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New Numerical Methods for Solving the Initial Value Problem Based on a Symmetrical Quadrature Integration Formula Using Hybrid Functions

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Abstract: In this study, we construct new numerical methods for solving the initial value problem (IVP) in ordinary differential equations based on a symmetrical quadrature integration formula using hybrid functions. The proposed methods are designed to provide an efficient and accurate solution to IVP and are more suitable for problems with non-smooth solutions. The key idea behind the proposed methods is to combine the advantages of traditional numerical methods, such as Runge–Kutta and Taylor's series methods, with the strengths of modern hybrid functions. Furthermore, we discuss the accuracy and stability analysis of these methods. The resulting methods can handle a wide range of problems, including those with singularities, discontinuities, and other non-smooth features. Finally, to demonstrate the validity of the proposed methods, we provide several numerical examples to illustrate the efficiency and accuracy of these methods.

Keywords: initial value problem; numerical method; hybrid function; local truncation errors; stability analysis

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1. Introduction

Differential equations are a fundamental tool in many fields of pure and applied science and are used to model a wide range of real-world phenomena [1,2]. While analytic methods exist for solving differential equations, many of the equations encountered in practice are too complex for a closed-form solution. Even when a solution formula is available, it may involve integrals that can only be approximated numerically. In such cases, numerical methods provide an alternative tool for solving differential equations under specified initial conditions. Initial value problems, which take the form of ordinary differential equations [3], are commonly encountered in science and engineering, and can be written in the form:

$$y' = f(x, y(x)), y(x_0) = y_0 \tag{1}$$

To solve the problem (1), various numerical methods with varying orders of convergence have been described and developed (see references [4–10]). The Runge–Kutta method is one of the most commonly used numerical methods for this purpose among the existing methods and has seen a growing interest in its development in recent years. The general m-stage Runge–Kutta method is given as follows:

$$y_{n+1} = y_n + h\varnothing(x_n, y_n; h) \tag{2}$$

where

$$\varnothing(x_n, y_n; h) = \sum_{i=1}^{m} w_i k_i,$$
$$k_1 = f(x_n, y_n),$$
$$k_i = f\left(x_n + c_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j\right),\ i = 2, 3, \ldots, m,$$

and

$$c_i = \sum_{j=1}^{i-1} a_{ij},\ i = 2, 3, \ldots, m.$$

In this paper, we propose two new numerical methods for solving initial value problems in ordinary differential equations. These methods are based on Newton's theorem in calculus, Taylor's series expansion, and the quadrature integration formula using hybrid functions [11]. We demonstrate that these methods have a second- and third-order convergence rate and are stable. We provide a comparison of these new methods with other relevant existing methods. Additionally, we present two specific initial value problems in ordinary differential equations to illustrate the efficiency of our proposed methods.

2. Derivation of New Methods

Consider the following formula of Newton's theorem of integration:

$$y(x) = y(x_n) + \int_{x_n}^{x} y'(t)dt \tag{3}$$

In Equation (3), we approximate the definite integral using the hybrid quadrature integration rule [11], as follows:

$$\int_{x_n}^{x} y'(t)dt \cong \frac{x - x_n}{m} \sum_{i=1}^{m} y'(x_n + (x - x_n)(\frac{2i-1}{2m})) \tag{4}$$

From Equations (1)–(4), we define the standard form of our proposed methods as:

$$y_{n+1} = y_n + h\varnothing(x_n, y_n; h) \tag{5}$$

where

$$\varnothing(x_n, y_n; h) = \sum_{i=1}^{m} w_i k_i \tag{6}$$

with

$$k_1 = f(x_n, y_n),\ k_i = f\left(x_n + c_i h, y_n + \sum_{\substack{j=1 \\ j \neq i}}^{i-1} a_{ij} k_j\right), i = 2, 3, \ldots m \tag{7}$$

and

$$\sum_{j=1}^{i-1} a_{ij} = c_i = \left(\frac{2i-1}{2m}\right), i = 2, 3, \ldots m \tag{8}$$

In the following, we present several numerical methods for solving Equation (1) using different values of m.

2.1. Method Based on Hybrid Quadrature Formula with Taylor's Expansion at $m = 1$

By using $m = 1$ in Equations (4)–(8), we define:

$$y_{n+1} = y_n + hw_1 k_2 \tag{9}$$

where

$$\left.\begin{array}{l} k_1 = f(x_n, y_n), \\ k_2 = f\left(x_n + \frac{1}{2}h, y_n + a_1 h k_1\right). \end{array}\right\} \tag{10}$$

In Equation (9), the unknowns, w_1 and $a_1 = a_{21}$, must be determined for the equation to agree with Taylor's series expansion to the highest possible order, see [12]. For this purpose, using Taylor's series expansion of $y(x_{n+1})$, we obtain:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + \frac{h^4}{24} y''''(x_n) \ldots \tag{11}$$

By expressing the derivatives of y in terms of f in Equation (1), we obtain:

$$y(x_{n+1}) = y_n + hf + h^2\left(\frac{1}{2}f_x + \frac{1}{2}ff_y\right) + h^3\left(\frac{1}{6}f_{xx} + \frac{1}{3}ff_{xy} + \frac{1}{6}f^{23y}f_{yy} + \frac{1}{6}f_xf_y + \frac{1}{64nd(1)orsereise}ff_y{}^2\right) + \ldots \quad (12)$$

From Equation (10), and using Taylor's series expansions of k_1 and k_2, we obtain:

$$k_1 = f \quad (13)$$

$$\begin{aligned} k_2 = f + h\left(\tfrac{1}{2}f_x + a_1k_1f_y\right) + \tfrac{h^2}{2}\left(\tfrac{1}{4}f_{xx} + a_1k_1f_{xy} + a_1{}^2k_1{}^2f_{yy}\right) + \\ \tfrac{h^3}{6}\left(\tfrac{1}{8}f_{xxx} + \tfrac{3}{4}a_1k_1f_{xxy} + \tfrac{3}{2}a_1{}^2k_1{}^2f_{xyy} + a_1{}^3k_1{}^3f_{yyy}\right) + \ldots \end{aligned} \quad (14)$$

Substituting Equations (13) and (14) into Equation (9), we obtain:

$$y_{n+1} = y_n + hfw_1 + \frac{h^2}{2}w_1\left(\frac{1}{2}f_x + 2a_1ff_y\right) + h^3w_1\left(\frac{1}{8}f_{xx} + \frac{1}{2}a_1ff_{xy} + \frac{1}{2}a_1{}^2f^2f_{yy}\right) + \ldots \quad (15)$$

By comparing Equation (15) with Equation (12), we have $w_1 = 1$ and $a_1 = \frac{1}{2}$. As a result, we obtain a special case, which is called the modified Euler's (midpoint integration) method [1,7,13–15], which is given by:

$$\left.\begin{aligned} y_{n+1} &= y_n + hk_2, \\ k_1 &= f(x_n, y_n), \\ k_2 &= f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right). \end{aligned}\right\} \quad (16)$$

2.2. Method Based on Hybrid Quadrature Formula with Taylor's Expansion at m = 2

By using $m = 2$ in Equations (4)–(8), we define:

$$y_{n+1} = y_n + h(w_1k_2 + w_2k_3) \quad (17)$$

where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\left(x_n + \tfrac{1}{4}h, y_n + a_1hk_1\right), \\ k_3 &= f\left(x_n + \tfrac{3h}{4}, y_n + h(a_2k_1 + a_3k_2)\right). \end{aligned}\right\} \quad (18)$$

Furthermore, Equation (17) must be such that $w_1, w_2, a_1 = a_{21}, a_2 = a_{31}$, and $a_3 = a_{32}$ are determined so that it is consistent with the highest possible order of Taylor's series expansion, see [12].

From Equation (18), using Taylor's series expansions of k_i, i = 1, 2, and 3, we have:

$$k_1 = f \quad (19)$$

$$\begin{aligned} k_2 = f + h\left(\tfrac{1}{4}f_x + a_1k_1f_y\right) + \tfrac{h^2}{2!}\left(\tfrac{1}{16}f_{xx} + \tfrac{1}{2}a_1k_1f_{xy} + a_1{}^2k_1{}^2f_{yy}\right) + \\ \tfrac{h^3}{3!}\left(\tfrac{1}{64}f_{xxx} + \tfrac{3}{16}a_1k_1f_{xxy} + \tfrac{3}{4}a_1{}^2k_1{}^2f_{xyy} + a_1{}^3k_1{}^3f_{yyy}\right) + \ldots \end{aligned} \quad (20)$$

and

$$\begin{aligned} k_3 = f + h\left(\tfrac{3}{4}f_x + (a_2k_1 + a_3k_2)f_y\right) + \tfrac{h^2}{2!}\left(\tfrac{91}{16}f_{xx} + \tfrac{6}{4}(a_2k_1 + a_3k_2)f_{xy} + (a_2k_1 + a_3k_2)^2f_{yy}\right) + \\ \tfrac{h^3}{3!}\left(\tfrac{27}{64}f_{xxx} + \tfrac{27}{16}(a_2k_1 + a_3k_2k_2)f_{xxy} + \tfrac{9}{4}(a_2k_1 + a_3k_2)^2f_{xyy} + (a_2k_1 + a_3k_2)^3f_{yyy}\right) + \ldots \end{aligned} \quad (21)$$

By plugging Equations (19)–(21) into Equation (17), we obtain:

$$\begin{aligned} y_{n+1} = y_n + hf(w_1 + w_2) + h^2(((a_2 + a_3)w_2 + a_1w_1)ff_y + \tfrac{1}{4}f_x(w_1 + 3w_2)) + \\ h^3(\tfrac{1}{32}(16f_{yy}(a_2 + a_3)^2f^2 + ((32a_1f_y{}^2 + 24f_{xy})a_3 + 24f_{xy}a_2)f + 8a_3f_xf_y + 9f_{xx})w_2 + \\ \tfrac{1}{2}(\tfrac{1}{16}f_{xx} + \tfrac{1}{2}f_{xy}a_1f + a_1{}^2f^2f_{yy})w_1) + \ldots \end{aligned} \quad (22)$$

By comparing the coefficients of h, h^2, and h^3 in Equation (22) with their counterparts in Equation (12), we obtain the following system of equations:

$$hf : w_1 + w_2 = 1 \quad (23)$$

$$h^2 f_x : 3w_2 + w_1 = 2 \tag{24}$$

$$h^2 f f_y : w_2 a_2 + w_2 a_3 + w_1 a_1 = \frac{1}{2} \tag{25}$$

$$h^3 f_{xx} : 9w_2 + w_1 = \frac{16}{3} \tag{26}$$

$$h^3 f f_{xy} : 3w_2 a_2 + 3w_2 a_3 + w_1 a_1 = \frac{4}{3} \tag{27}$$

$$h^3 f^2 f_{yy} : w_2 {a_2}^2 + 2w_2 a_2 a_3 + w_2 {a_3}^2 + w_1 {a_1}^2 = \frac{1}{3} \tag{28}$$

$$h^3 f_x f_y : w_2 a_3 = \frac{2}{3} \tag{29}$$

$$h^3 f {f_y}^2 : w_2 a_1 a_3 = \frac{1}{6} \tag{30}$$

By solving the above system of equations [16], we have $w_1 = \frac{1}{2}, w_2 = \frac{1}{2}, a_1 = \frac{1}{4}$, $a_2 = \frac{-7}{12}$, and $a_3 = \frac{4}{3}$. Consequently, the proposed method (17), referred to as HTM2, can be formulated as follows:

$$y_{n+1} = y_n + \frac{h}{2}(k_2 + k_3), \tag{31}$$

where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\Big(x_n + \tfrac{1}{4}h, y_n + \tfrac{1}{4}hk_1\Big), \\ k_3 &= f\Big(x_n + \tfrac{h}{2}, y_n + h\Big(\tfrac{-7}{12}k_1 + \tfrac{4}{3}k_2\Big)\Big). \end{aligned}\right\} \tag{32}$$

