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Analysis of the Competition System Using Parameterized Fractional Differential Equations: Application to Real Data

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Abstract: Natural symmetries exist in several processes of chemistry, physics, and biology. Symmetries possess interesting dynamical characteristics that cannot be seen in non-symmetric systems. The present paper investigates the competition between two banking systems, rural and commercial, in Indonesia, in parameterized fractional order Caputo derivative. A novel numerical method is used to discretize the competition system using the real data of rural and commercial banks in Indonesia for the period 2004–2014. The new scheme is more suitable and reliable for data fitting results and has good accuracy. The integer model is formulated in Caputo derivative and their stability results are presented. With the available parameters, the data for the model is analyzed using various scenarios. We shall compare the result with the previous method used in the literature and show that the present method is better than the previous method in the literature. It is shown that fractional order α and the parameter ρ involved in the numerical scheme provide excellent fitting.

Keywords: parameterized Caputo derivative; real data; 2004–2014; new numerical scheme; results and discussion

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1. Introduction

Many biological, chemical, and physical systems exhibit natural symmetry. Therefore, including these symmetries in the differential equations used to explain the systems is a sensible modeling assumption. Understanding the structure of differential equation solutions that adhere to the restrictions set by the symmetry group is essential to studying differential equations with symmetry, also known as equivalent differential equations. Symmetries can be temporal or geographical.

Mathematical models are useful when studying the practical problems arising in science and engineering and for determining the long-time behavior. The applications of the mathematical models to physics, mathematics, and other areas of science and engineering have already been documented in literature while in social sciences it is also been applied, and very interesting results have been obtained. Besides this, in financial areas, the application of mathematical models is getting more attention day by day. The banks, whether they are commercial or rural, are crucial in the development of the country's economic growth. Strong banking practices and their maximum advantages to the populace are essential to the development of a nation [1]. It should be highlighted that the banking sector can be either conventional or Islamic, or it can be both. The banks that adhere to Syrian law or do things traditionally are defined under Act No. 10 of 1998. The operations of commercial and rural banks in Indonesia have been examined, and it has been proven that the latter engage in more business activities than the former [2,3]. Additionally, there are more commercial banks in Indonesia than rural ones, however, both banks offer the same products [4]. Commercial banks are more numerous than rural banks, but rural banks are still enhancing their business operations to draw in more clients. Rural banks may face competition from commercial banks as a result of enhancements to their business products.

The rural banks in Indonesia may face intense competition if they keep working to improve their products. The Lotka–Volterra system of evolutionary differential equations can be used to study this competition efficiently [5]. To analyze the competition's numerous genuine word problems, the researchers used the Lotka–Volterra equations; for more information; see Refs. [6–13]. For instance, the competition system is used to examine the mobile company data of Korean companies [6]. Additional applications include modeling technology replacement (see Ref. [7]), and modeling of policy and its effects on the Korean stock market, and competition between banking sectors (see Refs. [8–11]). Additionally, some recent research on this topic can be found in Refs. [12–15].

It is important to note that the Lotka–Volterra system was employed in the aforementioned study to determine the dynamics of various problems with integer order, except for Refs. [12–15]. The application of the competition system to marine phage population dynamics has been considered in Ref. [16]. The Lotka–Volterra system applications to a three-level laser and their dynamics have been considered in Ref. [17]. The competition and sales of two competing retail formats using the Lotka–Volterra system have been analyzed in Ref. [18]. The seasonal succession using the Lokta–Volteerra model has been discussed by the authors in Ref. [19]. In Ref. [20], the authors considered the competition mixture and their complex fitness using the competition system.

In recent years, it has become clear that fractional calculus is crucial to the dynamic modeling of these kinds of real-world issues. Memory and hereditary properties are thought to be one of the causes. Real-world problems frequently have nonlinear models with crossover behavior, which makes it challenging to address them precisely. Such applications of the fractional order models with real-life problems may be shown in Refs. [21–27]. Recently, the authors in Ref. [28] considered a tumor disease study using fractional differential equations. A prey-predator system using fractional order derivative has been studied in Ref. [29]. The fractional order model for HIV of cell-to-cell dynamics has been analyzed in Ref. [30]. The discrete food chain model using fractional discrete derivative has been discussed in Ref. [31]. The important application of the fractional models is to fit well with the real data because there is more choice for the fractional order parameters, such as variable, or of constant order. Such remarkable benefits make it more powerful than the integer order systems; see Refs. [14,15,21]. For the fractional calculus used recently in literature when dealing the financial systems, see Refs. [32–34].

This research aims to investigate the banking data competition between two Indonesian banks using actual statistical data and a parameterized fractional order model. The Caputo derivative is the fractional derivative taken into account in this work. We employ the recently established numerical technique, utilizing the parameterized method for the numerical solution of the fractional model; see Ref. [35]. The method has been tested for various linear and nonlinear problems and found suitable fractional differential equations. Compared to existing techniques for fractional models that are accessible in the literature, this new method is more accurate and reasonable. We shall present the comparison of the present method with the FDE12 method. The FDE method was used for the fractional differential equation in the Caputo case. We shall see that the present method is better than FDE12 when considering the numerical value of ρ. The suggested method will demonstrate how the data fits nicely when the parameterized order and particular fractional orders are taken into account. The remaining findings in this research are as follows: Section 2 displays the model along with its descriptions. Section 3 displays the integral and related definitions. Section 4 illustrates the numerical results of the system with the new scheme that is designed for the fractional differential equation in the power law kernel by providing the algorithm. Section 5 presents a discussion of the results. Section 6 presents the achievements of the results of this work.

