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Dynamical Structures of Multi-Solitons and Interaction of Solitons to the Higher-Order KdV-5 Equation

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Abstract: In this study, we build multi-wave solutions of the KdV-5 model through Hirota's bilinear method. Taking complex conjugate values of the free parameters, various colliding exact solutions in the form of rogue wave, symmetric bell soliton and rogue waves form; breather waves, the interaction of a bell and rogue wave, and two colliding rogue wave solutions are constructed. To explore the characteristics of the breather waves, localized in any direction, the higher-order KdV-5 model, which describes the promulgation of weakly nonlinear elongated waves in a narrow channel, and ion-acoustic, and acoustic emission in harmonic crystals symmetrically is analyzed. With the appropriate parameters that affect and manage phase shifts, transmission routes, as well as energies of waves, a mixed solution relating to hyperbolic and sinusoidal expression are derived and illustrated by figures. All the single and multi-soliton appeared symmetric about an axis of the wave propagation. The analyzed outcomes are functional in achieving an understanding of the nonlinear situations in the mentioned fields.

Keywords: fifth-order KdV equations; Hirota's bilinear technique; lump; multiple solitons; breather solution

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1. Introduction

Nonlinearity in models has been an issue of focus in the study of the corporal characteristics in fields such as fluid motion, plasma, condensed themes in physics, aerographic, optics, atmospheric discipline, etc. Nonlinear disciplines have three types, namely soliton, fractal, and chaos. Currently, the major significant and efficient research arena in nonlinear discipline is the theory of solitons [1–8]. Lump, rogue, and breather waves are the main issues in such soliton theory. The term "lump wave" describes localized waves with one peak and one trough. It is localized in all directions of space and was first introduced by Manakov and others in 1977 [9]. Waves that appear suddenly out of nowhere in the sea and vanish without leaving any trace are called rogue waves. In 1965, Draper first introduced the concept [10]. Breathers are localized breathing waves with a periodic structure in a certain direction. It can also be used to explain rogue wave phenomena. We are going to discuss some aspects of the KdV-5 higher-order KdV-type equation in soliton physics, especially in the case of the interaction of soliton in nonlinear science.

To investigate the properties of solitons, a lot of powerful techniques (the extended hyperbolic function method [11], φ^6 model expansion scheme [12], the unified technique [13,14], the tanh technique [15], the exp $(-\Phi(\eta))$-expansion technique [16], the first integral technique [17], the inverse variational scheme [18], Darboux transformation [19], Hirota bilinear technique [20], etc.) have been applied by renowned researchers. The Hirota bilinear technique is more popular for its simplicity and directness. This method was first derived by Hirota [20]. Its simplified formula was extensively applied in the literature in different nonlinear studies [21–23]. This technique become operative and useable within a short time by dynamical researchers and was applied to obtain soliton, multi-soliton, lump wave, rogue wave, breather wave, and localized forms of wave solutions [24–30]. Besides this, dynamical researchers have derived the non-existence of global solutions of the time-fractional differential model [31], double Wronskian solutions [32], bright-dark and rogue soliton solutions [33], and Pfaffian solutions [34] with variable coefficients in the recent literature.

The KdV model was first derived to develop shallow water waves in narrow channels. The extended higher-order KdV model carries four non-linear terms and two linear dispersive terms compared with the standard KdV-5 models, and analyzes various types of dynamical solutions with properties arising in shallow-water or surface waves [35–40]. Besides this, a few KdV classes are investigated in [41,42].

In this paper, we aim to explain the physical shape of the following extended higher-order KdV-5 models [40] to analyze the rogue, lump, and breather pattern in symmetry, and the interaction of solitons in which few solutions come in symmetrical form. The mentioned model is

$$\begin{aligned} &\Psi_t + c\Psi_x + 3\Psi\Psi_x + 5\alpha^2(\Psi\Psi_{xxx} + 2\Psi_x\Psi_{xx}) + \tfrac{15\alpha^2}{2\beta}\Psi^2\Psi_x \\ &+\beta(\alpha^2\Psi_{xxxxx} + \Psi_{xxx}) = 0, \quad \beta \neq 0. \end{aligned} \tag{1}$$

We think this study will establish and clarify the physical structural solutions of the extended higher-order KdV-type KdV-5 equation, which will be helpful for further nonlinear study. To the best of the authors' knowledge, multi-solitons of the higher-order KdV-5 equation by Hirota's bilinear method have not been studied before.