2.3. Method Based on Hybrid Quadrature Formula with Taylor's Expansion at m = 3

By using $m = 3$ in Equations (4)–(8), we define:

$$y_{n+1} = y_n + h(w_1 k_2 + w_2 k_3 + w_3 k_4) \tag{33}$$

where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\Big(x_n + \tfrac{1}{6}h, y_n + a_1 h k_1\Big), \\ k_3 &= f\Big(x_n + \tfrac{3h}{6}, y_n + h(a_2 k_1 + a_3 k_2)\Big), \\ k_4 &= f\Big(x_n + \tfrac{5h}{6}, y_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3)\Big). \end{aligned}\right\} \tag{34}$$

To obtain the new formula, we must find the constants $a_i, (i = 1, \ldots, 6)$ and $w_j, (j = 1, 2, 3)$, which are required to ensure that Equation (33) satisfies Taylor's expansion to the highest possible order. For that k_j, j = 1, 2, 3, and 4 in Equation (34) were expanded using Taylor's series, and the following was obtained:

$$k_1 = f \tag{35}$$

$$\begin{aligned} k_2 = f + h\Big(\tfrac{1}{6}f_x + a_1 k_1 f_y\Big) + \tfrac{h^2}{2!}\Big(\tfrac{1}{36}f_{xx} + \tfrac{1}{3}a_1 k_1 f_{xy} + {a_1}^2 {k_1}^2 f_{yy}\Big) + \\ \tfrac{h^3}{3!}\Big(\tfrac{1}{216}f_{xxx} + \tfrac{1}{12}a_1 k_1 f_{xxy} + \tfrac{1}{2}{a_1}^2 {k_1}^2 f_{xyy} + {a_1}^3 {k_1}^3 f_{yyy}\Big) + \ldots \end{aligned} \tag{36}$$

$$\begin{aligned} k_3 = f + h(\tfrac{3}{6}f_x + (a_2 k_1 + a_3 k_2) f_y) + \tfrac{h^2}{2!}(\tfrac{9}{36}f_{xx} + (a_2 k_1 + a_3 k_2) f_{xy} + (a_2 k_1 + a_3 k_2)^2 f_{yy}) + \\ \tfrac{h^3}{3!}(\tfrac{27}{216}f_{xxx} + \tfrac{27}{36}(a_2 k_1 + a_3 k_2 k_2) f_{xxy} + \\ \tfrac{3}{2}(a_2 k_1 + a_3 k_2)^2 f_{xyy} + (a_2 k_1 + a_3 k_2)^3 f_{yyy}) + \ldots. \end{aligned} \tag{37}$$

and

$$\begin{aligned} k_4 = {} & f + h(\tfrac{5}{6} f_x + (a_4 k_1 + a_5 k_2 + a_6 k_3) f_y) + \tfrac{h^2}{2!}(\tfrac{25}{36} f_{xx} + \tfrac{5}{3}(a_4 k_1 + a_5 k_2 + a_6 k_3) f_{xy} + \\ & (a_4 k_1 + a_5 k_2 + a_6 k_3)^2 f_{yy}) + \tfrac{h^3}{3!}(\tfrac{125}{216} f_{xxx} + \tfrac{75}{36}(a_4 k_1 + a_5 k_2 + a_6 k_3) f_{xxy} + \binom{2}{} f_{xyy} + \\ & (a_2 k_1 + a_3 k_2)^3 f_{yyy}) + \ldots \end{aligned} \tag{38}$$

By substituting Equations (35)–(38) into Equation (33) to obtain an expression for y_{n+1}, we obtain:

$$\begin{aligned} y_{n+1} = {} & y_n + hf(w_1 + w_2 + w_3) + \\ & h^2(f((a_4 + a_5 + a_6)w_3 + (a_2 + a_3)w_2 + w_1 a_1) f_y + \tfrac{5}{6}(w_3 + \tfrac{1}{5} w_1 + \tfrac{3}{5} w_2) f_x) + \\ & h^3(\tfrac{1}{2} f_{yy}((a_4 + a_5 + a_6)^2 w_3 + (a_2 + a_3)^2 w_2 + {a_1}^2 w_1) f^2 + \\ & \tfrac{1}{72}((72a_2 + 72a_3)a_6 + 72a_1 a_5){f_y}^2 + 60 f_{xy}(a_4 + a_5 + a_6)w_3 + \\ & (72{f_y}^2 a_1 a_3 + 36 f_{xy}(a_2 + a_3)w_2 + 12 f_{xy} a_1 w_1) f + \\ & \tfrac{1}{72}(12 f_x f_y (a_5 + 3a_6) + 25 f_{xx}) w_3 + \\ & \tfrac{1}{27}(12 a_3 f_x f_y + 9 f_{xx}) w_2 + \tfrac{1}{72} w_1 f_{xx}) + \ldots \end{aligned} \tag{39}$$

By setting $a_2 + a_3 = \frac{1}{2}$ and $a_4 + a_5 + a_6 = \frac{5}{6}$, and by comparing the coefficients of $h^r, r = 1, 2,$ and 3, in Equation (39) with their equivalents in Equation (12), we obtain:

$$hf : w_1 + w_2 + w_3 = 1 \tag{40}$$

$$h^2 f_x : 5w_3 + w_1 + 3w_2 = 3 \tag{41}$$

$$h^2 f f_y : w_3 a_4 + w_3 a_5 + w_3 a_6 + w_2 a_2 + w_2 a_3 + w_1 a_1 = \frac{1}{2} \tag{42}$$

$$h^3 f_{xx} : w_1 + 9w_2 + 25w_3 = 12 \tag{43}$$

$$h^3 f f_{xy} : 60(a_4 + a_5 + a_6)w_3 + 36w_2(a_2 + a_3) + 12w_1 a_1 = 24 \tag{44}$$

$$h^3 f^2 f_{yy} : (a_4 + a_5 + a_6)^2 w_3 + w_2(a_2 + a_3)^2 + {a_1}^2 w_1 = \frac{1}{3} \tag{45}$$

$$h^3 f_x f_y : 12(a_5 + 3a_6)w_3 + 12w_2 a_3 = 12 \tag{46}$$

$$h^3 f {f_y}^2 : ((72a_2 + 72a_3)a_6 + 72a_1 a_5)w_3 + 72a_1 a_5 w_2 = 12 \tag{47}$$

Solving these equations simultaneously, the nine parameters are given as follows:

$$w_1 = \frac{3}{8},\ w_2 = \frac{2}{8}\ ,\ w_3 = \frac{3}{8} \text{ and } a_1 = \frac{1}{6}\ ,\ a_2 = \frac{-7}{2}\ ,\ a_3 = 4,\ a_4 = \frac{5}{6},\ a_5 = 0, \text{ and } a_6 = 0$$

Therefore, from Equation (33), the proposed new method, referred to as HTM3, can be written as follows:

$$y_{n+1} = y_n + \frac{h}{8}(3k_2 + 2k_3 + 3k_4) \tag{48}$$

where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\Big(x_n + \tfrac{1}{6}h, y_n + \tfrac{1}{6}hk_1\Big), \\ k_3 &= f\Big(x_n + \tfrac{h}{2}, y_n + h\big(\tfrac{-7}{2}k_1 + 4k_2\big)\Big), \\ k_4 &= f\Big(x_n + \tfrac{5h}{6}, y_n + \tfrac{5h}{6}k_1\Big). \end{aligned}\right\} \tag{49}$$

3. Accuracy of New Methods

In this section, we analyze the local truncation error (L.T.E.) of the newly proposed methods. The local truncation error of numerical methods used to solve Equation (1) is defined as:

$$L.T.E. = y(x_{n+1}) - y_{n+1} \tag{50}$$

where $y(x_{n+1})$ is the exact solution and y_{n+1} is the approximate solution.

3.1. Accuracy of HTM2 Method

From Equation (32), and by using Taylor's series expansion of k_j, j = 1, 2, and 3, we have:

$$k_1 = f, \tag{51}$$

$$\begin{gathered} k_2 = f + h\left(\tfrac{1}{4}f_x + \tfrac{1}{4}ff_y\right) + \tfrac{h^2}{2}\left(\tfrac{1}{16}f_{xx} + \tfrac{1}{8}ff_{xy} + \tfrac{1}{16}f^2 f_{yy}\right) + \\ \tfrac{h^3}{6}\left(\tfrac{1}{64}f_{xxx} + \tfrac{3}{64}ff_{xxy} + \tfrac{3}{64}f^2 f_{xyy} + \tfrac{1}{64}f^3 f_{yyy}\right) + \ldots \end{gathered} \tag{52}$$

$$\begin{gathered} k_3 = f + h(\tfrac{3}{4}f_x + \tfrac{3}{4}ff_y) + \tfrac{h^2}{2}(\tfrac{9}{32}f_{xx} + \tfrac{1}{96}(32{f_y}^2 + 54f_{xy})f + \tfrac{1}{3}f_x f_y + \tfrac{9}{32}f^2 f_{yy}) + \\ h^3(\tfrac{9}{128}f_{xxx} + \tfrac{1}{384}(112f_y f_{yy} + 81f_{xyy})f^2 + \tfrac{1}{384}(96f_x f_{yy} + 128f_{xy}f_y + 81f_{xxy})f + \\ \tfrac{1}{4}f_{xy}f_x + \tfrac{1}{24}f_y f_{xx} + \tfrac{9}{128}f^3 f_{yyy}) + \ldots \end{gathered} \tag{53}$$

Substituting Equations (51)–(53) into Equation (31), we have:

$$y_{n+1} = y_n + hf + h^2\left(\frac{1}{2}f_x + \frac{1}{2}ff_y\right) + h^3\left(\frac{1}{6}f_y f_x + \frac{1}{6}f{f_y}^2 + \frac{5}{35}f_{xx} + \frac{5}{16}ff_{xy} + \frac{5}{32}f^2 f_{yy}\right) + \ldots \tag{54}$$

By subtracting Equation (54) from Equation (12), we have:

$$L.T.E. = h^3\left(\frac{1}{96}f_{xx} + \frac{1}{48}ff_{xy} + \frac{1}{96}f^2 f_{yy}\right) + o\left(h^4\right) \tag{55}$$

As per Equation (55), the form of the HTM2 method is of second-order, with a local truncation error of third-order.