2. Basics of Fractional Operators

The development in fractional calculus is still underway with new discoveries in the form of analytical and numerical shapes. This section highlights the definitions that will be considered in this work, which are obtained from Ref. [35]. Let g be differentiable within $[0, T]$ then

$$ {}_0^{F_c}D_t^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g'(\beta) K(t-\beta) d\beta \tag{1} $$

where $K(t) = t^{-\alpha}$,

$$ \theta(\alpha) = \frac{1}{\Gamma(\alpha)} \text{ or } \frac{1}{1-\alpha}, 0 < \alpha \le 1. \tag{2} $$

The Caputo derivative is given by [36],

$$ {}_0^{C}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\beta)^{n-\alpha-1} g^n(\beta) d\beta. \tag{3} $$

The integral to (3) is given by

$$ {}_0^{C}I_t^{\alpha}(g(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t g(\beta)(t-\beta)^{\alpha-1} d\beta. \tag{4} $$

3. Mathematical Model

Mathematical models are useful to study the competition among species. Competition is not only among species, but also among other social sciences activities such as bank competitions or competition among different companies of their product. We shall consider the Indonesian banks that are rural and commercials using the well-known Lotkaâ€“Volterra system. We shall denote the profit of the commercial banks by $C(t)$ at any time t while the rural banks are given by $R(t)$. It is assumed that the banks have maximum profit and their limited funds behave as logistical growths. Thus, we can write the ordinary differential equation as

$$ \begin{aligned} \frac{dC}{dt} &= g_1 C\left(1 - \frac{C}{L_1}\right) - q_1 CR, \\ \frac{dR}{dt} &= g_2 R\left(1 - \frac{R}{L_2}\right) - q_2 CR, \end{aligned} \tag{5} $$

where the the initial conditions are $C(0) = C_0 \ge 0$, and $R(0) = R_0 \ge 0$. The parameter g_1 defines the growth rate of the commercial bank while g_2 stands for the growth rate of the rural banks. The parameter L_1 is the maximum profit gained by the commercial banks while L_2 is for the rural banks. q_1 and q_2 are the competition factors of commercial and rural banks. The parameters used in the system (5) are clearly positive.

3.1. Fractional Order Model

The model given by (5) is formulated in fractional differential equation using the Caputo definitions, and it is given by:

$$ \begin{aligned} {}_0^{C}D_t^{\alpha}\Big(C(t)\Big) &= g_1 C\left(1 - \frac{C}{L_1}\right) - q_1 CR, \\ {}_0^{C}D_t^{\alpha}\Big(R(t)\Big) &= g_2 R\left(1 - \frac{R}{L_2}\right) - q_2 CR, \end{aligned} \tag{6} $$

where α is defined to be the fractional order in the sense of Caputo derivative while the initial conditions are $C(0) = C_0 \ge 0$ and $R(0) = R_0 \ge 0$.

3.2. Existence and Uniqueness

First, we shall write the fractional order system (6) in the form below:

$$\begin{aligned} {}_0^C D_t^{\alpha}\Big(C(t)\Big) &= h_1(C,R,t), \\ {}_0^C D_t^{\alpha}\Big(R(t)\Big) &= h_2(C,R,t). \end{aligned} \tag{7}$$

Now, we define the norm,

$$||C||_{\infty} = \sup_{t\in D} |C((t)|, \tag{8}$$

where $D \in [0,T]$. We shall assume to prove that $C(t)$ and $R(t)$ are bounded within $[0,T]$, so for any $t \in [0,T]$ there exists U_1 and U_2 such that $||C||_{\infty} < U_1$ and $||R||_{\infty} < U_2$. We need to first show h_1 and h_2 are bounded.

$$\begin{aligned} |h_1(C,R,t)| &= |g_1 C\Big(1-\frac{C}{L_1}\Big) - q_1 CR|, \\ &\leq g_1|C\Big(1-\frac{C}{L_1}\Big)| + |q_1 CR|, \\ &\leq g_1|C||\Big(1-\frac{C}{L_1}\Big)| + |q_1 CR|, \\ &\leq g_1 \sup_{t\in[0,T]} |C(t)| \sup_{t\in[0,T]} |\Big(1-\frac{C(t)}{L_1}\Big)| + q_1 \sup_{t\in[0,T]} |C(t)| \sup_{t\in[0,T]} |R(t)|, \\ &\leq g_1||C||_{\infty}|\Big(1+\frac{||C||_{\infty}}{L_1}\Big)| + q_1||C||_{\infty}||R||_{\infty}, \\ &< g_1 U_1(1+\frac{U_1}{L_1}) + q_1 U_1 U_2, \\ &< \infty. \end{aligned} \tag{9}$$

Repeating the process again for the second equation of the mode (6), we shall have,

$$|h_2(C,R,t)| = g_2 U_2(1+\frac{U_2}{L_2}) + q_2 U_1 U_2 < \infty. \tag{10}$$

So, $C(t)$ and $R(t)$ are bounded, and then there exists F_1 and F_2 such that

$$\sup_{t\in[0,T]} |h_1(C,R,t)| < F_1, \quad \sup_{t\in[0,T]} |h_2(C,R,t)| < F_2. \tag{11}$$

On the other hand, if for all $t \in [0,T]$, $|h_1(C,R,t)| < F_1$ and $|h_2(C,R,t)| < F_2$, then $||C||_{\infty} < \infty$ and $||R||_{\infty} < \infty$.

