}_\mu(u_\mu) = \inf\{\widehat{\varphi}_\mu(u) : \ u \in W_0^{1,p(z)}(\Omega)\}. \tag{11}$$

By using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [13] (p. 274), we can find $t \in (0, 1)$ small, such that

$$0 \leqslant t\widehat{u}_1(z) \leqslant \min\{u_\lambda(z), \delta\} \quad \forall z \in \overline{\Omega}.$$

Then, as in the proof of Proposition 2, because $\tau_+ < p_-$ and taking $t \in (0, 1)$ even smaller if necessary, we have

$$\widehat{\varphi}_\mu(t\widehat{u}_1) < 0,$$

so

$$\widehat{\varphi}_\mu(u_\mu) < 0 = \widehat{\varphi}_\mu(0)$$

(see (11)); thus, $u_\mu \neq 0$.

From (11), we have

$$\langle \widehat{\varphi}_\mu'(u_\mu), h \rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle A_p(u_\mu), h\rangle = \int_\Omega \widehat{k}_\mu(z, u_\mu) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{12}$$

In (12), first we choose $h = -u_\mu^- \in W_0^{1,p(z)}(\Omega)$ and obtain $u_\mu \geqslant 0$, $u_\mu \neq 0$. Then, in (12) we use the test function $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} & \langle A_p(u_\mu), (u_\mu - u_\lambda)^+\rangle \\ = & \int_\Omega (\mu u_\lambda^{p(z)-1} + f(z, u_\lambda))(u_\mu - u_\lambda)^+ \, dz \\ \leqslant & \int_\Omega (\lambda u_\lambda^{p(z)-1} + f(z, u_\lambda))(u_\mu - u_\lambda)^+ \, dz \\ = & \langle A_\lambda(u_\lambda), (u_\mu - u_\lambda)^+\rangle \end{aligned}$$

(see (9) and use the fact that $u_\lambda \in S_\lambda$), so

$$u_\mu \leqslant u_\lambda$$

(see Theorem 2). Consequently, we have proved that

$$u_\mu \in [0, u_\lambda], \ u_\mu \neq 0. \tag{13}$$

From (13), (9) and (12), it follows that

$$u_\mu \in S_\mu \subseteq \operatorname{int} C_+,$$

and so $\mu \in \mathcal{L}$.

Let $\varrho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H_1(iv)$. We have

$$\begin{aligned} -\Delta_{p(z)} u_\mu + \widehat{\xi}_p u_\mu^{p(z)-1} & = \mu u_\lambda^{p(z)-1} + f(z, u_\mu) + \widehat{\xi}_\varrho u_\mu^{p(z)-1} \\ & \leqslant \lambda u_\lambda^{p(z)-1} + f(z, u_\lambda) + \widehat{\xi}_\varrho u_\lambda^{p(z)-1} \\ & = -\Delta_{p(z)} u_\lambda + \widehat{\xi}_\varrho u_\lambda^{p(z)-1} \end{aligned} \tag{14}$$

(see (13), hypothesis $H_1(iv)$, and use the fact that $u_\lambda \in S_\lambda$).

Note that $0 \prec (\lambda - \mu) u_\mu^{p(z)-1}$ (since $u_\mu \in \operatorname{int} C_+$). Consequently, from (14) and Proposition 2.5 of Papageorgiou–Rădulescu–Repovš [14], we infer that $u_\lambda - u_\mu \in \operatorname{int} C_+$. □

Next, we will produce a lower bound for the elements of S_λ. To this end, note that hypotheses $H_1(i), (iii)$ implies that we can find $c_5 > 0$, such that

$$f(z, x) \geqslant c_0 x^{\tau(z)-1} - c_5 x^{p(z)-1} \quad \text{for a.a. } z \in \Omega, \ x \geqslant 0. \tag{15}$$

This unilateral growth estimate leads to the following auxiliary problem

$$\begin{cases} -\Delta_{p(z)} u(z) = c_0 u(z)^{\tau(z)-1} - c_5 u(z)^{p(z)-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0. \end{cases} \tag{16}$$

In the next theorem, we show the existence and uniqueness of solutions for problem (16).