3.2. Accuracy of HTM3 Method

From Equation (49), and by using Taylor's series expansion of k_j, j = 1, 2, 3, and 4, we obtain:

$$k_1 = f \tag{56}$$

$$\begin{gathered} k_2 = f + h\left(\tfrac{1}{6}f_x + \tfrac{1}{6}ff_y\right) + \tfrac{h^2}{2}\left(\tfrac{1}{36}f_{xx} + \tfrac{1}{18}ff_{xy} + \tfrac{1}{36}f^2 f_{yy}\right) + \\ \tfrac{h^3}{6}\left(\tfrac{1}{216}f_{xxx} + \tfrac{1}{72}ff_{xxy} + \tfrac{1}{72}f^2 f_{xyy} + \tfrac{1}{216}f^3 f_{yyy}\right) + \ldots \end{gathered} \tag{57}$$

$$\begin{gathered} k_3 = f + h(\tfrac{1}{2}f_x + \tfrac{1}{2}ff_y) + \tfrac{h^2}{2}(\tfrac{1}{8}f_{xx} + \tfrac{1}{24}(16{f_y}^2 + 9f_{xy})f + \tfrac{2}{3}f_x f_y + \tfrac{1}{8}f^2 f_{yy}) + \\ h^3(\tfrac{1}{48}f_{xxx} + \tfrac{1}{144}(56f_y f_{yy} + 9f_{xyy})f^2 + \tfrac{1}{144}(48f_x f_{yy} + 64f_{xy}f_y + 9f_{xxy})f + \\ \tfrac{1}{3}f_{xy}f_x + \tfrac{1}{18}f_y f_{xx} + \tfrac{1}{48}f^3 f_{yyy}) + \ldots \end{gathered} \tag{58}$$

$$\begin{gathered} k_4 = f + h(\tfrac{5}{6}f_x + \tfrac{5}{6}ff_y) + h^2(\tfrac{25}{72}f_{xx} + \tfrac{25}{36}ff_{xy} + \tfrac{25}{72}f^2 f_{yy}) + \\ h^3\left(\tfrac{125}{1296}f_{xxx} + \tfrac{125}{432}ff_{xxy} + \tfrac{125}{432}f^2 f_{xyy} + \tfrac{125}{1296}f^3 f_{yyy}\right) + \ldots \end{gathered} \tag{59}$$

When we substitute Equations (56)–(59) into Equation (48), we obtain:

$$\begin{gathered} y_{n+1} = y_n + hf + h^2\left(\tfrac{1}{2}f_x + \tfrac{1}{2}ff_y\right) + h^3\left(\tfrac{1}{6}f^2 f_{yy} + \tfrac{1}{6}\left({f_y}^2 + 2f_{xy}\right)f + \tfrac{1}{6}f_y f_x + \tfrac{1}{6}f_{xx}\right) + \\ h^4\left(\tfrac{1}{24}f^3 f_{yyy} + \tfrac{1}{72}(7f_y f_{yy} + 9f_{xyy})f^2 + \tfrac{1}{72}(6f_x f_{yy} + 8f_{xy}f_y + 9f_{xxy})f + \tfrac{1}{12}f_{xy}f_x + \tfrac{1}{72}f_y f_{xx} + \tfrac{1}{24}f_{xxx}\right) + \ldots \end{gathered} \tag{60}$$

The local truncation error defined by Equation (50) can be evaluated by subtracting Equation (60) from Equation (12), resulting in:

$$L.T.E. = h^4\left(\frac{1}{24}f{f_y}^3 + \frac{1}{24}f_x{f_y}^2 + \frac{1}{72}\left(5f^2 f_{yy} + 7ff_{xy} + 2f_{xx}\right)f_y + \frac{1}{24}f_x\left(ff_{yy} + f_{xy}\right)\right) + o(h^5) \tag{61}$$

Thus, our new method, HTM3, has a convergence rate of third-order, indicating that the local truncation error is of the order $o\left(h^4\right)$.

4. Stability Analysis

In this section, we investigate the stability region of the newly proposed methods using Dahlquist's test problem, see [17]:

$$y\prime = \lambda y_n \, , \;\; y(x_0) = y_0$$

where the solution is given by $y = e^{\lambda y_n}$, and λ is a complex variable.

4.1. Absolute Stability of HTM2 Method

To study the absolute stability of the proposed method, HTM2, we employ Equation (32) to become the following:

$$\left.\begin{aligned} k_1 &= \lambda y_n, \\ k_2 &= \lambda y_n\left(1 + \tfrac{h\lambda}{4}\right), \\ k_3 &= \lambda y_n\left(1 + \tfrac{3h\lambda}{4} + \tfrac{h^2\lambda^2}{3}\right). \end{aligned}\right\} \tag{62}$$

By substituting (62) in (31), we obtain:

$$y_{n+1} = y_n + h\lambda y_n\left[1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{6}\right] \tag{63}$$

By evaluating $\frac{y_{n+1}}{y_n}$ from (63) and setting $z = h\lambda$, one can obtain the stability polynomial of the proposed method as:

$$R(z) = \frac{y_{n+1}}{y_n} = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right) + o(z^4) \tag{64}$$

Using MATLAB software, Figure 1 below shows that the stability region of the HTM2 method is wider than that of the other methods that have the same order.

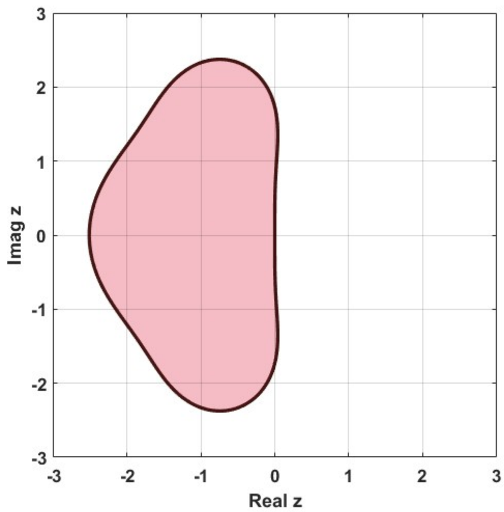

Figure 1. Absolute stability region of HTM2.

4.2. Absolute Stability of HTM3 Method

To study the absolute stability of the proposed method of HTM3, we employ Equation (49) to become the following:

$$\left.\begin{aligned} k_1 &= \lambda y_n, \\ k_2 &= \lambda y_n\left(1 + \tfrac{h\lambda}{6}\right), \\ k_3 &= \lambda y_n\left(1 - \tfrac{7h\lambda}{2} + 4h\lambda + \tfrac{2h^2\lambda^2}{3}\right), \\ k_4 &= \lambda y_n\left(1 + \tfrac{5h\lambda}{6}\right). \end{aligned}\right\} \tag{65}$$

By substituting Equation (65) in Equation (48) and setting $z = h\lambda$, we obtain:

$$y_{n+1} = y_n \left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} \right] \tag{66}$$

Thus, the stability polynomial of the proposed method becomes:

$$R(z) = \frac{y_{n+1}}{y_n} = \left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} \right] + o\left(z^4\right) \tag{67}$$

Utilizing the MATLAB software, the stability region of the above formula can be shown graphically in Figure 2 below:

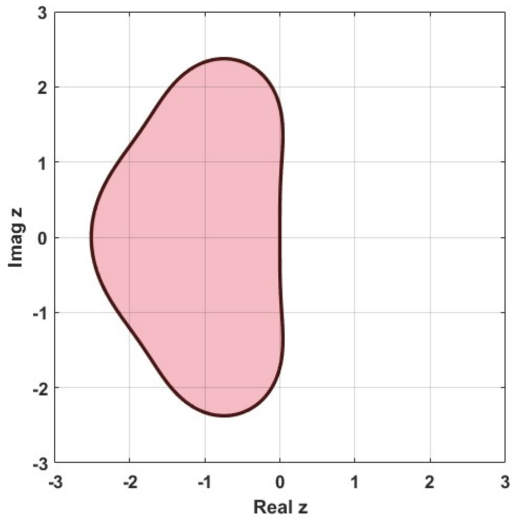

Figure 2. Absolute stability region of HTM3.

5. Numerical Experiments

In this section, we solve two initial value problems from the references [18,19] to demonstrate the accuracy and efficiency of the proposed methods in comparison to other relevant methods. We compare the proposed HTM2 method, which is a second-order method, with the RK2 method [15,20], Ralston's method [15,20,21], Heun's method [1,15], and Midpoint method [1,15,22]. In addition, we compare the proposed HTM3 method, which is a third-order method, with Ralston's method [15,20], RK3 method [15,21], and Heun's method [15,23], using different step sizes (h).

Problem 1. *Consider the IVP* $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right)$, $y(0) = 1$, *with the exact solution*

$$y = \frac{20}{1 + 19e^{\frac{-x}{4}}},\ 0 \le x \le 1.$$

To provide a comprehensive evaluation of the proposed methods, HTM2 and HTM3, Tables 1–3 and Tables 4–6 for different step sizes provide a numerical comparison of the exact solutions and approximate solutions of the new methods with other related methods of the same order of convergence. A numerical comparison of the absolute errors of our new methods with other relevant methods of the same order of convergence is presented in Tables 7–9 and Tables 10–12 using different step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$, respectively. The graphical representation of these results in the Figures 3–8 serves to supplement the numerical results and provide additional insight into the performance of the proposed methods.

Table 1. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 1 (h = 0.1).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	1.024018962351867	1.024016952473958	1.024016923095703	1.024016834960938	1.024017011230469	1.024018966423752
0.2	1.048582996382734	1.048578896534070	1.048578835761557	1.048578653444039	1.048579018079106	1.048583005159566
0.3	1.073702928838884	1.073696657191696	1.073696562908939	1.073696280060730	1.073696845757242	1.073702942998442
0.4	1.099389726731484	1.099381199729522	1.099381069716590	1.099380679677929	1.099381459755449	1.099389746998454
0.5	1.125654495329782	1.125643627697525	1.125643459626485	1.125642955413594	1.125643963839717	1.125654522478015
0.6	1.152508475906471	1.152495180661248	1.152494972091990	1.152494346384571	1.152495597799943	1.152508510761309
0.7	1.179963043224405	1.179947231691585	1.179946980067549	1.179946225195960	1.179947734939917	1.179963086665087
0.8	1.208029702753715	1.208011284585090	1.208010987228892	1.208010095161021	1.208011879297848	1.208029755715844
0.9	1.236720087608148	1.236698970803669	1.236698624912622	1.236697587240450	1.236699662586248	1.236720151086232
1.0	1.266045955189318	1.266022046122342	1.266021648763880	1.266020456689755	1.266022840839900	1.266046030239382