Proof. Consider that h_1 and h_2 are bounded, then

$$\begin{aligned} C(t) &= C(0) + \frac{1}{\Gamma(\alpha)}\int_0^t h_1(C,R,\psi)(t-\psi)^{\alpha-1} d\psi, \\ R(t) &= R(0) + \frac{1}{\Gamma(\alpha)}\int_0^t h_2(C,R,\psi)(t-\psi)^{\alpha-1} d\psi. \end{aligned} \tag{12}$$

$$\begin{aligned} |C(t)| &\leq |C(0)| + \frac{1}{\Gamma(\alpha)}\Big|\int_0^t h_1(C,R,\psi)(t-\psi)^{\alpha-1} d\psi\Big|, \\ |R(t)| &\leq |R(0)| + \frac{1}{\Gamma(\alpha)}\Big|\int_0^t h_2(C,R,\psi)(t-\psi)^{\alpha-1} d\psi\Big|. \end{aligned} \tag{13}$$

without loss of generality, we can show for $C(t)$,

$$\begin{aligned} |C(t)| &\leq |C(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t |h_1(C,R,\psi)|(t-\psi)^{\alpha-1}d\psi, \\ &< |C(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \sup_{t\in[0,T]} |h_1(C,R,\psi)|(t-\psi)^{\alpha-1}d\psi, \\ &< |C(0)| + \frac{F_1 t^\alpha}{\Gamma(\alpha+1)} < |C(0)| + \frac{F_1 T^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{14}$$

Thus,

$$|C(t)| \quad < \quad ||C(t)||_\infty < |C(0)| + \frac{F_1 T^\alpha}{\Gamma(\alpha+1)}. \tag{15}$$

We can also show,

$$|R(t)| \quad < \quad ||R(t)||_\infty < |R(0)| + \frac{F_2 T^\alpha}{\Gamma(\alpha+1)}. \tag{16}$$

□

To show the unique solution of the system proposed (6), the following conditions should be met,

1. for every $t \in [0,T]$, h_1 and h_2 satisfy $|h_1(C,R,t)|^2 < \mu_1(1+|C|^2)$, $|h_2(C,R,t)|^2 < \mu_2(1+|R|^2)$,
2. for every $t \in [0,T]$, h_1 and h_2 satisfy the Lipschitz condition $|h_1(C_1,R,t) - h_1(C_2,R,t)|^2 < \overline{\mu}_1|C_1 - C_2|^2$, $|h_2(C,R_1,t) - h_2(C,R_2,t)|^2 < \overline{\mu}_2|R_1 - R_2|^2$.

We shall start to prove for h_1,

$$\begin{aligned} |h_1(C_1,R,t) - h_1(C_2,R,t)|^2 &= \left|g_1C_1\left(1-\frac{C_1}{L_1}\right) - q_1C_1R - g_2C_2\left(1-\frac{C_2}{L_1}\right) + q_1C_2R\right|^2, \\ &= \left|g_1(C_1-C_2) - C_1\left(\frac{C_1^2}{L_1} - \frac{C_2^2}{L_1}\right) - q_1(C_1-C_2)R\right| \\ &< 3g_1^2|C_1-C_2|^2 + 3g_1^2\left|\frac{C_1^2}{L_1} - \frac{C_2^2}{L_1}\right|^2 + 3q_1^2|C_1-C_2|^2|R| \\ &< 3\left\{g_1^2 + 2g_1^2\left(\left|\frac{C_1}{L_1}\right|^2 + \left|\frac{C_2}{L_1}\right|^2\right) + q_1^2|R|^2\right\}|C_1-C_2|^2, \\ &< 3\left\{g_1^2 + 2g_1^2\left(\sup_{t\in[0,T]}\left|\frac{C_1}{L_1}\right|^2 + \sup_{t\in[0,T]}\left|\frac{C_2}{L_1}\right|^2\right) + q_1^2\sup_{t\in[0,T]}|R|^2\right\} \\ &\times|C_1-C_2|^2, \\ &< 3\left(g_1^2 + 2g_1^2\left(\frac{2U_1^2}{L_1^2}\right) + q_1^2U_2^2\right)|C_1-C_2|^2, \\ &< \overline{\mu_1}|C_1-C_2|^2, \end{aligned} \tag{17}$$

where $\overline{\mu_1} = 3\left(g_1^2 + 2g_1^2\left(\frac{2U_1^2}{L_1^2}\right) + q_1^2U_2^2\right)$. On the other hand,

$$|h_1(C,R,t)|^2 \quad = \quad |g_1C(1-\frac{C}{L_1}) - q_1CR|^2,$$

$$
\begin{aligned}
&\leq \quad 3g_1^2|C|^2 + 3g_1^2\left|\frac{C}{L_1}\right|^2 + 3q_1^2|C|^2|R|^2, \\
&\leq \quad \left(3g_1^2 + 3\frac{g_1^2}{L_1^2}|C|^2 + 3q_1^2|R|^2\right)|C|^2 \\
&\leq \quad \left(3g_1^2 + 3\frac{g_1^2}{L_1^2}\sup_{t\in[0,T]}\left|C(t)^2\right| + 3q_1^2\sup_{t\in[0,T]}\left|R(t)^2\right|\right)|C|^2, \\
&< \quad \left(3g_1^2 + \frac{3g_1^2}{L_1^2}U_1^2 + 3q_1^2U_2^2\right)|C|^2, \\
&< \quad \mu_1\left(1+|C|^2\right),
\end{aligned}
$$

where $\mu_1 = \left(3g_1^2 + \frac{3g_1^2}{L_1^2}U_1^2 + 3q_1^2U_2^2\right)$. So the fractional system (6) possesses a unique solution.