The structure of this paper is as follows: first, we constructed the multi soliton solutions of the higher-order KdV-5 model. Then, we used the complex conjugate values of free parameters into the established the multi-soliton solutions of this model and derived the rogue and breather interactions of them. Lastly, conclusions and some comments are provided.

2. The Extended Higher-Order KdV-5 Equations

It is important to find the collision shapes for the nonlinear waves. We are eager to find breather-type collision solutions to the extended KdV-5 equation in this segment. Here, we begin our analysis by studying the extended KdV-5 model in Equation (1).

In the beginning, suppose an auxiliary solution is

$$\Psi(x,t) = e^{\nu_i} = e^{\kappa_i x - \omega_i t}, \text{ where } \nu_i = \kappa_i x - \omega_i t. \tag{2}$$

From The linear term of Equation (1), to solve the dispersion relation

$$\omega_i = c\kappa_i + \beta\left(\kappa_i^3 + \alpha^2\kappa_i^5\right),\ i = 1,2,3,4. \tag{3}$$

Additionally, the corresponding phase variables are

$$\nu_i = \kappa_i x - \left\{c\kappa_i + \beta(\kappa_i^3 + \alpha^2\kappa_i^5)\right\}t. \tag{4}$$

The phase shift relation of Equation (1) is

$$A_{ij} = \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2}, \text{ when } i,j = 1,2,3,\ldots N.\ (i < j). \tag{5}$$

The multi-soliton solutions can take the transformation

$$\Psi(x,t) = h(\ln f(x,t))_{xx.} \tag{6}$$

For a different one, or more than one soliton, the solution is given by

$$\begin{aligned} f(x,t) &= 1 + \sum_{i=1}^{N} \exp(v_i) + \sum_{i<j}^{N} A_{ij} \exp(v_i + v_j) \\ &+ \sum_{i<j<k}^{N} A_{ij} A_{jk} A_{ik} \exp(v_i + v_j + v_k) + \ldots\ldots\ldots + \prod_{i<j}^{N} A_{ij} (\sum_{i}^{N} \exp(v_i) \end{aligned} \tag{7}$$

For single soliton, i.e., $N = 1$, then substituting Equation (6) into Equation (1) and solving for h, we find

$$h = 4\beta. \tag{8}$$

Case1: To find the lump wave, we consider 2-soliton solutions for $N = 2$. Then, the test function from Equation (7) is given by,

$$f(x,t) = 1 + e^{\nu_1} + e^{\nu_2} + A_{12} e^{\nu_1 + \nu_2} \tag{9}$$

with

$$\nu_1 = \kappa_1 x - \omega_1 t \text{ and } \nu_2 = \kappa_2 x - \omega_2 t,$$

$$\kappa_1 = p_1 + \mathrm{i} q_1 \text{ and } \kappa_2 = p_1 - i q_1 \text{ then}$$

$$\omega_1 = m + in \text{ and } \omega_2 = m - in.$$

From

$$\begin{aligned} \omega_1 = \mathrm{m} + \mathrm{in} \text{ and } \omega_2 = \mathrm{m} - \mathrm{in}, \text{ we have } \mathrm{m} &= \frac{\omega_1 + \omega_2}{2} = \frac{c\kappa_1 + \beta\left(\kappa_1^3 + \alpha^2 \kappa_1^5\right) + c\kappa_2 + \beta\left(\kappa_2^3 + \alpha^2 \kappa_2^5\right)}{2} \\ &= \frac{c(p_1 + iq_1) + \beta\left((p_1 + iq_1)^3 + \alpha^2 (p_1 + iq_1)^5\right) + c(p_1 - iq_1) + \beta\left((p_1 - iq_1)^3 + \alpha^2 (p_1 - iq_1)^5\right)}{2} \\ &= \alpha^2 \beta p_1{}^5 - 10\alpha^2 \beta p_1{}^3 q_1{}^2 + 5\alpha^2 \beta p_1 q_1{}^4 + \beta p_1{}^2 - \beta q_1{}^2 + cp_1{}^2 - cq_1{}^2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{n} &= \frac{\omega_1 - \omega_2}{2i} = \frac{c\kappa_1 + \beta\left(\kappa_1^3 + \alpha^2 \kappa_1^5\right) - c\kappa_2 - \beta\left(\kappa_2^3 + \alpha^2 \kappa_2^5\right)}{2i} \\ &= \frac{c(p_1 + iq_1) + \beta\left((p_1 + iq_1)^3 + \alpha^2 (p_1 + iq_1)^5\right) - c(p_1 - iq_1) - \beta\left((p_1 - iq_1)^3 + \alpha^2 (p_1 - iq_1)^5\right)}{2i} \\ &= q_1 (5\alpha^2 \beta p_1{}^4 - 10\alpha^2 \beta p_1{}^2 q_1{}^2 + \alpha^2 \beta q_1{}^4 + 2\beta p_1 + 2cp_1). \end{aligned}$$

