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Stability Properties of Self-Similar Solutions in Symmetric Teleparallel $f(Q)$-Cosmology

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Abstract: Self-similar cosmological solutions correspond to spacetimes that admit a homothetic symmetry. The physical properties of self-similar solutions can describe important eras of the cosmological evolution. Recently, self-similar cosmological solutions were derived for symmetric teleparallel $f(Q)$-theory with different types of connections. In this work, we study the stability properties of the self-similar cosmological solutions in order to investigate the effects of the different connections on the stability properties of the cosmic history. For the background geometry, we consider the isotropic Friedmann–Lemaître–Robertson–Walker space and the anisotropic and homogeneous Bianchi I space, for which we investigate the stability properties of Kasner-like universes.

Keywords: cosmology; symmetric teleparallel; self-similar solutions; Milne universe; Kasner universe

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1. Introduction

On the cosmological scale, General Relativity (GR) fails to provide an explanation for the observed phenomena [1–5]. At the present time, the universe is under in acceleration phase, which is attributed to a matter source called dark energy. Dark energy has not been directly observed until now, but new dynamical terms are attributed to it, which are necessarily introduced into the field equations of GR in order to provide anti-gravitational effects and explain the late-time acceleration. Because dark energy has been observed indirectly only, the nature and the origin of the dark energy is unknown to cosmologists. There is a specific approach adopted by cosmologists to explain the acceleration through the introduction of geometric invariants in the gravitational Action Integral [6], from which as a result the dark energy is a geometric effect of the gravitational theory. These new theories of gravity are known as modified theories of gravity [7–10]. The plethora of different models in the literature, which are based on the use of different geometric invariants for the modification of the Action Integral, form a "zoology" of gravitational theories, $f(R)$-theory [11,12], Gauss–Bonnet gravity [13,14], teleparallel theory of gravity [15–18] and others [19–21].

The fundamental invariant of GR is the curvature scalar R related to the Levi–Civita connection of the metric tensor which describes the physical space. However, an arbitrary connection can describe a manifold with a curvature scalar R, torsion scalar T and non-metricity scalar Q. These three scalars, R, T and Q, are also known as the geometrical trinity of gravity [22]. The Levi–Civita connection that is considered in GR is torsionless and with zero metricity parts. Thus, only the curvature scalar is used for the definition of the gravitational action. On the other hand, the existence of an unholonomic frame leads to a connection with a non-zero torsion scalar. The Weitzenböck connection [23] is a curvature-less connection, which leads to the Teleparallel Equivalent of General Relativity (TEGR), for which the torsion plays the role of the gravitational force [24]. Moreover, it has been found that a geometry with a torsion-free connection with non-zero scalar Q that describes a flat geometry, that is, the Riemann tensor is zero, is equivalent to GR and it is known as Symmetric Teleparallel General Relativity (STGR) [25].

When the Gravitational Action Integral is constructed by the scalars R, T and Q, the resulting three theories, GR, TEGR and STGR, are indeed equivalent. However, this equivalency is lost when additional invariants are introduced into the gravitational Action Integral in order to solve the dark energy problem. $f(R)$-gravity [9] is a fourth-order theory and includes various important models, which have been used to describe various epochs of the cosmological evolution such as the inflationary epoch [26,27] or the late-time acceleration phase related to the dark energy [28–30]. In a similar way, in teleparallelism a plethora of $f(T)-$gravitational models have been introduced for the description of the observable phenomena. For a review of teleparallelism, we refer the reader to [31]. Furthermore, in [32], it is shown that in teleparallel gravity inflation can occur without the existence of an inflaton field, while $f(T)$ theory can mimic the dark energy model [33]. See also [34–36] and references therein.

Recently, the $f(Q)$-gravity has drawn the attention of cosmologists. $f(Q)$-gravity can provide answers to various observation phenomena in cosmology, so as to provide dynamical terms in the field equations to explain the effects of dark energy, and to pass various tests by using the current observational data. In [37], the authors applied cosmological constraints for $f(Q)$ models, which can reproduce the ΛCDM expansion history; it was found that the $f(Q)$ theory is in agreement with the observations and it is supported by data over the ΛCDM model. A similar result was found dependently in [38]. The nonlinear $f(Q) = \alpha Q + \beta Q^n$ theory studied in [39] as dark energy model and the cosmographic parameters derived. In [40], the authors considered $f(Q)$ models, for which the equation of state parameter for the cosmological model can cross the phantom divine line. In [41–43] external and internal solutions of compact stars were studied. New cosmological asymptotic solutions in $f(Q)$ theory with initial singularity were derived in [44]. See also the results presented in [45–47]. Bouncing cosmological scenarios in $f(Q)$-gravity were investigated in [48]. Moreover, in [49], the method of statefinder diagnostics was applied in symmetric teleparallel theory and it was found that for two nonlinear $f(Q)$ models the field equations provide an evolution, which describes acceleration in the late time due to the dark energy component evolving from the non-metricity components. Finally, the ADM formulation and Hamiltonian analysis of $f(Q)$ theory are presented in [50].