3.3. Equilibrium Points and Its Stability

We shall obtain the equilibrium points of the model (6), by equating the equations in the steady-state below,

$$
{}_0^C D_t^\alpha\Big(C(t)\Big) = 0, \quad {}_0^C D_t^\alpha\Big(R(t)\Big) = 0. \tag{18}
$$

It follows from (18),

$$
\begin{aligned}
&g_1C\left(1-\frac{C}{L_1}\right) - q_1CR = 0, \\
&g_2R\left(1-\frac{R}{L_2}\right) - q_2CR = 0.
\end{aligned} \tag{19}
$$

The solution of Equation (19) gives the possible equilibrium points, given by:

$$
\begin{aligned}
P_0 &= (0,0), \ P_1 = (0,L_2), \ P_2 = (L_1,0), \\
P_3 &= \left(\frac{g_2L_1(L_2q_1-g_1)}{L_1L_2q_1q_2-g_1g_2}, \frac{g_1L_2(L_1q_2-g_2)}{L_1L_2q_1q_2-g_1g_2}\right).
\end{aligned}
$$

The above four equilibrium points shall be discussed in detail to obtain their stability results. We need to compute the Jacobian matrix associated with the system (6) given by:

$$
J = \begin{pmatrix} \left(1-\frac{2C^*}{L_1}\right)g_1 - q_1R^* & -q_1C^* \\ -q_2R^* & \left(1-\frac{2R^*}{L_2}\right)g_2 - q_2C^* \end{pmatrix}.
$$

We will study the stability analysis using the above equilibrium points. Let us start with $P_0 = (0,0)$. At P_0, the Jacobian matrix J gives the eigenvalues g_1, g_2, which is positive and hence at P_0 the system is unstable.

Consider the equilibrium point P_1, then the J gives the eigenvalues, $-g_2, g_1 - L_2q_1$. The first one is negative while the second can be negative if $g_1 < L_2q_1$. By fulfilling the condition in second eigenvalue, then the system becomes stable at P_1.

Consider the equilibrium point P_2, then the Jacobian matrix J provides the eigenvalues, $-g_1, g_2 - L_1q_2$. The first one is negative while the second one can be negative if $g_2 < L_1q_2$, then the system shall be stable at P_2.

The equilibrium point P_3 gives the following characteristics equation:

$$\Pi^2 + \eta_1 \Pi + \eta_2 = 0,$$

where

$$\begin{aligned} \eta_1 &= \frac{g_1 g_2 (g_1 - L_2 q_1 + g_2 - L_1 q_2)}{g_1 g_2 - L_1 L_2 q_1 q_2}, \\ \eta_2 &= \frac{g_1 g_2 (g_1 - L_2 q_1)(g_2 - L_1 q_2)}{g_1 g_2 - L_1 L_2 q_1 q_2}. \end{aligned}$$

The coefficients η_1 and η_2 can be positive if $(g_1 - L_2 q_1) > 0$, $(g_2 - L_1 q_2) >$ and $g_1 g_2 - L_1 L_2 q_1 q_2 > 0$. With these conditions, the system is stable locally asymptotically at P_3.

4. Scheme with Parameterized Caputo Fractional Derivative

In the last decades, several numerical methods for solving ordinary differential equations have been suggested, with some disadvantages and advantages. Different approaches were used to construct these powerful numerical schemes, for example, the nonlinear functions are approximated using polynomials, or, on the other hand, the function is approximated directly, as in the case of Euler. The parameterized method, although not very popular, can be considered a more general approximation that captures several other middle points. This method consists in introducing a parameter from 0 to 1. With some values of these parameters we can obtain other numerical schemes, as was clearly indicated in Ref. [35]. While this technique seems accurate, it was pointed out by Atangana and Seda that such techniques are more accurate only when the parameters are closer to 1. To solve this problem, Atangana and Seda in Ref. [35] modified this method and obtained a more accurate numerical scheme. In this paper, we shall use such a numerical scheme to solve our problems. The present section discusses in detail the parameterized method for the Caputo fractional differential equations. We adopt the method from Ref. [35] and shall present in detail the numerical scheme in the following. Let us consider the general Cauchy problem in Caputo derivative,

$$\begin{cases} {}_0^C D_t^\alpha z(t) = \theta(t, z(t)) \\ z(0) = z_0 \end{cases}. \tag{20}$$

Using the integration, the above equations lead to the following,

$$\begin{cases} z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \theta(\beta, z(\beta))(t-\beta)^{\alpha-1} d\beta, \\ z(0) = z_0 \end{cases}. \tag{21}$$

For $t = t_{m+1}$, we have

$$\begin{aligned} z(t_{m+1}) &= z(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{m+1}} \theta(\beta, z(\beta))(t_{m+1} - \beta)^{\alpha-1} d\beta, \\ &= z(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \theta(\beta, z(\beta))(t_{m+1} - \beta)^{\alpha-1} d\beta. \end{aligned} \tag{22}$$

Within $[t_j, t_{j+1}]$, the function $\theta(\beta, z(\beta))$ is approximated as

$$\theta(\beta, z(\beta)) \approx (1 - \frac{1}{2\rho})\theta(t_j, z_j) + \frac{1}{2\rho}\theta(t_{j+1}, z_{j+1}). \tag{23}$$

Replacing yields,

$$z(t_{m+1}) = z(0) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{m} \left\{ \left(1 - \frac{1}{2\rho}\right)\theta(t_j, z_j) + \frac{1}{2\rho}\theta(t_{j+1}, z_{j+1}) \right\}$$

$$\times\{(m-j+1)^{\alpha}-(m-j)^{\alpha}\}. \tag{24}$$

Further, we have

$$\begin{aligned} z(t_{m+1}) &= z(0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{\left(1-\frac{1}{2\rho}\right)\theta(t_j,z_j)+\frac{1}{2\rho}\theta(t_{j+1},z_{j+1})\right\}\delta^{\alpha}_{m,j} \\ &\quad+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{(1-\frac{1}{2\rho})\theta(t_m,z_m)+\frac{1}{2}\theta(t_{m+1},\widehat{z}_{m+1})\right\}, \end{aligned} \tag{25}$$

$\delta^{\alpha}_{m,m}=1$, and

$$\widehat{z}_{m+1}=z_0+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m}\theta(t_j,z_j)\delta^{n}_{m,j}, \tag{26}$$

where

$$\delta^{\alpha}_{m,j}=(m-j+1)^{\alpha}-(m-j)^{\alpha}.$$

Note that we used $\widehat{z}_{m+1}$ when $j=m$ to avoid an implicit case, so $\widehat{z}_{m+1}$ is obtained using Euler approximations.

