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On the P_3 Coloring of Graphs

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Abstract: The vertex coloring of graphs is a well-known coloring of graphs. In this coloring, all of the vertices are assigned colors in such a way that no two adjacent vertices have the same color. We can call this type of coloring P_2 coloring, where P_2 is a path graph. However, there are situations in which this type of coloring cannot give us the solution to the problem at hand. To answer such questions, in this article, we introduce a novel graph coloring called P_3 coloring. A graph is called P_3-colorable if we can assign colors to the vertices of the graph such that the vertices of every P_3 path are distinct. The minimum number of colors required for a graph to have P_3 coloring is called the P_3 chromatic number. The aim of this article is, in general, to prove some basic results concerning this coloring, and, in particular, to compute the P_3 chromatic number for different symmetric families of graphs.

Keywords: graph coloring; chromatic number; path graph; cycle graph; prism graph; ladder graph

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1. Introduction

The history of graph coloring started with a problem that was about the maps of some countries, such as the United States. A map was to be colored in such a way that any two countries with the same border could not be colored with the same color. In 1971, an article entitled "The Mathematics of Map Coloring" was published in the *Journal of Recreational Mathematics* by H.S.M. (Donald) Coxeter, who proved that a "minimum of four colors are required for the map of United States to color the common border states differently" (see [1]). Since then, graph coloring has progressed immensely. When we talk about graph theory and its applications, one of the most commonly used, studied, and applicable topics in graph theory is graph coloring (see [2]). Graph coloring has many applications in various fields of life, such as timetabling (see, for example, [3–6]), scheduling daily life activities, scheduling computer processes (see [7,8]), registering allocations to different institutions and libraries (see [6,9,10]), manufacturing tools (see [11]), printed circuit testing (see [12]), routing and wavelength assignment (see [6]), bag rationalization for a food manufacturer (see [13]), satellite range scheduling (see [14,15]), and frequency assignment (see [6,16]). These are some applications out of the many that already exist and many to come. In fact, coloring has inspired many other fields of graph theory.

Coloring theory is the theory of dividing sets with internally compatible conflicts, and there are many different types of graph coloring; the history of graph coloring is provided in a previous survey [2]. There are numerous conjectures about coloring problems that are still unsolved and are being researched by mathematicians and computer scientists internationally; some of these are noted in [17]. There are many research articles being published about graph coloring. There are two types of categories of such articles: One category gives different colors to a graph according to the rules of the topic of the article, and the other is about the colored structures of graphs whose coloring cannot be controlled. For further reading and a literature review about graph coloring, readers can refer to [18–21]. A historical review and some recent developments in graph coloring schemes are presented in [22–24].

One way to understand the coloring of the vertices of a graph G is that we can see it as a function f from the vertex set of G to positive integers such that if xy is an edge of G, then $f(x) \neq f(y)$. In other words, we assign colors to the vertices of G in such a way that adjacent vertices have different colors. This is called graph coloring or, more precisely, the vertex coloring of a graph. The minimum number of colors required for coloring a graph is called its chromatic number. Thus, in a sense, one can say that the assignment of colors to the vertices of a graph is called graph coloring if the colors of the vertices of all P_2 paths in the graph are distinct; that is, instead of using the word "edge", we can use the term "P_2 path".

Therefore, in this article, motivated by the above reasoning, we introduce the P_3-labeling of graphs, and we will discuss this labeling for some very well-known families of graphs, such as path graphs, wheel graphs, cycle graphs, complete graphs, prism graphs, ladder graphs, and star graphs.

Definition 1. *A P_3 coloring is a function f from the vertex set of G to the set of colors $\{c_1, c_2, c_3, \ldots, c_k\}$ such that for every P_3 path on graph G, the colors of its vertices are distinct, that is, if xyz is a P_3 path on G, then $f(x) \neq f(y) \neq f(z) \neq f(x)$. This is a natural generalization of P_2 coloring.*

Definition 2. *The minimum number of colors required for a graph G to have P_3 coloring is called the P_3 chromatic number, and it is denoted as $\chi_3(G)$.*

Note that every P_3 coloring of a graph is also its P_2 coloring. Therefore, we have

$$\chi(G) \leq \chi_3(G). \tag{1}$$

In addition, it is clear from the definition that for all graphs G,

$$\chi_3(G) \geq 3. \tag{2}$$

2. Motivation

To place more emphasis on the motivation for our introduction of this type of coloring, here, we will give an example in which we cannot apply P_2 coloring, but only P_3 coloring. In addition, much in the field of mathematics is produced by the curiosity of the minds of mathematicians when a question emerges while discussing something. In the field of computer science, there are graphs that are associated with bit strings. Suppose that we have a set S consisting of all bit strings of length $n > 1$. Then, a hypercube graph, which is also known as a cube graph and denoted as Q_n, is a graph consisting of the elements of S as vertices, and there is an edge between two strings if they differ at exactly one position. Now, if we want to assign different colors to the strings that are different by at most two positions (or someone challenges us to color the strings (vertices) of Q_n in such a way that any two strings that differ by at most two positions have different colors), then what type of coloring will we use? Moreover, what is the minimum number of colors required to achieve this? It is very easy to see that we cannot use P_2 coloring here, and we have to apply P_3 coloring to find a possible solution. For example, in Figure 1, we can see that the usual P_2 coloring of Q_2 requires only two colors to color this graph, but for the above question, we cannot color it with two colors. For this purpose, we have to use P_3 coloring, which gives us the solution with four colors for Q_2. For the case in which $n \geq 3$, $\chi_3(Q_n)$ is given in the section of this article on open questions.

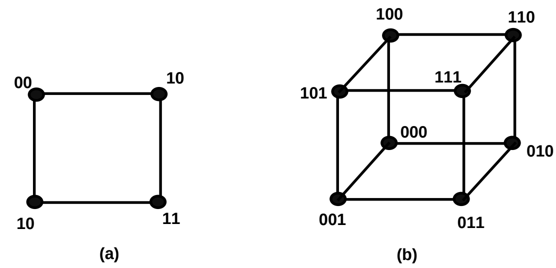

Figure 1. Hypercube graphs of strings. Part (**a**) is of Q_2 and part (**b**) is graph Q_3.

3. Main Results

In this section, we present the main results of this article. Theorem 1, Lemma 1, and its corollaries give us the P_3 chromatic number of some general graphs. In Theorems 2–5, closed formulas are found for the P_3 chromatic number of path graphs, cycle graphs, prism graphs, and ladder graphs. The next theorem gives us the P_3 chromatic number of path graph.