Unlike GR, in $f(Q)$-gravity it is possible to separate gravity from the inertial effects [51]. This is possible because we make use of a flat connection pertaining to the existence of affine coordinates in the coincident gauge in which the covariant derivative is reduced to the usual partial derivative. Indeed, because in STGR the curvature tensor is zero, there always exists a coordinate system in which the connection becomes zero, $\Gamma^{\lambda}_{\mu\nu} = 0$ [52].

The coincidence gauge can always be derived under a coordinate transformation. It has been found that the use of different connections for the definition of the non-metricity scalar, Q, affects the dynamics of $f(Q)$-gravity [53,54]. For the Friedmann–Lemaître–Robertson–Walker (FLRW) cosmological model four different families of connections have been found [55]. In [56], the existence of self-similar solutions was investigated in $f(Q)$-theory for FLRW cosmology. Self-similar solutions are of huge interest, because they can describe specific eras of the cosmological evolution. The self-similar solutions can be related to the asymptotic behaviour of the general dynamics [44]. In addition, the existence of the limit of solutions that are described by GR in $f(Q)$-theory is of special interest for the viability of the theory.

In this study, we are interested in the stability properties of the self-similar solutions in order to understand the effects of different connections during the different epochs of the cosmological history. Moreover, we are interested to investigate the existence of asymptotic scaling solutions that describe inflation to see if $f(Q)$ symmetric teleparallel theory for different kinds of connections can reproduce the inflationary epoch. Such analysis gives constraints on the viability of the different kind of connections. The structure of the paper is as follows:

In Section 2, we present the basic properties and definitions for the $f(Q)$ symmetric teleparallel theory. Section 3 includes the main results of this work in which we investigate the stability properties of self-similar cosmological solutions for the different connections in the coincidence gauge which generate the cosmological field equations of FLRW geometry in symmetric teleparallel theory. In Section 4, we discuss the existence of self-similar solutions in an anisotropic and homogeneous Bianchi I geometry in the context of $f(Q)$-theory. We find that a generalized Kasner solution is supported by $f(Q)$-theory in the coincidence gauge. Finally, in Section 5 we draw our conclusions.

2. Basic Properties and Definitions for the $f(Q)$ Symmetric Teleparallel Theory

Assume a four-dimensional manifold V^4 with metric tensor $g_{\mu\nu}$ and generic connection $\Gamma^{\kappa}_{\mu\nu}$. From the connection, we can define the Riemann tensor

$$R^{\kappa}{}_{\lambda\mu\nu} = \frac{\partial \Gamma^{\kappa}{}_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\kappa}{}_{\lambda\mu}}{\partial x^{\nu}} + \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\kappa}{}_{\mu\sigma},$$

the torsion tensor

$$\mathrm{T}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$$

and the non-metricity tensor

$$Q_{\lambda\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\sigma}{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma}.$$

In STGR, it follows that the torsion tensor vanishes, that is, $\mathrm{T}^{\lambda}_{\mu\nu} = 0$, and that the geometry is flat, i.e., $R^{\kappa}{}_{\lambda\mu\nu} = 0$, while only the non-metricity tensor survives. The non-metricity scalar is defined as

$$Q = Q_{\lambda\mu\nu} P^{\lambda\mu\nu},$$

where $P^{\lambda}_{\mu\nu}$ is the non-metricity conjugate tensor expressed as

$$P^{\lambda}_{\mu\nu} = -\frac{1}{4} Q^{\lambda}_{\mu\nu} + \frac{1}{2} Q_{(\mu}{}^{\lambda}{}_{\nu)} + \frac{1}{4}\left(Q^{\lambda} - \bar{Q}^{\lambda}\right) g_{\mu\nu} - \frac{1}{4}\delta^{\lambda}_{(\mu} Q_{\nu)}$$

are $Q_{\lambda} = Q_{\lambda}{}^{\mu}{}_{\mu}$, $\bar{Q}_{\lambda} = Q^{\mu}{}_{\lambda\mu}$.