Table 2. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 1 (h = 0.05).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	1.024018962351867	1.024018455637443	1.024018448167912	1.024018425759318	1.024018470576506	1.024018964748139
0.2	1.048582996382734	1.048581962772757	1.048581947321444	1.048581900967507	1.048581993675384	1.048583001384183
0.3	1.073702928838884	1.073701347715753	1.073701323745033	1.073701251832879	1.073701395657195	1.073702936666439
0.4	1.099389726731483	1.099387577042795	1.099387543988666	1.099387444826288	1.099387643151058	1.099389737618628
0.5	1.125654495329782	1.125651755590691	1.125651712861745	1.125651584674923	1.125651841048592	1.125654509523097
0.6	1.152508475906471	1.152505124202791	1.152505071179159	1.152504912108291	1.152505230250067	1.152508493666189
0.7	1.179963043224405	1.179959057216482	1.179958993248755	1.179958801345611	1.179959185151955	1.179963064824991
0.8	1.208029702753716	1.208025059681107	1.208024984089226	1.208024757313632	1.208025210864895	1.208029728484451
0.9	1.236720087608148	1.236714764295145	1.236714676367269	1.236714412583710	1.236714940150929	1.236720117773737
1.0	1.266045955189318	1.266039928051356	1.266039827042708	1.266039524016851	1.266040130068695	1.266045990110506

Table 3. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 1 (h = 0.025).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	1.024018962351867	1.024018835138843	1.024018833255648	1.024018827606067	1.024018838905230	1.024018963125439
0.2	1.048582996382734	1.048582736891561	1.048582732996068	1.048582721309593	1.048582744682544	1.048582997988505
0.3	1.073702928838884	1.073702531894975	1.073702525851689	1.073702507721835	1.073702543981544	1.073702931338594
0.4	1.099389726731484	1.099389187051215	1.099389178717997	1.099389153718346	1.099389203717649	1.099389730190115
0.5	1.125654495329782	1.125653807521091	1.125653796748901	1.125653764432330	1.125653829065472	1.125654499815693
0.6	1.152508475906471	1.152507634469638	1.152507621102265	1.152507581000150	1.152507661204383	1.152508481491529
0.7	1.179963043224405	1.179962042553033	1.179962026426831	1.179961978048225	1.179962074805438	1.179963049984132
0.8	1.208029702753716	1.208028537135910	1.208028518079512	1.208028460910319	1.208028575248708	1.208029710767429
0.9	1.236720087608148	1.236718751227908	1.236718729061944	1.236718662564059	1.236718795559836	1.236720096959107
1.0	1.266045955189318	1.266044442128150	1.266044416664960	1.266044340275396	1.266044493054533	1.266045965964875

Table 4. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 1 (h = 0.1).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	1.024018962351867	1.024018951072459	1.024018949714177	1.024018951961812	1.024018953256430
0.2	1.048582996382734	1.048582973405348	1.048582970599488	1.048582975242551	1.048582977917067
0.3	1.073702928838884	1.073702893737675	1.073702889390857	1.073702896583903	1.073702900727513
0.4	1.099389726731484	1.099389679073476	1.099389673088104	1.099389682992669	1.099389688698611
0.5	1.125654495329782	1.125654434675028	1.125654426949154	1.125654439733982	1.125654447099669
0.6	1.152508475906471	1.152508401808229	1.152508392235414	1.152508408076681	1.152508417203833
0.7	1.179963043224405	1.179962955229337	1.179962943698518	1.179962962780061	1.179962973774839
0.8	1.208029702753715	1.208029600402102	1.208029586797452	1.208029609310995	1.208029622284133
0.9	1.236720087608148	1.236719970434119	1.236719954634907	1.236719980780295	1.236719995847236
1.0	1.266045955189318	1.266045822721111	1.266045804601567	1.266045834586991	1.266045851868019

Table 5. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 1 (h = 0.05).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	1.024018962351867	1.024018960929323	1.024018960759068	1.024018961041800	1.024018961208012
0.2	1.048582996382734	1.048582993484895	1.048582993133194	1.048582993717242	1.048582994060601
0.3	1.073702928838884	1.073702924412083	1.073702923867236	1.073702924772033	1.073702925303975
0.4	1.099389726731483	1.099389720721158	1.099389719970937	1.099389721216793	1.099389721949271
0.5	1.125654495329782	1.125654487680494	1.125654486712123	1.125654488320255	1.125654489265756
0.6	1.152508475906471	1.152508466561927	1.152508465362070	1.152508467354628	1.152508468526188
0.7	1.179963043224405	1.179963032127484	1.179963030682225	1.179963033082322	1.179963034493546
0.8	1.208029702753716	1.208029689846496	1.208029688141323	1.208029690973061	1.208029692638139
0.9	1.236720087608148	1.236720072831934	1.236720070851721	1.236720074140224	1.236720076073946
1.0	1.266045955189318	1.266045938484675	1.266045936213665	1.266045939985103	1.266045942202880

Table 6. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 1 (h = 0.025).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	1.024018962351867	1.024018962173255	1.024018962151945	1.024018962187397	1.024018962208452
0.2	1.048582996382734	1.048582996018889	1.048582995974869	1.048582996048102	1.048582996091597
0.3	1.073702928838884	1.073702928283070	1.073702928214875	1.073702928328326	1.073702928395708
0.4	1.099389726731484	1.099389725976853	1.099389725882953	1.099389726039167	1.099389726131950
0.5	1.125654495329782	1.125654494369377	1.125654494248174	1.125654494449812	1.125654494569575
0.6	1.152508475906471	1.152508474733226	1.152508474583051	1.152508474832888	1.152508474981283
0.7	1.179963043224405	1.179963041831152	1.179963041650263	1.179963041951197	1.179963042129945
0.8	1.208029702753716	1.208029701133186	1.208029700919767	1.208029701274820	1.208029701485716
0.9	1.236720087608148	1.236720085752975	1.236720085505133	1.236720085917455	1.236720086162370
1.0	1.266045955189318	1.266045953092043	1.266045952807807	1.266045953280678	1.266045953561562

Table 7. Absolute errors for problem 1 using HTM2 and other methods for $h = 0.1$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	2.0099×10^{-6}	2.0393×10^{-6}	2.1274×10^{-6}	1.9511×10^{-6}	4.0719×10^{-9}
0.2	4.0998×10^{-6}	4.1606×10^{-6}	4.3429×10^{-6}	3.9783×10^{-6}	8.7768×10^{-9}
0.3	6.2716×10^{-6}	6.3659×10^{-6}	6.6488×10^{-6}	6.0831×10^{-6}	1.4160×10^{-8}
0.4	8.5270×10^{-6}	8.6570×10^{-6}	9.0471×10^{-6}	8.2670×10^{-6}	2.0267×10^{-8}
0.5	1.0868×10^{-5}	1.1036×10^{-5}	1.1540×10^{-5}	1.0531×10^{-5}	2.7148×10^{-8}
0.6	1.3295×10^{-5}	1.3504×10^{-5}	1.4130×10^{-5}	1.2878×10^{-5}	3.4855×10^{-8}
0.7	1.5812×10^{-5}	1.6063×10^{-5}	1.6818×10^{-5}	1.5308×10^{-5}	4.3441×10^{-8}
0.8	1.8418×10^{-5}	1.8716×10^{-5}	1.9608×10^{-5}	1.7823×10^{-5}	5.2962×10^{-8}
0.9	2.1117×10^{-5}	2.1463×10^{-5}	2.2500×10^{-5}	2.0425×10^{-5}	6.3478×10^{-8}
1.0	2.3909×10^{-5}	2.4306×10^{-5}	2.5498×10^{-5}	2.3114×10^{-5}	7.5050×10^{-8}

Table 8. Absolute errors for problem 1 using HTM2 and other methods for $h = 0.05$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	5.0671×10^{-7}	5.1418×10^{-7}	5.3659×10^{-7}	4.9178×10^{-7}	2.3963×10^{-9}
0.2	1.0336×10^{-6}	1.0491×10^{-6}	1.0954×10^{-6}	1.0027×10^{-6}	5.0014×10^{-9}
0.3	1.5811×10^{-6}	1.6051×10^{-6}	1.6770×10^{-6}	1.5332×10^{-6}	7.8276×10^{-9}
0.4	2.1497×10^{-6}	2.1827×10^{-6}	2.2819×10^{-6}	2.0836×10^{-6}	1.0887×10^{-8}
0.5	2.7397×10^{-6}	2.7825×10^{-6}	2.9107×10^{-6}	2.6543×10^{-6}	1.4193×10^{-8}
0.6	3.3517×10^{-6}	3.4047×10^{-6}	3.5638×10^{-6}	3.2457×10^{-6}	1.7760×10^{-8}
0.7	3.9860×10^{-6}	4.0500×10^{-6}	4.2419×10^{-6}	3.8581×10^{-6}	2.1601×10^{-8}
0.8	4.6431×10^{-6}	4.7187×10^{-6}	4.9454×10^{-6}	4.4919×10^{-6}	2.5731×10^{-8}
0.9	5.3233×10^{-6}	5.4112×10^{-6}	5.6750×10^{-6}	5.1475×10^{-6}	3.0166×10^{-8}
1.0	6.0271×10^{-6}	6.1281×10^{-6}	6.4312×10^{-6}	5.8251×10^{-6}	3.4921×10^{-8}

Table 9. Absolute errors for problem 1 using HTM2 and other methods for $h = 0.025$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	1.2721×10^{-7}	1.2910×10^{-7}	1.3475×10^{-7}	1.2345×10^{-7}	7.7357×10^{-10}
0.2	2.5949×10^{-7}	2.6339×10^{-7}	2.7507×10^{-7}	2.5170×10^{-7}	1.6058×10^{-9}
0.3	3.9694×10^{-7}	4.0299×10^{-7}	4.2112×10^{-7}	3.8486×10^{-7}	2.4997×10^{-9}
0.4	5.3968×10^{-7}	5.4801×10^{-7}	5.7301×10^{-7}	5.2301×10^{-7}	3.4586×10^{-9}
0.5	6.8781×10^{-7}	6.9858×10^{-7}	7.3090×10^{-7}	6.6626×10^{-7}	4.4859×10^{-9}
0.6	8.4144×10^{-7}	8.5480×10^{-7}	8.9491×10^{-7}	8.1470×10^{-7}	5.5851×10^{-9}
0.7	1.0007×10^{-6}	1.0168×10^{-6}	1.0652×10^{-6}	9.6842×10^{-7}	6.7597×10^{-9}
0.8	1.1656×10^{-6}	1.1847×10^{-6}	1.2418×10^{-6}	1.1275×10^{-6}	8.0137×10^{-9}
0.9	1.3364×10^{-6}	1.3585×10^{-6}	1.4250×10^{-6}	1.2920×10^{-6}	9.3510×10^{-9}
1.0	1.5131×10^{-6}	1.5385×10^{-6}	1.6149×10^{-6}	1.4621×10^{-6}	1.0776×10^{-8}