Lemma 1. *Let G be a simple graph on n vertices and assume that there is a vertex $v \in V(G)$ such that v is adjacent to every vertex of G, then $\chi_3(G) = n$.*

Proof. By contrast, suppose that $\chi_{P_3}(G) < n$. This means that there are two vertices in G having the same color, e.g., x, y are those vertices. However, every two vertices of G are on some P_3 path having v as the middle vertex. Thus, we have a P_3 path xvy, and this path has the same color as that of its end vertices. This is a contradiction. Therefore, $\chi_3(G)$ must be $|V(G)|$. □

The following corollaries directly result from the Lemma 1:

Corollary 1. *Let K_n be the complete graph then $\chi_3(K_n) = n$ for all $n \geq 3$;*

Corollary 2. *Let W_n be the wheel graph on n vertices, then $\chi_3(W_n) = n$ for all $n \geq 4$;*

Corollary 3. *Let S_n be the star graph on n vertices, then $\chi_3(S_n) = n$ for all $n \geq 3$.*

The following Theorem 1 also follows from the above definition:

Theorem 1. *Let G be a graph and H be a subgraph of G then $\chi_3(G) \geq \chi_3(H)$;*

Theorem 2. *Let P_n be the path graph, then $\chi_3(P_n) = 3$ for all $n \geq 3$;*

Proof. Let P_n be a path graph on n vertices, where $n \geq 3$ as shown in Figure 2.

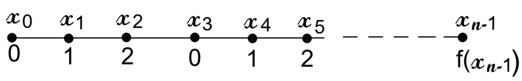

Figure 2. P_3 labeling of P_n.

To show that $\chi_{P_3}(P_n) = 3$, first, we show that the path graph has P_3 coloring. For this purpose, we define a function from the set of vertices of P_n to the set of colors $\{0, 1, 2\}$. Thus, lets define a function as follows:

$$f(x_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod{3}; \\ 1, & \text{if } i \equiv 1 \pmod{3}; \\ 2, & \text{if } i \equiv 2 \pmod{3}. \end{cases}$$

To prove that f is indeed a P_3 coloring, we show that all P_3 paths in P_n are of different colors. Let Q be a P_3 path in P_n as depicted in Figure 3; then, there are three possible cases.

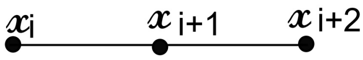

Figure 3. An arbitrary P_3 path in P_n.

Case I: If $j \equiv 0(\text{mod } 3)$, then $f(x_j) = 0,\ f(x_{j+1}) = 1,\ f(x_{j+2}) = 2$;
Case II: If $j \equiv 1(\text{mod } 3)$, then $f(x_{i+1}) = 1,\ f(x_{i+2}) = 2,\ f(x_{i+3}) = 0$;
Case III: If $j \equiv 2(\text{mod } 3)$, then $f(x_{i+2}) = 2,\ f(x_{i+3}) = 0,\ f(x_{i+4}) = 1$.

Thus, from all of the above possible cases, we can see that all P_3 paths Q have different colors of its vertices under the labeling f. Thus, f is a P_3 coloring and by using Equation (2), we reach our conclusion that is $chi_{P_3}(P_n) = 3$. □

Theorem 3. *Let C_n be the cycle graph and $n \neq 5$. Then, for all $n \geq 3$*

$$\chi_3(C_n) = \begin{cases} 3, & n \equiv 0(\text{mod } 3); \\ 4, & n \not\equiv 2(\text{mod } 3). \end{cases}$$

Proof. Let C_n be the cycle graph on n vertices and $n \geq 3$. This proof consists of three cases on three different values of n under mod 3.

Case I: Suppose that $n \equiv 0(\text{mod } 3)$. Let us define a function f on the vertices of C_n as follows:

$$f(x_i) = \begin{cases} 1, & \text{if } i \equiv 0\,(\text{mod } 3); \\ 2, & \text{if } i \equiv 1\,(\text{mod } 3); \\ 3, & \text{if } i \equiv 2\,(\text{mod } 3), \end{cases}$$

where $1 \leq i \leq n-4$. Figure 4 represents the P_3 coloring of C_9 to explain this labeling. Let $Q_1 : x_i x_{i+1} x_{i+2}$ be an arbitrary P_3 path in C_n, as shown in Figure 5, for $0 \leq i \leq n-1$.

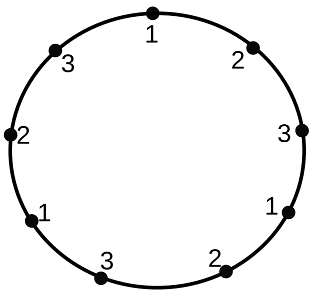

Figure 4. The labeling of the vertices of C_9 under f.

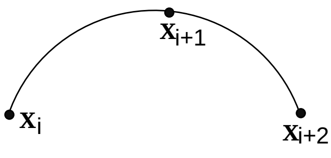

Figure 5. An arbitrary P_3 path of C_{15}.

Then, there are three possible cases.

(a). From Figure 5, we have,
if $i \equiv 0(\text{mod } 3)$, then $f(x_i) = 1, \quad f(x_{i+1}) = 2, \quad f(x_{i+2}) = 3$;
(b). If $i \equiv 1(\text{mod } 3)$, then $f(x_i) = 2, \quad f(x_{i+1}) = 3, \quad f(x_{i+2}) = 1$;
(c). If $i \equiv 2(\text{mod } 3)$, then $f(x_i) = 3, \quad f(x_{i+1}) = 1, \quad f(x_{i+2}) = 2$.

Thus, from all of these cases, we can see that all P_3 paths have different colors of their vertices. Thus, f is a P_3 coloring, and the result follows.

Case II: Suppose that $n \equiv 1 (\mathrm{mod}\ 3)$. In this case, when we start a P_3 coloring of C_n from any vertex, e.g., x_1, to the last, e.g., x_n, with at most three colors, then the last vertex x_n cannot be assigned any color from the given three colors. Thus, we need at least four colors to have a P_3 coloring of this graph. For the reverse case, we define the P_3-labeling function as follows:
$f(x_n) = 1, f(x_{n-1}) = 2, f(x_{n-2}) = 3, f(x_{n-3}) = 4$ and for all $1 \le i \le n-4$ we have

$$f(x_i) = \begin{cases} 4, & \text{if } i \equiv 1 \ (\mathrm{mod}\ 3); \\ 3, & \text{if } i \equiv 2 \ (\mathrm{mod}\ 3); \\ 2, & \text{if } i \equiv 0 \ (\mathrm{mod}\ 3). \end{cases}$$

Figure 6 represents the P_3 coloring of C_{10} to explain this labeling. Let Q_2 be an arbitrary path $x_i x_{i+1} x_{i+2}$ in C_n; then, there are four possible cases to discuss this labeling.