The gravitational Action Integral in In $f(Q)$-theory is as follows [57]

$$S = \frac{1}{2}\int d^4x \sqrt{-g} f(Q) + \int d^4x \sqrt{-g}\mathcal{L}_M + \lambda_{\kappa}{}^{\lambda\mu\nu} R^{\kappa}{}_{\lambda\mu\nu} + \tau_{\lambda}{}^{\mu\nu} \mathrm{T}^{\lambda}{}_{\mu\nu}, \tag{1}$$

in which $g = \det(g_{\mu\nu})$, $\mathcal{L}_M$ is the Lagrangian density for the matter source and $\lambda_{\kappa}{}^{\lambda\mu\nu}$, $\tau_{\lambda}{}^{\mu\nu}$ are two Lagrange multipliers that impose the flatness of the connection, i.e., $R^{\kappa}{}_{\lambda\mu\nu} = 0$, and that the connection is symmetric, that is, $\mathrm{T}^{\lambda}_{\mu\nu} = 0$.

The gravitational field equations are derived as

$$\frac{2}{\sqrt{-g}} \nabla_{\lambda}\left(\sqrt{-g} f'(Q) P^{\lambda}_{\mu\nu}\right) - \frac{1}{2} f(Q) g_{\mu\nu} + f'(Q)\left(P_{\mu\rho\sigma} Q_{\nu}^{\rho\sigma} - 2 Q_{\rho\sigma\mu} P^{\rho\sigma}{}_{\nu}\right) = T_{\mu\nu}, \tag{2}$$

where $f'(Q) = \frac{df(Q)}{dQ}$ and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L}_M)}{\partial g^{\mu\nu}}$ is the energy-momentum tensor, which describes the matter components of the gravitational fluid.

Moreover, variation of the Action Integral (1) with respect to the connection gives the field equations

$$\nabla_{\mu}\nabla_{\nu}\left(\sqrt{-g} f'(Q) P^{\mu\nu}{}_{\sigma}\right) = 0. \tag{3}$$

The field equations (2) can be written in the equivalent form

$$f'(Q)G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(f'(Q)Q - f(Q)\right) + 2f''(Q)(\nabla_\lambda Q)P^\lambda_{\ \mu\nu} = T_{\mu\nu}, \tag{4}$$

in which $G_{\mu\nu}$ is the Einstein-tensor, where $G_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}$, with $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ are the Riemannian Ricci tensor and scalar, respectively, which are constructed by the Levi–Civita connection.

We can easily define the effective energy-momentum tensor

$$\mathcal{T}_{\mu\nu} = -\frac{1}{f'(Q)}\left[\frac{1}{2}g_{\mu\nu}\left(f'(Q)Q - f(Q)\right) + 2f''(Q)(\nabla_\lambda Q)P^\lambda_{\ \mu\nu}\right]$$

which contributes the geometrodynamical degrees of freedom. Therefore, with the latter definition, the field equations can be written in the known form similar to that of general relativity

$$G_{\mu\nu} = \mathcal{T}_{\mu\nu} + \frac{1}{f'(Q)}T_{\mu\nu}.$$

3. FLRW Cosmology

According to the cosmological principle, at a large scale the universe is isotropic and homogeneous described by the FLRW line element

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where $a(t)$ is the scale factor and $N(t)$ is the lapse function. The Hubble function is defined as $H = \frac{1}{N}\frac{\dot{a}}{a}$, where $\dot{a} = \frac{da}{dt}$. Parameter k is the spatial curvature and $k = 0$ corresponds to a spatially flat space, $k = 1$ is a closed universe and $k = -1$ is an open universe. Another important characteristic of the FLRW geometry is that the three-dimensional hypersurface is maximally symmetric and it admits a six-dimensional Killing algebra.

All the compatible connections for the FLRW for the symmetric teleparallel theory by enforcing on a generic connection the six Killing symmetries of the background geometry and the requirement to be flat have been derived before in [53,54]. In the following, without loss of generalitym, we assume $N(t) = 1$.

For $k = 0$, there are three compatible connections with common components

$$\Gamma^r_{\ \theta\theta} = -r, \quad \Gamma^r_{\ \phi\phi} = -r\sin^2\theta,$$
$$\Gamma^\theta_{\ r\theta} = \Gamma^\theta_{\ \theta r} = \Gamma^\phi_{\ r\phi} = \Gamma^\phi_{\ \phi r} = \frac{1}{r}, \quad \Gamma^\theta_{\ \phi\phi} = -\sin\theta\cos\theta, \Gamma^\phi_{\ \theta\phi} = \Gamma^\phi_{\ \phi\theta} = \cot\theta.$$

The first connection, Γ_1, has the additional non-zero component

$$\Gamma^t_{\ tt} = \gamma(t),$$

where $\gamma(t)$ is a function of the time variable t.