with phase shift from Equation (5) given by

$$A_{12} = -\frac{q_1^2}{p_1^2}$$

Simplifying Equation (9) and applying the above relation gives

$$f(x,t) = 1 + 2e^{\sigma_1} \cos(\xi_1) + A_{12} e^{2\sigma_1}, \tag{10}$$

where $\sigma_1 = p_1 x - mt$ and $\xi_1 = q_1 x - nt$.

Substituting Equation (10) into Equation (6),

$$\Psi(x,t) = 4\beta \Big\{ \ln(1 + 2e^{\sigma_1} \cos(\xi_1) + A_{12} e^{2\sigma_1}) \Big\}_{xx} \tag{11}$$

The solution to Equation (11) comes via selecting the complex structure of the existing parametric values from 2 solitons and obtaining rogue shape breather waves.

The nature of the solution to Equation (11) is plotted in a 3D shape in Figure 1a and in a contour plot in Figure 1b with the parameters, $p_1 = -1.3$, $q_1 = 2$, $c = 0.02$, $\alpha = 0.2$, $\beta = 0.5$. Figure 1c represents 2D plots for time variations $t = -2$, $t = 0$ and $t = 2$. We see that the solution displays multirogue-shape breather waves along the paradox. Its speed, width, and channel remain the same for the whole dynamic procedure and periodic rogues occur at equal distances from one another. From its contour plots, the real shape and direction are observed. Again, if we assume purely imaginary parameters, then the solution to Equation (11) displays a breather line wave. Figure 1d–f portray 3D, contour, and 2D plots correspondingly for the parametric values $p_1 = 0,\ q_1 = 2,\ c = 0.02,\ \alpha = 0.2,\ \beta = 0.5$ and $t = -2,\ t = 0, t = 2$. All the figures are drawn using Maple software.