For our model, the scheme can be written as

$$\begin{aligned} C(t_{m+1}) &= C(0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{\left(1-\frac{1}{2\rho}\right)\theta(t_m,C_m,R_m)+\frac{1}{2\rho}\theta(t_{m+1},\tilde{C}_{m+1},\tilde{R}_{m+1})\right\} \\ &\quad+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{\left(1-\tfrac{1}{2\rho}\right)\theta(t_j,C_j,R_j)+\tfrac{1}{2\rho}\theta(t_{j+1},C_{j+1},R_{j+1})\right\} \\ &\quad\times\{(m-j+1)^{\alpha}-(m-j)^{\alpha}\}, \\ R(t_{m+1}) &= R(0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{\left(1-\frac{1}{2\rho}\right)\theta(t_m,C_m,R_m)+\frac{1}{2\rho}\theta(t_{m+1},\tilde{C}_{m+1},\tilde{R}_{m+1})\right\} \\ &\quad+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m-1}\left\{\left(1-\tfrac{1}{2\rho}\right)\theta(t_j,C_j,R_j)+\tfrac{1}{2\rho}\theta(t_{j+1},C_{j+1},R_{j+1})\right\} \\ &\quad\times\{(m-j+1)^{\alpha}-(m-j)^{\alpha}\}, \end{aligned} \tag{27}$$

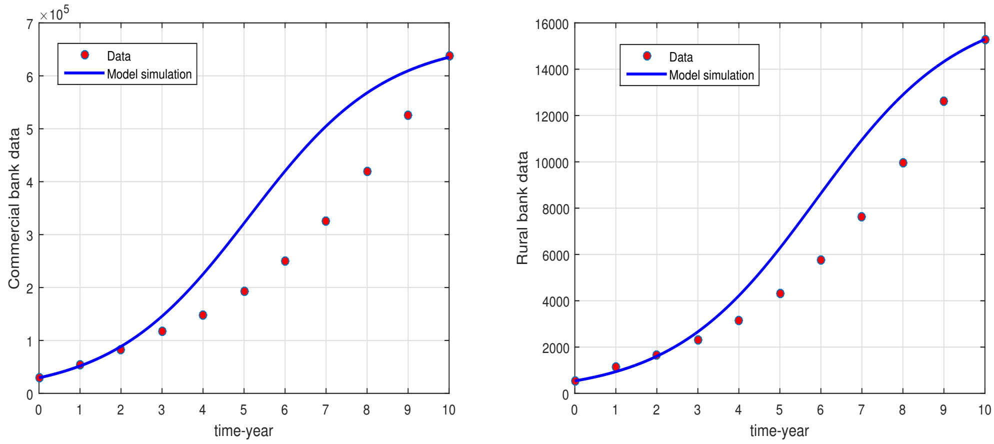
where

$$\begin{aligned} \tilde{C}_{m+1} &= C(0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m}\theta(t_j,C_j,R_j)\{(m-j+1)^{\alpha}-(m-j)^{\alpha}\}, \\ \tilde{R}_{m+1} &= R(0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{m}\theta(t_j,C_j,R_j)\{(m-j+1)^{\alpha}-(m-j)^{\alpha}\}. \end{aligned} \tag{28}$$

5. Numerical Results

We consider here the numerical solution of the competition model using the new and recent method introduced in the literature. One of the important advantages of this new method is the use of the ρ, which provides important results when varying its numerical values. In the present simulation, we consider the numerical values of the parameters $q_1=2.90\times10^{-10}$, $L_1=669{,}318.198$, $g_1=0.6$, $g_2=0.58$, $q_2=3.9\times10^{-8}$ and $L_2=17{,}540.6219$, see Ref. [37], while the initial conditions of the model variables are $C(0)=29{,}463$, $R(0)=539$. The data of the two banks in Indonesia is provided by Ref. [4], and the rural and commercial data used were for the period 2004–2014. We briefly study the numerical results with the above parameters and present its comparison with various values of α and ρ and determine the best values of the fractional and parameter ρ that have a good fitting with the data.

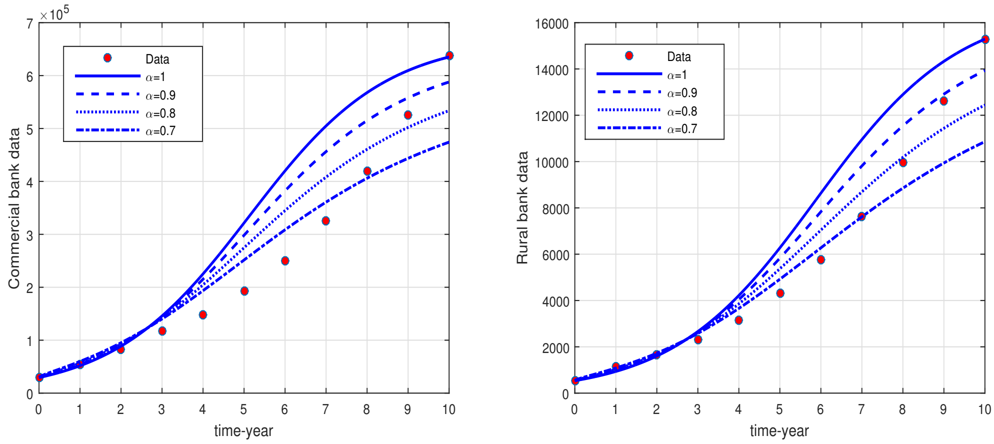
We give Figure 1 with sub-graphs (a) and (b) that show the simulation results of commercial and rural banks when $\rho = 1$ and fractional order $\alpha = 1$. It can be seen that the data have a normal fitting with the model. This data fitting was considered useful when there is no fractional order model or any other useful operator.