Theorem 4. *If $p \in C^{0,1}(\overline{\Omega})$ with $1 < p_- \leqslant p_+ < N$ and $\tau \in C(\overline{\Omega})$ with $\tau_+ < p_-$, then problem (16) has a unique positive solution $\overline{u} \in \operatorname{int} C_+$.*

Proof. First we show the existence of a positive solution. To this end, consider the C^1-functional $\sigma\colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{c_5}{p(z)}(u^+)^{p(z)}\,dz - \int_\Omega \frac{c_0}{\tau(z)}(u^+)^{\tau(z)}\,dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$. Because $\tau_+ < p_-$, it is clear that σ is coercive. Moreover, it is sequentially weakly lower semicontinuous. Consequently, we can find $\overline{u} \in W_0^{1,p(z)}(\Omega)$, such that

$$\sigma(\overline{u}) = \inf\{\sigma(u) : \ u \in W_0^{1,p(z)}(\Omega)\} < 0 = \sigma(0) \tag{17}$$

(because $\tau_+ < p_-$), so $\overline{u} \neq 0$.

From (17), we have

$$\langle \sigma'(\overline{u}), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle A_p(\overline{u}), h\rangle = \int_\Omega \big(c_0(\overline{u}^+)^{\tau(z)-1} - c_5(\overline{u}^+)^{p(z)-1}\big)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega).$$

Choosing $h = -\overline{u}^- \in W_0^{1,p(z)}(\Omega)$, we obtain $\overline{u} \geqslant 0$, $\overline{u} \neq 0$. Consequently, we have

$$-\Delta_{p(z)}\overline{u} = c_0\overline{u}^{\tau(z)-1} - c_5\overline{u}^{p(z)-1} \quad \text{in } \Omega.$$

From the anisotropic regularity theory, (see Fan [10]), we have $\overline{u} \in C_+ \setminus \{0\}$. We have

$$-\Delta_{p(z)}\overline{u} + c_5\overline{u}^{p(z)-1} \geqslant 0 \quad \text{in } \Omega,$$

so $\overline{u} \in \operatorname{int} C_+$ (see Papageorgiou–Rădulescu–Zhang [11]).

Now we will show the uniqueness of this positive solution. Suppose that $\overline{v} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of (16). For this solution, we also have $\overline{v} \in \operatorname{int} C_+$. Then, by using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [13] (p. 274), we have

$$\frac{\overline{u}}{\overline{v}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{v}}{\overline{u}} \in L^\infty(\Omega). \tag{18}$$

We introduce the integral functional $j\colon L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)}|Du^{\frac{1}{\tau_+}}|^{p(z)}\,dz & \text{if } u \geqslant 0,\ u^{\frac{1}{\tau_+}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : \ j(u) < +\infty\}$ (the effective domain of j). From Takáč–Giacomoni [15], we know that j is convex. Let $h = \overline{u}^{\tau_+} - \overline{v}^{\tau_+} \in C_0^1(\overline{\Omega})$. Then, by using (18) we see that for $t \in (0,1)$ small we have

$$\overline{u}^{\tau_+} + th \in \operatorname{dom} j, \quad \overline{v}^{\tau_+} + th \in \operatorname{dom} j.$$

Consequently, we can compute the directional derivatives of j at $\overline{u}^{\tau_+}$ and at $\overline{v}^{\tau_+}$ in the direction h. A direct computation using the chain rule and Green's identity gives

$$\begin{aligned} j'(\overline{u}^{\tau_+})(h) &= \frac{1}{\tau_+}\int_\Omega \frac{-\Delta_{p(z)}\overline{u}}{\overline{u}^{\tau_+-1}}h\,dz \\ &= \frac{1}{\tau_+}\int_\Omega \big(c_0 - c_5\overline{u}^{p(z)-\tau_+}\big)h\,dz, \end{aligned}$$

$$\begin{aligned} j'(\overline{v}^{\tau^+})(h) &= \frac{1}{\tau_+}\int_\Omega \frac{-\Delta_{p(z)}\overline{v}}{\overline{v}^{\tau_+-1}}h\,dz \\ &= \frac{1}{\tau_+}\int_\Omega \big(c_0 - c_5\overline{v}^{p(z)-\tau_+}\big)h\,dz. \end{aligned}$$