The second connection, Γ_2, has the extra non-zero components

$$\Gamma^t_{\ tt} = \frac{\dot{\gamma}(t)}{\gamma(t)} + \gamma(t), \quad \Gamma^r_{\ tr} = \Gamma^r_{\ rt} = \Gamma^\theta_{\ t\theta} = \Gamma^\theta_{\ \theta t} = \Gamma^\phi_{\ t\phi} = \Gamma^\phi_{\ \phi t} = \gamma(t),$$

while the third connection, Γ_3, has the non-zero components for the connection

$$\Gamma^t_{\ tt} = -\frac{\dot{\gamma}(t)}{\gamma(t)}, \quad \Gamma^t_{\ rr} = \gamma(t), \quad \Gamma^t_{\ \theta\theta} = \gamma(t)r^2, \quad \Gamma^t_{\ \phi\phi} = \gamma(t)r^2\sin^2\theta.$$

For an FLRW geometry with non-zero spatial curvature, that is, $k \neq 0$, there exists only one compatible connection, namely Γ_4^k with non-zero coefficients

$$\Gamma^t{}_{tt} = -\frac{k + \dot{\gamma}(t)}{\gamma(t)}, \quad \Gamma^t{}_{rr} = \frac{\gamma(t)}{1 - kr^2} \quad \Gamma^t{}_{\theta\theta} = \gamma(t) r^2, \quad \Gamma^t{}_{\phi\phi} = \gamma(t) r^2 \sin^2(\theta)$$
$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = -\frac{k}{\gamma(t)}, \quad \Gamma^r{}_{rr} = \frac{kr}{1 - kr^2},$$
$$\Gamma^r{}_{\theta\theta} = -r\left(1 - kr^2\right), \quad \Gamma^r{}_{\phi\phi} = -r \sin^2(\theta)\left(1 - kr^2\right), \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r},$$
$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta.$$

We remark that the latter connection in the limit $k = 0$ reduces to the third connection for the flat case. That is an important observation because these two connections may relate the field equations for the flat and non-flat spatial curvature cases.

Indeed, for each different connection the resulting cosmological field equations are different. The existence of self-similar solutions was the subject of study in [56]. For each connection, the functional form of the arbitrary function $f(Q)$ is explicitly determined by the assumption that the background space and the connection admit a homothetic symmetry vector related to the existence of the self-similar solution, similarly to the case of GR.

Below, we present the self-similar solutions determined before and we investigate the stability of the solutions. Such an analysis provides important information about the evolution of the dynamical variables. Simultaneously, constraints can be constructed for the free parameters of the model and for the initial conditions of the theory.

3.1. First Connection

For the first connection, Γ_1, we derive the non-metricity scalar

$$Q = -\frac{6\dot{a}^2}{a^2} = -6H^2. \tag{5}$$

Moreover, the gravitational field equations are (4)

$$\begin{aligned} 3H^2 f'(Q) + \tfrac{1}{2}(f(Q) - Qf'(Q)) &= \rho\,, \\ -\tfrac{2}{N}\tfrac{d}{dt}(f'(Q)H) - 3H^2 f'(Q) - \tfrac{1}{2}(f(Q) - Qf'(Q)) &= p. \end{aligned} \tag{6}$$

The parameters ρ and p correspond to the energy density and pressure components of the energy-momentum tensor for an external fluid. As it was found before in [56], there does not exist any self-similar solution for the first-connection without an external matter source.

Assume now that the energy-momentum tensor describes an ideal gas with a constant equation of state parameter w, that is $p = w\rho$, and the background geometry describes a self-similar universe with $a(t) = a_0 t^\lambda$. Then, from the conservation law of energy for the energy-momentum tensor, it follows that $\rho(t) = \rho_0 t^{-3\lambda(1+w)}$, where ρ_0 is a constant.

By substituting in the field equation we find

$$f(Q) = f_0\sqrt{-Q} + f_1(-Q)^{\frac{3}{2}\lambda(w+1)}, \tag{7}$$

where $f_1 = \frac{2\lambda^{-3\lambda(1+w)}\rho_0}{6^{\frac{3}{2}\lambda(w+1)}(1-3\lambda(1+w))}$.