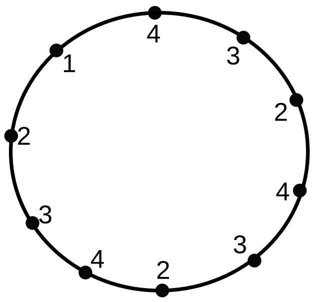

Figure 6. P_3 labeling of C_{10}.

(a). If $i \equiv 0 (\mathrm{mod}\ 3)$, then $f(x_i) = 2$, $f(x_{i+1}) = 4$, $f(x_{i+2}) = 3$;
(b). If $i \equiv 1 (\mathrm{mod}\ 3)$, then $f(x_i) = 4$, $f(x_{i+1}) = 3$, $f(x_{i+2}) = 2$;
(c). If $i \equiv 2 (\mathrm{mod}\ 3)$, then $f(x_i) = 3$, $f(x_{i+1}) = 2$, $f(x_{i+2}) = 4$;
(d). When the paths are of the forms $x_1 x_2 x_n$, $x_1 x_n x_{n-1}$, $x_{n-2} x_{n-3} x_{n-4}$ and $x_{n-3} x_{n-4} x_{n-5}$;

- **(i)** For the path $x_1 x_2 x_n$, we have the labeling $f(x_1) = 4$, $f(x_2) = 3$, $f(x_n) = 1$;
- **(ii)** For the path $x_1 x_n x_{n-1}$, we have the labeling $f(x_1) = 4$, $f(x_n) = 1$, $f(x_{n-1}) = 2$;
- **(iii)** For the path $x_{n-2} x_{n-3} x_{n-4}$, we have the labeling $f(x_{n-2}) = 3$, $f(x_{n-3}) = 4$, $f(x_{n-4}) = 2$ because $n \equiv 1 (\mathrm{mod}\ 3)$;
- **(iv)** For the path $x_{n-3} x_{n-4} x_{n-5}$, we have the labeling $f(x_{n-3}) = 4$, $f(x_{n-4}) = 2$, $f(x_{n-5}) = 3$ because $n \equiv 1 (\mathrm{mod}\ 3)$.
  Thus, from all of these cases, we can see that all P_3 paths have different colors of their vertices. Thus, f is indeed a P_3 coloring. This shows that $\chi_{P_3}(C_n) \le 4$. This concludes the result.

Case III: When $n \equiv 2 (\mathrm{mod}\ 3)$. In this case, to start a P_3 coloring of C_n from vertex x_1 to the last with at most three colors, the last two vertices x_{n-1}, x_n cannot be assigned any color from the given three colors. Therefore, we need at least four colors to have P_3 coloring of this graph. For the reverse case, we will define P_3 labeling function as follows:
$f(x_1) = 1$, $f(x_2) = 2$, $f(x_3) = 3$, $f(x_4) = 4$, $f(x_n) = 4$, $f(x_{n-1}) = 3$, $f(x_{n-2}) = 2$, $f(x_{n-3}) = 1$, and for all $5 \le i \le n-4$, the function is defined by

$$f(x_i) = \begin{cases} 2, & \text{if } i \equiv 2 \ (\mathrm{mod}\ 3); \\ 3, & \text{if } i \equiv 0 \ (\mathrm{mod}\ 3); \\ 4, & \text{if } i \equiv 1 \ (\mathrm{mod}\ 3). \end{cases}$$

To explain this labeling, Figure 7 shows a P_3 coloring of C_{11} under f. Let Q_3 be any arbitrary P_3 path in C_n; then, we have the following cases to discuss for the assertion of P_3 coloring.

(a). If $i \equiv 0(\text{mod } 3)$ and $5 \leq i \leq n-4$, then $f(x_i) = 3, f(x_{i+1}) = 4, f(x_{i+2}) = 2$;
(b). If $i \equiv 1(\text{mod } 3)$ and $5 \leq i \leq n-4$, then $f(x_i) = 4, f(x_{i+1}) = 2, f(x_{i+2}) = 3$;
(c). If $i \equiv 2(\text{mod } 3)$ and $5 \leq i \leq n-4$, then $f(x_i) = 2, f(x_{i+1}) = 3, f(x_{i+2}) = 4$;
(d). For the following paths, we have different labeling:

(i) For path $x_3x_4x_5$, we have the labeling $f(x_3) = 3,\ f(x_4) = 4,\ f(x_5) = 2$;
(ii) For path $x_4x_5x_6$, we have the labeling $f(x_4) = 4,\ f(x_5) = 2,\ f(x_6) = 3$;
(iii) For path $x_{n-2}x_{n-3}x_{n-4}$, we have the labeling $f(x_{n-2}) = 2,\ f(x_{n-3}) = 1,\ f(x_{n-4}) = 4$ because $n \equiv 2(\text{mod}3)$;
(iv) For path $x_{n-3}x_{n-4}x_{n-5}$, we have the labeling $f(x_{n-3}) = 1,\ f(x_{n-4}) = 4,\ f(x_{n-5}) = 3$ because $n \equiv 2(\text{mod}3)$.

Therefore, from all of these cases, we can see that all P_3 paths have different colors of their vertices. Thus, f is a P_3 coloring and $\chi_3(C_n) \leq 4$ for all $n \geq 8$. Hence, the proof is completed.

□

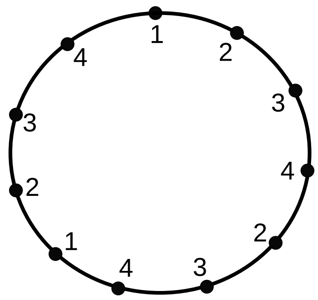

Figure 7. P_3 labeling of C_{11}.