The convexity of j implies the monotonicity of j'. Consequently, we have

$$0 \leqslant \int_\Omega c_5\left(\overline{v}^{p(z)-\tau_+} - \overline{u}^{p(z)-\tau_+}\right)\left(\overline{u}^{\tau_+} - \overline{v}^{\tau_+}\right) dz,$$

and thus $\overline{u} = \overline{v}$. This implies the uniqueness of the positive solution of problem (16). $\square$

This solution provides a lower bound for the elements of S_λ, $\lambda \in \mathcal{L}$, as shown in the next proposition.

Proposition 3. *If hypotheses H_0, H_1 hold, then $\overline{u} \leqslant u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda \subseteq \operatorname{int} C_+$ and introduce the Carathéodory function $l(z,x)$ defined by

$$l(z,x) = \begin{cases} c_0(x^+)^{\tau(z)-1} - c_5(x^+)^{p(z)-1} & \text{if} \quad x \leqslant u(z), \\ c_0 u(z)^{p(z)-1} - c_5 u(z)^{p(z)-1} & \text{if} \quad u(z) < x. \end{cases} \tag{19}$$

We set

$$L(z,x) = \int_0^x l(z,s)\, ds$$

and consider the C^1-functional $w\colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} w(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz - \int_\Omega L(z,u)\, dz \\ &\geqslant \frac{1}{p} \varrho_p(Du) - c_6 \end{aligned}$$

for some $c_6 > 0$ (see (19)); consequently, w is coercive (see Theorem 1).

Moreover, w is sequentially weakly lower semicontinuous. Consequently, we can find $\widetilde{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$w(\widetilde{u}) = \inf\{w(u) :\ u \in W_0^{1,p(z)}(\Omega)\}. \tag{20}$$

Given $v \in \operatorname{int} C_+$, because $\tau_+ < p_-$ for $t \in (0,1)$ small so that $tv \leqslant u$ (recall that $u \in \operatorname{int} C_+$ and use Proposition 4.1.22 of Papageorgiou–Rădulescu–Repovš [13] (p. 274)), we obtain

$$w(tv) < 0,$$

so

$$w(\widetilde{u}) < 0 = w(0)$$

(see (20)), and thus $\widetilde{u} \neq 0$.

From (20), we have

$$\langle w'(\widetilde{u}), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle A_p(\widetilde{u}), h\rangle = \int_\Omega l(z,\widetilde{u}) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{21}$$

We choose $h = -\widetilde{u}^- \in W_0^{1,p(z)}(\Omega)$ and obtain $\widetilde{u} \geqslant 0$, $\widetilde{u} \neq 0$.

Next, in (21) we use the test function $h = (\widetilde{u} - u)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} \langle A_p(\widetilde{u}), (\widetilde{u} - u)^+\rangle &= \int_\Omega \left(c_0 u^{\tau(z)-1} - c_5 u^{p(z)-1}\right)(\widetilde{u} - u)^+\, dz \\ &\leqslant \int_\Omega f(z,u)(\widetilde{u} - u)^+\, dz \\ &\leqslant \langle A_p(u), (\widetilde{u} - u)^+\rangle \end{aligned}$$

(see (15), (19), and use the fact that $u \in S_\lambda$), so $\widetilde{u} \leqslant u$ (see Theorem 2). Consequently, we have

$$\widetilde{u} \in [0,u], \quad \widetilde{u} \neq 0. \tag{22}$$

Then, (19), (21), (22) and Theorem 4 imply that $\widetilde{u} = \overline{u}$, so

$$\overline{u} \leqslant u \quad \forall u \in S_\lambda.$$

□

Let $\lambda^* = \sup \mathcal{L}$. From Propositions 1 and 2, we see that

$$\eta^* = \frac{p_-}{p_+}\widetilde{\mu}_1 \leqslant \lambda^* \leqslant \widehat{\lambda}_1.$$

It is natural to ask about the admissibility of the critical parameter value. On this issue, we have only some partial answers.