The first coefficient of (7) does not contribute to the field equations. Indeed, the term $\sqrt{|gQ|}$is a total derivative and when it is replaced by (5) in the Action Integral (1) it can be eliminated. Thus, the important coefficient is the second one, the power $f(Q) = f_1(-Q)^{\frac{3}{2}\lambda(w+1)}$. The effective cosmological fluid for the self-similar solution has a constant effective equation of the state parameter $w_{eff} = -1 + \frac{2}{3\lambda}$, which does not depend upon the external fluid. That means that the gravitational effects of external fluid are overlapped or neutralized by the $f(Q)$ terms.

With the use of (7) it is easy to see that the scaling solution $a(t) = a_0 t^{\lambda}$ is the analytic solution to the problem, which means that the solution is stable, because it is the unique solution. We remark that the stability analysis presented in [44] is for $f(Q) = Q + f_1 Q^n$ and the results there should not be confused with our conclusion for this work.

3.2. Second Connection

For the second connection we derive the non-metricity scalar

$$Q = -6H^2 + 9\gamma H + 3\dot{\gamma} \tag{8}$$

in which the gauge function $\gamma(t)$ is involved and it contributes to the cosmological dynamics.

The field equations in the case of a vacuum read

$$3H^2 f'(Q) + \frac{1}{2}\left(f(Q) - Qf'(Q)\right) + \frac{3\gamma \dot{Q} f''(Q)}{2} = 0, \tag{9}$$

$$2\frac{d}{dt}\left(f'(Q)H\right) - 3\gamma \dot{Q} f''(Q) = 0, \tag{10}$$

where there exists the constraint equation for the function $f(Q)$ given by

$$\dot{Q}^2 f'''(Q) + \left[\ddot{Q} + 3H\dot{Q}\right] f''(Q) = 0. \tag{11}$$

The self-similar solution [56] $a(t) = a_0 t^{\frac{1}{3}}$m which describes a universe dominated by stiff fluid, i.e., $w_{eff} = 1$, exists for $f(Q) = f_1\, Q + f_2\, Q \ln(-Q)$, and the gauge function $\gamma(t) = \frac{C}{t} - \frac{2}{9t} \ln t$, where $C = \frac{f_1 + f_2\left(3 + \ln\left(\frac{4}{3}\right)\right)}{9 f_2}$. We remark that $f(Q) = f_1\, Q + f_2\, Q \ln(-Q)$ for $f_2 = -\bar{f}_2 f_1$ and very small values of $\bar{f}_2$ it can be seen as the first term of the Taylor expansion of the $f(Q) = -f_1 (-Q)^{1+\bar{f}_2}$ theory. This means that for small deviation from GR, the resulting solution describes a stiff fluid source and not the Minkowski spacetime. Recall that we do not consider the existence of an external matter source. This result differs from that found in the similar case of $f(R)$-cosmology [58].

A more general self-similar scaling solution, $a(t) = a_0 t^{\lambda}$, $\lambda \neq 0, \frac{1}{3}$, is supported by the cosmological field equations for $f(Q) = f_1 Q + f_2 Q^{\frac{2}{3(1-\lambda)}}$ and gauge function $\gamma = \frac{q_0 t^{3\lambda} + 12\lambda^2 t}{6(3\lambda - 1)t^2}$, where $f_2 = -6 f_1 (\lambda - 1)\lambda q_0^{\frac{2}{3(\lambda-1)}}$.

Stability Properties

We proceed with the investigation of the stability properties for the self-similar solution with $\lambda \neq 0, \frac{1}{3}$. We consider perturbations only on the scale factors and not on the gauge function $\gamma(t)$.

Thus, for the scale factor $a(t) = a_0 t^{\lambda}$ we derive the Hubble function $H(t) = \frac{\lambda}{t}$. We assume now that $H(t) = \frac{\lambda}{t} + \varepsilon \delta H(t)$. We substitute into (9) and the first-order perturbations give the first-order differential equation

$$\begin{aligned} 0 = 3q_0^2 t^{6\lambda} &\left(\left(8 - 39\lambda + 36\lambda^2\right)\delta H - t\delta\dot{H}\right) \\ &+ 16t^2\lambda^3(3\lambda - 2)\left((2 - 3\lambda)\delta H + t\delta\dot{H}\right) - 8q_0 t^{1+3\lambda}\lambda\left((6\lambda - 1)\delta H + t\delta\dot{H}\right). \end{aligned}$$