Table 10. Absolute errors for problem 1 using HTM3 and other methods for $h = 0.1$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	1.1279×10^{-8}	1.2638×10^{-8}	1.0390×10^{-8}	9.0954×10^{-9}
0.2	2.2977×10^{-8}	2.5783×10^{-8}	2.1140×10^{-8}	1.8466×10^{-8}
0.3	3.5101×10^{-8}	3.9448×10^{-8}	3.2255×10^{-8}	2.8111×10^{-8}
0.4	4.7658×10^{-8}	5.3643×10^{-8}	4.3739×10^{-8}	3.8033×10^{-8}
0.5	6.0655×10^{-8}	6.838×10^{-8}	5.5596×10^{-8}	4.8230×10^{-8}
0.6	7.4098×10^{-8}	8.3671×10^{-8}	6.7830×10^{-8}	5.8703×10^{-8}
0.7	8.7995×10^{-8}	9.9526×10^{-8}	8.0444×10^{-8}	6.9450×10^{-8}
0.8	1.0235×10^{-7}	1.1596×10^{-7}	9.3443×10^{-8}	8.0470×10^{-8}
0.9	1.1717×10^{-7}	1.3297×10^{-7}	1.0683×10^{-7}	9.1761×10^{-8}
1.0	1.3247×10^{-7}	1.5059×10^{-7}	1.2060×10^{-7}	1.0332×10^{-7}

Table 11. Absolute errors for problem 1 using HTM3 and other methods for $h = 0.05$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	1.4225×10^{-9}	1.5928×10^{-9}	1.3101×10^{-9}	1.1439×10^{-9}
0.2	2.8978×10^{-9}	3.2495×10^{-9}	2.6655×10^{-9}	2.3221×10^{-9}
0.3	4.4268×10^{-9}	4.9716×10^{-9}	4.0669×10^{-9}	3.5349×10^{-9}
0.4	6.0103×10^{-9}	6.7605×10^{-9}	5.5147×10^{-9}	4.7822×10^{-9}
0.5	7.6493×10^{-9}	8.6177×10^{-9}	7.0000×10^{-9}	6.0640×10^{-9}
0.6	9.3445×10^{-9}	1.0544×10^{-9}	8.5518×10^{-9}	7.3803×10^{-9}
0.7	1.1097×10^{-8}	1.2542×10^{-8}	1.0142×10^{-8}	8.7309×10^{-9}
0.8	1.2907×10^{-8}	1.4612×10^{-8}	1.1781×10^{-8}	1.0116×10^{-8}
0.9	1.4776×10^{-8}	1.6756×10^{-8}	1.3468×10^{-8}	1.1534×10^{-8}
1.0	1.6705×10^{-8}	1.8976×10^{-8}	1.5204×10^{-8}	1.2986×10^{-8}

Table 12. Absolute errors for problem 1 using HTM3 and other methods for $h = 0.025$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	1.7861×10^{-10}	1.9992×10^{-10}	1.6447×10^{-10}	1.4342×10^{-10}
0.2	3.6385×10^{-10}	4.0786×10^{-10}	3.3463×10^{-10}	2.9114×10^{-10}
0.3	5.5581×10^{-10}	6.2401×10^{-10}	5.1056×10^{-10}	4.4318×10^{-10}
0.4	7.5463×10^{-10}	8.4853×10^{-10}	6.9232×10^{-10}	5.9953×10^{-10}
0.5	9.6041×10^{-10}	1.0816×10^{-9}	8.7997×10^{-10}	7.6021×10^{-10}
0.6	1.1732×10^{-9}	1.3234×10^{-9}	1.0736×10^{-9}	9.2519×10^{-10}
0.7	1.3933×10^{-9}	1.5741×10^{-9}	1.2732×10^{-9}	1.0945×10^{-9}
0.8	1.6205×10^{-9}	1.8339×10^{-9}	1.4789×10^{-9}	1.2680×10^{-9}
0.9	1.8552×10^{-9}	2.1030×10^{-9}	1.6907×10^{-9}	1.4458×10^{-9}
1.0	2.0973×10^{-9}	2.3815×10^{-9}	1.9086×10^{-9}	1.6278×10^{-9}

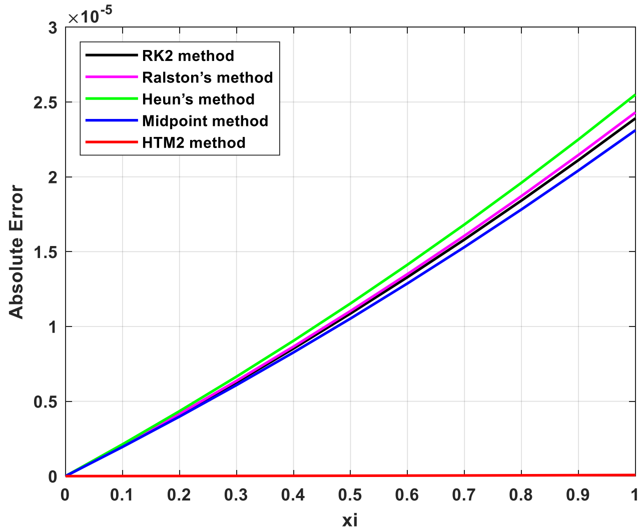

Figure 3. Comparison of HTM2 method with relevant methods for problem 1 at $h = 0.1$.

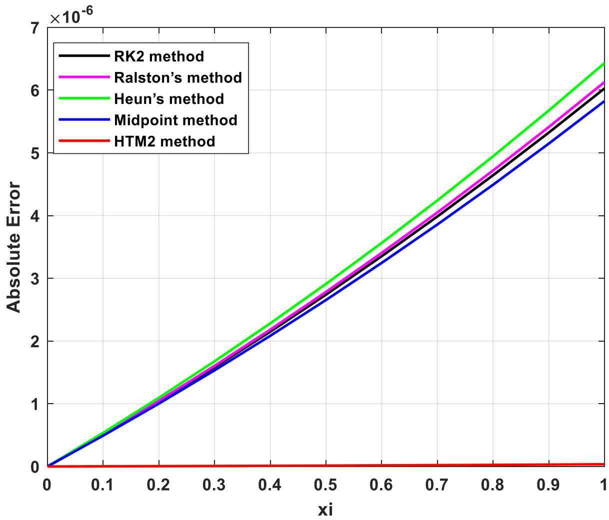

Figure 4. Comparison of HTM2 method with relevant methods for problem 1 at $h = 0.05$.

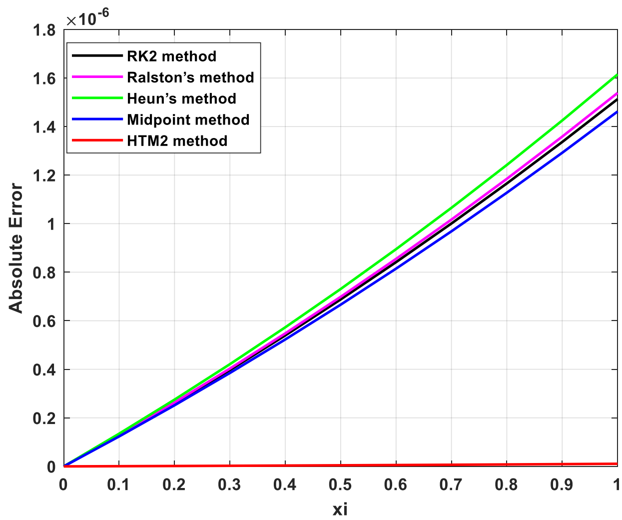

Figure 5. Comparison of HTM2 method with relevant methods for problem 1 at $h = 0.025$.

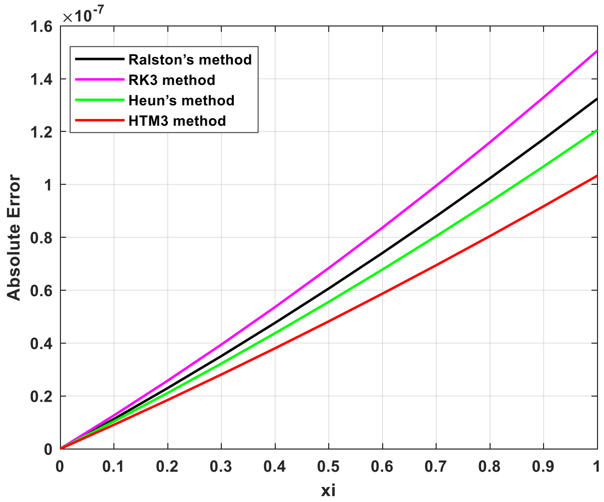

Figure 6. Comparison of HTM3 method with relevant methods for problem 1 at $h = 0.1$.

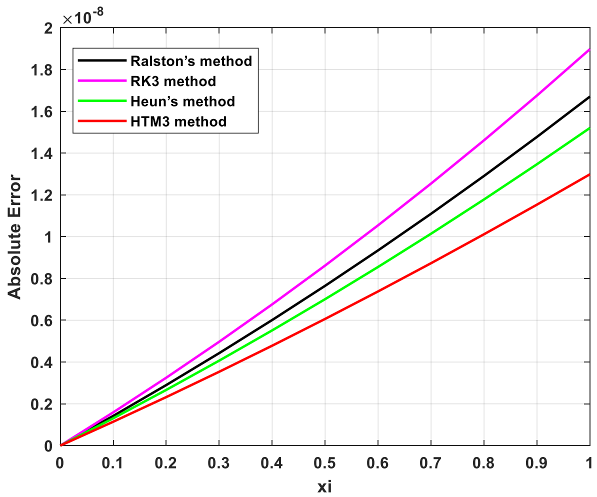

Figure 7. Comparison of HTM3 method with relevant methods for problem 1 at $h = 0.05$.

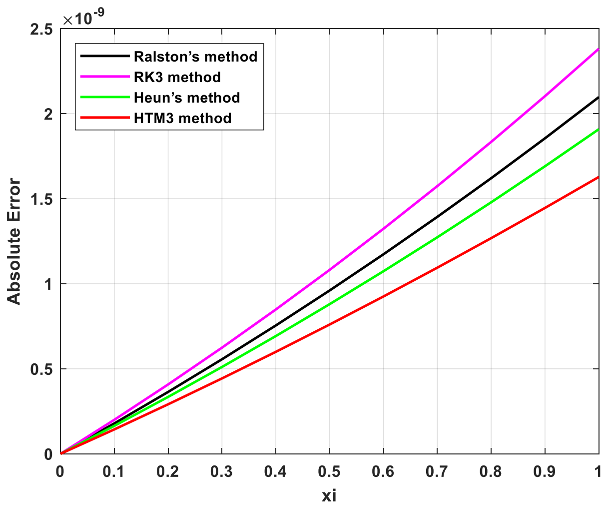

Figure 8. Comparison of HTM3 method with relevant methods for problem 1 at $h = 0.025$.

Problem 2. *Consider the IVP* $y' = 80 - \frac{45y}{(2000-5x)}$, $y(0) = 100$, *with the exact solution*

$$y = 2(2000 - 5x) - \frac{3900}{(2000)^9}(2000 - 5x)^9,\ 0 \le x \le 1$$

Tables 13–18 provide a numerical comparison of the exact solutions and approximate solutions of the proposed methods, HTM2 and HTM3, with other related methods of the same order of convergence using different step sizes $h = 0.1$, $h = 0.05$ and $h = 0.025$, respectively. In addition, the numerical comparison of the absolute errors of the proposed methods, HTM2 and HTM3, and other relevant methods of the same order of convergence is presented in Tables 19–24. Furthermore, Figures 9–14 show the graphical representations of these results, which provide additional support to the numerical results.