First, directly from Proposition 1, we have the following result.

Proposition 4. *If hypotheses H_0, H_1 hold and $\lambda^* = \widehat{\lambda}_1$, then $\lambda^* \notin \mathcal{L}$.*

Another result in this direction is the following one.

Proposition 5. *If hypotheses H_0, H_1 hold and $\lambda^* < \mu_1$, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ be such that $\lambda_n \nearrow \lambda^*$. From the proof of Theorem 3, we know that we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$, such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \forall n \in \mathbb{N},$$

so

$$\int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)}\,dz \leqslant \int_\Omega \frac{\lambda_n}{p(z)}u_n^{p(z)}\,dz + \int_\Omega F(z,u_n)\,dz \quad \forall n \in \mathbb{N}. \tag{23}$$

Hypotheses $H_1(i),(ii)$ imply that given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$, such that

$$F(z,x) \leqslant \frac{\varepsilon}{p(z)}x^{p(z)} + c_7 x^{q(z)} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{24}$$

We use (24) in (23) and obtain

$$\int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)}\,dz \leqslant \int_\Omega \frac{\lambda^* + \varepsilon}{p(z)}u_n^{p(z)} + c_8\varrho_q(u) \quad \forall n \in \mathbb{N},$$

for some $c_8 > 0$, so

$$\int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)}\,dz \leqslant \int_\Omega \frac{1}{p(z)}\frac{\lambda^* + \varepsilon}{\mu_1}|Du_n|^{p(z)}\,dz + c_8\varrho_q(u_n)$$

(see (1)), and thus

$$\left(1 - \frac{\lambda^* + \varepsilon}{\mu_1}\right)\int_\Omega \frac{1}{p(z)}|Du_n|^{p(z)}\,dz \leqslant c_8\varrho_q(u_n). \tag{25}$$

Our aim is to show that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. So, without any loss of generality, we may assume that

$$\|u_n\| > 1, \quad \|u_n\|_{q(z)} > 1 \quad \forall n \in \mathbb{N}.$$

Recall that by hypothesis $\lambda^* < \mu_1$. Consequently, we can choose $\varepsilon > 0$ small so that $\lambda^* + \varepsilon < \mu_1$. Then, from (25), Theorem 1 and because the embedding $W_0^{1,p(z)}(\Omega) \subseteq L^{q(z)}(\Omega)$ is continuous, we have

$$\|u_n\|^{p_-} \leqslant c_9 \|u_n\|^{q_+} \quad \forall n \in \mathbb{N},$$

for some $c_9 > 0$. Thus the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded (since $q_+ < p_-$).

Consequently, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p(z)}(\Omega), \quad u_n \longrightarrow u_* \quad \text{in } L^{p(z)}(\Omega). \tag{26}$$

Because $u_n \in S_{\lambda_n}$, $n \in \mathbb{N}$, we have

$$\langle A_p(u_n), h \rangle = \int_\Omega \left(\lambda_n u_n^{p(z)-1} + f(z, u_n)\right) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega). \tag{27}$$

In (27), we use the test function $h = u_n - u_* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (26). We obtain

$$\lim_{n\to+\infty} \langle A_p(u_n), u_n - u_* \rangle = 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{28}$$

(see Theorem 2). From Proposition 3, we know that

$$\overline{u} \leqslant u_n \quad \forall n \in \mathbb{N},$$

so

$$\overline{u} \leqslant u_*, \tag{29}$$

and so $u_* \neq 0$.