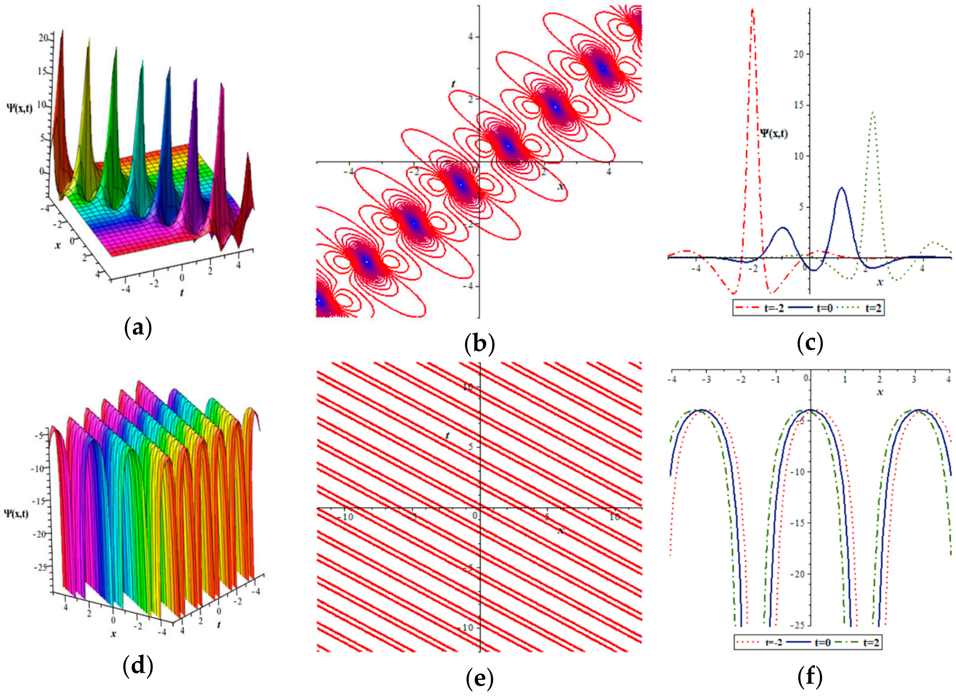

Figure 1. (**a**,**b**) are outlooks of Equation (11) in 3D and contour plots, respectively, with the parametric values $p_1 = -1.3,\ q_1 = 2,\ c = 0.02,\ \alpha = 0.2,\ \beta = 0.5$. (**c**) signifies the corresponding 2D plot for Equation (11) for $t = -2,\ t = 0,\ t = 2$; (**d**,**e**) represent the outlook of the breather waves for the parameters $p_1 = 0,\ q_1 = 2,\ c = 0.02,\ \alpha = 0.2,\ \beta = 0.5$. (**f**) is a corresponding 2D plot of breather waves for $t = -2,\ t = 0,\ t = 2$.

Case2: Consider $N = 3$ in Equation (6) to find three soliton solutions. Then, the function $f(x,t)$ can be written as

$$f(x,t) = 1 + e^{\nu_1} + e^{\nu_2} + e^{\nu_3} + A_{12}e^{\nu_1+\nu_2} + A_{23}e^{\nu_2+\nu_3} + A_{13}e^{\nu_1+\nu_3} + A_{123}e^{\nu_1+\nu_2+\nu_3} \quad (12)$$

where

$$\nu_1 = \kappa_1 x - \omega_1 t, \quad \nu_2 = \kappa_2 x - \omega_2 t \text{ and } \nu_3 = \kappa_3 x - \omega_3 t.$$

Let

$$\kappa_1 = p_1 x + iq_1, \quad \kappa_2 = p_1 x - iq_1, \text{ and } \kappa_3 = \tau,$$

$$\omega_1 = m + in, \omega_2 = m - in \text{ and } \omega_3 = c + \alpha c^3 + \alpha^3 \beta c^5,$$

where

$$\mathrm{m} = \alpha^2\beta {p_1}^5 - 10\alpha^2\beta {p_1}^3{q_1}^2 + 5\alpha^2\beta p_1{q_1}^4 + \beta{p_1}^2 - \beta{q_1}^2 + c{p_1}^2 - c{q_1}^2,$$

$$\mathrm{n} = q_1(5\alpha^2\beta{p_1}^4 - 10\alpha^2\beta{p_1}^2{q_1}^2 + \alpha^2\beta{q_1}^4 + 2\beta p_1 + 2cp_1),$$

and phase shift terms from Equation(5) are

$$\mathrm{A}_{12} = -\frac{q_1^2}{p_1^2},\ A_{23} = p_1 + iq_1 = \rho_1 \exp(i\theta_1) \text{ and } A_{13} = p_1 - iq_1 = \rho_1 \exp(-i\theta_1),$$

where

$$\rho_1 = \sqrt{{p_1}^2 + {q_1}^2} \text{ and } \theta_1 = \tan^{-1}(\frac{q_1}{p_1}).$$

Using the above relations from Equation (12) we have

$$\begin{aligned} f(x,t) &= 1 + 2e^{\sigma_1}\cos(\xi_1) + e^{\Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t} + A_{12}e^{2\sigma_1} \\ &+2\rho_1 e^{(\sigma_1 + \Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t}\cos(\xi_1 + \theta_1) + \rho_1{}^2 A_{12}e^{\{2\sigma_1 + \Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t\}}, \end{aligned} \tag{13}$$

where $\sigma_1 = p_1 x - mt$ and $\xi_1 = q_1 x - nt$.