(**a**) (**b**)

Figure 1. Real data of commercial and rural banks versus model for $\rho = 1$ (**a**) commercial, (**b**) rural.

In Figure 2, we can see that the fractional order α provides little change compared to the integer case. Regarding the variation in fractional order α and keeping $\rho = 1$, the results are a little close in the case of Figure 2a, while they are a bit better for the case in Figure 2b.



(**a**) (**b**)

Figure 2. Fitting of data versus the model for various values α and $\rho = 1$, (**a**) commercial, (**b**) rural.

In Figure 3, we consider the fractional order $\alpha = 1$ and use $\rho = 0.01$. We see that the results obtained in Figure 3a,b are better than the results given in Figures 1 and 2. For example, we can compare the result given in Figures 1a and 2a for the commercial data with the result in Figure 3a, which is obviously better from Figures 1a and 2a. Similarly, the results are given in subfigures b of Figures 1 and 2, and its comparison with Figure 3a, where one can see that the result given in Figure 3a is fine and better.

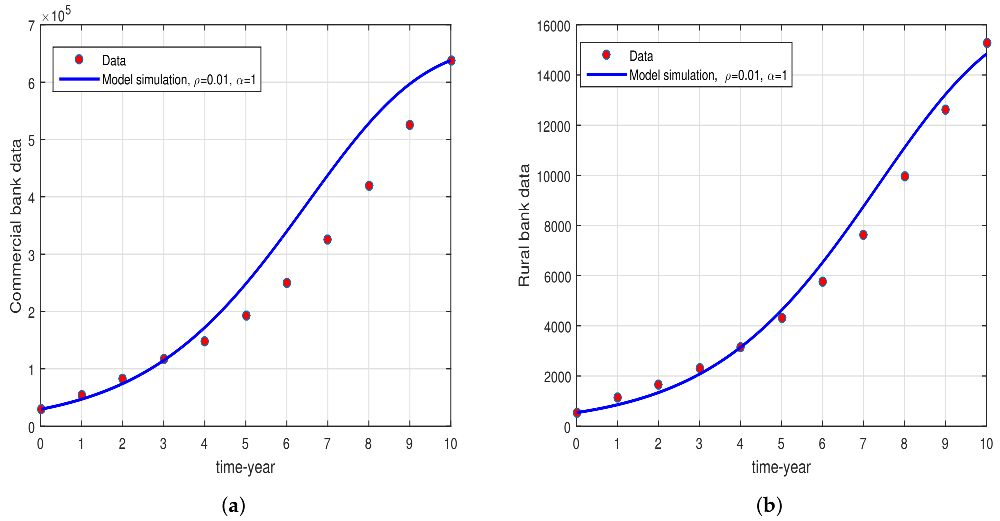


Figure 3. Data versus model comparison when $\alpha = 1$, and $\rho = 0.01$ (**a**) commercial banks, (**b**) rural banks.

If we choose different values of fractional order $\alpha = 0.97, 0.94$ and keep the parameter $\rho = 0.01$ fixed , we get the result given in Figure 4. In subfigures a,b in Figure 4, we use the fractional order $\alpha = 0.97$ and $\rho = 0.01$, which indicates that the results are refining more. Similarly, for the case in subfigures c,d of Figure 4, the results of the model are much closer to the real data. It means that the variation in fractional order and the parameter ρ provides better data fitting results.

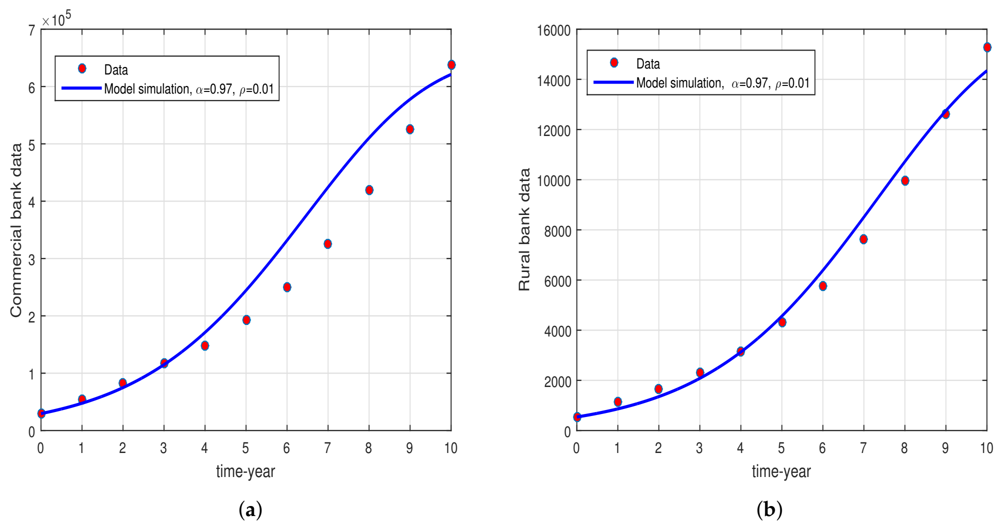
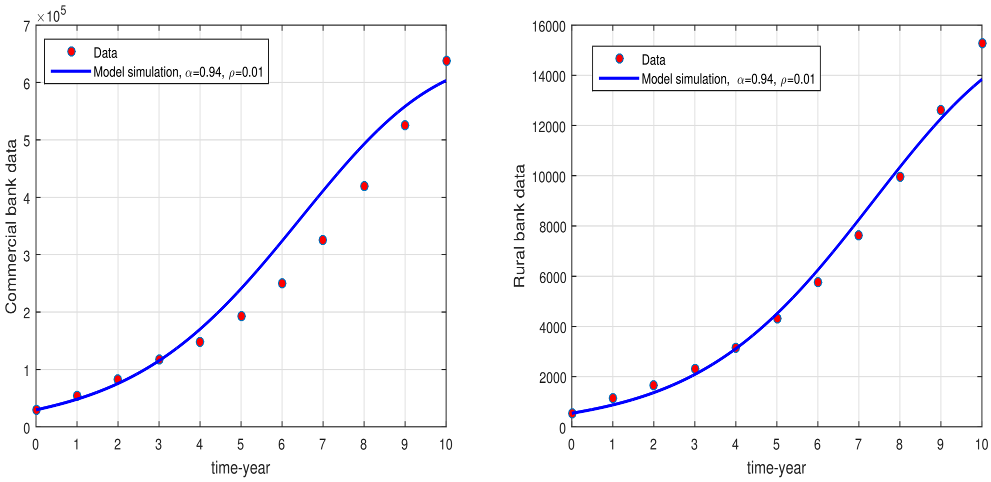