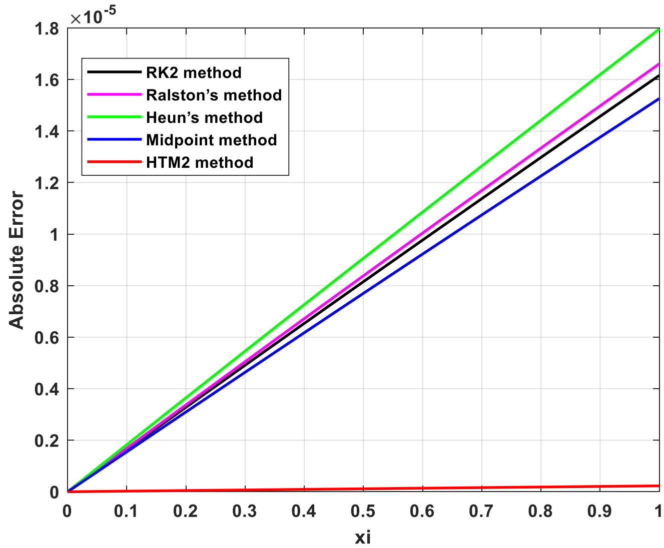

Figure 9. Comparison of HTM2 method with relevant methods for problem 2 at $h = 0.1$.

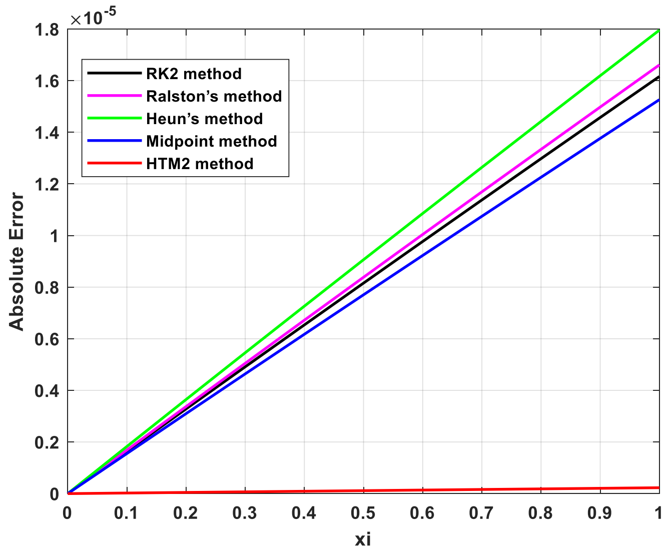

Figure 10. Comparison of HTM2 method with relevant methods for problem 2 at $h = 0.05$.

Table 13. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 2 (h = 0.1).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	$1.077662301168311\times10^2$	$1.077662235372562\times10^2$	$1.077662233543790\times10^2$	$1.077662228057014\times10^2$	$1.077662239029879\times10^2$	$1.077662302115619\times10^2$
0.2	$1.155149409193027\times10^2$	$1.155149277848099\times10^2$	$1.155149274197407\times10^2$	$1.155149263244417\times10^2$	$1.155149285149027\times10^2$	$1.155149411084106\times10^2$
0.3	$1.232461630508842\times10^2$	$1.232461433860695\times10^2$	$1.232461428394920\times10^2$	$1.232461411996225\times10^2$	$1.232461444791564\times10^2$	$1.232461633340158\times10^2$
0.4	$1.309599271090915\times10^2$	$1.309599009384900\times10^2$	$1.309599002110860\times10^2$	$1.309598980286920\times10^2$	$1.309599023932070\times10^2$	$1.309599274858934\times10^2$
0.5	$1.386562636455415\times10^2$	$1.386562309936298\times10^2$	$1.386562300860797\times10^2$	$1.386562273632022\times10^2$	$1.386562328086166\times10^2$	$1.386562641156632\times10^2$
0.6	$1.463352031660152\times10^2$	$1.463351640572091\times10^2$	$1.463351629701914\times10^2$	$1.463351597088663\times10^2$	$1.463351662311085\times10^2$	$1.463352037291060\times10^2$
0.7	$1.539967761305115\times10^2$	$1.539967305891663\times10^2$	$1.539967293233580\times10^2$	$1.539967255256163\times10^2$	$1.539967331206245\times10^2$	$1.539967767862210\times10^2$
0.8	$1.616410129533037\times10^2$	$1.616409610037156\times10^2$	$1.616409595597920\times10^2$	$1.616409552276596\times10^2$	$1.616409638913823\times10^2$	$1.616410137012834\times10^2$
0.9	$1.692679440029992\times10^2$	$1.692678856694046\times10^2$	$1.692678840480392\times10^2$	$1.692678791835372\times10^2$	$1.692678889119325\times10^2$	$1.692679448429009\times10^2$
1.0	$1.768775996025961\times10^2$	$1.768775349091707\times10^2$	$1.768775331110355\times10^2$	$1.768775277161798\times10^2$	$1.768775385052160\times10^2$	$1.768776005340717\times10^2$

Table 14. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 2 (h = 0.05).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	$1.077662301168311\times10^2$	$1.077662284732696\times10^2$	$1.077662284276013\times10^2$	$1.077662282905904\times10^2$	$1.077662285646036\times10^2$	$1.077662301400763\times10^2$
0.2	$1.155149409193027\times10^2$	$1.155149376383397\times10^2$	$1.155149375471741\times10^2$	$1.155149372736659\times10^2$	$1.155149378206653\times10^2$	$1.155149409657066\times10^2$
0.3	$1.232461630508842\times10^2$	$1.232461581386640\times10^2$	$1.232461580021718\times10^2$	$1.232461575926784\times10^2$	$1.232461584116397\times10^2$	$1.232461631203599\times10^2$
0.4	$1.309599271090915\times10^2$	$1.309599205717424\times10^2$	$1.309599203900941\times10^2$	$1.309599198451264\times10^2$	$1.309599209350277\times10^2$	$1.309599272015514\times10^2$
0.5	$1.386562636455415\times10^2$	$1.386562554891789\times10^2$	$1.386562552625442\times10^2$	$1.386562545826120\times10^2$	$1.386562559424340\times10^2$	$1.386562637609003\times10^2$
0.6	$1.463352031660152\times10^2$	$1.463351933967384\times10^2$	$1.463351931252869\times10^2$	$1.463351923108986\times10^2$	$1.463351939396244\times10^2$	$1.463352033041868\times10^2$
0.7	$1.539967761305115\times10^2$	$1.539967647544044\times10^2$	$1.539967644383051\times10^2$	$1.539967634899678\times10^2$	$1.539967653865831\times10^2$	$1.539967762914094\times10^2$
0.8	$1.616410129533037\times10^2$	$1.616409999764360\times10^2$	$1.616409996158576\times10^2$	$1.616409985340773\times10^2$	$1.616410006975703\times10^2$	$1.616410131368426\times10^2$
0.9	$1.692679440029992\times10^2$	$1.692679294314255\times10^2$	$1.692679290265361\times10^2$	$1.692679278118176\times10^2$	$1.692679302411788\times10^2$	$1.692679442090935\times10^2$
1.0	$1.768775996025961\times10^2$	$1.768775834423550\times10^2$	$1.768775829933226\times10^2$	$1.768775816461693\times10^2$	$1.768775843403918\times10^2$	$1.768775998311597\times10^2$

Table 15. Comparison of Analytical and Approximate Solutions for HTM2 and Relevant Methods in Problem 2 (h = 0.025).

x_i	Exact Solution	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0	0
0.1	$1.077662301168311 \times 10^2$	$1.077662297061071 \times 10^2$	$1.077662296946963 \times 10^2$	$1.077662296604634 \times 10^2$	$1.768772168318756 \times 10^2$	$1.077662301225877 \times 10^2$
0.2	$1.155149409193027 \times 10^2$	$1.155149400993944 \times 10^2$	$1.155149400766157 \times 10^2$	$1.155149400082781 \times 10^2$	$2.520298931317081 \times 10^2$	$1.155149409307948 \times 10^2$
0.3	$1.232461630508842 \times 10^2$	$1.232461618233270 \times 10^2$	$1.232461617892230 \times 10^2$	$1.232461616869088 \times 10^2$	$3.254880571751630 \times 10^2$	$1.232461630680901 \times 10^2$
0.4	$1.309599271090915 \times 10^2$	$1.309599254754162 \times 10^2$	$1.309599254300294 \times 10^2$	$1.309599252938663 \times 10^2$	$3.972812879500018 \times 10^2$	$1.309599271319889 \times 10^2$
0.5	$1.386562636455415 \times 10^2$	$1.386562616072772 \times 10^2$	$1.386562615506501 \times 10^2$	$1.386562613807653 \times 10^2$	$4.674387207720501 \times 10^2$	$1.386562636741100 \times 10^2$
0.6	$1.463352031660152 \times 10^2$	$1.463352007246861 \times 10^2$	$1.463352006568610 \times 10^2$	$1.463352004533817 \times 10^2$	$5.359890528448672 \times 10^2$	$1.463352032002334 \times 10^2$
0.7	$1.539967761305115 \times 10^2$	$1.539967732876376 \times 10^2$	$1.539967732086569 \times 10^2$	$1.539967729717097 \times 10^2$	$6.029605487635397 \times 10^2$	$1.539967761703577 \times 10^2$
0.8	$1.616410129533037 \times 10^2$	$1.616410097104022 \times 10^2$	$1.616410096203079 \times 10^2$	$1.616410093500192 \times 10^2$	$6.683810459630230 \times 10^2$	$1.616410129987570 \times 10^2$
0.9	$1.692679440029992 \times 10^2$	$1.692679403615832 \times 10^2$	$1.692679402604174 \times 10^2$	$1.692679399569134 \times 10^2$	$7.322779601114496 \times 10^2$	$1.692679440540384 \times 10^2$
1.0	$1.768775996025961 \times 10^2$	$1.768775955641741 \times 10^2$	$1.768775954519786 \times 10^2$	$1.768775951153851 \times 10^2$	$7.946782904488222 \times 10^2$	$1.768775996591993 \times 10^2$