Substituting Equation (13) into Equation (6),

$$\begin{aligned} \Psi(x,t) &= 4\beta\Big\{\ln(1 + 2e^{\sigma_1}\cos(\xi_1) + e^{\Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t} + A_{12}e^{2\sigma_1} \\ &+2\rho_1 e^{(\sigma_1 + \Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t}\cos(\xi_1 + \theta_1) + \rho_1{}^2 A_{12}e^{\{2\sigma_1 + \Lambda x - (c\Lambda + \beta\Lambda^3 + \alpha^2\beta\Lambda^5)t\}})\Big\}_{xx}. \end{aligned} \tag{14}$$

In Equation (14), the solution comes from the combination of exp and sinusoidal functions exhibiting the interaction of a periodic rogue and bell shape line solitons, as viewed in Figure 2 for $p_1 = -1.3,\ q_1 = 0.2,\ c = 0.02,\ \alpha = 0.05,\ \beta = 2,\ \Lambda = -0.5$. It is interesting that before the ($t < 0$) interaction there are 2 waves (periodic rogue and bell shape line solitons) interacting at $t = 0$ and then rogue type soliton split into two parts of rogue type waves (See Figure 2). Therefore, a completely non-elastic collision solution is obtained. The actual structure of the collision is shown from its 3D plot in Figure 2a and contour plots in Figure 2b. A corresponding 2D plot is shown in Figure 2c for $t = -2,\ t = 0,\ t = 2$. Again, for the purely imaginary values of the parameters, the solution to Equation (14) displays the collision of breather line waves with kinky waves, providing kinky type breather waves. It is portrayed in Figure 2d,e for the parametric values $p_1 = 0,\ q_1 = 0.35,\ c = 0.5,\ \alpha = 0.2,\ \beta = 0.5, \Lambda = 0.5$. The corresponding 2D plot is displayed in Figure 2f for $t = -3,\ t = 0,\ t = 3$. All the figures are drawn using Maple software.

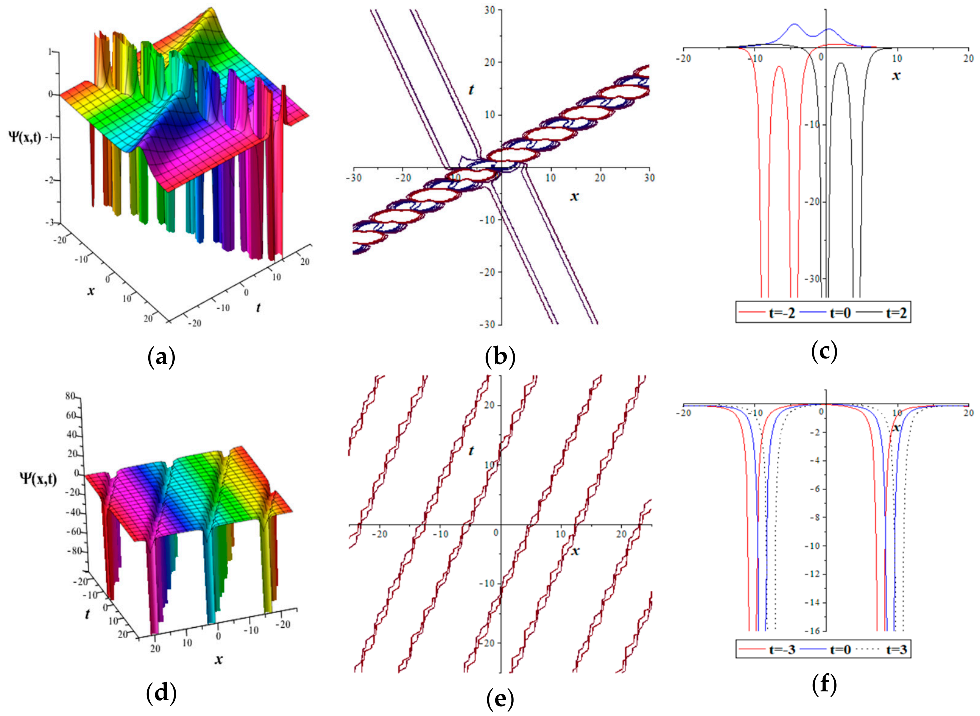

Figure 2. (**a**,**b**) are the outlooks of Equation (14) in 3D and contour plots, respectively, with the parametric values $p_1 = -1.2,\ q_1 = 0.15,\ c = -1,\ \alpha = 0.005,\ \beta = 2,\ \Lambda = -0.5$. (**c**) signifies the corresponding 2D plot for Equation (14) for $t = -2,\ t = 0,\ t = 2$; (**d**,**e**) represent the outlook of the breather waves for the parameters $p_1 = 0,\ q_1 = 0.35,\ c = 0.5,\ \alpha = 0.2,\ \beta = 0.5,\ \Lambda = 0.5$. (**f**) is the corresponding 2D shape of the breather wave for $t = -3,\ t = 0,\ t = 3$.