Remark 1. *The* $\chi_3(C_5) = 5$*, and it is very easy to see that we cannot have* P_3*-coloring of* C_5 *with less than* 5 *colors.*

Remark 2. *Note that for* $\chi_3(C_n) = 3 = \chi_3(C_n)$ *for* $n = 3$ *and* $\chi_3(P_n) = 3 > 2 = \chi(P_n)$ *this shows that for some graphs, the* P_3*-chromatic number is equal to their chromatic number, and for some graphs, this relation is strict.*

Theorem 4. *Let* $G \cong D_n$ *be the prism graph. Then*

$$\chi_3(D_n) = \begin{cases} 4, & \text{if } n \equiv 0(\text{mod } 4) \text{ and } n \geq 4; \\ 5, & \text{if } n \equiv 1(\text{mod } 4) \text{ and } n \geq 9; \\ 5, & \text{if } n \equiv 2(\text{mod } 4) \text{ and } n \geq 10; \\ 5, & \text{if } n \equiv 3(\text{mod } 4) \text{ and } n \geq 15. \end{cases}$$

Proof. Let D_n be the prism graph as depicted in Figure 8. We shall discuss the proof in four cases.

Case I. Assume that $n \equiv 0(\text{mod } 4)$ and $n \geq 4$. Since C_4 is a subgraph of D_n, then from Theorem 1, we have $\chi_3(D_n) \geq 4$. To prove the reverse, we shall define a function $g : V(Dn) \to \{1,2,3,4\}$ as follows:

$$g(\alpha_i) = \begin{cases} 1, & \text{if } i \equiv 1(\text{mod } 4), \quad 1 \leq i \leq n; \\ 2, & \text{if } i \equiv 2(\text{mod } 4), \quad 1 \leq i \leq n; \\ 3, & \text{if } i \equiv 3(\text{mod } 4), \quad 1 \leq i \leq n; \\ 4, & \text{if } i \equiv 0(\text{mod } 4), \quad 1 \leq i \leq n. \end{cases}$$

$$g(\beta_i) = \begin{cases} 3, & \text{if } i \equiv 1(\text{mod } 4), \quad 1 \leq i \leq n; \\ 4, & \text{if } i \equiv 2(\text{mod } 4), \quad 1 \leq i \leq n; \\ 1, & \text{if } i \equiv 3(\text{mod } 4), \quad 1 \leq i \leq n; \\ 2, & \text{if } i \equiv 0(\text{mod } 4), \quad 1 \leq i \leq n. \end{cases}$$

We will show that f is a P_3 coloring. Let Q_1 be any arbitrary P_3 path in D_n; then, there are ten possible types of P_3 paths in D_n, and they are as follows: The paths are $\alpha_i\alpha_{i+1}a_{i+2}$, $\alpha_i\alpha_{i+1}\beta_{i+1}$, $\alpha_i\beta_i\beta_{i+1}$, $\beta_i\beta_{i+1}b_{i+2}$, $\beta_i\alpha_i\alpha_{i+1}$, $\beta_i\beta_{i+1}\alpha_{i+1}$, $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$ and $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$.

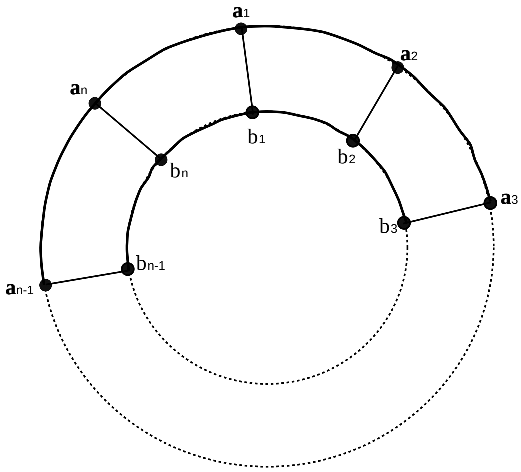

Figure 8. The prism graph of D_n, where the vertices $b_i = \beta_i$.

(a). For $i \equiv 0(\text{mod } 4)$, we have ten possibilities of the induced coloring of Q_1 from g as follows:

- **(i)** If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 1$, $g(\alpha_{i+2}) = 2$;
- **(ii)** If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 1$, $g(\beta_{i+1}) = 3$;
- **(iii)** If the path is $\alpha_i b_i\beta_{i+1}$, then we have $g(\alpha_i) = 4$, $g(\beta_i) = 2$, $g(\beta_{i+1}) = 3$;
- **(iv)** If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 2$, $g(\beta_{i+1}) = 3$, $g(\beta_{i+2}) = 4$;
- **(v)** If the path is $\beta_i a_i\alpha_{i+1}$, then we have $g(\beta_i) = 2$, $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 1$;
- **(vi)** If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 2$, $g(\beta_{i+1}) = 3$, $g(\alpha_{i+1}) = 1$;
- **(vii)** If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 1, g(\alpha_{i+2}) = 2$, $g(\beta_{i+2}) = 3$;
- **(viii)** If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 3, g(\alpha_{i+1}) = 1$, $g(\alpha_{i+2}) = 2$;
- **(ix)** If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 2, g(\beta_{i+2}) = 4$, $g(\beta_{i+1}) = 3$;
- **(x)** If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 4, g(\beta_{i+1}) = 3$, $g(\alpha_{i+1}) = 1$.

(b). For $i \equiv 1(\text{mod } 4)$, we have ten possibilities of the induced coloring of Q_1 from g as follows:

- **(i)** If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 1$, $g(\alpha_{i+1}) = 2$, $g(\alpha_{i+2}) = 3$;
- **(ii)** If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 1$, $g(\alpha_{i+1}) = 2$, $g(\beta_{i+1}) = 4$;
- **(iii)** If the path is $\alpha_i b_i\beta_{i+1}$, then we have $g(\alpha_i) = 1$, $g(\beta_i) = 3$, $g(\beta_{i+1}) = 4$;
- **(iv)** If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 3$, $g(\beta_{i+1}) = 4$, $g(\beta_{i+2}) = 1$;
- **(v)** If the path is $\beta_i a_i\alpha_{i+1}$, then we have $g(\beta_i) = 3$, $g(\alpha_i) = 1$, $g(\alpha_{i+1}) = 2$;
- **(vi)** If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 3$, $g(\beta_{i+1}) = 4$, $g(\alpha_{i+1}) = 2$;
- **(vii)** If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 4, g(\alpha_{i+2}) = 3$, $g(\beta_{i+2}) = 1$;
- **(viii)** If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 4, g(\alpha_{i+1}) = 2$, $g(\alpha_{i+2}) = 3$;
- **(ix)** If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 2, g(\beta_{i+2}) = 1$, $g(\beta_{i+1}) = 4$;
- **(x)** If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 1, g(\beta_{i+1}) = 4$, $g(\alpha_{i+1}) = 2$.