Table 16. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 2 (h = 0.1).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	$1.077662301168311 \times 10^2$	$1.077662301205557 \times 10^2$	$1.077662301209444 \times 10^2$	$1.077662301201518 \times 10^2$	$1.077662301195728 \times 10^2$
0.2	$1.155149409193027 \times 10^2$	$1.155149409267394 \times 10^2$	$1.155149409275154 \times 10^2$	$1.155149409259330 \times 10^2$	$1.155149409247769 \times 10^2$
0.3	$1.232461630508842 \times 10^2$	$1.232461630620198 \times 10^2$	$1.232461630631818 \times 10^2$	$1.232461630608124 \times 10^2$	$1.232461630590812 \times 10^2$
0.4	$1.309599271090915 \times 10^2$	$1.309599271239121 \times 10^2$	$1.309599271254587 \times 10^2$	$1.309599271223051 \times 10^2$	$1.309599271200008 \times 10^2$
0.5	$1.386562636455415 \times 10^2$	$1.386562636640352 \times 10^2$	$1.386562636659651 \times 10^2$	$1.386562636620299 \times 10^2$	$1.386562636591547 \times 10^2$
0.6	$1.463352031660152 \times 10^2$	$1.463352031881691 \times 10^2$	$1.463352031904810 \times 10^2$	$1.463352031857670 \times 10^2$	$1.463352031823227 \times 10^2$
0.7	$1.539967761305115 \times 10^2$	$1.539967761563123 \times 10^2$	$1.539967761590047 \times 10^2$	$1.539967761535147 \times 10^2$	$1.539967761495035 \times 10^2$
0.8	$1.616410129533037 \times 10^2$	$1.616410129827389 \times 10^2$	$1.616410129858106 \times 10^2$	$1.616410129795472 \times 10^2$	$1.616410129749710 \times 10^2$
0.9	$1.692679440029992 \times 10^2$	$1.692679440360560 \times 10^2$	$1.692679440395056 \times 10^2$	$1.692679440324717 \times 10^2$	$1.692679440273324 \times 10^2$
1.0	$1.768775996025961 \times 10^2$	$1.768775996392609 \times 10^2$	$1.768775996430871 \times 10^2$	$1.768775996352853 \times 10^2$	$1.768775996295850 \times 10^2$

Table 17. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 2 (h = 0.05).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	$1.077662301168311 \times 10^2$	$1.077662301172961 \times 10^2$	$1.077662301173447 \times 10^2$	$1.077662301172457 \times 10^2$	$1.077662301171734 \times 10^2$
0.2	$1.155149409193027 \times 10^2$	$1.155149409202316 \times 10^2$	$1.155149409203285 \times 10^2$	$1.155149409201309 \times 10^2$	$1.155149409199866 \times 10^2$
0.3	$1.232461630508842 \times 10^2$	$1.232461630522752 \times 10^2$	$1.232461630524203 \times 10^2$	$1.232461630521245 \times 10^2$	$1.232461630519083 \times 10^2$
0.4	$1.309599271090915 \times 10^2$	$1.309599271109421 \times 10^2$	$1.309599271111352 \times 10^2$	$1.309599271107414 \times 10^2$	$1.309599271104537 \times 10^2$
0.5	$1.386562636455415 \times 10^2$	$1.386562636478511 \times 10^2$	$1.386562636480920 \times 10^2$	$1.386562636476007 \times 10^2$	$1.386562636472416 \times 10^2$
0.6	$1.463352031660152 \times 10^2$	$1.463352031687821 \times 10^2$	$1.463352031690708 \times 10^2$	$1.463352031684822 \times 10^2$	$1.463352031680521 \times 10^2$
0.7	$1.539967761305115 \times 10^2$	$1.539967761337337 \times 10^2$	$1.539967761340699 \times 10^2$	$1.539967761333844 \times 10^2$	$1.539967761328835 \times 10^2$
0.8	$1.616410129533037 \times 10^2$	$1.616410129569800 \times 10^2$	$1.616410129573635 \times 10^2$	$1.616410129565815 \times 10^2$	$1.616410129560100 \times 10^2$
0.9	$1.692679440029992 \times 10^2$	$1.692679440071280 \times 10^2$	$1.692679440075588 \times 10^2$	$1.692679440066805 \times 10^2$	$1.692679440060387 \times 10^2$
1.0	$1.768775996025961 \times 10^2$	$1.768775996071750 \times 10^2$	$1.768775996076527 \times 10^2$	$1.768775996066786 \times 10^2$	$1.768775996059668 \times 10^2$

Table 18. Comparison of Analytical and Approximate Solutions for HTM3 and Relevant Methods in Problem 2 (h = 0.025).

x_i	Exact Solution	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0	0
0.1	$1.077662301168311 \times 10^2$	$1.077662301168891 \times 10^2$	$1.077662301168951 \times 10^2$	$1.077662301168828 \times 10^2$	$1.077662301168737 \times 10^2$
0.2	$1.155149409193027 \times 10^2$	$1.155149409194189 \times 10^2$	$1.155149409194310 \times 10^2$	$1.155149409194063 \times 10^2$	$1.155149409193883 \times 10^2$
0.3	$1.232461630508842 \times 10^2$	$1.232461630510583 \times 10^2$	$1.232461630510764 \times 10^2$	$1.232461630510395 \times 10^2$	$1.232461630510124 \times 10^2$
0.4	$1.309599271090915 \times 10^2$	$1.309599271093224 \times 10^2$	$1.309599271093465 \times 10^2$	$1.309599271092973 \times 10^2$	$1.309599271092613 \times 10^2$
0.5	$1.386562636455415 \times 10^2$	$1.386562636458300 \times 10^2$	$1.386562636458600 \times 10^2$	$1.386562636457987 \times 10^2$	$1.386562636457538 \times 10^2$
0.6	$1.463352031660152 \times 10^2$	$1.463352031663611 \times 10^2$	$1.463352031663971 \times 10^2$	$1.463352031663236 \times 10^2$	$1.463352031662698 \times 10^2$
0.7	$1.539967761305115 \times 10^2$	$1.539967761309141 \times 10^2$	$1.539967761309561 \times 10^2$	$1.539967761308705 \times 10^2$	$1.539967761308079 \times 10^2$
0.8	$1.616410129533037 \times 10^2$	$1.616410129537632 \times 10^2$	$1.616410129538111 \times 10^2$	$1.616410129537134 \times 10^2$	$1.616410129536420 \times 10^2$
0.9	$1.692679440029992 \times 10^2$	$1.692679440035155 \times 10^2$	$1.692679440035693 \times 10^2$	$1.692679440034596 \times 10^2$	$1.692679440033793 \times 10^2$
1.0	$1.768775996025961 \times 10^2$	$1.768775996031681 \times 10^2$	$1.768775996032277 \times 10^2$	$1.768775996031060 \times 10^2$	$1.768775996030170 \times 10^2$

Table 19. Absolute errors for problem 2 using HTM2 and other methods for $h = 0.1$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	6.5796×10^{-6}	6.7625×10^{-6}	7.3111×10^{-6}	6.2138×10^{-6}	9.4731×10^{-8}
0.2	1.3134×10^{-5}	1.3500×10^{-5}	1.4595×10^{-5}	1.2404×10^{-5}	1.8911×10^{-7}
0.3	1.9665×10^{-5}	2.0211×10^{-5}	2.1851×10^{-5}	1.8572×10^{-5}	2.8313×10^{-7}
0.4	2.6171×10^{-5}	2.6898×10^{-5}	2.9080×10^{-5}	2.4716×10^{-5}	3.7680×10^{-7}
0.5	3.2652×10^{-5}	3.3559×10^{-5}	3.6282×10^{-5}	3.0837×10^{-5}	4.7012×10^{-7}
0.6	3.9109×10^{-5}	4.0196×10^{-5}	4.3457×10^{-5}	3.6935×10^{-5}	5.6309×10^{-7}
0.7	4.5541×10^{-5}	4.6807×10^{-5}	5.0605×10^{-5}	4.3010×10^{-5}	6.5571×10^{-7}
0.8	5.1950×10^{-5}	5.3394×10^{-5}	5.7726×10^{-5}	4.9062×10^{-5}	7.4798×10^{-7}
0.9	5.8334×10^{-5}	5.9955×10^{-5}	6.4819×10^{-5}	5.5091×10^{-5}	8.3990×10^{-7}
1.0	6.4693×10^{-5}	6.6492×10^{-5}	7.1886×10^{-5}	6.1097×10^{-5}	9.3148×10^{-7}

Table 20. Absolute errors for problem 2 using HTM2 and other methods for $h = 0.05$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	1.6436×10^{-6}	1.6892×10^{-6}	1.8262×10^{-6}	1.5522×10^{-6}	2.3245×10^{-8}
0.2	3.2810×10^{-6}	3.3721×10^{-6}	3.6456×10^{-6}	3.0986×10^{-6}	4.6404×10^{-8}
0.3	4.9122×10^{-6}	5.0487×10^{-6}	5.4582×10^{-6}	4.6392×10^{-6}	6.9476×10^{-8}
0.4	6.5373×10^{-6}	6.7190×10^{-6}	7.2640×10^{-6}	6.1741×10^{-6}	9.2460×10^{-8}
0.5	8.1564×10^{-6}	8.3830×10^{-6}	9.0629×10^{-6}	7.7031×10^{-6}	1.1536×10^{-7}
0.6	9.7693×10^{-6}	1.0041×10^{-5}	1.0855×10^{-5}	9.2264×10^{-6}	1.3817×10^{-7}
0.7	1.1376×10^{-5}	1.1692×10^{-5}	1.2641×10^{-5}	1.0744×10^{-5}	1.6090×10^{-7}
0.8	1.2977×10^{-5}	1.3337×10^{-5}	1.4419×10^{-5}	1.2256×10^{-5}	1.8354×10^{-7}
0.9	1.4572×10^{-5}	1.4976×10^{-5}	1.6191×10^{-5}	1.3762×10^{-5}	2.0609×10^{-7}
1.0	1.6160×10^{-5}	1.6609×10^{-5}	1.7956×10^{-5}	1.5262×10^{-5}	2.2856×10^{-7}

Table 21. Absolute errors for problem 2 using HTM2 and other methods for $h = 0.025$.

x_i	RK2 Method	Ralston's Method	Heun's Method	Midpoint Method	HTM2 Method
0	0	0	0	0	0
0.1	4.1072×10^{-7}	4.2213×10^{-7}	4.5637×10^{-7}	3.8790×10^{-7}	5.7565×10^{-9}
0.2	8.1991×10^{-7}	8.4269×10^{-7}	9.1102×10^{-7}	7.7435×10^{-7}	1.1492×10^{-8}
0.3	1.2276×10^{-6}	1.2617×10^{-6}	1.3640×10^{-6}	1.1594×10^{-6}	1.7206×10^{-8}
0.4	1.6337×10^{-6}	1.6791×10^{-6}	1.8152×10^{-6}	1.5429×10^{-6}	2.2897×10^{-8}
0.5	2.0383×10^{-6}	2.0949×10^{-6}	2.2648×10^{-6}	1.9250×10^{-6}	2.8569×10^{-8}
0.6	2.4413×10^{-6}	2.5092×10^{-6}	2.7126×10^{-6}	2.3057×10^{-6}	3.4218×10^{-8}
0.7	2.8429×10^{-6}	2.9219×10^{-6}	3.1588×10^{-6}	2.6849×10^{-6}	3.9846×10^{-8}
0.8	3.2429×10^{-6}	3.3330×10^{-6}	3.6033×10^{-6}	3.0627×10^{-6}	4.5453×10^{-8}
0.9	3.6414×10^{-6}	3.7426×10^{-6}	4.0461×10^{-6}	3.4391×10^{-6}	5.1039×10^{-8}
1.0	4.0384×10^{-6}	4.1506×10^{-6}	4.4872×10^{-6}	3.8140×10^{-6}	5.6603×10^{-8}