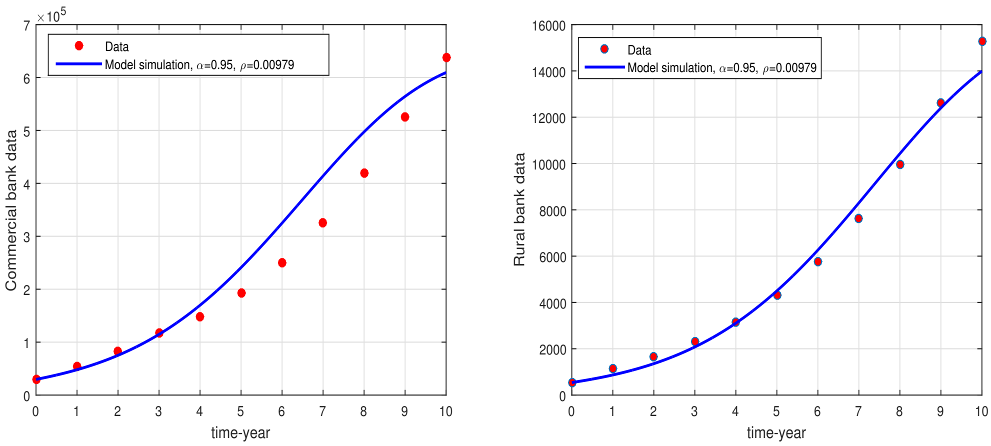
Figure 4. *Cont.*



(**c**) (**d**)

Figure 4. Comparison results of data versus model comparison when $\alpha = 0.97, 0.94$, and $\rho = 0.01$ sub-graph (**a**,**c**) describes the commercial banks, while (**b**,**d**) is for rural banks.

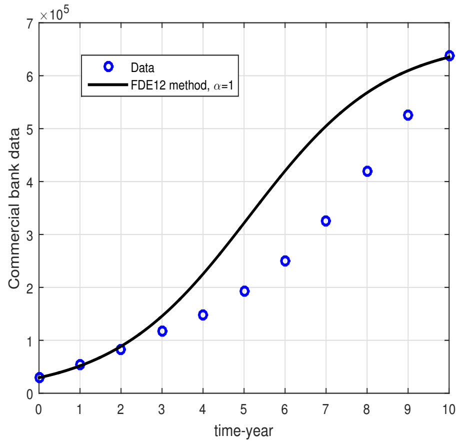
Figure 5 is given for fractional order $\alpha = 0.95$ and $\rho = 0.00979$. It can be observed from the graphical results that commercial and rural data have good fitting to the data and it is considered the best for this case.



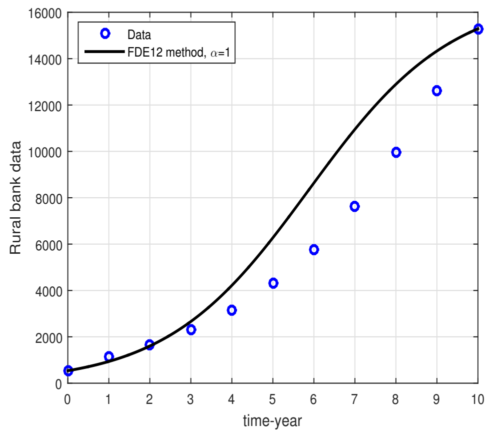
(**a**) (**b**)

Figure 5. Comparison results of data versus model when $\alpha = 0.95$ and $\rho = 0.00979$. (**a**) gives the comparison of model versus commercial bank data when $\alpha = 0.95$ and $\rho = 0.00979$ while (**b**) gives the comparison of model versus rural bank data when $\alpha = 0.95$ and $\rho = 0.00979$.

We shall compare the present method with the FDE12 method used in literature for the Caputo differential equation; see Ref. [38]. We numerically solve the fractional order system (6) and show the graphical results in Figures 6–8. In Figure 6, we compare the data of rural and commercial banks with the FDE12 method when $\alpha = 1$, which matched perfectly with the case in Figure 1.