(c). For $i \equiv 2(\text{mod } 4)$, we again have ten possibilities of the induced coloring of Q_1 from g as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$, $g(\alpha_{i+2}) = 4$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$, $g(\beta_{i+1}) = 1$;
(iii) If the path is $\alpha_i b_i\beta_{i+1}$, then we have $g(\alpha_i) = 2$, $g(\beta_i) = 4$, $g(\beta_{i+1}) = 1$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 4$, $g(\beta_{i+1}) = 1$, $g(\beta_{i+2}) = 2$;
(v) If the path is $\beta_i a_i\alpha_{i+1}$, then we have $g(\beta_i) = 4$, $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$;
(vi) If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 4$, $g(\beta_{i+1}) = 1$, $g(\alpha_{i+1}) = 3$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 3$, $g(\alpha_{i+2}) = 4$, $g(\beta_{i+2}) = 2$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 1$, $g(\alpha_{i+1}) = 3$, $g(\alpha_{i+2}) = 4$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 4$, $g(\beta_{i+2}) = 2$, $g(\beta_{i+1}) = 1$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 2$, $g(\beta_{i+1}) = 1$, $g(\alpha_{i+1}) = 3$.

(d). For $i \equiv 3(\text{mod } 4)$, we have ten possibilities of the induced coloring of Q_1 from g as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$, $g(\beta_{i+1}) = 2$;
(iii) If the path is $\alpha_i b_i\beta_{i+1}$, then we have $g(\alpha_i) = 3$, $g(\beta_i) = 1$, $g(\beta_{i+1}) = 2$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 2$, $g(\beta_{i+2}) = 3$;
(v) If the path is $\beta_i a_i\alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$;
(vi) If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 3$, $g(\alpha_{i+1}) = 4$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 3$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 3$, $g(\beta_{i+1}) = 2$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 3$, $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$.

In all of these subcases, we can see that g is indeed a P_3 coloring. Therefore, the P_3-chromatic number of prism graph D_n is 4 for all $n \geq 4$ and $n \equiv 0(\text{mod } 4)$.
Case II. Assume that $n \equiv 1(\text{mod } 4)$ and $n \geq 9$. Note that in any P_3 coloring, if we color the vertices of the D_n graph from a set of only four colors $\{a, b, c, d\}$, because graph C_4 is a subgraph of D_n, then all of the vertices of every C_4 have different colors, as shown in Figure 9.

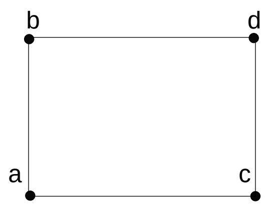

Figure 9. An arbitrary C_4 in D_n, when $n \equiv 1(\text{mod } 4)$.

Therefore, when we apply any P_3 coloring on D_n, for any C_4 subgraph, its left and right adjacent C_4's have the only possible P_3 coloring, as shown in Figure 10.

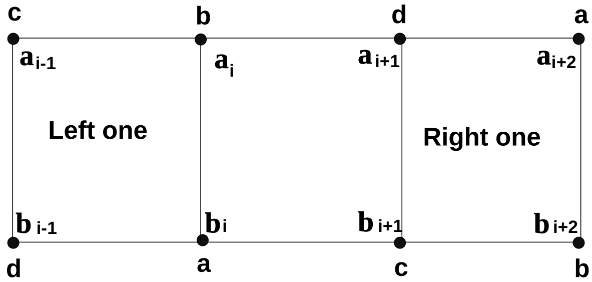

Figure 10. Adjacent C_4s when $n \equiv 1 (\bmod 4)$, where the vertices $b_i = \beta_i$.

Now, this process is continued until a P_3-coloring with only four colors $\{a, b, c, d\}$ is completed or produced. We define a C_4 to be correctly colored if all its vertices have different colors. Because $n \not\equiv 0 (\bmod 4)$, and we have n number of C_4s that covers D_n, eventually, we will arrive at a situation displayed in Figure 11.

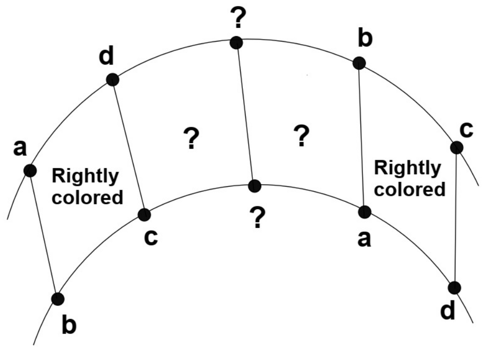

Figure 11. Covering of D_n by correctly colored C_4s when $n \equiv 1 (\bmod 4)$.

Since $n \equiv 1 (\bmod 4)$, from Figure 11, we can easily see that the remaining two C_4s cannot be correctly colored with only four given colors. Thus, we have $\chi_3(D_n) \geq 5$, and this shows that we need at least five colors to produce a P_3 coloring of D_n. For the reverse case, let us define a function g from $V(D_n)$ to $\{1, 2, 3, 4, 5\}$ as follows:

$$g(\alpha_i) = \begin{cases} i, & \text{if } 1 \leq i \leq 5; \\ 1, & \text{if } i \equiv 2 \mod 4, \quad 6 \leq i \leq n; \\ 2, & \text{if } i \equiv 3 \mod 4, \quad 6 \leq i \leq n; \\ 3, & \text{if } i \equiv 0 \mod 4, \quad 6 \leq i \leq n; \\ 4, & \text{if } i \equiv 1 \mod 4, \quad 6 \leq i \leq n, \end{cases}$$

$g(\beta_1) = 3, g(\beta_2) = 5,$

$$g(\beta_i) = \begin{cases} i-2, & \text{if } 3 \leq i \leq 7; \\ 1, & \text{if } i \equiv 0 \mod 4, \quad 8 \leq i \leq n; \\ 2, & \text{if } i \equiv 1 \mod 4, \quad 8 \leq i \leq n; \\ 3, & \text{if } i \equiv 2 \mod 4, \quad 8 \leq i \leq n; \\ 5, & \text{if } i \equiv 3 \mod 4, \quad 8 \leq i \leq n. \end{cases}$$

Let Q_2 be an arbitrary P_3 path; then, as before, there will be ten possible P_3 paths for any given $i \in \{1, 2, \dots n\}$. It is enough to discuss the following possible P_3 paths to prove that g is indeed a P_3 coloring: $\alpha_i\alpha_{i+1}\alpha_{i+2}$, $\alpha_i\alpha_{i+1}\beta_{i+1}$, $\alpha_i, \beta_i, \beta_{i+1}$, $\beta_i\beta_{i+1}\beta_{i+2}$, $\beta_i\alpha_i\alpha_{i+1}$, $\beta_i\beta_{i+1}\alpha_{i+1}$, $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$ and the last one is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, for all $9 \leq i \leq n-1$.
It is clear from the above definition that the remaining paths satisfy the P_3 coloring, as depicted in Figure 12.