Case3: For $N = 4$, we can find the four soliton solutions. The test function from Equation (6) is

$$\begin{aligned} f(x,t) = 1 + e^{\nu_1} + e^{\nu_2} + e^{\nu_3} + e^{\nu_4} + A_{12}e^{\nu_1+\nu_2} + A_{13}e^{\nu_1+\nu_3} + \\ A_{14}e^{\nu_1+\nu_4} + A_{23}e^{\nu_2+\nu_3} + A_{24}e^{\nu_2+\nu_4} + A_{34}e^{\nu_{31}+\nu_4} + A_{123}e^{\nu_1+\nu_2+\nu_3} + \\ A_{234}e^{\nu_2+\nu_3+\nu_4} + A_{134}e^{\nu_1+\nu_3+\nu_4} + A_{1234}e^{\nu_1+\nu_2+\nu_3+\nu_4} \end{aligned} \tag{15}$$

where

$$\nu_1 = \kappa_1 x - \omega_1 t, \nu_2 = \kappa_2 x - \omega_2 t, \nu_3 = \kappa_3 x - \omega_3 t \text{ and } \nu_4 = \kappa_4 x - \omega_4 t.$$

Let

$$\kappa_1 = p_1 + iq_1, \kappa_2 = p_1 - iq_1, \kappa_3 = p_2 + iq_2 \text{ and } \kappa_4 = p_2 - iq_2,$$

$$\omega_1 = m_1 + in_1, \omega_2 = m_1 - in_1, \omega_3 = m_2 + in_2 \text{ and } \omega_4 = m_2 - in_2,$$

where

$$\mathrm{m}_1 = \alpha^2\beta {p_1}^5 - 10\alpha^2\beta {p_1}^3{q_1}^2 + 5\alpha^2\beta p_1{q_1}^4 + \beta{p_1}^2 - \beta{q_1}^2 + c{p_1}^2 - c{q_1}^2 \mathrm{n}_1 = q_1(5\alpha^2\beta{p_1}^4 - 10\alpha^2\beta{p_1}^2{q_1}^2 + \alpha^2\beta{q_1}^4 + 2\beta p_1 + 2cp_1),$$

$$\mathrm{m}_2 = \alpha^2\beta {p_2}^5 - 10\alpha^2\beta {p_2}^3{q_2}^2 + 5\alpha^2\beta p_2{q_2}^4 + \beta{p_2}^2 - \beta{q_2}^2 + c{p_2}^2 - c{q_2}^2,\ \mathrm{n}_2 = q_2(5\alpha^2\beta{p_2}^4 - 10\alpha^2\beta{p_2}^2{q_2}^2 + \alpha^2\beta{q_2}^4 + 2\beta p_2 + 2cp_2).$$

The term phase shift terms of Equation(5) are

$$\mathrm{A}_{12} = -\frac{q_1^2}{p_1^2},\ A_{23} = p_1 + iq_1 = \rho_1 \exp(i\theta_1) \text{ and } A_{13} = p_1 - iq_1 = \rho_1 \exp(-i\theta_1).$$

$$\mathrm{A}_{14} = -\frac{q_2^2}{p_2^2},\ A_{14} = p_2 + iq_2 = \rho_2 \exp(i\theta_2) \text{ and } A_{23} = p_2 - iq_2 = \rho_2 \exp(-i\theta_2).$$

To find the values of ρ_1, ρ_2, θ_1 and θ_2, we will apply $\rho_i = \sqrt{{p_i}^2 + {q_i}^2}$ and $\theta_i = \tan^{-1}(\frac{q_i}{p_i})$, where $i = 1, \quad 2$.