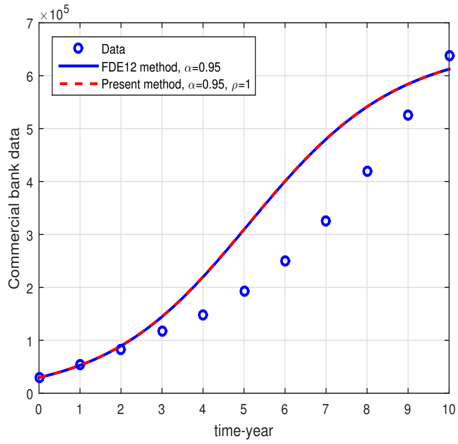
(**a**)



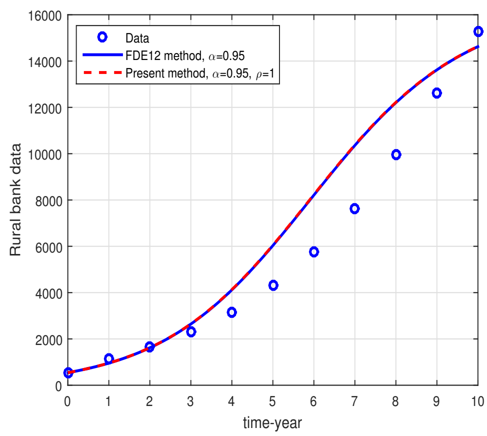
(**b**)

Figure 6. FDE12 method comparison with real data of commercial and rural. Figures (**a**,**b**) respectively give the comparison of the FDE12 method with data when $\alpha = 1$.

The comparison of the present method with the FDE12 method for data comparison when $\alpha = 0.95$ and $\rho = 1$ is given in Figure 7. One can see that both the methods for $\alpha = 0,97$ provide the same results. It means that the present method is perfectly matched with the FDE-12 for $\alpha = 0.95$ and for any fractional order α one can have the same result for both methods.



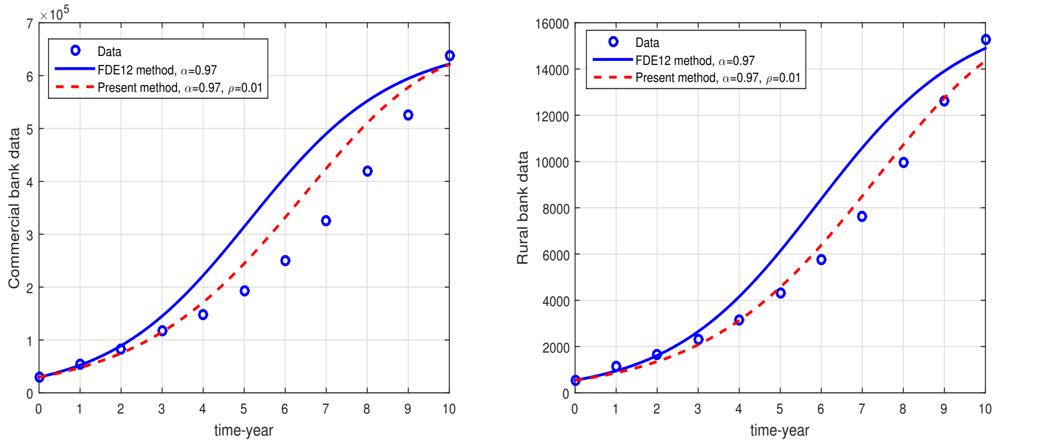
(**a**)



(**b**)

Figure 7. Comparison of FDE12 and the present method for data comparison when $\alpha = 0.95$ and $\rho = 1$, where (**a**) denotes commercial data while (**b**) rural data.

The comparison of the present method with FDE12 when $\alpha = 0.97$ and $\rho = 0.01$ is given in Figure 8. Here, one can see the importance of the ρ that provides the best result compared to the FDE12 method. So, we can say that the ρ used in the present method is more powerful than FDE12.



(**a**) (**b**)

Figure 8. Comparison of FDE12 and present method for data comparison when $\alpha = 0.97$ and $\rho = 0.01$, where (**a**) defines commercial banks while (**b**) defines rural banks.

6. Conclusions

In the present work, we considered the data of the two banks, rural and commercial, in Indonesia, and describe the profit competition among the two with the same products of the bank. We considered the previously published values of the parameters and the initial values of the variables. We considered a new method that uses an extra parameter for the factional order differential equation in Caputo case. We focused on the importance of the parameter ρ in the present work and discussed their simulation results in comparison with the real data. We presented the detailed numerical results and their discussion of the results. We utilized various aspects of the method and provided the results. We have shown that the results for $\rho = 0.01$ and $\alpha = 1$ are more powerful than the integer order. Similarly, the results with the variations in ρ and α are better than integer orders. Here, $\rho = 0.00979$ and $\alpha = 0.95$ are more powerful than the previous and fit the data well. This application case of banking data and its comparison using the new method of the fractional differential is reliable and useful. We compared the present method with the available method in literature, called the FDE12 method. We compared both methods for some fractional order α and found that both methods matched well. We found significantly different results for the same fractional order α and $\rho = 0.01$ (present method). The result of the present method was found to be better than the previous method FDE12 due to the involvement of the parameter ρ. One can utilize the method used in this work for other nonlinear problems arising in science and engineering with real data. The present method shall be considered for the case of nonlinear fractional differential equations in the case of Caputo–Fabrizio and other fractional operators for real-life problems in the future. We believe that this method is more powerful than the previous methods used in the literature. In the future, we aim to provide the result for the given model in the sense of the Atangana–Baleanu derivative and their comparison with other schemes available in the literature.

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