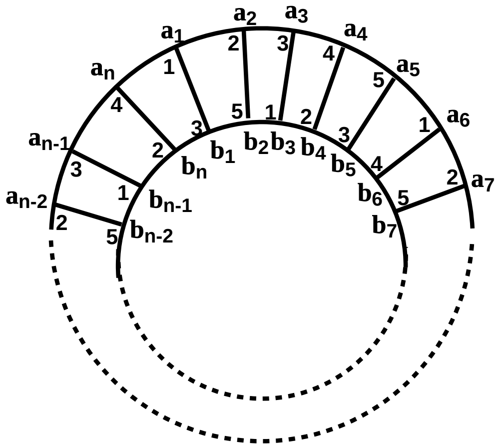

Figure 12. P_3 coloring of D_n when $n \equiv 1(\text{mod } 4)$, where the vertices $b_i = \beta_i$.

Now, we will show that g is indeed P_3 coloring in four cases, which are discussed below.

(a). For $i \equiv 0(\text{mod } 4)$ and $9 \le i \le n-1$, all possibilities of Q_2 are discussed as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$, $g(\beta_{i+1}) = 2$;
(iii) If the path is $\alpha_i b_i \beta_{i+1}$, then we have $g(\alpha_i) = 3$, $g(\beta_i) = 1$, $g(\beta_{i+1}) = 2$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 2$, $g(\beta_{i+2}) = 3$;
(v) If the path is $\beta_i a_i \alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\alpha_i) = 3$, $g(\alpha_{i+1}) = 4$;
(vi) If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 3$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 3$, $g(\beta_{i+1}) = 2$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 3$, $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$.

(b). For $i \equiv 1(\text{mod } 4)$, all possibilities of Q_2 are discussed as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 5$, $g(\alpha_{i+2}) = 2$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 5$, $g(\beta_{i+1}) = 1$;
(iii) If the path is $\alpha_i b_i \beta_{i+1}$, then we have $g(\alpha_i) = 4$, $g(\beta_i) = 3$, $g(\beta_{i+1}) = 1$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 3$, $g(\beta_{i+1}) = 1$, $g(\beta_{i+2}) = 4$;
(v) If the path is $\beta_i a_i \alpha_{i+1}$, then we have $g(\beta_i) = 3$, $g(\alpha_i) = 4$, $g(\alpha_{i+1}) = 5$;
(vi) If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 3$, $g(\beta_{i+1}) = 1$, $g(\alpha_{i+1}) = 5$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 5$, $g(\alpha_{i+2}) = 2$, $g(\beta_{i+2}) = 4$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 2$, $g(\beta_{i+2}) = 4$, $g(\beta_{i+1}) = 1$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 4$, $g(\beta_{i+1}) = 1$, $g(\alpha_{i+1}) = 5$.

(c). For $i \equiv 2(\text{mod } 4)$, all possibilities of Q_2 are discussed as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 5$, $g(\alpha_{i+1}) = 2$, $g(\alpha_{i+2}) = 3$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 5$, $g(\alpha_{i+1}) = 2$, $g(\beta_{i+1}) = 4$;
(iii) If the path is $\alpha_i b_i \beta_{i+1}$, then we have $g(\alpha_i) = 5$, $g(\beta_i) = 1$, $g(\beta_{i+1}) = 4$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 4$, $g(\beta_{i+2}) = 5$;

(v) If the path is $\beta_i a_i \alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\alpha_i) = 5$, $g(\alpha_{i+1}) = 2$;
(vi) If the path is $\beta_i \beta_{i+1} \alpha_{i+1}$, then we have $g(\beta_i) = 1$, $g(\beta_{i+1}) = 4$, $g(\alpha_{i+1}) = 2$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 2$, $g(\alpha_{i+2}) = 3$, $g(\beta_{i+2}) = 5$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 3$, $g(\beta_{i+2}) = 5$, $g(\beta_{i+1}) = 4$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 5$, $g(\beta_{i+1}) = 4$, $g(\alpha_{i+1}) = 2$.

(d). For $i \equiv 3 (\bmod 4)$, all possibilities of Q_2 are discussed as follows:

(i) If the path is $\alpha_i\alpha_{i+1}\alpha_{i+2}$, then we have $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$, $g(\alpha_{i+2}) = 1$;
(ii) If the path is $\alpha_i\alpha_{i+1}\beta_{i+1}$, then we have $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$, $g(\beta_{i+1}) = 5$;
(iii) If the path is $\alpha_i b_i \beta_{i+1}$, then we have $g(\alpha_i) = 2$, $g(\beta_i) = 4$, $g(\beta_{i+1}) = 5$;
(iv) If the path is $\beta_i\beta_{i+1}\beta_{i+2}$, then we have $g(\beta_i) = 4$, $g(\beta_{i+1}) = 5$, $g(\beta_{i+2}) = 2$;
(v) If the path is $\beta_i a_i \alpha_{i+1}$, then we have $g(\beta_i) = 4$, $g(\alpha_i) = 2$, $g(\alpha_{i+1}) = 3$;
(vi) If the path is $\beta_i\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_i) = 4$, $g(\beta_{i+1}) = 5$, $g(\alpha_{i+1}) = 3$;
(vii) If the path is $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, then we have $g(\alpha_{i+1}) = 3$, $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 2$;
(viii) If the path is $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, then we have $g(\beta_{i+1}) = 2$, $g(\alpha_{i+1}) = 4$, $g(\alpha_{i+2}) = 1$;
(ix) If the path is $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, then we have $g(\alpha_{i+2}) = 1$, $g(\beta_{i+2}) = 2$, $g(\beta_{i+1}) = 5$;
(x) If the path is $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$, then we have $g(\beta_{i+2}) = 2$, $g(\beta_{i+1}) = 5$, $g(\alpha_{i+1}) = 3$.

Thus, in this case, all of these subcases proved that g is a P_3 coloring of D_n for $n \equiv 1 (\bmod 4)$. Therefore, $\chi_{P_3}(D_n) = 5$.