Simplifying Equation (15) and applying the above relations, we have

$$\begin{aligned} f(x,t) = 1 + 2e^{\sigma_1}\cos(\xi_1) + A_{12}e^{2\sigma_1} + 2e^{\sigma_2}\cos(\xi_2) + A_{34}e^{2\sigma_2} + \\ 2\rho_1 e^{(\sigma_1+\sigma_2)}\cos(\xi_1+\xi_2+\theta_1) + 2\rho_2 e^{(\sigma_1+\sigma_2)}\cos(\xi_1-\xi_2+\theta_2) + \\ 2a_{12}\rho_1\rho_2 e^{(2\sigma_1+\sigma_2)}\cos(\xi_2+\theta_1+\theta_2) + 2A_{34}\rho_1\rho_2 e^{(\sigma_1+2\sigma_2)}\cos(\xi_1+\theta_1-\theta_2) + \\ a_{12}a_{34}{\rho_1}^2\rho_2^2 e^{(2\sigma_1+2\sigma_2)}, \end{aligned} \tag{16}$$

Here $\sigma_1 = a_1 x - m_1 t, \xi_1 = b_1 x - n_1 t, \sigma_2 = a_2 x - m_2 t$ and $\xi_2 = b_2 x - n_2 t$.

Substituting Equation (16) into Equation (6),

$$\begin{aligned} \Psi(x,t) = 4\beta\ln(1 + 2e^{\sigma_1}\cos(\xi_1) + A_{12}e^{2\sigma_1} + 2e^{\sigma_2}\cos(\xi_2) + A_{34}e^{2\sigma_2} + \\ 2\rho_1 e^{(\sigma_1+\sigma_2)}\cos(\xi_1+\xi_2+\theta_1) + 2\rho_2 e^{(\sigma_1+\sigma_2)}\cos(\xi_1-\xi_2+\theta_2) + \\ 2a_{12}\rho_1\rho_2 e^{(2\sigma_1+\sigma_2)}\cos(\xi_2+\theta_1+\theta_2) + 2A_{34}\rho_1\rho_2 e^{(\sigma_1+2\sigma_2)}\cos(\xi_1+\theta_1-\theta_2) \\ + a_{12}a_{34}{\rho_1}^2\rho_2^2 e^{(2\sigma_1+2\sigma_2)})_{xx}. \end{aligned} \tag{17}$$

In Equation (17), the solution comes from the mixture of exp and periodic sinusoidal functions exhibiting the interaction of double periodic rogue solitons, as shown in Figure 3 with $p_1 = -1.3, p_2 = 1, q_1 = 0.8, \quad q_2 = 0.45, \quad c = 2, \quad \alpha = 0.25, \quad \beta = 1$. It is interesting that in earlier ($t < 0$) and later ($t > 0$) interactions, each rogue remains in the same solitonic nature and interacts at $t = 0$ coming along the reverse paradox (See Figure 3). We see that some rogue waves periodically get into each soliton, being equidistant from one another. The real structure, direction, and collision solutions are observed from its 3D plot in Figure 3a and contour plots in Figure 3b. The 2D plot is represented in Figure 3c with $t = -5, t = 0, t = 5$. Again, the solution to Equation (17) shows the interaction of double-breather shape line solitons with an acute angle for the purely imaginary parameters. It's 3D and contour shapes are portrayed in Figure 3d,e for the parametric values $p_1 = 0$,

$p_2 = 0,\ q_1 = 1,\ q_2 = 1,\ c = 1,\ \alpha = -1,\ \beta = 1$ and the 2D plot is represented in Figure 3f with $t = -1, t = 0, t = 1$. All the figures are drawn usingMaple software.