In (27), we pass to the limit as $n \to +\infty$ and use (28). We have

$$\langle A_p(u_*), h \rangle = \int_\Omega \left(\lambda^* u_*^{p(z)-1} + f(z, u_*)\right) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so $u_* \in S_{\lambda^*} \subseteq C_+$ (see (29)) and so $\lambda^* \in \mathcal{L}$. □

Remark 3. *Propositions 4 and 5 are consistent with what is known in the isotropic case (when p is constant). For that case, we have* $0 < \widehat{\lambda}_1 = \mu_1 = \widetilde{\mu}_1$ *and* $\mathcal{L} = (0, \widehat{\lambda}_1)$ *(using Picone's identity, see Papageorgiou–Rădulescu–Repovš [1]).*

It is an open question what can be said about $\lambda^* > 0$ *when*

$$\mu_1 < \widehat{\lambda}_1 \quad \textit{and} \quad \mu_1 \leqslant \lambda^* < \widehat{\lambda}_1.$$

Consequently, we can state the following existence theorem for problem (P_λ).

Theorem 5. *If hypotheses* H_0 *and* H_1 *hold, then there exists* $\lambda^* \in [\frac{p_-}{p_+}\widetilde{\mu}_1, \widehat{\lambda}_1]$, *such that*
(a) for all $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *has at least one positive solution* $u_\lambda \in \operatorname{int} C_+$;
(b) for all $\lambda > \lambda^*$, *problem* (P_λ) *has no positive solution;*
(c) if $\lambda^* = \widehat{\lambda}_1$, *then* $\lambda^* \notin \mathcal{L}$ *and if* $\lambda^* < \mu_1$, *then* $\lambda^* \in \mathcal{L}$.

4. Minimal Positive Solution

In this section, we show that for every $\lambda \in \mathcal{L}$, problem (P_λ) has a smallest positive solution and we determine the continuity and monotonicity properties of the minimal solution map.

Theorem 6. *If hypotheses* H_0, H_1 *hold and* $\lambda \in \mathcal{L}$, *then problem* (P_λ) *has a smallest positive solution* $\widehat{u}_\lambda \in \operatorname{int} C_+$.

Proof. From Filippakis–Papageorgiou [16] (Lemma 4.1), we know that S_λ is downward-directed (that is, if $u_1, u_2 \in S_\lambda$, then we can find $\widetilde{u} \in S_\lambda$, such that $\widetilde{u} \leqslant u_1$, $\widetilde{u} \leqslant u_2$). Consequently, invoking Theorem 5.109 of Hu–Papageorgiou [17] (p. 309), we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$, such that

$$\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle A_p(u_n), h\rangle = \int_\Omega \left(\lambda u_n^{p(z)-1} + f(z, u_n(z))\right) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}, \tag{30}$$

so

$$\overline{u} \leqslant u_n \leqslant u_1 \quad \forall n \in \mathbb{N} \tag{31}$$

(see Proposition 3). From (30) and (31), we see that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. Consequently, we may assume that

$$u_n \xrightarrow{w} \widehat{u}_\lambda \quad \text{in } W_0^{1,p(z)}(\Omega), \quad u_n \longrightarrow \widehat{u}_\lambda \quad \text{in } L^{p(z)}(\Omega). \tag{32}$$

In (30), we use the test function $h = u_n - \widehat{u}_\lambda \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (32). We have

$$\lim_{n\to+\infty} \langle A_p(u_n), u_n - \widehat{u}_\lambda\rangle = 0$$

so

$$u_n \longrightarrow \widehat{u}_\lambda \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{33}$$

(see Theorem 2). From (30), in the limit as $n \to +\infty$, we have

$$\langle A_p(\widehat{u}_\lambda), h\rangle = \int_\Omega (\lambda \widehat{u}_\lambda^{p(z)-1} + f(z, \widehat{u}_\lambda) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega)$$

(see (33)), so

$$\overline{u} \leqslant \widehat{u}_\lambda$$

(see (31)) and hence

$$\widehat{u}_\lambda \in S_\lambda \subseteq \operatorname{int} C_+, \quad \widehat{u}_\lambda = \inf S_\lambda.$$

□

We consider the minimal solution map $\widehat{\vartheta}\colon \mathcal{L} \longrightarrow C_0^1(\overline{\Omega})$ defined by

$$\widehat{\vartheta}(\lambda) = \widehat{u}_\lambda \quad \forall \lambda \in \mathcal{L}.$$

We say that $\widehat{\vartheta}$ is strictly increasing if

$$0 < \mu < \lambda \implies \widehat{u}_\lambda - \widehat{u}_\mu \in \operatorname{int} C_+.$$

The next theorem indicates some monotonicity properties of $\widehat{\vartheta}$.