Case III. Assume that $n \equiv 2 (\bmod 4)$ and $n \geq 10$. Similarly, as in case II, when we apply any P_3 coloring on D_n with only four given colors $\{a, b, c, d\}$, for any C_4 subgraph in D_n, its left and right adjacent C_4's have the only possible P_3 coloring labels, as shown in Figure 13.

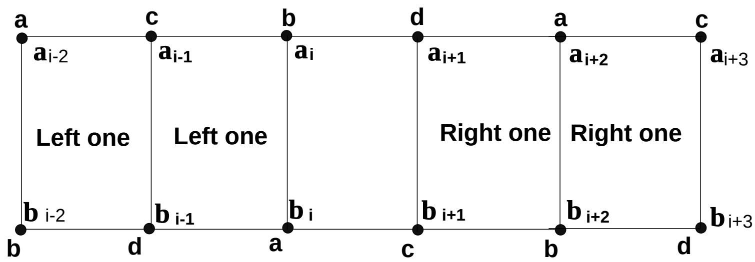

Figure 13. Adjacent C_4s when $n \equiv 2 (\bmod 4)$, where the vertices $b_i = \beta_i$.

We continue this process to complete (or produce) a P_3 coloring with only four colors a, b, c and d. Because $n \equiv 2 (\bmod 4)$, and we have n number of C_4 subgraphs that covers D_n, eventually, we will reach the situation displayed in Figure 14.

Thus, in the case when $n \equiv 2 (\bmod 4)$, from Figure 14, we can easily see that the remaining three C_4s cannot be correctly colored with only four given colors. Thus, we have $\chi_3(D_n) \geq 5$. Therefore, we need at least five colors to produce a P_3 coloring of D_n. For the reverse case, let us define a function g from $V(D_n)$ to $\{1, 2, 3, 4, 5\}$ as follows:

$$g(\alpha_i) = \begin{cases} 1, & \text{if } i \equiv 1 \mod 5, & 1 \leq i \leq 10; \\ 2, & \text{if } i \equiv 2 \mod 5, & 1 \leq i \leq 10; \\ 3, & \text{if } i \equiv 3 \mod 5, & 1 \leq i \leq 10; \\ 4, & \text{if } i \equiv 4 \mod 5, & 1 \leq i \leq 10; \\ 5, & \text{if } i \equiv 0 \mod 5, & 1 \leq i \leq 10; \\ 1, & \text{if } i \equiv 3 \mod 4, & 11 \leq i \leq n; \\ 2, & \text{if } i \equiv 0 \mod 4, & 11 \leq i \leq n; \\ 3, & \text{if } i \equiv 1 \mod 4, & 11 \leq i \leq n; \\ 4, & \text{if } i \equiv 2 \mod 4, & 11 \leq i \leq n. \end{cases}$$

$g(\beta_1) = 3, g(\beta_2) = 5.$

$$g(\beta_i) = \begin{cases} 1, & \text{if } i \equiv 3 \mod 5, & 3 \leq i \leq 12; \\ 2, & \text{if } i \equiv 4 \mod 5, & 3 \leq i \leq 12; \\ 3, & \text{if } i \equiv 0 \mod 5, & 3 \leq i \leq 12; \\ 4, & \text{if } i \equiv 1 \mod 5, & 3 \leq i \leq 12; \\ 5, & \text{if } i \equiv 2 \mod 5, & 3 \leq i \leq 12; \\ 1, & \text{if } i \equiv 1 \mod 4, & 13 \leq i \leq n; \\ 2, & \text{if } i \equiv 2 \mod 4, & 13 \leq i \leq n; \\ 3, & \text{if } i \equiv 3 \mod 4, & 13 \leq i \leq n; \\ 5, & \text{if } i \equiv 0 \mod 4, & 13 \leq i \leq n. \end{cases}$$

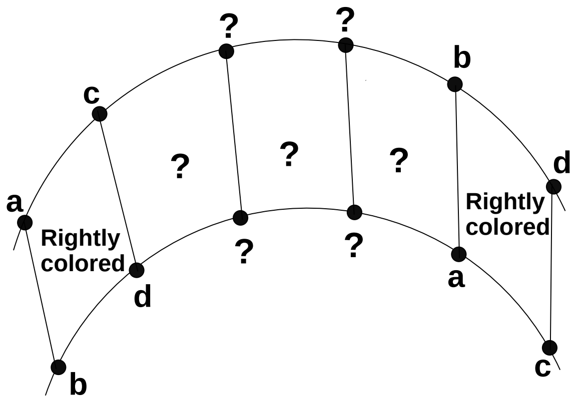

Figure 14. Covering of D_n by rightly colored C_4s when $n \equiv 2(\text{mod } 4)$.

Let Q_3 be an arbitrary P_3 path; then, similarly as in case I and case II, it is very easy to see that g is indeed a P_3 coloring. Together with the argument at the beginning of this case, this proves that the P_3-chromatic number of a prism graph D_n is equal to 5 for $n \equiv 2(\text{mod } 4)$ for all $n \geq 10$.

Case IV. Assume that $n \equiv 3(\text{mod } 4)$ and $n \geq 15$. Similarly to case II, whenever we apply any P_3 coloring on D_n with only four colors $\{a, b, c, d\}$, for any C_4 subgraph in D_n, its left and right adjacent C_4's have the only possible P_3-coloring labels, as shown in Figure 10.

We continue this process to complete (or produce) a P_3 coloring with only four colors a, b, c, and d. Because $n \equiv 3(\text{mod } 4)$, and we have the n number of C_4 subgraphs that covers D_n, eventually, we will reach the situation displayed in Figure 11.