This has the analytic solution

$$\delta H(t) = \delta H_0 t^{8-9\lambda}\left(3q_0 t^{3\lambda} + 4t(2 - 3\lambda)\lambda\right)^{6\lambda-7}\left(q_0 t^{3\lambda} + 4\lambda^2 t\right)^{6\lambda-3}.$$

Therefore, for $\lambda > \frac{1}{3}$ and $8 + 36\lambda^2 - 39\lambda < 0$, the perturbations decay for large values of t, i.e., $\lim_{t\to+\infty} \delta H(t) = 0$, which means that the self-similar solution $a(t) = a_0 t^\lambda$ is a stable solution and it is an attractor. For these values of the free parameter λ, it follows that $-0.1754 < w_{eff} < 1$, that is, there is no stable self-similar solution which describes an accelerated universe.

However, we have not considered the equation of motion (11) in the stability analysis. We remark that we have not perturbed the gauge. In this case, the first order perturbations of (11) decay when $4 + 18\lambda^2 - 21\lambda$. Thus, the field equations are stable when $\frac{1}{3} < \lambda < \frac{13+\sqrt{41}}{24}$.

3.3. Third Connection

For the third connection the non-metricity scalar is calculated to be

$$Q = -6H^2 + \frac{3\gamma}{a^2}H + \frac{3\dot{\gamma}}{a^2}, \tag{12}$$

while cosmological field equations are

$$3H^2 f'(Q) + \frac{1}{2}(f(Q) - Qf'(Q)) - \frac{3\gamma\dot{Q}f''(Q)}{2a^2} = 0, \tag{13}$$

$$2\frac{d}{dt}(f'(Q)H) + \frac{\gamma\dot{Q}f''(Q)}{a^2} = 0, \tag{14}$$

with the constraint

$$\dot{Q}^2 f'''(Q) + \left[\ddot{Q} + \dot{Q}\left(H + \frac{2\dot{\gamma}}{\gamma}\right)\right] f''(Q) = 0. \tag{15}$$

For the field equations of the third connection, we found that they support a self-similar universe with $a(t) = a_0 t^\lambda$, for which the connection is also self-similar under a homothetic transformation, when $\gamma(t) = \gamma_0 t^{2\lambda-1}$, for the function [56]

$$f(Q) = f_1 Q^{\frac{5\lambda-1}{2}}, \tag{16}$$

with the constraint equation $\gamma_0 = \frac{2\lambda(2-5\lambda)}{5\lambda-3}$ and $\lambda \neq \frac{1}{5}, \frac{3}{5}$ or $\gamma_0 = \frac{2\lambda}{3\lambda-1}$ and $\lambda > \frac{3}{5}$.

We proceed with the study of the perturbations for the two exact scaling solutions.

Stability Properties

Similarly to the previous connection, we performed a perturbation on the Hubble function, $H(t) = \frac{\lambda}{t} + \varepsilon\delta H(t)$ and we substitute in (13) for the power-law function (16).

For $\gamma_0 = \frac{2\lambda(2-5\lambda)}{5\lambda-3}$, the first-order perturbations provide the ordinary differential equation

$$t(8 - 15\lambda)\delta\dot{H} + 32 + \lambda(30\lambda - 77)\delta H = 0,$$

with analytic solution $\delta H(t) = \delta H_0 t^{\frac{32-77\lambda+30\lambda^2}{15\lambda-8}}$, from which it follows that the perturbations decay and the self-similar solution is stable for $\frac{32-77\lambda+30\lambda^2}{15\lambda-8} < 0$, i.e., $\lambda < 0.52$ and $0.53 < \lambda < 2.04$.

For the second case with $\gamma_0 = \frac{2\lambda}{3\lambda-1}$ the first-order perturbations are always zero which means that the self-similar solution is always stable.

For the first case and the perturbations of the constraint equation, (15) we find that the field equations admit the self-similar solution $a(t) = a_0 t^\lambda$ as an attractor when $\frac{32-77\lambda+30\lambda^2}{15\lambda-8} < 0$ and $\frac{16-\lambda(7+45\lambda)}{15\lambda-8} < 0$, that is, $-0.68 < \lambda < 0.52$ and $0.53 < \lambda < 2.04$.