Theorem 7. *If hypotheses H_0 and H_1 hold, then*
(a) $\widehat{\vartheta}$ is strictly increasing; and
(b) $\widehat{\vartheta}$ is left continuous.

Proof. (a) Let $0 < \mu < \lambda \leqslant \lambda^*$ and let $\widehat{u}_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ be the minimal positive solution of (P_λ). According to Theorem 3, we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$, such that

$$\widehat{u}_\lambda - u_\mu \in \operatorname{int} C_+.$$

Because $\widehat{u}_\mu \leqslant u_\mu$, we have

$$\widehat{u}_\lambda - \widehat{u}_\mu \in \operatorname{int} C_+,$$

so $\widehat{\vartheta}$ is strictly increasing.
(b) Let $\lambda_n > 0$, $\lambda_n \nearrow \lambda \leqslant \lambda^*$. Consider the corresponding minimal solutions $\widehat{u}_{\lambda_n} \subseteq S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$. From (a), we know that the sequence $\{\widehat{u}_{\lambda_n}\}_{n\in\mathbb{N}}$ is increasing. We have

$$\langle A_p(\widehat{u}_{\lambda_n}), h\rangle = \int_\Omega \left(\lambda_n \widehat{u}_{\lambda_n}^{p(z)-1} + f(z, \widehat{u}_{\lambda_n})\right) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}, \tag{34}$$

and

$$\overline{u} \leqslant \widehat{u}_{\lambda_n} \leqslant \widehat{u}_\lambda \quad \forall n \in \mathbb{N} \tag{35}$$

(see Proposition 3). From (34) and (35), it follows that the sequence $\{\widehat{u}_{\lambda_n}\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded.

Then, from Fan–Zhao [18] (Theorem 4.1) (see also Papageorgiou–Rădulescu–Zhang [11]) (Proposition A1), we have that

$$\{\widehat{u}_{\lambda_n}\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega), \quad \|\widehat{u}_{\lambda_n}\|_\infty \leqslant c_{10} \quad \forall n \in \mathbb{N},$$

for some $c_{10} > 0$. The anisotropic regularity theory of Fan [10], implies that there exists $\alpha \in (0,1)$ and $c_{11} > 0$, such that

$$\widehat{u}_{\lambda_n} \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|\widehat{u}_{\lambda_n}\|_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{11} \quad \forall n \in \mathbb{N}. \tag{36}$$

From (36), the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$ and the monotonicity of the sequence $\{\widehat{u}_{\lambda_n}\}_{n\in\mathbb{N}}$, we have

$$\widehat{u}_{\lambda_n} \longrightarrow \widetilde{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega}). \tag{37}$$

We claim that $\widetilde{u}_\lambda = \widehat{u}_\lambda$. If not, then there is $z_0 \in \Omega$, such that

$$\widehat{u}_\lambda(z_0) < \widetilde{u}_\lambda(z_0),$$

so

$$\widehat{u}_\lambda(z_0) < \widehat{u}_{\lambda_n}(z_0) \quad \forall n \geqslant n_0$$

(see (37)), which contradicts part (a). Consequently, $\widetilde{u}_\lambda = \widehat{u}_\lambda$, and we conclude that $\widehat{\vartheta}$ is left continuous. □

Remark 4. *Closing this work, we mention an open problem. Are the results of this paper valid if we replace hypothesis $H_1(iii)$ by the weaker one*

$$\lim_{x\to 0^+} \frac{f(z,x)}{x^{p(z)-1}} = 0 \text{ uniformly for a.a. } z \in \Omega?$$

5. Conclusions

In this paper, we have examoned a superlinear perturbation of the anisotropic eigenvalue problem, and we were able to provide a complete description of the set of positive solutions as the parameter changes. Moreover, we established the existence of a minimal positive solution $\widehat{u}_\lambda$ and examined the continuity and monotonicity properties of the map $\lambda \mapsto \widehat{u}_\lambda$.

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