Therefore, in the case when $n \equiv 3(\text{mod } 4)$, from Figure 11 and the above argument, we can easily see that the remaining two C_4 cannot be correctly colored with only four colors. Thus, we must have $\chi_3(D_n) \geq 5$, that is, we need at least five colors to produce (or obtain) a P_3 coloring of D_n. For the reverse case, let us define a function g from $V(D_n)$ to

$\{1,2,3,4,5\}$ as follows:

$$g(\alpha_i) = \begin{cases} 1, & \text{if } i \equiv 1 \mod 5, & 1 \le i \le 15; \\ 2, & \text{if } i \equiv 2 \mod 5, & 1 \le i \le 15; \\ 3, & \text{if } i \equiv 3 \mod 5, & 1 \le i \le 15; \\ 4, & \text{if } i \equiv 4 \mod 5, & 1 \le i \le 15; \\ 5, & \text{if } i \equiv 0 \mod 5, & 1 \le i \le 15; \\ 1, & \text{if } i \equiv 0 \mod 4, & 16 \le i \le n; \\ 2, & \text{if } i \equiv 1 \mod 4, & 16 \le i \le n; \\ 3, & \text{if } i \equiv 2 \mod 4, & 16 \le i \le n; \\ 4, & \text{if } i \equiv 3 \mod 4, & 16 \le i \le n. \end{cases}$$

$g(\beta_1) = 3$, $g(\beta_2) = 5$.

$$g(\beta_i) = \begin{cases} 1, & \text{if } i \equiv 3 \mod 5, & 3 \le i \le 17; \\ 2, & \text{if } i \equiv 4 \mod 5, & 3 \le i \le 17; \\ 3, & \text{if } i \equiv 0 \mod 5, & 3 \le i \le 17; \\ 4, & \text{if } i \equiv 1 \mod 5, & 3 \le i \le 17; \\ 5, & \text{if } i \equiv 2 \mod 5, & 3 \le i \le 17; \\ 1, & \text{if } i \equiv 2 \mod 4, & 18 \le i \le n; \\ 2, & \text{if } i \equiv 3 \mod 4, & 18 \le i \le n; \\ 3, & \text{if } i \equiv 0 \mod 4, & 18 \le i \le n; \\ 5, & \text{if } i \equiv 1 \mod 4, & 18 \le i \le n. \end{cases}$$

Let Q_4 be an arbitrary P_3 path in D_n; then, as in case I and case II, it is very easy to see that g is indeed a P_3 coloring. This, combined with the argument at the beginning of this case, proves that the P_3-chromatic number of a prism graph D_n is equal to 5 for $n \equiv 3 (\text{mod } 4), \forall n \ge 15$.

Thus, the proof of this theorem is completed. □

Theorem 5. *Let L_n be the ladder graph; then, $\chi_3(L_n) = 4$ for all $n \ge 2$.*

Proof. Let L_n be the ladder graph with $n \ge 2$. Then, by Theorem 1 and Theorem 3, we have $\chi_3(L_n) \ge 4$. To show that $\chi_3(L_n) \le 4$, we will define a labeling g on $V(L_n)$ to the color set $\{1,2,3,4\}$.

$$g(\alpha_i) = \begin{cases} 1, & \text{if } i \equiv 1 \mod 4, & 1 \le i \le n; \\ 2, & \text{if } i \equiv 2 \mod 4, & 1 \le i \le n; \\ 3, & \text{if } i \equiv 3 \mod 4, & 1 \le i \le n; \\ 4, & \text{if } i \equiv 0 \mod 4, & 1 \le i \le n. \end{cases}$$

$$g(\beta_i) = \begin{cases} 3, & \text{if } i \equiv 1 \mod 4, & 1 \le i \le n; \\ 4, & \text{if } i \equiv 2 \mod 4, & 1 \le i \le n; \\ 1, & \text{if } i \equiv 3 \mod 4, & 1 \le i \le n; \\ 2, & \text{if } i \equiv 0 \mod 4, & 1 \le i \le n. \end{cases}$$

For the reader, this labeling is explained in Figure 15. Let Q be any arbitrary path of L_n; then, there are ten possible types of P_3 paths in L_n, and they are as follows. The paths are $\alpha_i\alpha_{i+1}\alpha_{i+2}$, $\alpha_i\alpha_{i+1}\beta_{i+1}$, $\alpha_i\beta_i\beta_{i+1}$, $\beta_i\beta_{i+1}\beta_{i+2}$, $\beta_i\alpha_i\alpha_{i+1}$, $\beta_i\beta_{i+1}\alpha_{i+1}$, $\alpha_{i+1}\alpha_{i+2}\beta_{i+2}$, $\beta_{i+1}\alpha_{i+1}\alpha_{i+2}$, $\alpha_{i+2}\beta_{i+2}\beta_{i+1}$, and $\beta_{i+2}\beta_{i+1}\alpha_{i+1}$. Then, similarly to the first case of Theorem 4, its clear that f is indeed a P_3 coloring. Thus, we achieve the result. □

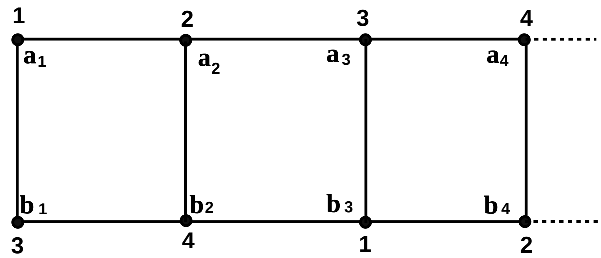

Figure 15. A P_3 coloring of L_n, where the vertices $b_i = \beta_i$.

4. Open Problems and Time Complexity

It is well known that graph coloring and, in particular, vertex coloring is an NP-hard problem. Since our P_3 coloring is also a vertex coloring problem, one could speculate that this is also an NP-hard problem. We proved that for some families of graphs, this problem can be solved in polynomial time, which is clear from Theorem 2 to Theorem 5. For future research, we plan to list a few of the open problems concerning this new coloring scheme, which include the following objectives:

Problem 1: Prove or disprove the conjecture that "The P_3 coloring is an NP-hard problem";
Problem 2: Discuss all types of time complexity of P_3 coloring as an NP-hard problem;
Problem 3: Find stronger upper and lower bounds of the P_3 coloring;
Problem 4: Find $\chi_3(Q_n)$ for $n \geq 3$.

5. Conclusions

In this article, we introduced a new type of graph coloring scheme, and we called it P_3 coloring. A graph having P_3 coloring is also a vertex-colorable graph (in a normal sense). We proved that when a graph has a vertex that is adjacent to all other vertices of the graph, its P_3-chromatic number is the cardinality of the vertices of the graph. Then, we computed the P_3 chromatic number of some well-known families of graphs. To this end, we computed the P_3-chromatic number of path, cycle, prism, and ladder graphs. It is clear from the above discussion and results that the P_3-chromatic number of a graph is greater than or equal to its chromatic number. Graph coloring always raises strong interest among mathematicians and other related researchers. Therefore, any new and novel coloring technique is always followed by numerous graph-coloring studies in the literature. We gave an example of a challenge that can be countered only by using P_3 coloring. This also gives us the reasons for why this type of coloring should exist. We presented some future plans and open problems to further advance in this topic and extend its application.

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