Table 22. Absolute errors for problem 2 using HTM3 and other methods for $h = 0.1$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	3.7246×10^{-9}	4.1133×10^{-9}	3.3207×10^{-9}	2.7416×10^{-9}
0.2	7.4366×10^{-9}	8.2127×10^{-9}	6.6303×10^{-9}	5.4741×10^{-9}
0.3	1.1136×10^{-8}	1.2298×10^{-8}	9.9282×10^{-9}	8.1970×10^{-9}
0.4	1.4821×10^{-8}	1.6367×10^{-8}	1.3214×10^{-8}	1.0909×10^{-8}
0.5	1.8494×10^{-8}	2.0424×10^{-8}	1.6488×10^{-8}	1.3613×10^{-8}
0.6	2.2154×10^{-8}	2.4466×10^{-8}	1.9752×10^{-8}	1.6308×10^{-8}
0.7	2.5801×10^{-8}	2.8493×10^{-8}	2.3003×10^{-8}	1.8992×10^{-8}
0.8	2.9435×10^{-8}	3.2507×10^{-8}	2.6243×10^{-8}	2.1667×10^{-8}
0.9	3.3057×10^{-8}	3.6506×10^{-8}	2.9472×10^{-8}	2.4333×10^{-8}
1.0	3.6665×10^{-8}	4.0491×10^{-8}	3.2689×10^{-8}	2.6989×10^{-8}

Table 23. Absolute errors for problem 2 using HTM3 and other methods for $h = 0.05$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	4.6501×10^{-10}	5.1354×10^{-10}	4.1457×10^{-10}	3.4225×10^{-10}
0.2	9.2889×10^{-10}	1.0258×10^{-9}	8.2821×10^{-10}	6.8384×10^{-10}
0.3	1.3910×10^{-9}	1.5361×10^{-9}	1.2403×10^{-9}	1.0241×10^{-9}
0.4	1.8506×10^{-9}	2.0437×10^{-9}	1.6499×10^{-9}	1.3622×10^{-9}
0.5	2.3096×10^{-9}	2.5506×10^{-9}	2.0592×10^{-9}	1.7002×10^{-9}
0.6	2.7670×10^{-9}	3.0556×10^{-9}	2.4671×10^{-9}	2.0370×10^{-9}
0.7	3.2222×10^{-9}	3.5584×10^{-9}	2.8729×10^{-9}	2.3720×10^{-9}
0.8	3.6763×10^{-9}	4.0598×10^{-9}	3.2778×10^{-9}	2.7063×10^{-9}
0.9	4.1288×10^{-9}	4.5595×10^{-9}	3.6813×10^{-9}	3.0395×10^{-9}
1.0	4.5789×10^{-9}	5.0566×10^{-9}	4.0826×10^{-9}	3.3707×10^{-9}

Table 24. Absolute errors for problem 2 using HTM3 and other methods for $h = 0.025$.

x_i	Ralston's Method	RK3 Method	Heun's Method	HTM3 Method
0	0	0	0	0
0.1	5.7938×10^{-11}	6.4006×10^{-11}	5.1628×10^{-11}	4.2590×10^{-11}
0.2	1.1617×10^{-10}	1.2828×10^{-10}	1.0360×10^{-10}	8.5564×10^{-11}
0.3	1.7408×10^{-10}	1.9220×10^{-10}	1.5525×10^{-10}	1.2824×10^{-10}
0.4	2.3087×10^{-10}	2.5497×10^{-10}	2.0580×10^{-10}	1.6982×10^{-10}
0.5	2.8851×10^{-10}	3.1858×10^{-10}	2.5724×10^{-10}	2.1231×10^{-10}
0.6	3.4589×10^{-10}	3.8193×10^{-10}	3.0846×10^{-10}	2.5466×10^{-10}
0.7	4.0254×10^{-10}	4.4454×10^{-10}	3.5894×10^{-10}	2.9632×10^{-10}
0.8	4.5941×10^{-10}	5.0736×10^{-10}	4.0967×10^{-10}	3.3825×10^{-10}
0.9	5.1622×10^{-10}	5.7003×10^{-10}	4.6035×10^{-10}	3.8011×10^{-10}
1.0	5.7196×10^{-10}	6.3162×10^{-10}	5.0994×10^{-10}	4.2095×10^{-10}

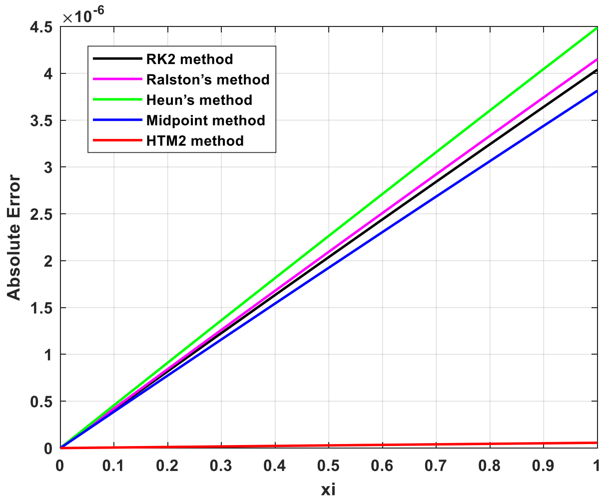

Figure 11. Comparison of HTM2 method with relevant methods for problem 2 at $h = 0.025$.

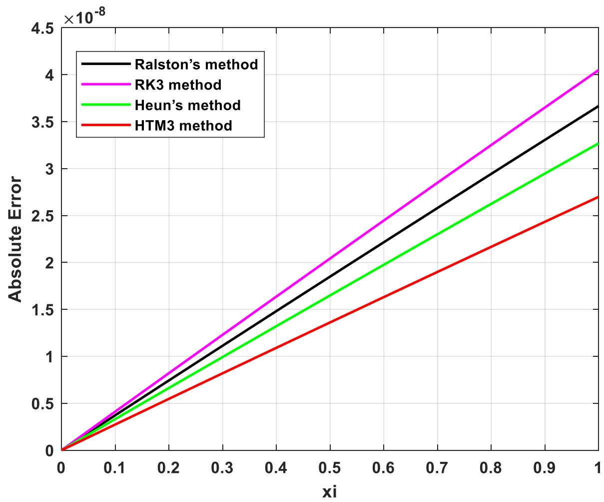

Figure 12. Comparison of HTM3 method with relevant methods for problem 2 at $h = 0.1$.

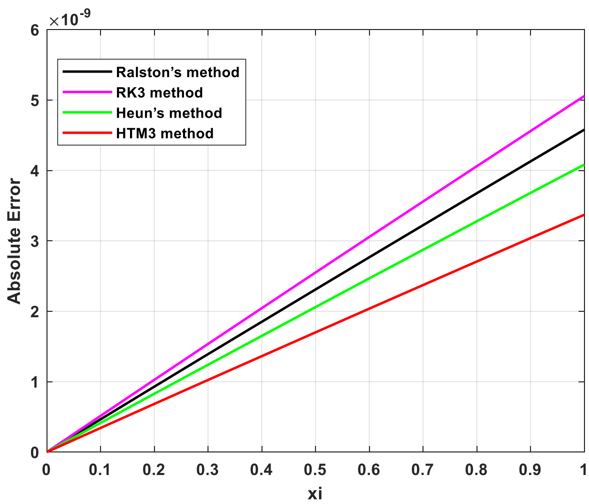

Figure 13. Comparison of HTM3 method with relevant methods for problem 2 at $h = 0.05$.

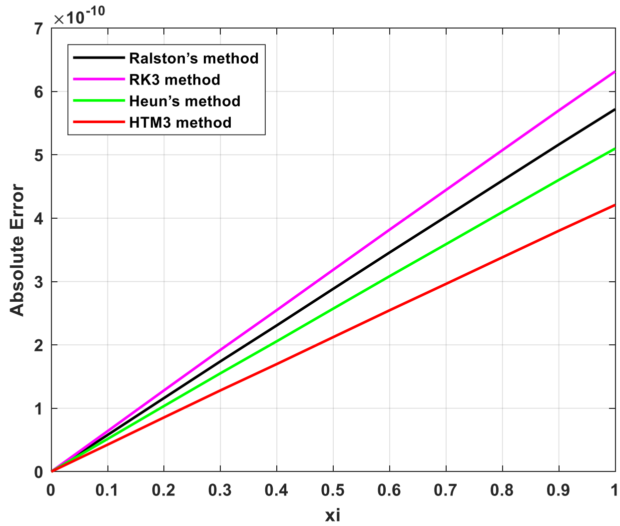

Figure 14. Comparison of HTM3 method with relevant methods for problem 2 at $h = 0.025$.

6. Discussion and Conclusions

In this section, we discuss and conclude the two new methods for solving first-order ordinary differential equations presented in the previous section: the second-order HTM2 method and the third-order HTM3 method. Both methods were based on Newton's theorem in calculus, Taylor's series expansion, and the quadrature integration formula using hybrid functions. To demonstrate the competency of these new methods, we used two examples. It is worth mentioning that the numerical results, tables, and figures were obtained using the software MATLAB (R2022a) on a specific computer machine with the following specifications: Windows 11 Pro has an 11th Gen Intel(R) Core (TM) i7-11800H @ 2.30 GHz processor and 16.0 GB RAM storage (15.7 GB usable).

In the comparison of numerical results obtained from solving the two initial value problems, Problem 1 and Problem 2, it was observed that the HTM2 method outperformed its corresponding methods in terms of proximity to the analytical solution for various step sizes ($h = 0.1$, $h = 0.05$, and $h = 0.025$). This is evident from the comparison of the approximate and analytical solutions presented in Tables 1–3 and Tables 7–9 for Problem 1. Similarly, the HTM3 method was found to be more efficient than its counterparts in terms of closeness to the exact solution, as demonstrated in Tables 13–15 and Tables 19–21 for Problem 2, for the same step sizes. These results indicate that the proposed methods, HTM2 and HTM3, are highly competent in solving the initial value problems.

The results obtained from solving the two initial value problems, Problem 1 and Problem 2, as shown in Tables 4–6 and Tables 10–12, respectively, demonstrate the advantage of the HTM2 method over other relevant methods in terms of lower absolute error for different step sizes ($h = 0.1$, $h = 0.05$, and $h = 0.025$). This is further validated by the plots in Figures 3–5 and Figures 6–8, which show a clear preference of the HTM2 method over other relevant methods. Additionally, the numerical results in Tables 16–18 and Tables 22–24, and the plots in Figures 9–11 and Figures 12–14, demonstrate the superiority of the HTM3 method over Ralston's method, RK3 method, and Heun's method. It is also observed that as the step size decreases, the error approaches zero, indicating that reducing the step size leads to greater accuracy. The stability region of the new second-order HTM2 method, as shown in Figure 1, is wider than the stability regions of other relevant methods, while the stability region of the new third-order HTM3 method, as indicated in Figure 2, is the same as that of other corresponding methods.

We conclude the new methods presented in the paper are more efficient and accurate for solving IVPs in ordinary differential equations compared to other known and relevant methods. In addition, these methods will provide a new computational tool for solving IVPs in ordinary differential equations and can be applied in various fields of science and engineering. Further research can be conducted to improve these methods and apply them to more complex problems.

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