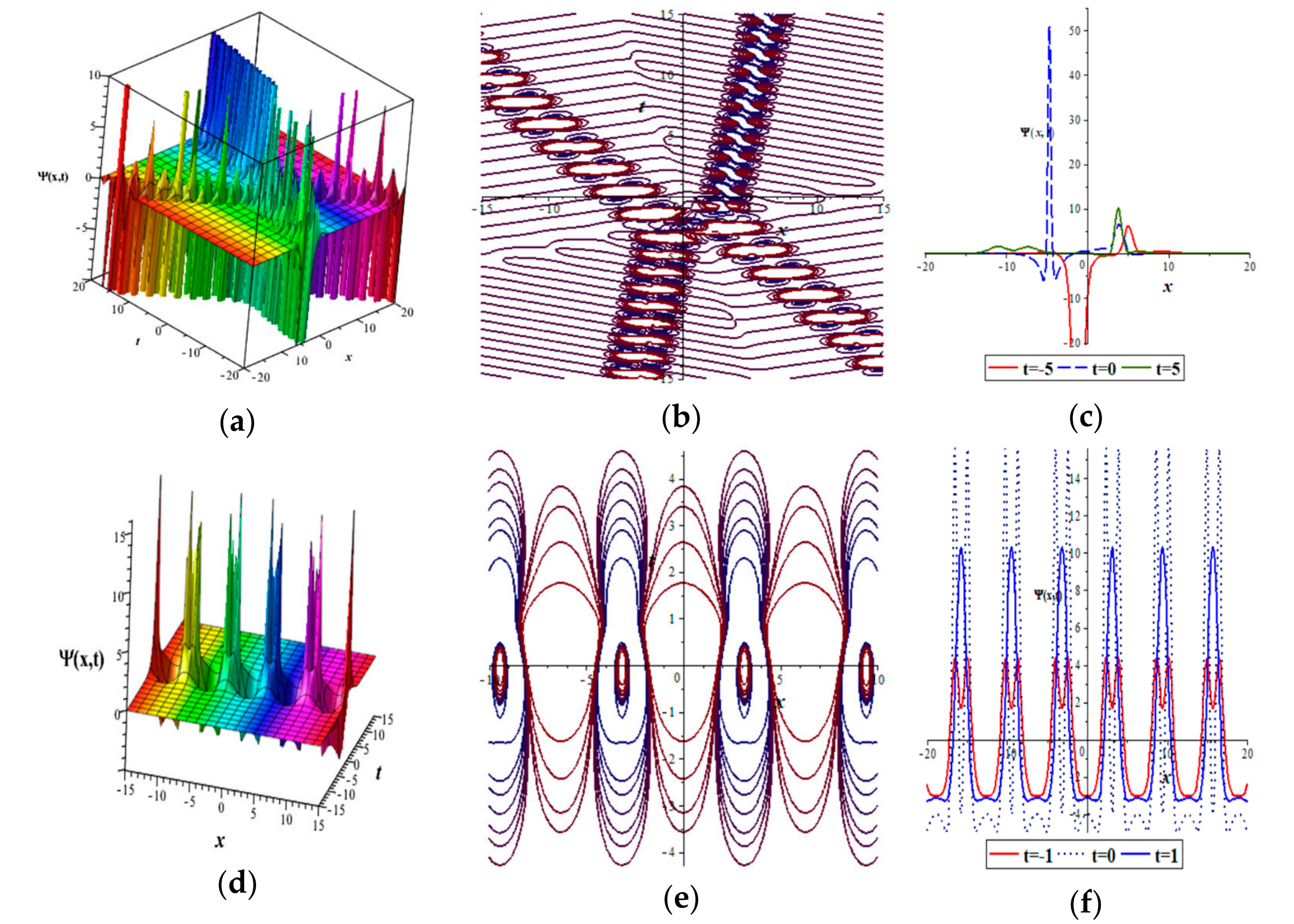

Figure 3. (**a**,**b**) are the outlooks of Equation (17) in 3D and contour plots, respectively, with the parametric values $p_1 = -1.3,\ p_2 = 1,\ q_1 = 0.8,\ q_2 = 0.45,\ c = 2,\ \alpha = 0.25,\ \beta = 1$. (**c**) signifies the corresponding 2D plot for Equation (17) for the values of $t = -5,\ t = 0,\ t = 5$; (**d**,**e**) represent the outlook of the breather waves for the parameters $p_1 = 0, p_2 = 0,\ q_1 = 1,\ q_2 = 1,\ c = 1,\ \alpha = -1,\ \beta = 1$. (**f**) is a corresponding 2D plot of breather wave for $t = -1,\ t = 0,\ t = 1$.

3. Conclusions

In this research, we have successfully employed the Hirota bilinear technique to analyze exact multi-wave solutions of the KdV-5model. Diverse types of parameters have been nominated to obtain and distinguish the dynamical properties of the multi-wave solutions for nonlinear systems. Consequently, we derived periodic lump waves with a rogue-type line wave, which is symmetric with the line, periodic cross-lump wave, lump-type periodic breather wave, and kinky-lump-type periodic breather waves setting complex values of free parameters. This technique has some advantages as it is easy, recognizable, and elementary, and it can also be used for other nonlinear models.

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Conflicts of Interest: The authors declare no conflict of interest.

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