3.4. Fourth Connection

For a non-zero spatial curvature, k, the non-metricity scalar becomes

$$Q = -6H^2 + \frac{3\gamma}{a^2}H + \frac{3\dot{\gamma}}{a^2} + k\left[\frac{6}{a^2} + \frac{3}{\gamma}\left(\frac{\dot{\gamma}}{\gamma} - 3H\right)\right]. \tag{17}$$

The gravitational field equations in a vacuum are

$$3H^2 f'(Q) + \frac{1}{2}(f(Q) - Qf'(Q)) - \frac{3\gamma\dot{Q}f''(Q)}{2a^2} + 3k\left(\frac{f'(Q)}{a^2} - \frac{\dot{Q}f''(Q)}{2\gamma}\right) = 0, \tag{18}$$

$$2\frac{d}{dt}(f'(Q)H) + \frac{\gamma\dot{Q}f''(Q)}{a^2} + 4\frac{k}{a^2}f'(Q) = 0$$

and the equation of motion for the connection is written as

$$\dot{Q}^2 f'''(Q)\left(1 + \frac{ka^2}{\gamma^2}\right) + \left[\ddot{Q}\left(1 + \frac{ka^2}{\gamma^2}\right) + \dot{Q}\left(\left(1 + \frac{3ka^2}{\gamma^2}\right)H + \frac{2\dot{\gamma}}{\gamma}\right)\right]f''(Q) = 0. \tag{19}$$

For the last connection, which describes an FLRW universe with non-zero spatial curvature, for an arbitrary curvature the self-similar solution was found to be $a(t) = a_0 t$ [56] for the gauge function $\gamma(t) = \gamma_0 t$, and

$$f(Q) = Q^2,$$

with the constraint equation $\gamma_0 = -3a_0^2$.

On the other hand, for the open universe, i.e., $k = -1$, there exists the additional power theory

$$f(Q) = q_0 Q^{\frac{a_0 \mp 1}{2a_0}},$$

where q_0 is a constant and it is required in order $a_0 \neq \pm 1$ and $\gamma(t) = \pm a_0 t$. The case $a_0^2 = 1$ describes the Milne spacetime while the latter function, $f(Q)$, is reduced to that of STGR, i.e., $f(Q) = Q$.

Stability Properties

We now study the stability properties of the scaling-solution $a(t) = a_0 t$ for the quadratic $f(Q) = Q^2$ model. In a similar approach to the previous one, we do not change the gauge $\gamma(t)$. Thus, we substitute $H = \frac{1}{t} + \varepsilon\delta H(t)$ into (18) and from the first-order perturbations we define the ordinary differential equation

$$3t\delta\dot{H} - 5\delta H = 0,$$

that is, $\delta H(t) = \delta H_0 t^{\frac{5}{3}}$. Hence, the scaling solution is always unstable.

For the open universe and the power-law theory $f(Q) = q_0 Q^{\frac{a_0 \mp 1}{2a_0}}$ it follows that the first-order perturbations of Equation (18) do not decay, that is, the Milne-line scaling solution is always unstable.

4. Existence of Self-Similar Solutions in an Anisotropic Bianchi I Cosmology in the Context of $f(Q)$-Gravity

Consider now the Bianchi I line element

$$ds^2 = -N(t)^2 dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \tag{20}$$

in which a, b and c are the three scale factors. When $a = b$ and $a = c$, the Bianchi I line-element reduces to that of the spatially flat FLRW cosmology. Without loss of generality we have selected a constant lapse function, i.e., $N(t) = 1$.

For the Bianchi I geometry we follow [59] and in the coincidence gauge we assume the following metricity scalar

$$Q=-2\left(\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{a}\dot{c}}{ab}+\frac{\dot{b}\dot{c}}{bc}\right),$$

where in the limit of isotropization the metricity scalar for the connection Γ_1 of FLRW spacetime is recovered. In [41] the isotropization of the Bianchi I spacetime in $f(Q)$ theory was investigated; specifically, it was found that due to inflation which is related with the non-metricity components the Bianchi I spacetime can become isotropic. However, in this study, we are interested in a Kasner-like solution which always describes an anisotropic universe.

In the case in which there is no additional matter source, the gravitational field equations are

$$\left(\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{a}\dot{c}}{ac}+\frac{\dot{b}\dot{c}}{bc}\right)f'(Q)+\frac{1}{2}\left(f(Q)-Qf'(Q)\right)=0, \tag{21}$$

$$2\frac{d}{dt}\left(f'(Q)\left(\dot{b}c+\dot{c}\right)\right)+bc\left(f(Q)-Qf'(Q)\right)=0, \tag{22}$$

$$2\frac{d}{dt}\left(f'(Q)\left(\dot{a}c+\dot{c}\right)\right)+ac\left(f(Q)-Qf'(Q)\right)=0, \tag{23}$$

$$2\frac{d}{dt}\left(f'(Q)\left(\dot{a}\dot{b}+\dot{a}\right)\right)+ab\left(f(Q)-Qf'(Q)\right)=0. \tag{24}$$

Consider now the self-similar Kasner-like spacetime with scale-factors

$$a(t)=t^{p_1}\ ,\ b(t)=t^{p_2}\ \text{and}\ c(t)=t^{p_3}, \tag{25}$$

for the lapse function $N(t)=1$.

For these scale-factors the Bianchi I spacetime (20) admits the proper homothetic vector field [60]

$$X=t\partial_t+(p_1-1)x\partial_x+(p_2-1)y\partial_y+(p_3-1)z\partial_z.$$

In the case of General Relativity, the exponents $p_1,\ p_2$ and p_3 satisfy the Kasner relations. They are [61]

$$p_1+p_2+p_3=1$$

and

$$(p_1)^2+(p_2)^2+(p_3)^2=1$$

For a discussion and the importance of the Kasner spacetime we refer the reader to [62].

Hence, by replacing the scale factors (25) in the field Equations (21)–(24) and the power-law theory $f(Q)=f_0Q^{\mu}$, it follows that the exponents $p_1,\ p_2$ and p_3 satisfy the generalized Kasner relations

$$p_1+p_2+p_3=2\mu-1,$$

and

$$(p_1)^2+(p_2)^2+(p_3)^2=(2\mu-1)^2.$$

However, in that case $Q=0$. Indeed, for any non-singular function $f(Q)$, where $f(Q\to 0)=0$, this Kasner-like solution solves the field equations. For the power-law $f(Q)=f_0Q^{\mu}$ theory, the Kasner-like solution is the analytic solution. Hence there is no reason to study the stability of the self-similar solution. We remark that the generalized Kasner relations can be written in the form of the Kasner relations

$$P_1 + P_2 + P_3 = 1$$

$$(P_1)^2 + (P_2)^2 + (P_3)^3 = 1,$$

where $P_1 = \frac{p_1}{2\mu-1}$, $P_2 = \frac{p_2}{2\mu-1}$ and $P_3 = \frac{p_3}{2\mu-1}$.

It is important to mention that in terms of dynamics the field Equations (21)–(24) are of the same form as those of $f(T)$ teleparallel theory of gravity which means that the algebraic results for the stability analysis presented in [63] are valid.

In a future work it will be of special interest to investigate the effects of different connections in Bianchi geometries.

5. Discussion

In this study, we investigated the stability properties of some self-similar cosmological solutions in symmetric teleparallel gravity. In particular, for the case of isotropic FLRW geometry we derived the field equations and self-similar solutions for the four different connections of the coincidence gauge. Three of the different corrections describe a spatially flat FLRW geometry, while the fourth connection describes an FLRW geometry with non-zero spatial curvature. The families of the connections introduce a gauge function, which affects the dynamics of the gravitational field equations for a non-linear $f(Q)$-theory.

We considered self-similar solutions that describe ideal gas solutions in the case of spatially flat FLRW geometry, and Milne-like geometries for non-zero spatially FLRW spacetime. The solutions exist for specific functions $f(Q)$ and gauge functions for the corresponding connection. The self-similar solutions do not describe the full degrees of freedom of the field equations which means that they are special solutions and not the general solutions. In particular, they are asymptotic exact solutions of the generic solutions in each case.

Because of the complexity of the field equations, we were not able to study the phase-space by using dimensionless variables in the H-normalization approach. Instead, we took the standard approach to consider small perturbations of the scale factor, i.e., of the Hubble function, in the region of the similarity solutions. For each connection, we derived the differential equation for the perturbations and we solved each one of them. It was found that the behaviour of the perturbations depends upon the free parameters of the self-similar solution. Hence, the free parameter for the $f(Q)$-theory was a constraint for each connection with the requirement that the self-similar solution be an attractor for the field equations. The case for which the asymptotic self-similar solutions describe accelerated universes was studied.

Finally, we consider the anisotropic and homogeneous Bianchi I geometry and for the connection in the coincidence gauge we proved that Kasner-like solutions exist for a power-law theory $f(Q) = Q^\mu$, and for any non-singular function $f(Q)$ with $f(Q \to 0) = 0$. Finally, the field equations for the background space are dynamically equivalent with the field equation with that of $f(T)$-theory in teleparallelism. Thus, the stability properties of the Kasner-like solutions are similar to those of the study [63].

In a future study, we plan to investigate further the self-similar solutions for anisotropic and homogeneous cosmological spacetimes.

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